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Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems

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ABSTRACT

We consider initial value/boundary value problems for fractional diffusion-wave equation: $\partial_t^\alpha u(x,t) = Lu(x,t)$, where $0 < \alpha \leqslant 2$, where L is a symmetric uniformly elliptic operator with t-independent smooth coefficients. First we establish the unique existence of the weak solution and the asymptotic behavior as the time t goes to ∞ and the proofs are based on the eigenfunction expansions. Second for $\alpha \in (0,1)$, we apply the eigenfunction expansions and prove (i) stability in the backward problem in time, (ii) the uniqueness in determining an initial value and (iii) the uniqueness of solution by the decay rate as $t \to \infty$, (iv) stability in an inverse source problem of determining t-dependent factor in the source by observation at one point over $(0,T)$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$. We consider a partial differential equation with the fractional derivative in time t:

$$\partial_t^\alpha u(x,t) = (Lu)(x,t) + F(x,t), \quad x \in \Omega,\ t \in (0,T),\ 0 < \alpha \leqslant 2. \tag{1.1}$$

Here ∂_t^α denotes the Caputo fractional derivative with respect to t and is defined by

$$\partial_t^\alpha g(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} g(\tau)\,d\tau, & n-1 < \alpha < n,\ n \in \mathbb{N}, \\ \frac{d^n}{dt^n} g(t), & \alpha = n \in \mathbb{N}, \end{cases}$$

Γ is the Gamma function and the operator L is a symmetric uniformly elliptic operator and F is a given function in $\Omega \times (0,T)$ and $T > 0$ is a fixed value. Note that if $\alpha = 1$ and $\alpha = 2$, then Eq. (1.1) represents a parabolic equation and a hyperbolic equation respectively. Since we are interested mainly in the fractional cases, we restrict the order α to the two cases $0 < \alpha < 1$ and $1 < \alpha < 2$.

We will solve Eq. (1.1) satisfying the following initial/boundary value conditions:

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t \in (0,T), \tag{1.2}$$

$$u(x,0) = a(x), \quad x \in \Omega, \tag{1.3}$$

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and

$$\partial_t u(x,0) = b(x), \quad x \in \Omega, \text{ if } 1 < \alpha < 2. \tag{1.4}$$

In the case of $0 < \alpha < 1$, Eq. (1.1) is called a fractional diffusion equation, while the equation is called a fractional diffusion-wave equation or a fractional wave equation in the case $1 < \alpha < 2$. The fractional diffusion equation has been introduced in physics by Nigmatullin [34] to describe diffusions in media with fractal geometry. Adams and Gelhar [1] pointed out that field data show anomalous diffusion in a highly heterogeneous aquifer. Hatano and Hatano [15] applied the continuous-time random walk for better simulations for the anomalous diffusion in an underground environmental problem. One can regard (1.1) as a macroscopic model derived from the continuous-time random walk. Metzler and Klafter [30] demonstrated that a fractional diffusion equation describes a non-Markovian diffusion process with a memory. Roman and Alemany [41] investigated continuous-time random walks on fractals and showed that the average probability density of random walks on fractals obeys a diffusion equation with a fractional time derivative asymptotically. Ginoa, Cerbelli and Roman [13] presented a fractional diffusion equation describing relaxation phenomena in complex viscoelastic materials. Mainardi [27] pointed out that the fractional wave equation governs the propagation of mechanical diffusive waves in viscoelastic media.

Here we refer to several works on the mathematical treatments for Eq. (1.1). Kochubei [19,20] applied the semigroup theory in Banach spaces, and Eidelman and Kochubei [9] constructed the fundamental solution in $\mathbb{R}^d$ and proved the maximum principle for the Cauchy problem. Schneider and Wyss [46] used the Mellin transform and Fox H-functions for an integrodifferential equation which is equivalent to the fractional diffusion equation (1.1). However, these mathematical treatments are made in unbounded domain. Mainardi [26,28] solved a fractional diffusion-wave equation using the Laplace transform in a one-dimensional bounded domain. See also Mainardi [25]. Gejji and Jafari [11] solved a nonhomogeneous fractional diffusion-wave equation in a one-dimensional bounded domain. Fujita [10] discussed an integrodifferential equation which interpolates the heat equation and the wave equation in an unbounded domain. Agarwal [3] solved a fractional diffusion equation using a finite sine transform technique and presented numerical results in a one-dimensional bounded domain. As for an inverse problem of determining a coefficient and the order α in the case where the spatial dimension is one, see Cheng, Nakagawa, Yamamoto and Yamazaki [6].

As source books related with fractional derivatives, see Samko, Kilbas and Marichev [44] which is an encyclopedic treatment of the fractional calculus and also Gorenflo and Mainardi [14], Kilbas, Srivastava and Trujillo [18], Mainardi [29], Miller and Ross [31], Oldham and Spanier [35], Podlubny [37].

In spite of the importance, to the authors' best knowledge, there are not many works published concerning the unique existence of the solution to (1.1)–(1.4) and the properties which are remarkably different from the standard diffusion and wave equations. In Prüss [40] (especially in Chapter I.3), one can refer to the methods for (1.1). In particular, Theorem 2.4 (p. 62) in [40] gives the regularity of solution for Hölder continuous F in t and see also Theorem 3.3 (pp. 77–78) in [40]. Also see [7].

In Luchko [22], the maximum principle for an initial value/boundary value problem is established. In Luchko [23] and [24], the author constructed solutions by the eigenfunction expansion in the case of $F = 0$ and $0 < \alpha \leqslant 1$ and discussed the unique existence of the generalized solution to (1.1)–(1.3).

For discussions on inverse problems and qualitative properties of solutions to (1.1)–(1.4), representation formulae of solutions by the eigenfunctions, are very convenient, and we need the regularity property of solutions given by the eigenfunctions. See [6] for example as a paper where the eigenfunction expansions of solutions to (1.1)–(1.3) are used for the study of an inverse problem. To the authors' best knowledge, except for [23] and [24], there are no works published concerning the regularity properties of the eigenfunction expansions of the solutions and the regularity should correspond to the results in Chapter 3 of Lions and Magenes [21] and Pazy [36] for example. The first purpose of this paper is to prove the well-posedness and the regularity of the solution given by the eigenfunction expansions. Second we establish several uniqueness results for related inverse problems.

The remainder of this paper is composed of three sections. In Section 2, we state the main results on the eigenfunction expansions of solutions to (1.1)–(1.4) and properties such as a priori estimates, asymptotic behavior, which mean the well-posedness of (1.1)–(1.4). In Section 3, we prove them by means of the eigenfunction expansion, and in Section 4, we apply the results in Section 2 to inverse problems.

2. Well-posedness of the initial value/boundary value problems

Let $L^2(\Omega)$ be a usual L^2-space with the scalar product $(\cdot,\cdot)$, and $H^\ell(\Omega)$, $H_0^m(\Omega)$ denote Sobolev spaces (e.g., Adams [2], Gilbarg and Trudinger [12]). In what follows, let L be given by

$$\mathcal{L}u(x) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{d} A_{ij}(x)\frac{\partial}{\partial x_j}u(x)\right) + C(x)u(x), \quad x \in \Omega,$$

where $A_{ij} = A_{ji}$, $1 \leqslant i, j \leqslant d$. Moreover, we assume that the operator $\mathcal{L}$ is uniformly elliptic on $\overline{\Omega}$ and that its coefficients are smooth: there exists a constant $\nu > 0$ such that

$$\nu \sum_{i=1}^{d} \xi_i^2 \leqslant \sum_{i,j=1}^{d} A_{ij}(x)\xi_i\xi_j, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^d,$$

and the coefficients satisfy

$$A_{ij} \in C^1(\overline{\Omega}), \quad C \in C(\overline{\Omega}), \quad C(x) \leqslant 0, \quad x \in \overline{\Omega}.$$

We define an operator L in $L^2(\Omega)$ by

$$(Lu)(x) = (\mathcal{L}u)(x), \quad x \in \Omega, \qquad \mathcal{D}(-L) = H^2(\Omega) \cap H_0^1(\Omega).$$

Then the fractional power $(-L)^\gamma$ is defined for $\gamma \in \mathbb{R}$ (e.g., [36]) and $\mathcal{D}((-L)^{\frac{1}{2}}) = H_0^1(\Omega)$ for example. Henceforth we set $\|u\|_{\mathcal{D}((-L)^\gamma)} = \|(-L)^\gamma u\|_{L^2(\Omega)}$. We note that the norm $\|u\|_{\mathcal{D}((-L)^\gamma)}$ is stronger than $\|u\|_{L^2(\Omega)}$ for $\gamma > 0$.

Since $-L$ is a symmetric uniformly elliptic operator, the spectrum of $-L$ is entirely composed of eigenvalues and counting according to the multiplicities, we can set: $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$. By $\varphi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ we denote the orthonormal eigenfunction corresponding to $-\lambda_n$: $L\varphi_n = -\lambda_n\varphi_n$. Then the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is orthonormal basis in $L^2(\Omega)$. Then we see that

$$\mathcal{D}\big((-L)^\gamma\big) = \left\{\psi \in L^2(\Omega);\ \sum_{n=1}^{\infty} \lambda_n^{2\gamma} \big|(\psi, \varphi_n)\big|^2 < \infty\right\}$$

and that $\mathcal{D}((-L)^\gamma)$ is a Hilbert space with the norm:

$$\|\psi\|_{\mathcal{D}((-L)^\gamma)} = \left\{\sum_{n=1}^{\infty} \lambda_n^{2\gamma} \big|(\psi, \varphi_n)\big|^2\right\}^{\frac{1}{2}}.$$

We have $\mathcal{D}((-L)^\gamma) \subset H^{2\gamma}(\Omega)$ for $\gamma > 0$. In particular, $\mathcal{D}((-L)^{\frac{1}{2}}) = H_0^1(\Omega)$. Since $\mathcal{D}((-L)^\gamma) \subset L^2(\Omega)$, identifying the dual $(L^2(\Omega))'$ with itself, we have $\mathcal{D}((-L)^\gamma) \subset L^2(\Omega) \subset (\mathcal{D}((-L)^\gamma))'$. Henceforth we set $\mathcal{D}((-L)^{-\gamma}) = (\mathcal{D}((-L)^\gamma))'$, which consists of bounded linear functionals on $\mathcal{D}((-L)^\gamma)$. For $f \in \mathcal{D}((-L)^{-\gamma})$ and $\psi \in \mathcal{D}((-L)^\gamma)$, by ${}_{-\gamma}\langle f, \psi\rangle_\gamma$, we denote the value which is obtained by operating f to ψ. We note that $\mathcal{D}((-L)^{-\gamma})$ is a Hilbert space with the norm:

$$\|\psi\|_{\mathcal{D}((-L)^{-\gamma})} = \left\{\sum_{n=1}^{\infty} \lambda_n^{-2\gamma} \big|{}_{-\gamma}\langle f, \varphi_n\rangle_\gamma\big|^2\right\}^{\frac{1}{2}}.$$

We further note that

$${}_{-\gamma}\langle f, \psi\rangle_\gamma = (f, \psi) \quad \text{if } f \in L^2(\Omega) \text{ and } \psi \in \mathcal{D}\big((-L)^\gamma\big)$$

(e.g., Chapter V in Brezis [4]).

Henceforth C_j denote positive constants which are independent of F in (1.1), a, b in (1.3) and (1.4), but may depend on α and the coefficients of the operator L. The numbering in C_j can be independent in the succeeding different sections.

Moreover we define the Mittag–Leffler function by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants. By the power series, we can directly verify that $E_{\alpha,\beta}(z)$ is an entire function of $z \in \mathbb{C}$.

Definition 2.1. We call u a weak solution to (1.1)–(1.3) if (1.1) holds in $L^2(\Omega)$ and $u(\cdot,t) \in H_0^1(\Omega)$ for almost all $t \in (0,T)$ and $u \in C([0,T]; \mathcal{D}((-L)^{-\gamma}))$,

$$\lim_{t\to 0}\big\|u(\cdot,t) - a\big\|_{\mathcal{D}((-L)^{-\gamma})} = 0$$

with some $\gamma > 0$. Moreover we call u a weak solution to (1.1)–(1.4) if (1.1) holds in $L^2(\Omega)$ and $u(\cdot,t) \in H_0^1(\Omega)$ for almost all $t \in (0,T)$ and $u, \partial_t u \in C([0,T]; \mathcal{D}((-L)^{-\gamma}))$,

$$\lim_{t\to 0}\big\|u(\cdot,t) - a\big\|_{\mathcal{D}((-L)^{-\gamma})} = \lim_{t\to 0}\big\|\partial_t u(\cdot,t) - b\big\|_{\mathcal{D}((-L)^{-\gamma})} = 0$$

with some $\gamma > 0$. Here $\gamma > 0$ may depend on a, b.

We are ready to state our main theorems on the unique existence of solution to (1.1)–(1.4).

Theorem 2.1. *Let $0 < \alpha < 1$ and let $F = 0$.*

(i) *Let $a \in L^2(\Omega)$. Then there exists a unique weak solution $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ to (1.1)–(1.3) such that $\partial_t^\alpha u \in C((0,T]; L^2(\Omega))$. Moreover there exists a constant $C_1 > 0$ such that*

$$\begin{cases} \|u\|_{C([0,T];L^2(\Omega))} \leqslant C_1 \|a\|_{L^2(\Omega)}, \\ \|u(\cdot,t)\|_{H^2(\Omega)} + \|\partial_t^\alpha u(\cdot,t)\|_{L^2(\Omega)} \leqslant C_1 t^{-\alpha} \|a\|_{L^2(\Omega)}, \end{cases} \tag{2.1}$$

and we have

$$u(x,t) = \sum_{n=1}^{\infty} (a, \varphi_n) E_{\alpha,1}\left(-\lambda_n t^\alpha\right) \varphi_n(x) \tag{2.2}$$

in $C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$. Moreover $u : (0,T] \longrightarrow L^2(\Omega)$ is analytically extended to a sector $\{z \in \mathbb{C};\ z \neq 0,\ |\arg z| < \frac{1}{2}\pi\}$.

(ii) *We assume that $a \in H_0^1(\Omega)$. Then the unique weak solution u further belongs to $L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$, $\partial_t^\alpha u \in L^2(\Omega \times (0,T))$ and there exists a constant $C_2 > 0$ satisfying the following inequality:*

$$\|u\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t^\alpha u\|_{L^2(\Omega \times (0,T))} \leqslant C_2 \|a\|_{H^1(\Omega)} \tag{2.3}$$

and we have (2.2) in the corresponding space on the right-hand side of (2.3).

(iii) *We assume that $a \in H^2(\Omega) \cap H_0^1(\Omega)$. Then the unique weak solution u belongs to $C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$, $\partial_t^\alpha u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H_0^1(\Omega))$ and the following inequality holds:*

$$\|u\|_{C([0,T];H^2(\Omega))} + \|\partial_t^\alpha u\|_{C([0,T];L^2(\Omega))} \leqslant C_3 \|a\|_{H^2(\Omega)} \tag{2.4}$$

and we have (2.2) in the corresponding space on the right-hand side of (2.4).

Theorem 2.2.

(i) *Let $0 < \alpha < 1$ and let $a = 0$. Let $F \in L^\infty(0,T; L^2(\Omega))$. Then there exists a unique weak solution $u \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$ to (1.1)–(1.3) such that $\partial_t^\alpha u \in L^2(\Omega \times (0,T))$. In particular, for any γ satisfying $\gamma > \frac{d}{4} - 1$, we have $u \in C([0,T]; \mathcal{D}((-L)^{-\gamma}))$,*

$$\lim_{t \to 0} \|u(\cdot,t) - a\|_{\mathcal{D}((-L)^{-\gamma})} = 0,$$

and if $d = 1, 2, 3$, then

$$\lim_{t \to 0} \|u(\cdot,t) - a\|_{L^2(\Omega)} = 0.$$

Moreover there exists a constant $C_4 > 0$ such that

$$\|u\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t^\alpha u\|_{L^2(\Omega \times (0,T))} \leqslant C_4 \|F\|_{L^2(\Omega \times (0,T))} \tag{2.5}$$

and we have

$$u(x,t) = \sum_{n=1}^{\infty} \left(\int_0^t \left(F(\cdot,\tau), \varphi_n\right)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n (t-\tau)^\alpha\right) d\tau \right) \varphi_n(x), \tag{2.6}$$

in the corresponding space on the right-hand side of (2.5).

(ii) *Let $1 < \alpha < 2$ and let $a = b = 0$. Let $F \in L^2(0,T; \mathcal{D}((-L)^{\frac{1}{\alpha}})) \cap L^\infty(0,T; L^2(\Omega))$. Let $\gamma > \frac{d}{4} + 1$. Then there exists a unique weak solution $u \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ to (1.1)–(1.4) such that $\partial_t^\alpha u \in C([0,T]; L^2(\Omega))$. In particular,*

$$\lim_{t \to 0} \|u(\cdot,t)\|_{H^2(\Omega)} = \lim_{t \to 0} \|\partial_t u(\cdot,t)\|_{\mathcal{D}((-L)^{-\gamma})} = 0.$$

Moreover there exists a constant $C_4 > 0$ such that

$$\|u\|_{C([0,T];H^2(\Omega))} + \|\partial_t^\alpha u\|_{C([0,T];L^2(\Omega))} \leqslant C_4 \|F\|_{L^2(0,T;\mathcal{D}((-L)^{\frac{1}{\alpha}})) \cap L^\infty(0,T;L^2(\Omega))},$$

and the series (2.6) holds in the corresponding space.

Remark. As shown in Lemma 3.3 below, for $0 < \alpha < 1$, the function $E_{\alpha,1}(-t^\alpha)$ in $t > 0$ is completely monotonic (e.g., Gorenflo and Mainardi [14]), while for $1 < \alpha < 2$, the function $E_{\alpha,1}(-t^\alpha)$ in $t > 0$ is not completely monotonic. As a consequence, the regularity requirements for F in Theorem 2.2(i) and (ii) are different.

We do not exploit the maximal regularity of u for $F \in L^\infty(0,T;L^2(\Omega))$ or $F \in L^2(0,T;\mathcal{D}((-L)^{\frac{1}{\alpha}})) \cap L^\infty(0,T;L^2(\Omega))$. As for other maximal regularity, see Theorem 2.4 and Corollary 2.5.

Theorem 2.3. *Let $1 < \alpha < 2$ and $F = 0$.*

(i) *Let $a \in L^2(\Omega)$ and $b \in \mathcal{D}((-L)^{-\frac{1}{\alpha}})$. Then there exists a unique weak solution $u \in C([0,T];L^2(\Omega)) \cap C((0,T];H^2(\Omega) \cap H_0^1(\Omega))$ to* (1.1)–(1.4) *with $\partial_t^\alpha u \in C((0,T];L^2(\Omega))$. Moreover there exist constants $C_5 > 0$ and $C_6 > 0$ satisfying*

$$\|u\|_{C([0,T];L^2(\Omega))} + \|\partial_t u\|_{C([0,T];\mathcal{D}((-L)^{-\frac{1}{\alpha}}))} \leqslant C_5\big(\|a\|_{L^2(\Omega)} + \|b\|_{\mathcal{D}((-L)^{-\frac{1}{\alpha}})}\big),$$
$$\lim_{t\to 0}\big\|u(\cdot,t) - a\big\|_{L^2(\Omega)} = \lim_{t\to 0}\big\|\partial_t u(\cdot,t) - b\big\|_{\mathcal{D}((-L)^{-\frac{1}{\alpha}})} = 0 \tag{2.7}$$

and

$$\begin{cases} \big\|\partial_t u(\cdot,t)\big\|_{L^2(\Omega)} \leqslant C_6\big(t^{-1}\|a\|_{L^2(\Omega)} + \|b\|_{L^2(\Omega)}\big), \\ \big\|\partial_t^\alpha u(\cdot,t)\big\|_{L^2(\Omega)} \leqslant C_6\big(t^{-\alpha}\|a\|_{L^2(\Omega)} + t^{1-\alpha}\|b\|_{L^2(\Omega)}\big). \end{cases} \tag{2.8}$$

Moreover $u : (0,T] \longrightarrow L^2(\Omega)$ is analytically extended to $\{z \in \mathbb{C};\ z \neq 0,\ |\arg z| < \frac{1}{2}\pi\}$.

(ii) *Let $a \in H^2(\Omega) \cap H_0^1(\Omega)$ and $b \in H_0^1(\Omega)$. Then there exists a unique weak solution $u \in C([0,T];H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ to* (1.1)–(1.4) *and $\partial_t^\alpha u \in C([0,T];L^2(\Omega))$. Moreover there exists a constant $C_7 > 0$ satisfying*

$$\|u\|_{C^1([0,T];L^2(\Omega))} + \|u\|_{C([0,T];H^2(\Omega))} + \big\|\partial_t^\alpha u\big\|_{C([0,T];L^2(\Omega))} \leqslant C_7\big(\|a\|_{H^2(\Omega)} + \|b\|_{H^1(\Omega)}\big). \tag{2.9}$$

Then we have

$$u(x,t) = \sum_{n=1}^\infty \big\{(a,\varphi_n)E_{\alpha,1}\big(-\lambda_n t^\alpha\big) + (b,\varphi_n)tE_{\alpha,2}\big(-\lambda_n t^\alpha\big)\big\}\varphi_n(x),$$
$$\partial_t u(x,t) = \sum_{n=1}^\infty \big\{-\lambda_n t^{\alpha-1}(a,\varphi_n)E_{\alpha,\alpha}\big(-\lambda_n t^\alpha\big) + (b,\varphi_n)E_{\alpha,1}\big(-\lambda_n t^\alpha\big)\big\}\varphi_n(x) \tag{2.10}$$

in the corresponding spaces in (i) *and* (ii).

In Theorem 2.2, if F is smoother, then the regularity of $\partial_t^\alpha u$ is improved. We set

$$C^\theta\big([0,T];L^2(\Omega)\big) = \left\{F \in C\big([0,T];L^2(\Omega)\big);\ \sup_{0\leqslant t<s\leqslant T}\frac{\|F(\cdot,t) - F(\cdot,s)\|_{L^2(\Omega)}}{|t-s|^\theta} < \infty\right\}$$

and

$$\|F\|_{C^\theta([0,T];L^2(\Omega))} = \|F\|_{C([0,T];L^2(\Omega))} + \sup_{0\leqslant t<s\leqslant T}\frac{\|F(\cdot,t) - F(\cdot,s)\|_{L^2(\Omega)}}{|t-s|^\theta}.$$

For $F \in C^\theta([0,T];L^2(\Omega))$, we can state the same maximal regularity for the solution to (1.1)–(1.4) for any $\alpha \in (0,2)$.

Theorem 2.4. *Let $0 < \alpha < 2$ and let $a \in H^2(\Omega) \cap H_0^1(\Omega)$, $b = 0$ if $1 < \alpha < 2$, $F \in C^\theta([0,T];L^2(\Omega))$. Then for the solution u given by*

$$u(x,t) = \sum_{n=1}^\infty \left\{(a,\varphi_n)E_{\alpha,1}\big(-\lambda_n t^\alpha\big) + \int_0^t \big(F(\cdot,\tau),\varphi_n\big)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n(t-\tau)^\alpha\big)\,d\tau\right\}\varphi_n(x), \tag{2.11}$$

we have:

(1) *For every $\delta > 0$,*

$$\|Lu\|_{C^\theta([\delta,T];L^2(\Omega))} + \big\|\partial_t^\alpha u\big\|_{C^\theta([\delta,T];L^2(\Omega))} \leqslant \frac{C_8}{\delta}\big(\|F\|_{C^\theta([\delta,T];L^2(\Omega))} + \|a\|_{H^2(\Omega)}\big).$$

(2) $$\|Lu\|_{C([0,T];L^2(\Omega))} + \big\|\partial_t^\alpha u\big\|_{C([0,T];L^2(\Omega))} \leqslant C_9\big(\|a\|_{H^2(\Omega)} + \|F\|_{C^\theta([0,T];L^2(\Omega))}\big).$$

(3) *If* $a = 0$ *and* $F(\cdot, 0) = 0$*, then*

$$\|Lu\|_{C^\theta([0,T];L^2(\Omega))} + \left\|\partial_t^\alpha u\right\|_{C^\theta([0,T];L^2(\Omega))} \leqslant C_{10}\|F\|_{C^\theta([0,T];L^2(\Omega))}.$$

Corollary 2.5. *Let* $1 < \alpha < 2$*,* $a = b = 0$ *and* $F \in L^2(\Omega \times (0, T))$*. Then for u given by* (2.6)*, we have*

$$u \in C\big([0, T]; \mathcal{D}\big((-L)^{1-\frac{1}{\alpha}}\big)\big) \tag{2.12}$$

and

$$\|u\|_{C([0,T];\mathcal{D}((-L)^{1-\frac{1}{\alpha}}))} \leqslant C_{11}\|F\|_{L^2(\Omega\times(0,T))}. \tag{2.13}$$

Corollary 2.6. *Let* $0 < \alpha < 1$*,* $a \in L^2(\Omega)$ *and* $F = 0$*. Then for the unique weak solution* $u \in C([0,\infty); L^2(\Omega)) \cap C((0,\infty); H^2(\Omega) \cap H_0^1(\Omega))$ *to* (1.1)–(1.3)*, there exists a constant* $C_{12} > 0$ *such that*

$$\big\|u(\cdot, t)\big\|_{L^2(\Omega)} \leqslant \frac{C_{12}}{1 + \lambda_1 t^\alpha}\|a\|_{L^2(\Omega)}, \quad t \geqslant 0. \tag{2.14}$$

Moreover there exists a constant $C_{12} > 0$ *such that*

$$u \in C^\infty\big((0,\infty); L^2(\Omega)\big), \qquad \big\|\partial_t^m u(\cdot, t)\big\|_{L^2(\Omega)} \leqslant \frac{C_{13}}{t^m}\|a\|_{L^2(\Omega)}, \quad t > 0,\ m \in \mathbb{N}. \tag{2.15}$$

Corollary 2.7. *Let* $1 < \alpha < 2$*,* $a \in H^2(\Omega) \cap H_0^1(\Omega)$*,* $b \in H_0^1(\Omega)$ *and* $F = 0$*. Then for the unique weak solution* $u \in C([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty) : L^2(\Omega))$ *to* (1.1)–(1.4)*, there exists a constant* $C_{14} > 0$ *satisfying*

$$\big\|u(\cdot, t)\big\|_{L^2(\Omega)} \leqslant \frac{C_{14}}{1 + \lambda_1 t^\alpha}\big\{\|a\|_{L^2(\Omega)} + t\|b\|_{L^2(\Omega)}\big\}, \quad t \geqslant 0 \tag{2.16}$$

and

$$\big\|\partial_t u(\cdot, t)\big\|_{L^2(\Omega)} \leqslant \frac{C_{14}}{1 + \lambda_1 t^\alpha}\big(t^{\alpha-1}\|a\|_{H^2(\Omega)} + \|b\|_{L^2(\Omega)}\big), \quad t \geqslant 0. \tag{2.17}$$

Moreover, for some $C_{15} > 0$*, we have*

$$\big\|\partial_t^m u(\cdot, t)\big\|_{L^2(\Omega)} \leqslant C_{15}\big(t^{-m}\|a\|_{L^2(\Omega)} + t^{-m+1}\|b\|_{L^2(\Omega)}\big), \quad t > 0,\ m \in \mathbb{N}. \tag{2.18}$$

The eigenfunction expansions (2.2), (2.6) and (2.11) of the solutions to (1.1)–(1.4) can be derived by the Fourier method. That is, we multiply both sides of (1.1) by $\varphi_n(x)$ and integrate the equation with respect to x. Using the Green formula and $\varphi_n|_{\partial\Omega} = 0$, we obtain

$$\partial_t^\alpha u_n(t) = -\lambda_n u_n(t) + F_n(t), \quad t > 0, \tag{2.19}$$

and

$$u_n(0) = (a, \varphi_n) \quad \text{in the case } 0 < \alpha < 1,$$
$$u_n(0) = (a, \varphi_n), \qquad \frac{du_n}{dt}(0) = (b, \varphi_n) \quad \text{in the case } 1 < \alpha < 2,$$

where $u_n(t) = (u(\cdot, t), \varphi_n)$ and $F_n(t) = (F(\cdot, t), \varphi_n)$. The formulae of solutions to the initial value problem for (2.19) are given in [14,18,37] for example, and we can formally obtain the expansions.

2.1. Comparison of our results with standard results for the case of $\alpha = 1, 2$

(1) In the case of $0 < \alpha < 1$, we have no smoothing property like the classical diffusion equation (i.e., $\alpha = 1$). For $F = 0$, there is the smoothing property in space with order 2 which means that $u(\cdot, t) \in H^2(\Omega)$ for any $t > 0$ and any $u(\cdot, 0) \in L^2(\Omega)$, while (2.15) means that the regularity in time immediately becomes stronger in t, and is of infinity order (i.e., u is of C^∞ for $t > 0$). In Section 4, we show that the smoothing in $H^2(\Omega)$ is the best possible and the solution cannot be smoother than $H^2(\Omega)$ at $t > 0$ if $u(\cdot, 0) \in L^2(\Omega)$.
(2) In Theorem 2.3(i), estimate (2.7) generalizes the result in the case of $\alpha = 2$ which is proved e.g., in [21].
(3) In the case of $0 < \alpha < 1$ and $a = 0$, estimate (2.5) in Theorem 2.2 is the corresponding regularity of solution to the case of $\alpha = 1$ (e.g., Theorem 1.1 (p. 5) of Chapter 4 in [21]).
(4) Theorem 2.4 means that for $0 < \alpha < 2$, the same regularity properties hold for the nonhomogeneous equation in the case of $\alpha = 1$ (i.e., Theorem 3.5 (p. 114) in [36]). Theorem 2.4(3) is proved in Theorem 2.4 (p. 62) and Theorem 3.3 (pp. 77–78) in [40] by a different method.
(5) Corollary 2.5 gives a well-known result for $\alpha = 2$ (e.g., [21]).
(6) Corollaries 2.6 and 2.7 show the decay of solution with order $t^{-\alpha}$ as $t \to \infty$, which is slower than the exponential decay in the case of $\alpha = 1$. In Section 4, we state other property on the decay.

3. Proof of Theorems 2.1–2.4 and Corollaries 2.5–2.6

We first state two lemmata.

Lemma 3.1. *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that μ is such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C_1 = C_1(\alpha, \beta, \mu) > 0$ such that*

$$\left|E_{\alpha,\beta}(z)\right| \leqslant \frac{C_1}{1+|z|}, \qquad \mu \leqslant \left|\arg(z)\right| \leqslant \pi. \tag{3.1}$$

The proof can be found on p. 35 in Podlubny [37].

Lemma 3.2. *For $\lambda > 0$, $\alpha > 0$ and positive integer $m \in \mathbb{N}$, we have*

$$\frac{d^m}{dt^m} E_{\alpha,1}\left(-\lambda t^\alpha\right) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}\left(-\lambda t^\alpha\right), \quad t > 0 \tag{3.2}$$

and

$$\frac{d}{dt}\left(t E_{\alpha,2}\left(-\lambda t^\alpha\right)\right) = E_{\alpha,1}\left(-\lambda t^\alpha\right), \quad t \geqslant 0. \tag{3.3}$$

Proof. Since $E_{\alpha,\beta}(z)$ is an entire function of z, the function $E_{\alpha,\beta}(x)$ is real analytic and the series $\sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)} = E_{\alpha,\beta}(z)$ is termwise differentiable in $\mathbb{R}$. Since t^α is also real analytic in $t > 0$, so is $E_{\alpha,\beta}(-\lambda t^\alpha)$ in $t > 0$. Therefore the equations above obtained by termwise differentiation are valid. □

We proceed to the proof of the theorems and the corollaries stated in Section 2.

Proof of Theorem 2.1. (i). We will show that (2.2) certainly gives the weak solution to (1.1)–(1.3). We first have

$$\left\|u(\cdot,t)\right\|^2_{L^2(\Omega)} = \sum_{n=1}^\infty \left|(a,\varphi_n)E_{\alpha,1}\left(-\lambda_n t^\alpha\right)\right|^2 \leqslant \sum_{n=1}^\infty C_2'^2(a,\varphi_n)^2 \leqslant C_2\|a\|^2_{L^2(\Omega)}. \tag{3.4}$$

Moreover by Lemma 3.1, we have

$$\left\|Lu(\cdot,t)\right\|^2_{L^2(\Omega)} \leqslant C_3\|a\|^2_{L^2(\Omega)} t^{-2\alpha}, \quad t > 0. \tag{3.5}$$

In (3.4), since $\sum_{n=1}^\infty (a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n$ is convergent in $L^2(\Omega)$ uniformly in $t \in [0,T]$, we see that $u \in C([0,T]; L^2(\Omega))$. Moreover in (3.5), since $\sum_{n=1}^\infty \lambda_n(a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n$ is convergent in $L^2(\Omega)$ uniformly in $t \in [\delta, T]$ with any given $\delta > 0$, we see that $Lu \in C((0,T]; L^2(\Omega))$, that is, $u \in C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$. Therefore we obtain that $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$.

By (1.1) we see that $\partial_t^\alpha u \in C((0,T]; L^2(\Omega))$ and estimate (2.1).

We have to prove

$$\lim_{t\to 0}\left\|u(\cdot,t) - a\right\|_{L^2(\Omega)} = 0. \tag{3.6}$$

In fact,

$$\left\|u(\cdot,t) - a\right\|^2_{L^2(\Omega)} = \sum_{n=1}^\infty \left|(a,\varphi_n)\right|^2 \left(E_{\alpha,\alpha}\left(-\lambda_n t^\alpha\right) - 1\right)^2$$

and $\lim_{t\to 0}(E_{\alpha,\alpha}(-\lambda_n t^\alpha) - 1) = 0$ for each $n \in \mathbb{N}$ and

$$\sum_{n=1}^\infty \left|(a,\varphi_n)\right|^2 \left|E_{\alpha,\alpha}\left(-\lambda_n t^\alpha\right) - 1\right|^2 \leqslant 2\sum_{n=1}^\infty \left\{\left(\frac{C_1}{1+\lambda_n t^\alpha}\right)^2 + 1\right\}\left|(a,\varphi_n)\right|^2 < \infty$$

for $0 \leqslant t \leqslant T$. The Lebesgue theorem yields (3.6).

Next we prove the uniqueness of the weak solution to (1.1)–(1.3) within the class given in Definition 2.1. Under the conditions $a = 0$ and $F = 0$, we have to prove that system (1.1)–(1.3) has only a trivial solution. Since $\varphi_n(x)$ is the eigenfunctions to the following eigenvalue problem:

$$(L\varphi_n)(x) = -\lambda_n\varphi_n(x), \quad x \in \Omega, \qquad \varphi_n(x) = 0, \quad x \in \partial\Omega,$$

in terms of the regularity of u, taking the duality pairing ${}_{-\gamma}\langle\cdot,\cdot\rangle_\gamma$ of (1.1) with φ_n and setting $u_n(t) = {}_{-\gamma}\langle u(\cdot,t),\varphi_n\rangle_\gamma$, we obtain

$$\partial_t^\alpha u_n(t) = -\lambda_n u_n(t), \quad \text{almost all } t \in (0,T).$$

Since $u(\cdot,t) \in L^2(\Omega)$ for almost all $t \in (0,T)$ and $u_n(t) \equiv {}_{-\gamma}\langle u(\cdot,t),\varphi_n\rangle_\gamma = (u(\cdot,t),\varphi_n)$ where ${}_{-\gamma}\langle\cdot,\cdot\rangle_\gamma$ denotes the duality pairing between $\mathcal{D}((-L)^{-\gamma})$ and $\mathcal{D}((-L)^{\gamma})$, it follows from $\lim_{t\to 0}\|u(\cdot,t)\|_{\mathcal{D}((-L)^{-\gamma})} = 0$ that $u_n(0) = 0$. Due to the existence and uniqueness of the ordinary fractional differential equation (e.g., Chapter 3 in [18], [37]), we obtain that $u_n(t) = 0$, $n = 1,2,3,\dots$. Since $\{\varphi_n\}_{n\in\mathbb{N}}$ is a complete orthonormal system in $L^2(\Omega)$, we have $u = 0$ in $\Omega \times (0,T)$.

Finally we prove the analyticity of $u(\cdot,t)$ in $S \equiv \{z \in \mathbb{C};\ z \neq 0,\ |\arg z| < \frac{1}{2}\pi\}$. It follows that $E_{\alpha,1}(-\lambda_n t^\alpha)$ is analytic in S because $E_{\alpha,1}(-\lambda_n z)$ is an entire function (e.g., Section 1.8 in [18], [37]). Therefore $u_N(\cdot,t) = \sum_{n=1}^N (a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n$ is analytic in S. Furthermore by (3.1)

$$\|u_N(\cdot,z) - u(\cdot,z)\|_{L^2(\Omega)}^2 = \sum_{n=N+1}^\infty \left|(a,\varphi_n)E_{\alpha,1}(-\lambda_n z^\alpha)\right|^2 \leqslant C_3 \sum_{n=N+1}^\infty \left|(a,\varphi_n)\right|^2, \quad z \in \bar{S}.$$

Hence $\lim_{n\to\infty}\|u_N - u\|_{L^\infty(S_\alpha;L^2(\Omega))} = 0$, so that also u is analytic in S. Thus the proof of Theorem 2.1(i) is complete.

(ii). By (3.1), we have

$$\begin{aligned}
\|u(\cdot,t)\|_{H^2(\Omega)}^2 &\leqslant C_4'\|Lu(\cdot,t)\|_{L^2(\Omega)}^2 \leqslant C_4'\sum_{n=1}^\infty \lambda_n^2\left|(a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\right|^2 \\
&= C_4'\sum_{n=1}^\infty \left|\lambda_n^{\frac{1}{2}}(a,\varphi_n)(\lambda_n t^\alpha)^{\frac{1}{2}}E_{\alpha,1}(-\lambda_n t^\alpha)\right|^2 t^{-\alpha} \\
&\leqslant C_4'\sum_{n=1}^\infty \left|\left((-L)^{\frac{1}{2}}a,\varphi_n\right)\frac{C_1(\lambda_n t^\alpha)^{\frac{1}{2}}}{1+\lambda_n t^\alpha}\right|^2 t^{-\alpha} \leqslant C_4\|a\|_{H^1(\Omega)}t^{-\alpha}.
\end{aligned}$$

By $0 < \alpha < 1$, we see $\|u\|_{L^2(0,T;H^2(\Omega))} \leqslant C_4\|a\|_{H^1(\Omega)}$. Therefore we have $u \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$.

We have

$$\begin{aligned}
\int_0^T \|\partial_t^\alpha u(\cdot,t)\|_{L^2(\Omega)}^2\,dt &= \int_0^T \sum_{n=1}^\infty \left|(a,\varphi_n)\right|^2\lambda_n^2\left|E_{\alpha,1}(-\lambda_n t^\alpha)\right|^2 dt \\
&\leqslant \frac{C_8^2 T^{1-\alpha}}{1-\alpha}\sum_{n=1}^\infty (a,\varphi_n)^2\lambda_n \\
&\leqslant C_5\|a\|_{H^1(\Omega)}^2.
\end{aligned}$$

By (1.1) we have $\partial_t^\alpha u = Lu$, which yields $\partial_t^\alpha u \in L^2(\Omega\times(0,T))$ and the proof of Theorem 2.1(ii) is complete.

(iii). Let $a \in H^2(\Omega)\cap H_0^1(\Omega)$. Then we have

$$\begin{aligned}
\|u(\cdot,t)\|_{H^2(\Omega)}^2 &\leqslant C_6'\|Lu(\cdot,t)\|_{L^2(\Omega)}^2 \\
&\leqslant \sum_{n=1}^\infty (a,\varphi_n)^2 E_{\alpha,1}(-\lambda_n t^\alpha)^2\lambda_n^2 \leqslant C_6\|a\|_{H^2(\Omega)}^2, \quad t \geqslant 0.
\end{aligned}$$

By (1.1) we have

$$\|\partial_t^\alpha u(\cdot,t)\|_{L^2(\Omega)}^2 \leqslant C_7\|a\|_{H^2(\Omega)}^2, \quad t > 0.$$

Similarly to the proof of Theorem 2.1 (i), we can prove (2.4), and the proof of Theorem 2.1 (iii) is complete. □

Proof of Theorem 2.2. (i). First we show

Lemma 3.3. *For* $0 < \alpha < 1$, *we have*

$$E_{\alpha,\alpha}(-\eta) \geqslant 0, \quad \eta \geqslant 0.$$

As for the proof, see Miller and Samko [32], Schneider [47], and also see Pollard [38]. Lemma 3.3 is also seen from (3.2) and the fact that $E_{\alpha,1}(-t^\alpha)$ with $0<\alpha<1$ is completely monotonic (e.g., Gorenflo and Mainardi [14]), that is, $(-1)^n \frac{d^n}{dt^n} E_{\alpha,1}(-t^\alpha) \geqslant 0$ for all $t>0$ and $n=0,1,2,\dots$. We can refer to Section 4 of Chapter 1 in Prüss [40] for completely monotonic functions.

By Lemmata 3.2 and 3.3, we have

$$\int_0^\eta \left|t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n t^\alpha\right)\right| dt = \int_0^\eta t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n t^\alpha\right) dt$$
$$= -\frac{1}{\lambda_n}\int_0^\eta \frac{d}{dt}E_{\alpha,1}\left(-\lambda_n t^\alpha\right) dt = \frac{1}{\lambda_n}\left(1-E_{\alpha,1}\left(-\lambda_n \eta^\alpha\right)\right), \quad \eta>0. \tag{3.7}$$

In [14], pp. 140–141 in [18], p. 140 in [37], by means of the Laplace transform, we can see that

$$\partial_t^\alpha \int_0^t \left(F(\cdot,\tau),\varphi_n\right)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^\alpha\right) d\tau$$
$$= -\lambda_n \int_0^t \left(F(\cdot,\tau),\varphi_n\right)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^\alpha\right) d\tau + \left(F(\cdot,t),\varphi_n\right). \tag{3.8}$$

By (3.7), (3.8) and the Young inequality for the convolution, we have

$$\left\|\partial_t^\alpha \int_0^t \left(F(\cdot,\tau),\varphi_n\right)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^\alpha\right) d\tau\right\|_{L^2(0,T)}^2$$
$$\leqslant C_8\int_0^T \left|\left(F(\cdot,t),\varphi_n\right)\right|^2 dt + C_8\left(\int_0^T \left|\left(F(\cdot,t),\varphi_n\right)\right|^2 dt\right)\left(\int_0^T \left|\lambda_n t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n t^\alpha\right)\right| dt\right)^2$$
$$\leqslant C_9\int_0^T \left|\left(F(\cdot,t),\varphi_n\right)\right|^2 dt.$$

Hence

$$\left\|\partial_t^\alpha u\right\|_{L^2(\Omega\times(0,T))}^2 = \sum_{n=1}^\infty \int_0^T \left|\partial_t^\alpha\left(\int_0^t \left(F(\cdot,\tau),\varphi_n\right)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^\alpha\right) d\tau\right)\right|^2 dt$$
$$\leqslant C_9\sum_{n=1}^\infty \int_0^T \left|\left(F(\cdot,t),\varphi_n\right)\right|^2 dt = C_9\|F\|_{L^2(\Omega\times(0,T))}^2.$$

By (1.1), we see also $\|Lu\|_{L^2(\Omega\times(0,T))} \leqslant C_9\|F\|_{L^2(\Omega\times(0,T))}$, which implies (2.5).

Finally we have to prove

$$\lim_{t\to 0}\left\|u(\cdot,t)\right\|_{\mathcal{D}((-L)^{-\gamma})} = 0.$$

In fact, by (3.7) we have

$$\left\|u(\cdot,t)\right\|_{\mathcal{D}((-L)^{-\gamma})}^2 = \sum_{n=1}^\infty \frac{1}{\lambda_n^{2\gamma}}\left|\int_0^t \left(F(\cdot,\tau),\varphi_n\right)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^\alpha\right) d\tau\right|^2$$
$$= \sum_{n=1}^\infty \frac{1}{\lambda_n^{2\gamma}}\sup_{0\leqslant\tau\leqslant T}\left|\left(F(\cdot,\tau),\varphi_n\right)\right|^2\left|\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^\alpha\right) d\tau\right|^2$$
$$\leqslant C_8\|F\|_{L^\infty(0,T;L^2(\Omega))}\sum_{n=1}^\infty \frac{1}{\lambda_n^{2\gamma+2}}\left(1-E_{\alpha,1}\left(-\lambda_n t^\alpha\right)\right).$$

Since

$$\lambda_n \geqslant C_8' n^{\frac{2}{d}}, \quad n \in \mathbb{N}$$

(e.g., Courant and Hilbert [8]), we have

$$\frac{1}{\lambda_n^{2\gamma+2}} \leqslant \frac{C_8''}{n^{\frac{4(\gamma+1)}{d}}}.$$

By $\gamma > \frac{d}{4} - 1$, we have $\frac{4(\gamma+1)}{d} > 1$, and $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2\gamma+2}}(1 - E_{\alpha,1}(-\lambda_n t^\alpha)) < \infty$. Since $\lim_{t\to 0}(1 - E_{\alpha,1}(-\lambda_n t^\alpha)) = 0$ for each $n \in \mathbb{N}$, the Lebesgue theorem implies $\lim_{t\to 0} \|u(\cdot,t)\|_{\mathcal{D}((-L)^{-\gamma})} = 0$. The uniqueness of weak solution is already proved in the proof of Theorem 2.1. Thus the proof of Theorem 2.2(i) is complete.

(ii). First by $F \in L^2(0,T;\mathcal{D}((-L)^{\frac{1}{\alpha}})$ we have

$$\begin{aligned}
\|Lu(\cdot,t)\|_{L^2(\Omega)}^2 &= \sum_{n=1}^{\infty} \lambda_n^2 \left| \int_0^t \big(F(\cdot,\tau),\varphi_n\big)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n(t-\tau)^\alpha\big)\,d\tau \right|^2 \\
&\leqslant \sum_{n=1}^{\infty} \lambda_n^2 \int_0^t \big|\big(F(\cdot,\tau),\varphi_n\big)\big|^2\,d\tau \int_0^t (t-\tau)^{2\alpha-2}\big|E_{\alpha,\alpha}\big(-\lambda_n(t-\tau)^\alpha\big)\big|^2\,d\tau \\
&\leqslant C_9 \sum_{n=1}^{\infty} \lambda_n^2 \lambda_n^{-\frac{2}{\alpha}} \int_0^t \big|\big((-L)^{\frac{1}{\alpha}} F(\cdot,\tau),\varphi_n\big)\big|^2\,d\tau \int_0^t \left|\frac{(\lambda_n \tau^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n\tau^\alpha}\right|^2 d\tau\, \lambda_n^{-\frac{2\alpha-2}{\alpha}} \\
&\leqslant C_9' t \|F\|_{L^2(0,T;\mathcal{D}((-L)^{\frac{1}{\alpha}}))}^2.
\end{aligned} \tag{3.9}$$

By (3.9) we can estimate also $\|\partial_t^\alpha u\|_{C([0,T];L^2(\Omega))}$ and we have $\lim_{t\to 0} \|u(\cdot,t)\|_{L^2(\Omega)} = 0$. Next applying $\frac{d}{dt}(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)) = t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha)$ (e.g., formula (1.83) on p. 22 of [37]) and $\lambda_n^{2\gamma-2} \geqslant C_9' n^{\gamma_1}$ with $\gamma_1 > 1$ by $\gamma > \frac{d}{4} + 1$ and $\lambda_n \geqslant C_8' n^{\frac{2}{d}}$, we have

$$\begin{aligned}
\|\partial_t u(\cdot,t)\|_{\mathcal{D}((-L)^{-\gamma})}^2 &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2\gamma}} \left| \int_0^t \big(F(\cdot,\tau),\varphi_n\big)\lambda_n(t-\tau)^{\alpha-2}E_{\alpha,\alpha-1}\big(-\lambda_n(t-\tau)^\alpha\big)\,d\tau \right|^2 \\
&\leqslant \sum_{n=1}^{\infty} \sup_{0\leqslant\tau\leqslant T} \big|\big(F(\cdot,\tau),\varphi_n\big)\big|^2 \left| \int_0^t \tau^{\alpha-2}E_{\alpha,\alpha-1}\big(-\lambda_n\tau^\alpha\big)\,d\tau\right|^2 \frac{1}{\lambda_n^{2\gamma-2}} \leqslant C_9 \|F\|_{L^\infty(0,T;L^2(\Omega))}^2 t^{2\alpha-2}.
\end{aligned}$$

Therefore $\lim_{t\to 0} \|\partial_t u(\cdot,t)\|_{\mathcal{D}((-L)^{-\gamma})}^2 = 0$. Thus the proof of Theorem 2.2(ii) is complete. □

Proof of Theorem 2.3. (i). The uniqueness of weak solution is verified similarly to Theorems 2.1 and 2.2. As for the initial condition, we first consider

$$\begin{aligned}
\|u(\cdot,t) - a\|_{L^2(\Omega)}^2 &= \sum_{n=1}^{\infty} \big|(a,\varphi_n)\big(E_{\alpha,1}\big(-\lambda_n t^\alpha\big) - 1\big) + t(b,\varphi_n)E_{\alpha,2}\big(-\lambda_n t^\alpha\big)\big|^2 \\
&\leqslant 2\sum_{n=1}^{\infty} \big|(a,\varphi_n)\big|^2\big|E_{\alpha,1}\big(-\lambda_n t^\alpha\big) - 1\big|^2 + 2\sum_{n=1}^{\infty} \big|\big((-L)^{-\frac{1}{\alpha}}b,\varphi_n\big)\big|^2\big(\lambda_n t^\alpha\big)^{\frac{2}{\alpha}}\left(\frac{C_1}{1+\lambda_n t^\alpha}\right)^2 \\
&\equiv S_1(t) + S_2(t)
\end{aligned}$$

where we have used Lemma 3.1. Therefore similarly to Theorem 2.1, we can see that $\lim_{t\to 0} S_1(t) = 0$. Since

$$\sup_{\eta>0} \frac{\eta^{\frac{1}{\alpha}}}{1+\eta} = \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}}, \tag{3.10}$$

we see that $S_2(t) \leqslant 2\|(-L)^{-\frac{1}{\alpha}}b\|_{L^2(\Omega)}^2$. Therefore

$$\|u\|_{C([0,T];L^2(\Omega))} \leqslant C_5\big(\|a\|_{L^2(\Omega)} + \|b\|_{\mathcal{D}((-L)^{-\frac{1}{\alpha}})}\big).$$

Since $\lim_{t\to 0}\frac{(\lambda_n t^\alpha)^{\frac{1}{\alpha}}}{1+\lambda_n t^\alpha}=0$, by (3.10), the Lebesgue theorem yields $\lim_{t\to 0} S_2(t)=0$, that is, $\lim_{t\to 0}\|u(\cdot,t)-a\|_{L^2(\Omega)}=0$. Next we have

$$\begin{aligned}\left\|\partial_t u(\cdot,t)-b\right\|^2_{\mathcal{D}((-L)^{-\frac{1}{\alpha}})} &= \sum_{n=1}^\infty \left|-\lambda_n t^{\alpha-1}(a,\varphi_n)\lambda_n^{-\frac{1}{\alpha}}E_{\alpha,\alpha}\left(-\lambda_n t^\alpha\right)+\lambda_n^{-\frac{1}{\alpha}}\left(E_{\alpha,1}\left(-\lambda_n t^\alpha\right)-1\right)(b,\varphi_n)\right|^2\\ &\leqslant 2\sum_{n=1}^\infty\left|(a,\varphi_n)\right|^2\left(\frac{C_1(\lambda_n t^\alpha)^{\frac{\alpha-1}{2\alpha}}}{1+\lambda_n t^\alpha}\right)^2+2\sum_{n=1}^\infty\left|\left((-L)^{-\frac{1}{\alpha}}b,\varphi_n\right)\right|^2\left|E_{\alpha,1}\left(-\lambda_n t^\alpha\right)-1\right|^2.\end{aligned}$$

By the Lebesgue theorem, we see that $\lim_{t\to 0}\|\partial_t u(\cdot,t)-b\|_{\mathcal{D}((-L)^{-\frac{1}{\alpha}})}=0$ and

$$\|\partial_t u\|_{C([0,T];\mathcal{D}((-L)^{-\frac{1}{\alpha}}))}\leqslant C_5\left(\|a\|_{L^2(\Omega)}+\|b\|_{\mathcal{D}((-L)^{-\frac{1}{\alpha}})}\right).$$

By Lemma 3.1, we have

$$\begin{aligned}\left\|u(\cdot,t)\right\|^2_{L^2(\Omega)} &= \sum_{n=1}^\infty\left|(a,\varphi_n)E_{\alpha,1}\left(-\lambda_n t^\alpha\right)+(b,\varphi_n)tE_{\alpha,2}\left(-\lambda_n t^\alpha\right)\right|^2\\ &\leqslant 2\sum_{n=1}^\infty (a,\varphi_n)^2\left(\frac{C_1}{1+\lambda_n t^\alpha}\right)^2+2\sum_{n=1}^\infty(b,\varphi_n)^2\left(\frac{C_1 t}{1+\lambda_n t^\alpha}\right)^2\leqslant C_{11}\left(\|a\|^2_{L^2(\Omega)}+\|b\|^2_{L^2(\Omega)}\right).\end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned}\left\|\partial_t u(\cdot,t)\right\|^2_{L^2(\Omega)} &= \sum_{n=1}^\infty\left|(a,\varphi_n)(-\lambda_n)t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n t^\alpha\right)+(b,\varphi_n)E_{\alpha,1}\left(-\lambda_n t^\alpha\right)\right|^2\\ &\leqslant 2\sum_{n=1}^\infty\left\{(a,\varphi_n)^2t^{-2}\left(\frac{C_1\lambda_n t^\alpha}{1+\lambda_n t^\alpha}\right)^2+\sum_{n=1}^\infty(b,\varphi_n)^2\left(\frac{C_1}{1+\lambda_n t^\alpha}\right)^2\right\}.\end{aligned}\tag{3.11}$$

Since $\partial_t^\alpha(E_{\alpha,1}(-\lambda_n t^\alpha))=-\lambda_n E_{\alpha,1}(-\lambda_n t^\alpha)$ and $\partial_t^\alpha(tE_{\alpha,2}(-\lambda_n t^\alpha))=-\lambda_n tE_{\alpha,2}(-\lambda_n t^\alpha)$ (e.g., [14,18]), we have

$$\partial_t^\alpha u(x,t)=\sum_{n=1}^\infty\left\{-\lambda_n(a,\varphi_n)E_{\alpha,1}\left(-\lambda_n t^\alpha\right)-\lambda_n(b,\varphi_n)tE_{\alpha,2}\left(-\lambda_n t^\alpha\right)\right\}\tag{3.12}$$

and similarly we can prove

$$\left\|\partial_t^\alpha u(\cdot,t)\right\|_{L^2(\Omega)}\leqslant C_{10}\left(t^{-\alpha}\|a\|_{L^2(\Omega)}+t^{1-\alpha}\|b\|_{L^2(\Omega)}\right).$$

The analyticity of $u(\cdot,t)$ is proved similarly to Theorem 2.1. Thus the proof of Theorem 2.3(i) is complete.

(ii). By Lemma 3.1, we have

$$\begin{aligned}\left(Lu(\cdot,t),Lu(\cdot,t)\right) &= \sum_{n=1}^\infty\left|\lambda_n(a,\varphi_n)E_{\alpha,1}\left(-\lambda_n t^\alpha\right)+\lambda_n(b,\varphi_n)tE_{\alpha,2}\left(-\lambda_n t^\alpha\right)\right|^2\\ &\leqslant 2C_1^2\sum_{n=1}^\infty\left\{\lambda_n^2(a,\varphi_n)^2+\lambda_n(b,\varphi_n)^2\frac{\lambda_n t^\alpha}{(1+\lambda_n t^\alpha)^2}t^{2-\alpha}\right\}\\ &\leqslant C_{11}\left(\|a\|^2_{H^2(\Omega)}+T^{2-\alpha}\|b\|^2_{H^1(\Omega)}\right).\end{aligned}$$

Similarly to (3.11) and (3.12), we can argue to complete the proof. □

Proof of Theorem 2.4. It is sufficient to prove the theorem in the case of $0<\alpha<1$, because the case of $\alpha=1$ is similar to Section 3 of Chapter 4 in [36] for example. We first prove

Lemma 3.4. *Let $F\in C^\theta([0,T];L^2(\Omega))$. We set*

$$v(x,t)=\sum_{n=1}^\infty\lambda_n\left(\int_0^t\left(F(\cdot,\tau)-F(\cdot,t),\varphi_n\right)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^\alpha\right)d\tau\right)\varphi_n(x).$$

Then $v\in C^\theta([0,T];L^2(\Omega))$ and

$$\|v\|_{C^\theta([0,T];L^2(\Omega))}\leqslant C_{12}\|F\|_{C^\theta([0,T];L^2(\Omega))}.$$

Proof. We take $0 \leqslant t < t+h \leqslant T$. Then

$$\begin{aligned}
v(x,t+h)-v(x,t) &= \sum_{n=1}^{\infty}\lambda_n\Biggl\{\int_0^{t+h}\bigl(F(\cdot,\tau)-F(\cdot,t+h),\varphi_n\bigr)(t+h-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t+h-\tau)^\alpha\bigr)\,d\tau \\
&\quad -\int_0^{t}\bigl(F(\cdot,\tau)-F(\cdot,t),\varphi_n\bigr)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t-\tau)^\alpha\bigr)\,d\tau\Biggr\}\varphi_n(x) \\
&= \sum_{n=1}^{\infty}\lambda_n\Biggl\{\int_0^{t}\bigl(F(\cdot,\tau)-F(\cdot,t),\varphi_n\bigr)\bigl((t+h-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t+h-\tau)^\alpha\bigr) \\
&\quad -(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t-\tau)^\alpha\bigr)\bigr)d\tau\Biggr\}\varphi_n(x) \\
&\quad +\sum_{n=1}^{\infty}\lambda_n\Biggl\{\int_0^{t}\bigl(F(\cdot,t)-F(\cdot,t+h),\varphi_n\bigr)(t+h-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t+h-\tau)^\alpha\bigr)\,d\tau\Biggr\}\varphi_n(x) \\
&\quad +\sum_{n=1}^{\infty}\lambda_n\Biggl\{\int_t^{t+h}\bigl(F(\cdot,\tau)-F(\cdot,t+h),\varphi_n\bigr)(t+h-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t+h-\tau)^\alpha\bigr)\,d\tau\Biggr\}\varphi_n(x) \\
&= I_1(x,t)+I_2(x,t)+I_3(x,t).
\end{aligned}$$

We estimate each of the three terms separately.

For $0 < t-\tau < t-\tau+h \leqslant T$, by Lemma 3.1 we have

$$\begin{aligned}
&\bigl|\lambda_n\bigl\{(t+h-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t+h-\tau)^\alpha\bigr)-(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t-\tau)^\alpha\bigr)\bigr\}\bigr| \\
&\quad =\lambda_n\left|\int_{t-\tau}^{t-\tau+h}s^{\alpha-2}E_{\alpha,\alpha-1}\bigl(-\lambda_n s^\alpha\bigr)\,ds\right|\leqslant\lambda_n\int_{t-\tau}^{t-\tau+h}\frac{C_1 s^{\alpha-2}}{1+\lambda_n s^\alpha}\,ds \\
&\quad \leqslant C_1\int_{t-\tau}^{t-\tau+h}s^{-2}\,ds=\frac{C_1 h}{(t-\tau+h)(t-\tau)}.
\end{aligned}$$

At first equality, we have used formula (1.83) on p. 22 in [37]: $\frac{d}{dt}(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha))=t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)$.

We set $C_{13}=\|F\|_{C^\theta([0,T];L^2(\Omega))}$. Then by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\bigl\|I_1(\cdot,t)\bigr\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}\lambda_n^2\Biggl\{\int_0^{t}\bigl(F(\cdot,\tau)-F(\cdot,t),\varphi_n\bigr)\bigl((t+h-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t+h-\tau)^\alpha\bigr) \\
&\quad -(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t-\tau)^\alpha\bigr)\bigr)d\tau\Biggr\}^2 \\
&\leqslant C_1^2h^2\sum_{n=1}^{\infty}\lambda_n^2\left|\int_0^t\bigl\{\bigl|\bigl(F(\cdot,\tau)-F(\cdot,t),\varphi_n\bigr)\bigr|(t+h-\tau)^{-\frac12}(t-\tau)^{-\frac{\theta+1}{2}}\bigr\}\bigl\{(t+h-\tau)^{-\frac12}(t-\tau)^{\frac{\theta-1}{2}}\bigr\}\,d\tau\right|^2 \\
&\leqslant C_1^2h^2\sum_{n=1}^{\infty}\int_0^t\bigl(F(\cdot,\tau)-F(\cdot,t),\varphi_n\bigr)^2(t+h-\tau)^{-1}(t-\tau)^{-\theta-1}\,d\tau\int_0^t(t+h-\tau)^{-1}(t-\tau)^{\theta-1}\,d\tau \\
&\leqslant C_{13}^2C_1^2h^2\left(\int_0^t(t+h-\tau)^{-1}(t-\tau)^{-1+\theta}\,d\tau\right)^2.
\end{aligned}$$

On the other hand, by $0<\theta<1$, we have

$$\int_0^\infty \frac{\eta^{\theta-1}}{\eta+h}\,d\eta = \frac{\pi h^{\theta-1}}{\sin(\theta\pi)}$$

(e.g., Prudnikov, Brychkov and Marichev [39, vol. I, formula 2.2.4-25 in Chapter 2]). Hence

$$\left(\int_0^t (t+h-\tau)^{-1}(t-\tau)^{-1+\theta}\,d\tau\right)^2 \leqslant \left(\int_0^t \frac{\eta^{\theta-1}}{\eta+h}\,d\eta\right)^2 \leqslant \left(\int_0^\infty \frac{\eta^{\theta-1}}{\eta+h}\,d\eta\right)^2 \leqslant C_{14}h^{2\theta-2}.$$

Hence

$$\left\|I_1(\cdot,t)\right\|_{L^2(\Omega)} \leqslant C_{15}C_{13}^2h^{2\theta}.$$

By Lemma 3.2, we have

$$\begin{aligned}
\left\|I_2(\cdot,t)\right\|^2_{L^2(\Omega)} &= \sum_{n=1}^\infty \bigl(F(\cdot,t)-F(\cdot,t+h),\varphi_n\bigr)^2\left(\int_0^t \lambda_n(t+h-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t+h-\tau)^\alpha\bigr)\,d\tau\right)^2\\
&= \sum_{n=1}^\infty \bigl(F(\cdot,t)-F(\cdot,t+h),\varphi_n\bigr)^2\bigl(E_{\alpha,1}\bigl(-\lambda_n h^\alpha\bigr)-E_{\alpha,1}\bigl(-\lambda_n(t+h)^\alpha\bigr)\bigr)^2\\
&\leqslant C_{16}^2C_{13}^2h^{2\theta},
\end{aligned}$$

and

$$\begin{aligned}
\left\|I_3(\cdot,t)\right\|^2_{L^2(\Omega)} &= \sum_{n=1}^\infty \lambda_n^2\left(\int_t^{t+h} \bigl(F(\cdot,\tau)-F(\cdot,t+h),\varphi_n\bigr)(t+h-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t+h-\tau)^\alpha\bigr)\,d\tau\right)^2\\
&\leqslant \sum_{n=1}^\infty \int_t^{t+h} \bigl(F(\cdot,\tau)-F(\cdot,t+h),\varphi_n\bigr)^2(t+h-\tau)^{-\theta-1}\,d\tau\\
&\quad\times\int_t^{t+h}(t+h-\tau)^{2\alpha+\theta-1}\left(\frac{C_1\lambda_n}{1+\lambda_n(t+h-\tau)^\alpha}\right)^2d\tau\\
&\leqslant \sum_{n=1}^\infty\left(\int_t^{t+h} C_{13}^2(t+h-\tau)^{2\theta}(t+h-\tau)^{-\theta-1}\,d\tau\right)\left(\int_t^{t+h}(t+h-\tau)^{\theta-1}\left(\frac{C_1\lambda_n(t+h-\tau)^\alpha}{1+\lambda_n(t+h-\tau)^\alpha}\right)^2d\tau\right)\\
&\leqslant C_1^2C_{13}^2\left(\int_t^{t+h}(t+h-\tau)^{\theta-1}\,d\tau\right)^2 = C_{17}C_{13}^2h^{2\theta}.
\end{aligned}$$

Thus the proof of Lemma 3.4 is complete. □

Now we complete the proof of Theorem 2.4(i). By (3.8) and Lemma 3.2, we have

$$\begin{aligned}
\partial_t^\alpha u(x,t) = &-\sum_{n=1}^\infty \lambda_n\left\{(a,\varphi_n)E_{\alpha,1}\bigl(-\lambda_n t^\alpha\bigr)\right.\\
&\left.+\int_0^t \bigl(F(\cdot,\tau),\varphi_n\bigr)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t-\tau)^\alpha\bigr)\,d\tau\right\}\varphi_n(x)+\sum_{n=1}^\infty\bigl(F(\cdot,t),\varphi_n\bigr)\varphi_n(x)\\
= &-\sum_{n=1}^\infty \lambda_n(a,\varphi_n)E_{\alpha,1}\bigl(-\lambda_n t^\alpha\bigr)\varphi_n(x)+\sum_{n=1}^\infty\bigl(F(\cdot,t),\varphi_n\bigr)\varphi_n(x)\\
&-\sum_{n=1}^\infty\lambda_n\left(\int_0^t\bigl(F(\cdot,\tau)-F(\cdot,t),\varphi_n\bigr)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t-\tau)^\alpha\bigr)\,d\tau\right)\varphi_n(x)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{n=1}^{\infty}\lambda_n\left(\int_0^t \big(F(\cdot,t),\varphi_n\big)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n(t-\tau)^{\alpha}\big)\,d\tau\right)\varphi_n(x)\\
&= -\sum_{n=1}^{\infty}\lambda_n(a,\varphi_n)E_{\alpha,1}\big(-\lambda_n t^{\alpha}\big)\varphi_n(x) + \sum_{n=1}^{\infty}\big(F(\cdot,t),\varphi_n\big)\varphi_n(x)\\
&\quad -\sum_{n=1}^{\infty}\lambda_n\left(\int_0^t \big(F(\cdot,\tau)-F(\cdot,t),\varphi_n\big)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n(t-\tau)^{\alpha}\big)\,d\tau\right)\varphi_n(x)\\
&\quad -\sum_{n=1}^{\infty}\big(F(\cdot,t),\varphi_n\big)\big(1-E_{\alpha,1}\big(-\lambda_n t^{\alpha}\big)\big)\varphi_n(x)\\
&= \left\{-\sum_{n=1}^{\infty}\lambda_n(a,\varphi_n)E_{\alpha,1}\big(-\lambda_n t^{\alpha}\big)\varphi_n(x)\right\} + \left\{\sum_{n=1}^{\infty}\big(F(\cdot,t),\varphi_n\big)E_{\alpha,1}\big(-\lambda_n t^{\alpha}\big)\varphi_n(x)\right\}\\
&\quad + \left\{-\sum_{n=1}^{\infty}\lambda_n\left(\int_0^t \big(F(\cdot,\tau)-F(\cdot,t),\varphi_n\big)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n(t-\tau)^{\alpha}\big)\,d\tau\right)\varphi_n(x)\right\}\\
&= v_1(x,t)+v_2(x,t)-v(x,t).
\end{aligned}
\tag{3.13}
$$

From Lemma 3.4, it follows that $\|v_3\|_{C^{\theta}([0,T];L^2(\Omega))} \leqslant C_{18}\|F\|_{C^{\theta}([0,T];L^2(\Omega))}$. We have

$$
\begin{aligned}
v_2(x,t+h)-v_2(x,t) &= \sum_{n=1}^{\infty}\big(F(\cdot,t+h)-F(\cdot,t),\varphi_n\big)E_{\alpha,1}\big(-\lambda_n(t+h)^{\alpha}\big)\varphi_n(x)\\
&\quad -\sum_{n=1}^{\infty}\big(F(\cdot,t),\varphi_n\big)\big\{E_{\alpha,1}\big(-\lambda_n t^{\alpha}\big)-E_{\alpha,1}\big(-\lambda_n(t+h)^{\alpha}\big)\big\}\varphi_n(x) \equiv I_4(x,t)+I_5(x,t),
\end{aligned}
$$

and by Lemma 3.1 we obtain

$$
\begin{aligned}
\big\|I_4(\cdot,t)\big\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}\big(F(\cdot,t+h)-F(\cdot,t),\varphi_n\big)^2 E_{\alpha,1}\big(-\lambda_n(t+h)^{\alpha}\big)^2\\
&\leqslant C_1^2C_{13}^2h^{2\theta}.
\end{aligned}
$$

In order to estimate I_5, by Lemmata 3.1 and 3.2, we have

$$
\begin{aligned}
\big|E_{\alpha,1}\big(-\lambda_n t^{\alpha}\big)-E_{\alpha,1}\big(-\lambda_n(t+h)^{\alpha}\big)\big| &= \left|\int_{t+h}^{t}\lambda_n\tau^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n\tau^{\alpha}\big)\,d\tau\right|\\
&\leqslant \int_t^{t+h}\lambda_n\tau^{\alpha-1}\frac{C_1}{1+\lambda_n\tau^{\alpha}}\,d\tau \leqslant C_1\int_t^{t+h}\tau^{-1}\,d\tau.
\end{aligned}
\tag{3.14}
$$

Then for $\delta \leqslant t \leqslant T$, we have

$$
\begin{aligned}
\big\|I_5(\cdot,t)\big\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}\big(F(\cdot,t),\varphi_n\big)^2\big|E_{\alpha,1}\big(-\lambda_n t^{\alpha}\big)-E_{\alpha,1}\big(-\lambda_n(t+h)^{\alpha}\big)\big|^2\\
&\leqslant C_1^2\|F\|^2_{C([0,T];L^2(\Omega))}\left(\int_t^{t+h}\tau^{-1}\,d\tau\right)^2 = C_1^2C_{13}^2\left(\log\left(1+\frac{h}{t}\right)\right)^2\\
&\leqslant \frac{C_1^2C_{13}^2h^2}{\delta^2} \leqslant \frac{C_{19}C_{13}^2h^{2\theta}}{\delta^2}.
\end{aligned}
$$

Here we use also $\log(1+\eta) \leqslant \eta$ for $\eta > 0$.

Finally we will estimate $v_1(x,t)$. By Lemmata 3.1 and 3.2, we have

$$\begin{aligned}\bigl\|v_1(\cdot,t+h)-v_1(\cdot,t)\bigr\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}\lambda_n^2(a,\varphi_n)^2\bigl(E_{\alpha,1}\bigl(-\lambda_n t^\alpha\bigr)-E_{\alpha,1}\bigl(-\lambda_n(t+h)^\alpha\bigr)\bigr)^2\\ &\leqslant \sum_{n=1}^{\infty}\lambda_n^2(a,\varphi_n)^2\left(\int_t^{t+h}\lambda_n\tau^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n\tau^\alpha\bigr)\,d\tau\right)^2\\ &\leqslant C_1^2\sum_{n=1}^{\infty}\lambda_n^2(a,\varphi_n)^2\left(\int_t^{t+h}\frac{\lambda_n\tau^{\alpha-1}}{1+\lambda_n\tau^\alpha}\,d\tau\right)^2\\ &\leqslant C_1^2\sum_{n=1}^{\infty}\bigl((-L)a,\varphi_n\bigr)^2\left(\int_t^{t+h}\lambda_n\tau^{\alpha-1}\frac{1}{\lambda_n\tau^\alpha}\,d\tau\right)^2\\ &\leqslant \frac{C_1^2\|a\|^2_{H^2(\Omega)}h^2}{\delta^2}.\end{aligned}$$

Thus the proof of (i) is complete. The proof of Theorem 2.4(ii) follows from (3.13) and Lemma 3.4.

Finally we will complete the proof of Theorem 2.4(iii). From (3.13) and Lemma 3.4, it is sufficient to prove that $I_5 \in C^\theta([0,T];L^2(\Omega))$. Since $F(\cdot,0)=0$ implies $\|F\|_{L^2(\Omega)} \leqslant C_{13}t^\theta$, by (3.14) we have

$$\begin{aligned}\bigl\|I_5(\cdot,t)\bigr\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}\bigl(F(\cdot,t),\varphi_n\bigr)^2\bigl|E_{\alpha,1}\bigl(-\lambda_n t^\alpha\bigr)-E_{\alpha,1}\bigl(-\lambda_n(t+h)^\alpha\bigr)\bigr|^2\\ &\leqslant C_1^2C_{13}^2t^{2\theta}\left(\int_t^{t+h}\tau^{-1}\,d\tau\right)^2 \leqslant C_1^2C_{13}^2\left(\int_t^{t+h}t^\theta\tau^{-1}\,d\tau\right)^2\\ &\leqslant C_1^2C_{13}^2\left(\int_t^{t+h}\tau^{\theta-1}\,d\tau\right)^2 = \frac{C_1^2C_{13}^2}{\theta^2}\bigl\{(t+h)^\theta-t^\theta\bigr\}^2 \leqslant \frac{C_1^2C_{13}^2h^{2\theta}}{\theta^2}.\end{aligned}$$

Thus the proof of (iii) is complete. □

Proof of Corollary 2.5. We first have

$$\begin{aligned}(-L)^{\frac{\alpha-1}{\alpha}}u(\cdot,t) &= \sum_{n=1}^{\infty}\left(\int_0^t\bigl(F(\cdot,\tau),\varphi_n\bigr)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t-\tau)^\alpha\bigr)d\tau\right)(-L)^{\frac{\alpha-1}{\alpha}}\varphi_n\\ &= \sum_{n=1}^{\infty}\left(\int_0^t\bigl(F(\cdot,\tau),\varphi_n\bigr)\bigl(\lambda_n(t-\tau)^\alpha\bigr)^{\frac{\alpha-1}{\alpha}}E_{\alpha,\alpha}\bigl(-\lambda_n(t-\tau)^\alpha\bigr)\,d\tau\right)\varphi_n.\end{aligned}$$

On the other hand, by $1<\alpha\leqslant 2$, we see from Lemma 3.1 that

$$\bigl|\bigl(\lambda_n(t-\tau)^\alpha\bigr)^{\frac{\alpha-1}{\alpha}}E_{\alpha,\alpha}\bigl(-\lambda_n(t-\tau)^\alpha\bigr)\bigr| \leqslant C_1\sup_{\eta>0}\frac{\eta^{\frac{\alpha-1}{\alpha}}}{1+\eta} \leqslant \frac{C_1(\alpha-1)^{\frac{\alpha-1}{\alpha}}}{\alpha}.$$

Therefore, in terms of the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\bigl\|(-L)^{\frac{\alpha-1}{\alpha}}u(\cdot,t)\bigr\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}\left|\int_0^t\bigl(F(\cdot,\tau),\varphi_n\bigr)\bigl(\lambda_n(t-\tau)^\alpha\bigr)^{\frac{\alpha-1}{\alpha}}E_{\alpha,\alpha}\bigl(-\lambda_n(t-\tau)^\alpha\bigr)\,d\tau\right|^2\\ &\leqslant \frac{C_1^2(\alpha-1)^{\frac{2(\alpha-1)}{\alpha}}}{\alpha^2}\sum_{n=1}^{\infty}\left|\int_0^t\bigl(F(\cdot,\tau),\varphi_n\bigr)\,d\tau\right|^2 \leqslant \frac{C_1^2T(\alpha-1)^{\frac{2(\alpha-1)}{\alpha}}}{\alpha^2}\int_0^T\sum_{n=1}^{\infty}\bigl|\bigl(F(\cdot,\tau),\varphi_n\bigr)\bigr|^2\,d\tau.\end{aligned}$$

Therefore estimate (2.13) is seen, and the proof of Corollary 2.5 is complete. □

Proof of Corollary 2.6. By Lemma 3.1, we have

$$\begin{aligned}\|u(\cdot,t)\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}(a,\varphi_n)^2 E_{\alpha,1}\big(-\lambda_n t^\alpha\big)^2\\ &\leqslant \sum_{n=1}^{\infty}(a,\varphi_n)^2\left(\frac{C_1}{1+\lambda_n t^\alpha}\right)^2 \leqslant \left(\frac{C_1}{1+\lambda_1 t^\alpha}\|a\|_{L^2(\Omega)}\right)^2, \quad t \geqslant 0.\end{aligned}$$

By Lemma 3.2, we have

$$\partial_t^m u(\cdot,t) = -\sum_{n=1}^{\infty}\lambda_n t^{\alpha-m}(a,\varphi_n)E_{\alpha,\alpha-m+1}\big(-\lambda_n t^\alpha\big)\varphi_n$$

for $m \in \mathbb{N}$, so that

$$\big\|\partial_t^m u(\cdot,t)\big\|^2_{L^2(\Omega)} \leqslant \frac{C_{21}}{t^{2m}}\|a\|^2_{L^2(\Omega)}. \qquad \square$$

Proof of Corollary 2.7. By Lemma 3.2, for $m \geqslant 2$, we have

$$\partial_t^m u(\cdot,t) = \sum_{n=1}^{\infty}\big\{-\lambda_n(a,\varphi_n)t^{\alpha-m}E_{\alpha,\alpha-m+1}\big(-\lambda_n t^\alpha\big) - \lambda_n(b,\varphi_n)t^{\alpha-(m-1)}E_{\alpha,\alpha-(m-1)+1}\big(-\lambda_n t^\alpha\big)\big\}\varphi_n.$$

Henceforth, in terms of Lemma 3.1, we can argue to complete the proof. $\square$

4. Applications of the eigenfunction expansion

We apply the eigenfunction expansion of the solution only in the case of $0 < \alpha < 1$. The arguments in the case of $1 < \alpha < 2$ are similar. Let L be the same elliptic operator defined in Section 2.

4.1. Backward problem in time

Theorem 4.1. *Let $T > 0$ be arbitrarily fixed. For any given $a_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique weak solution $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ to (1.1) and (1.2) with $F = 0$ such that $u(\cdot,T) = a_1$. Moreover there exist constants $C_1, C_2 > 0$ such that*

$$C_1\big\|u(\cdot,0)\big\|_{L^2(\Omega)} \leqslant \big\|u(\cdot,T)\big\|_{H^2(\Omega)} \leqslant C_2\big\|u(\cdot,0)\big\|_{L^2(\Omega)}. \tag{4.1}$$

Here C_1, C_2 are independent of choices of a_1.

The backward problem of the classical diffusion equation (e.g., $\alpha = 1$) is severely ill-posed (e.g., Isakov [17]), and any estimate of Lipschitz type by Sobolev norm is impossible.

Proof of Theorem 4.1. By (2.2), we have

$$u(x,T) = \sum_{n=1}^{\infty}(a,\varphi_n)E_{\alpha,1}\big(-\lambda_n T^\alpha\big)\varphi_n(x).$$

Hence we note that $u(\cdot,T) \in H^2(\Omega)$ if and only if

$$\sum_{n=1}^{\infty}(a,\varphi_n)^2\lambda_n^2 E_{\alpha,1}\big(-\lambda_n T^\alpha\big)^2 < \infty.$$

Since $E_{\alpha,1}(-\lambda_n t^\alpha)$, $t > 0$, is completely monotonic (e.g., [14]),

$$\frac{d}{dt}E_{\alpha,1}\big(-\lambda_n t^\alpha\big) \leqslant 0, \quad t > 0 \tag{4.2}$$

and

$$E_{\alpha,1}\big(-\lambda_n t^\alpha\big) \geqslant 0, \quad t > 0. \tag{4.3}$$

Hence by (4.2) and (4.3), we obtain

$$E_{\alpha,1}\big(-\lambda_n t^\alpha\big) > 0, \quad t \geqslant 0. \tag{4.4}$$

In fact, we assume that there exists $t_0 > 0$ such that $E_{\alpha,1}(-\lambda_n t_0^\alpha) = 0$. Then $E_{\alpha,1}(-\lambda_n t^\alpha) = 0$ for all $t \geqslant t_0$ by (4.2) and (4.3). Therefore the analyticity in t implies that $E_{\alpha,1}(-\lambda_n t^\alpha) = 0$ for all $t \geqslant 0$, which contradicts $E_{\alpha,1}(0) \neq 0$.

For $a_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$C_1' \|a_1\|_{H^2(\Omega)}^2 \leqslant \sum_{n=1}^{\infty} \lambda_n^2 (a_1, \varphi_n)^2 \leqslant C_2' \|a_1\|_{H^2(\Omega)}^2.$$

By (4.4) we can set

$$c_n = \frac{(a_1, \varphi_n)}{E_{\alpha,1}(-\lambda_n T^\alpha)}.$$

In terms of (4.4), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} c_n^2 &= \sum_{n=1}^{\infty} \frac{(a_1, \varphi_n)^2}{E_{\alpha,1}(-\lambda_n T^\alpha)^2} \\ &= \sum_{n=1}^{\infty} \lambda_n^2 T^{2\alpha} \Gamma(1-\alpha)^2 (a_1, \varphi_n)^2 \left(\frac{1}{1 + O(\lambda_n^{-1} t^{-\alpha})} \right)^2 \leqslant C_3 T^{2\alpha} \sum_{n=1}^{\infty} \lambda_n^2 (a_1, \varphi_n)^2.\end{aligned}$$

Setting $a = \sum_{n=1}^{\infty} c_n \varphi_n$ and denoting the solution to (1.1)–(1.3) with this initial value a by $u(x,t)$, we have $a_1 = u(\cdot, T)$ and $\|a\|_{L^2(\Omega)} \leqslant C_4 \|u(\cdot, T)\|_{H^2(\Omega)}$. The second inequality in (4.1) is already proved in Theorem 2.1. $\square$

4.2. Uniqueness of solution to a boundary value problem

We note that $-L$ defines the fractional power $(-L)^\beta$ with $\beta \in \mathbb{R}$ and

$$\|u\|_{H^{2\beta}(\Omega)} \leqslant C_5 \left\| (-L)^\beta u \right\|_{L^2(\Omega)}$$

(e.g., [36]).

Theorem 4.2. *Let $a \in \mathcal{D}((-L)^{2\beta})$ with $\beta > \frac{d}{4}$. Let $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ satisfy* (1.1) *and* (1.2) *with $F = 0$. Let $\omega \subset \Omega$ be an arbitrarily chosen subdomain and let $T > 0$. Then $u(x,t) = 0$, $x \in \omega$, $0 < t < T$, implies $u = 0$ in $\Omega \times (0,T)$.*

This theorem corresponds to Corollary 2.3 in Schmidt and Weck [45] and see Nakagiri [33] for similar arguments for other inverse problems. For $\alpha = 1$, we have that the uniqueness holds without (1.2), which is the unique continuation (e.g., [17]). However for $\alpha \neq 1$, we do not know whether the uniqueness holds without (1.2).

Proof of Theorem 4.2. By $\lambda_n = O(n^{\frac{2}{d}})$ and $a \in \mathcal{D}((-L)^{2\beta})$ and the Sobolev embedding theorem, we have

$$\|\varphi_n\|_{L^\infty(\Omega)} \leqslant C_6'' \|\varphi_n\|_{H^{2\beta}(\Omega)} \leqslant C_6' \left\| (-L)^\beta \varphi_n \right\|_{L^2(\Omega)} \leqslant C_6 |\lambda_n|^\beta$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} \left|(a, \varphi_n)\right| \|\varphi_n\|_{L^\infty(\Omega)} &\leqslant C_6 \sum_{n=1}^{\infty} \left|(a, \varphi_n)\right| |\lambda_n|^\beta = C_6 \sum_{n=1}^{\infty} \left|(a, \varphi_n)\right| |\lambda_n|^{2\beta} |\lambda_n|^{-\beta} \\ &\leqslant C_6 \left(\sum_{n=1}^{\infty} \left|(a, \varphi_n)\right|^2 |\lambda_n|^{4\beta} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2\beta}} \right)^{\frac{1}{2}} \\ &\leqslant C_7 \left(\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4\beta}{d}}} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \left|(a, \varphi_n)\right|^2 \left(|\lambda_n|^{2\beta}\right)^2 \right)^{\frac{1}{2}} < \infty. \end{aligned} \tag{4.5}$$

Then, by Lemma 3.1, $\sum_{n=1}^{\infty} (a, \varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n(x)$ can be extended analytically in t to $\{z \in \mathbb{C};\ z \neq 0,\ |\arg z| \leqslant \mu_0\}$ with some $\mu_0 > 0$. Therefore, since

$$u(x,t) = \sum_{n=1}^{\infty} (a, \varphi_n) E_{\alpha,1}\left(-\lambda_n t^\alpha\right) \varphi_n(x) = 0, \quad x \in \omega,\ 0 < t < T,$$

we have

$$\sum_{n=1}^{\infty} (a, \varphi_n) E_{\alpha,1}\left(-\lambda_n t^\alpha\right) \varphi_n(x) = 0, \quad x \in \omega,\ t > 0. \tag{4.6}$$

We set $\sigma(-L) = \{\mu_k\}_{k\in\mathbb{N}}$ and we denote by $\{\varphi_{kj}\}_{1\leqslant j\leqslant m_k}$ an orthonormal basis of $\mathrm{Ker}(\mu_k + L)$. Note that we consider $\sigma(-L)$ as set, not as sequence with multiplicities. Therefore we can rewrite (4.6) by

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m_k}(a,\varphi_{kj})\varphi_{kj}(x)\right)E_{\alpha,1}\left(-\mu_k t^{\alpha}\right) = 0, \quad x\in\omega,\ t>0. \tag{4.7}$$

By (4.5) and Lemma 3.3, we have

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\left|(a,\varphi_{kj})\varphi_{kj}(x)\right|\left|E_{\alpha,1}\left(-\mu_k t^{\alpha}\right)\right| \leqslant C_8\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\left|(a,\varphi_{kj})\right|\|\varphi_{kj}\|_{L^{\infty}(\Omega)} < \infty.$$

Hence the Lebesgue convergence theorem yields that

$$\int_0^{\infty} e^{-zt}\left(\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}(a,\varphi_{kj})\varphi_{kj}(x)E_{\alpha,1}\left(-\mu_k t^{\alpha}\right)\right)dt$$
$$= \sum_{k=1}^{\infty}\sum_{j=1}^{m_k}(a,\varphi_{kj})\left(\int_0^{\infty} e^{-zt}E_{\alpha,1}\left(-\mu_k t^{\alpha}\right)dt\right)\varphi_{kj}(x), \quad x\in\omega,\ \mathrm{Re}\, z>0. \tag{4.8}$$

We take the Laplace transform to have

$$\int_0^{\infty} e^{-zt}E_{\alpha,1}\left(-\mu_k t^{\alpha}\right)dt = \frac{z^{\alpha-1}}{z^{\alpha}+\mu_k}, \quad \mathrm{Re}\, z>0. \tag{4.9}$$

In fact, we can take the Laplace transforms termwise in the power series defining $E_{\alpha,1}(z)$ to obtain

$$\int_0^{\infty} e^{-zt}E_{\alpha,1}\left(-\mu_k t^{\alpha}\right)dt = \frac{z^{\alpha-1}}{z^{\alpha}+\mu_k}, \quad \mathrm{Re}\, z>\mu_k^{\frac{1}{\alpha}}$$

(cf. formula (1.80) on p. 21 in [37]). Since $\sup_{t\geqslant 0,k\in\mathbb{N}}|E_{\alpha,1}(-\mu_k t^{\alpha})| < \infty$ by Lemma 3.1, we see that $\int_0^{\infty} e^{-zt}E_{\alpha,1}(-\mu_k t^{\alpha})\,dt$ is analytic with respect to z in $\mathrm{Re}\, z>0$. Therefore the analytic continuation yields (4.9) for $\mathrm{Re}\, z>0$.

Hence (4.8) and (4.9) yield

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}(a,\varphi_{kj})\frac{z^{\alpha-1}}{z^{\alpha}+\mu_k}\varphi_{kj}(x) = 0, \quad x\in\omega,\ \mathrm{Re}\, z>0,$$

that is,

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}(a,\varphi_{kj})\frac{1}{\eta+\mu_k}\varphi_{kj}(x) = 0, \quad x\in\omega,\ \mathrm{Re}\,\eta>0. \tag{4.10}$$

By (4.5), we can analytically continue both sides of (4.10) in η, so that (4.10) holds for $\eta\in\mathbb{C}\setminus\{-\mu_k\}_{k\in\mathbb{N}}$. We can take a suitable disk which includes $-\mu_\ell$ and does not include $\{-\mu_k\}_{k\neq\ell}$. Integrating (4.10) in a disk, we have

$$u_\ell(x) \equiv \sum_{j=1}^{m_\ell}(a,\varphi_{\ell j})\varphi_{\ell j}(x) = 0, \quad x\in\omega.$$

Since $(L+\mu_\ell)u_\ell = 0$ in Ω, and $u_\ell = 0$ in ω, the unique continuation (e.g., Isakov [17]) implies $u_\ell = 0$ in Ω for each $\ell\in\mathbb{N}$. Since $\{\varphi_{\ell j}\}_{1\leqslant j\leqslant m_\ell}$ is linearly independent in Ω, we see that $(a,\varphi_{\ell j}) = 0$ for $1\leqslant j\leqslant m_\ell$, $\ell\in\mathbb{N}$. Therefore $u = 0$ in $\Omega\times(0,T)$. Thus the proof of Theorem 4.2 is complete. □

4.3. Decay rate at $t = \infty$

We state a different version of Corollary 2.6. In fact, the following theorem asserts that the solution cannot decay faster than $\frac{1}{t^m}$ with any $m\in\mathbb{N}$ if the solution does not vanish identically. It is a remarkable property of the fractional diffusion equation because the classical diffusion equation with $\alpha = 1$ admits non-zero solutions decaying exponentially. This is one description of the slower diffusion, compared to the classical one.

Theorem 4.3. *Let $a \in \mathcal{D}((-L)^{2\beta})$ with $\beta > \frac{d}{4}$ and let $\omega \subset \Omega$ be an arbitrary subdomain. Let $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ satisfy (1.1) and (1.2) with $F = 0$. We assume that for any $m \in \mathbb{N}$, there exists a constant $C(m) > 0$ such that*

$$\left\| u(\cdot,t) \right\|_{L^\infty(\omega)} \leqslant \frac{C(m)}{t^m} \quad \text{as } t \to \infty. \tag{4.11}$$

Then $u = 0$ in $\Omega \times (0,\infty)$.

Proof. By (4.5), the series

$$u(x,t) = \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} (a, \varphi_{kj}) E_{\alpha,1}\left(-\mu_k t^\alpha\right) \varphi_{kj}(x)$$

converges uniformly for $x \in \overline{\Omega}$ and $\delta \leqslant t \leqslant T$ with any $\delta, T > 0$. Hence, by Theorem 1.4 (pp. 33–34) in [37], for any $p \in \mathbb{N}$, we have

$$u(x,t) = -\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \sum_{\ell=1}^{p} \frac{(-1)^\ell}{\Gamma(1-\alpha\ell)\mu_k^\ell t^{\alpha\ell}} (a, \varphi_{kj}) \varphi_{kj}(x) + \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} O\left(\frac{1}{\mu_k^{p+1} t^{\alpha(p+1)}}\right) (a, \varphi_{kj}) \varphi_{kj}(x) \quad \text{as } t \to \infty.$$

We note that $\Gamma(1-\alpha) \neq 0$ by $1 - \alpha > 0$. Setting $m = 1$ in (4.11) and $p = 1$, multiplying t^α and letting $t \to \infty$, we have

$$\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \frac{1}{\Gamma(1-\alpha)\mu_k} (a, \varphi_{kj}) \varphi_{kj}(x) = 0, \quad x \in \omega.$$

By $0 < \alpha < 1$, there exists $\{\ell_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ such that $\lim_{j\to\infty} \ell_j = \infty$ and $\alpha\ell_j \notin \mathbb{N}$. In fact, let $\alpha \notin \mathbb{Q}$. Then $\ell\alpha \notin \mathbb{N}$ for any $\ell \in \mathbb{N}$. Let $\alpha \in \mathbb{Q}$. Set $\alpha = \frac{n_1}{m_1}$ where $m_1, n_1 \in \mathbb{N}$ have no common divisors except for 1. There exist infinitely many $\ell \in \mathbb{N}$ possessing no common divisors with m_1, and $\ell\alpha \in \mathbb{Q} \backslash \mathbb{N}$. Then $\Gamma(1 - \alpha\ell_j) \neq 0$.

Therefore, setting $p = 2, 3, \ldots$ and repeating the above argument, we obtain

$$\sum_{k=1}^{\infty} \frac{1}{\mu_k^{\ell_i}} \left(\sum_{j=1}^{m_k} (a, \varphi_{kj}) \varphi_{kj}(x) \right) = 0, \quad x \in \omega,\ \ell_i \in \mathbb{N}.$$

Hence

$$\sum_{j=1}^{m_1} (a, \varphi_{1j}) \varphi_{1j}(x) + \sum_{k=2}^{\infty} \left(\frac{\mu_1}{\mu_k}\right)^{\ell_i} \sum_{j=1}^{m_k} (a, \varphi_{kj}) \varphi_{kj}(x) = 0, \quad x \in \omega,\ \ell_i \in \mathbb{N}.$$

By (4.5) and $0 < \mu_1 < \mu_2 < \cdots$, we have

$$\begin{aligned} \left\| \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} \left(\frac{\mu_1}{\mu_k}\right)^{\ell_i} (a, \varphi_{kj}) \varphi_{kj} \right\|_{L^\infty(\Omega)} &\leqslant \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} \left|\frac{\mu_1}{\mu_k}\right|^{\ell_i} \left|(a, \varphi_{kj})\right| \|\varphi_{kj}\|_{L^\infty(\Omega)} \\ &\leqslant \left|\frac{\mu_1}{\mu_2}\right|^{\ell_i} \sum_{k=2}^{\infty} \sum_{j=1}^{m_k} \left|(a, \varphi_{kj})\right| \|\varphi_{kj}\|_{L^\infty(\Omega)} \leqslant C_9 \left|\frac{\mu_1}{\mu_2}\right|^{\ell_i}. \end{aligned}$$

Letting $\ell_i \to \infty$ and $|\frac{\mu_1}{\mu_2}| < 1$, we see that

$$\sum_{j=1}^{m_1} (a, \varphi_{1j}) \varphi_{1j}(x) = 0, \quad x \in \omega.$$

Similarly we obtain

$$\sum_{j=1}^{m_k} (a, \varphi_{kj}) \varphi_{kj}(x) = 0, \quad x \in \omega,\ k \in \mathbb{N}.$$

Since $a = \sum_{k=1}^{\infty} (\sum_{j=1}^{m_k} (a, \varphi_{kj}) \varphi_{kj})$ in $L^2(\Omega)$, we can conclude that $u = 0$ in $\Omega \times (0,\infty)$. Thus the proof of Theorem 4.3 is complete. □

4.4. Inverse source problem

For

$$\begin{cases} \partial_t^\alpha u(x,t) = Lu(x,t) + f(x)p(t), & x \in \Omega,\ 0 < t < T, \\ u(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T, \\ u(x,0) = 0, & x \in \Omega, \end{cases} \tag{4.12}$$

we discuss

Inverse source problem. Let f be given and $x_0 \in \Omega$ be given. Determine $p(t)$, $0 < t < T$, by $u(x_0,t)$, $0 < t < T$.

In this inverse problem, given a spatial distribution of a source, we are required to determine a time varying factor $p(t)$. As for this kind of inverse problem for parabolic equation, see e.g., Cannon and Esteva [5], Saitoh, Tuan and Yamamoto [42,43] for example. Here we prove a stability estimate in one simple case:

Theorem 4.4. *Let $f \in \mathcal{D}((-L)^\beta)$ with $\beta > 1 + \frac{3d}{4}$ and let u satisfy* (4.12) *for $p \in C[0,T]$. We assume that*

$$f(x_0) \neq 0.$$

Then there exist constants $C_{10}, C_{11} > 0$ such that

$$C_{10}\left\|\partial_t^\alpha u(x_0,\cdot)\right\|_{C[0,T]} \leqslant \|p\|_{C[0,T]} \leqslant C_{11}\left\|\partial_t^\alpha u(x_0,\cdot)\right\|_{C[0,T]}. \tag{4.13}$$

In the theorem, the condition $f(x_0) \neq 0$ yields the both-sided Lipschitz stability, and $f(x_0) \neq 0$ means that the observation point is on the inside of the source, and the choice as observation point is not realistic because in practical inverse source problems, it is assumed that one cannot have access to the source and has to determine by data away from the source. In the case of $f(x_0) = 0$, the stability estimate is expected to be worse (e.g., [5,42,43] for the parabolic case) and for the fractional diffusion equation, we can discuss the case of $f(x_0) = 0$, but here we discuss only the case $f(x_0) \neq 0$.

Proof of Theorem 4.4. By $p \in C[0,T]$ and $f \in \mathcal{D}((-L)^\beta)$, we apply Theorem 2.2 to obtain

$$u(x,t) = \sum_{n=1}^\infty \left(\int_0^t p(\tau)(f,\varphi_n)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n(t-\tau)^\alpha\big)\,d\tau\right)\varphi_n(x)$$

in $L^2(0,T;H^2(\Omega))$ and

$$\partial_t^\alpha u(x,t) = p(t)f(x) + \sum_{n=1}^\infty -\lambda_n\left(\int_0^t p(\tau)(f,\varphi_n)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n(t-\tau)^\alpha\big)\,d\tau\right)\varphi_n(x) \tag{4.14}$$

in $L^2(\Omega \times (0,T))$. By $f \in \mathcal{D}((-L)^\beta)$ with $\beta > 1 + \frac{3d}{4}$ and the Sobolev embedding theorem, we have

$$\begin{aligned} \left\|\lambda_n(f,\varphi_n)\varphi_n\right\|_{L^\infty(\Omega)} &\leqslant C_{12}\left\|\lambda_n(f,\varphi_n)\varphi_n\right\|_{H^{2\beta-2-d}(\Omega)} \\ &\leqslant C_{13}\left\|\lambda_n(f,\varphi_n)(-L)^{\beta-1-\frac{d}{2}}\varphi_n\right\|_{L^2(\Omega)} \\ &= C_{13}\left\|\lambda_n^{\beta-\frac{d}{2}}(f,\varphi_n)\varphi_n\right\|_{L^2(\Omega)} \leqslant C_{13}\lambda_n^{-\frac{d}{2}}\left|\big((-L)^\beta f,\varphi_n\big)\right|. \end{aligned}$$

Hence, by [8], we see that $\lambda_n \geqslant C'_{13}n^{\frac{2}{d}}$, for $(x,t) \in \overline{\Omega} \times [0,T]$ we obtain

$$\begin{aligned} &\sum_{n=1}^\infty\left|\int_0^t \lambda_n p(\tau)(f,\varphi_n)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n(t-\tau)^\alpha\big)\,d\tau\,\varphi_n(x)\right| \\ &\quad\leqslant C_{13}\sum_{n=1}^\infty \|p\|_{C[0,T]}\frac{1}{n}\left|\big((-L)^\beta f,\varphi_n\big)\right|\int_0^t (t-\tau)^{\alpha-1}\,d\tau \\ &\quad\leqslant C_{14}\|p\|_{C[0,T]}\left(\sum_{n=1}^\infty \frac{1}{n^2}\right)^{\frac{1}{2}}\left(\sum_{n=1}^\infty\left|\big((-L)^\beta f,\varphi_n\big)\right|^2\right)^{\frac{1}{2}} \leqslant C_{15}\|p\|_{C[0,T]}\left\|(-L)^\beta f\right\|_{L^2(\Omega)} \\ &\quad\leqslant C_{15}\|p\|_{C[0,T]}. \end{aligned} \tag{4.15}$$

Therefore we see that $\partial_t^\alpha u \in C(\overline{\Omega} \times [0,T])$, the series (4.14) is convergent in $C(\overline{\Omega} \times [0,T])$ and

$$\left\|\partial_t^\alpha u\right\|_{C(\overline{\Omega}\times[0,T])} \leqslant C_{15}\|p\|_{C[0,T]}.$$

Hence the first inequality in (4.13) is proved.

Since the series (4.14) is convergent in $C(\overline{\Omega} \times [0,T])$, we have

$$\partial_t^\alpha u(x_0,t) = p(t)f(x_0) + \sum_{n=1}^{\infty}\int_0^t p(\tau)\bigl\{-\lambda_n(f,\varphi_n)E_{\alpha,\alpha}\bigl(-\lambda_n(t-\tau)^\alpha\bigr)\varphi_n(x_0)\bigr\}(t-\tau)^{\alpha-1}\,d\tau$$

for $0 < t < T$. Setting

$$Q(t) = \sum_{n=1}^{\infty} -\lambda_n(f,\varphi_n)E_{\alpha,\alpha}\bigl(-\lambda_n t^\alpha\bigr)\varphi_n(x_0),$$

similarly to (4.15) we can see that $Q \in C[0,T]$. Therefore

$$\partial_t^\alpha u(x_0,t) = p(t)f(x_0) + \int_0^t (t-\tau)^{\alpha-1}Q(t-\tau)p(\tau)\,d\tau, \quad 0 < t < T,$$

that is,

$$p(t) = \frac{\partial_t^\alpha u(x_0,t)}{f(x_0)} - \frac{1}{f(x_0)}\int_0^t (t-\tau)^{\alpha-1}Q(t-\tau)p(\tau)\,d\tau, \quad 0 < t < T$$

by $f(x_0) \neq 0$. Hence

$$\bigl|p(t)\bigr| \leqslant C_{16}\left\|\partial_t^\alpha u(x_0,\cdot)\right\|_{C[0,T]} + C_{16}\|Q\|_{C[0,T]}\int_0^t (t-\tau)^{\alpha-1}\bigl|p(\tau)\bigr|\,d\tau, \quad 0 < t < T.$$

Applying an inequality of Gronwall type with weakly singular kernel $(t-\tau)^{\alpha-1}$ (e.g., Lemma 7.1.1 (pp. 188–189) in [16]), we see

$$\bigl|p(t)\bigr| \leqslant C_{17}\left\|\partial_t^\alpha u(x_0,\cdot)\right\|_{C[0,T]}, \quad 0 < t < T,$$

that is, the second inequality in (4.13) is proved. Thus the proof of Theorem 4.4 is complete. □

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