



Regularity of solutions to coercive and self-controlling viscoplastic problems

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ABSTRACT

We prove higher regularity of solutions to viscoplastic problems. The main idea is to use the finite difference method, which allows us to cancel the monotone nonlinearities. Our main assumption is that velocity is in $L^2(H^1)$. This is true for all coercive and some special (for example linear self-controlling) non-coercive models.

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1. Introduction

We will prove regularity for the solutions of the equations from inelastic deformations theory. For a presentation of this theory we refer to [2], where existence of solutions with homogeneous boundary values for coercive models is proved (for the definition of coercivity see Eq. (1.4) in the present paper). Papers [3] and [9] discuss non-homogeneous problems as well as non-coercive ones.

The idea of this paper is to use finite differences approximation and apply monotonicity argument to get rid of nonlinear terms coming from the constitutive relation (cf. Eq. (D)₃ or (Q)₃ in the present paper). The method of finite differences was used to prove regularity of stresses in the Prandtl–Reuss and Norton–Hoff models, cf. [5,6,11,16]. Recently, H.-D. Alber and S. Nesenenko [4] have proven regularity of solutions to general coercive models of viscoplasticity with variable coefficients using a perturbation argument. D. Knees in [13] improved their result using reflection over the boundary technique, obtaining stresses in the space $H^{\frac{1}{2}-\delta}$. To our knowledge the best result so far is that of $H^{\frac{1}{2}+\delta}$ regularity with a very small $\delta > 0$ which has been obtained in the PhD thesis of Löbach [14] (for the equations of isotropic and kinematic hardening).

Our paper gives a simpler approach, while generalizing the results also for non-coercive models,¹ both quasistatic and dynamic, possessing sufficiently regular solutions (for example the *linear self-controlling* models, studied in [9]). Our proofs work for general, n -dimensional problems (of course $n = 3$ is of particular interest).

We are also able to show partial regularity of normal derivatives with the methods inspired by those in [4], with the aid of the Fourier transform.

The basic assumption we will use is that the speed of displacements, u_t , has regularity $L^2(H^1)$. This is true for coercive models and also linear self-controlling. Unfortunately many models do not satisfy this condition (for example Prandtl–Reuss) and so our method fails in this case.

In fact, for the case of non-coercive models the gradients of the displacement field (and so also the strain tensor) need not be regular, even in a simple, 1-dimensional case (cf. [17, p. 27], and references therein). For example, the classical theory

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¹ The methods presented in [4] rely heavily on the coercivity assumption.

of existence for the Prandtl–Reuss model gives that in fact the strain is only a Young measure (and so not even a function). Therefore for such problems one only hopes to improve regularity of the stresses and not of the strain.

The paper is organized as follows: first we present the problem being studied. Then we describe some function spaces and introduce the notation we will be using. Next we study local regularity properties of solutions to quasistatic and dynamic problems. The main method is outlined here. We then show how the proofs can be extended to the case of variable coefficients, but only with the assumption that the model is coercive. After that we present some boundary regularity results. For tangential derivatives we treat first the case of straight boundary, and then show how, using ideas from [4] general boundaries can be treated. Next we study the regularity of normal derivatives for the case of quasistatic models. These always pose some problems when dealing with nonlinear systems. Again, the idea from [4] works, but we present a simpler approach, which unfortunately gives also a worse result. Finally, we show how our ideas can be used, for coercive and non-coercive problems, if the regularity assumptions on u_t or just u are weakened.

1.1. The equations of viscoplasticity

The object of this paper is to study the regularity of solutions to the following equations from viscoplasticity theory:

$$\begin{aligned} \rho v_t(t, x) - \operatorname{div}_x \mathcal{D}(\varepsilon(t, x) - \varepsilon^p(t, x)) &= f(t, x), \quad t > 0, x \in \Omega, \\ \varepsilon_t(t, x) &= \frac{1}{2}(\nabla v(t, x) + \nabla^T v(t, x)), \quad t > 0, x \in \Omega, \\ z_t(t, x) &\in g(-\rho \nabla_z \psi(\varepsilon(t, x), z(t, x))), \quad t > 0, x \in \Omega, \\ v(t, x) &= g_D(t, x), \quad t > 0, x \in \partial\Omega, \\ v(0, x) &= v^0(x), \quad \varepsilon(0, x) = \varepsilon^0(x), \quad z(0, x) = z^0(x), \quad x \in \Omega. \end{aligned} \quad (D)$$

The set $\Omega \subset \mathbb{R}^n$ is assumed to be open, bounded, with boundary of class C^2 .

Eq. (D)₁ is the so-called *balance equation*. It is just Newton's second law of dynamics written in differential form (which is applicable to continuous media). ρ denotes the density of the material. $v(t, x)$ is the velocity at time t of the material point $x \in \Omega$.

\mathcal{D} is called the *elasticity tensor*. It is a positive definite, linear operator acting on the space $S(n)$ of symmetric $n \times n$ matrices.

ε denotes the *strain*. We assume small deformations, i.e. that

$$\varepsilon = \varepsilon(\nabla u) = \frac{1}{2}(\nabla u + \nabla^T u) \quad (1.1)$$

where u is the *displacement*, $u_t = v$. Algebraically, Eq. (1.1) means that ε is the symmetric part of ∇u (in this paper by $\varepsilon(A)$ we will thus denote the symmetric part of matrix A). The general assumption of viscoplasticity is that the strain can be decomposed into the *elastic* and *plastic* parts:

$$\varepsilon = \varepsilon^e + \varepsilon^p$$

(usually this decomposition is only formal and physically unobservable). Hooke's law for elastic material then states that the stream of the forces acting on the boundary of some small subset of Ω is proportional to the elastic part of the strain, i.e. to $\varepsilon^e = \varepsilon - \varepsilon^p$. From this one can then deduce the term $-\operatorname{div}_x \mathcal{D}(\varepsilon - \varepsilon^p)$ appearing in (D)₁. The quantity $T = \mathcal{D}(\varepsilon - \varepsilon^p)$ is called the *stress tensor*.

f in (D)₁ denotes the *external forces* acting on the body. This is usually gravity, but in more complicated situations this can also denote electromagnetic forces, for example.

Eq. (D)₂ is just (1.1) differentiated with respect to time.

In Eq. (D)₃, z denotes the vector of *internal variables*, $z \in \mathbb{R}^N$. We assume that the plastic strain ε^p is an internal variable. Since ε^p is in $S(n)$, it has $\frac{n(n+1)}{2}$ degrees of freedom² and so the first $\frac{n(n+1)}{2}$ components of z are the entries of the matrix ε^p . To simplify the notation, one uses the projection operator $B: \mathbb{R}^N \rightarrow S(n)$ so that $Bz = \varepsilon^p$ and $B^T \varepsilon^p = (\varepsilon^p, 0) \in \mathbb{R}^N$. For example, if $n = 3$ then

$$B(\alpha, \beta, \gamma, \delta, \xi, \eta, \bar{z}) = \begin{pmatrix} \alpha & \delta & \xi \\ \delta & \beta & \eta \\ \xi & \eta & \gamma \end{pmatrix}$$

and

² In fact, ε^p has $\frac{n(n+1)}{2} - 1$ degrees of freedom, since it is assumed that $\operatorname{tr} \varepsilon^p = 0$, i.e. only shearing forces produce plastic strain.

$$B^T \begin{pmatrix} \alpha & \delta & \xi \\ \delta & \beta & \eta \\ \xi & \eta & \gamma \end{pmatrix} = (\alpha, \beta, \gamma, \delta, \xi, \eta, \vec{0}).$$

Eq. (D)₁ is often written with the help of B as

$$\rho v_t(t, x) - \operatorname{div}_x \mathcal{D}(\varepsilon(t, x) - Bz(t, x)) = f(t, x), \quad t > 0, x \in \Omega.$$

Eq. (D)₃ is called the *constitutive law*. It describes the evolution in time of the internal variables. The function g is often multi-valued (a so-called *multi-function*), meaning that its values are subsets of \mathbb{R}^N , but in our considerations it is not important whether g is single- or multi-valued. Therefore, not to overburden our presentation with too much notation (which is necessary to precisely present results for multi-functions) we assume that g is single-valued and so we have equality in (D)₃ instead of inclusion.

The function g can only be determined experimentally, which is the reason why so many different models exist, depending on the material being studied. Some thermodynamical considerations lead to the fact that g satisfies

$$g(s) \cdot s \geq 0$$

which is sometimes called “monotonicity at 0” (see [2] or [8, Eq. (2.1)]). Often more is assumed, namely that g is *monotone*:

$$(g(s) - g(t)) \cdot (s - t) \geq 0. \quad (1.2)$$

In general, it is not believed that a uniform existence theory exists for all functions g . Thus different classes of models have been introduced. In this paper we consider *coercive* and *self-controlling* models. To describe them briefly, let us consider the argument of g , i.e. the function $-\rho \nabla_z \psi(\varepsilon, z)$. The bilinear form

$$\rho \psi(\varepsilon, z) = \frac{1}{2} \mathcal{D}(\varepsilon - Bz) \cdot (\varepsilon - Bz) + \frac{1}{2} Lz \cdot z$$

is called the *free energy*. L is a symmetric, positive semi-definite matrix. When L is positive definite, then

$$C_L |z|^2 \geq Lz \cdot z \geq c_L |z|^2 \quad (1.3)$$

for some $c_L > 0$ and taking into account that $|Bz| \leq c|z|$ we have

$$\rho \psi(\varepsilon, z) \geq c(|\varepsilon|^2 + |z|^2), \quad (1.4)$$

i.e. ψ is a coercive bilinear form. Thus one defines *coercive models* as those for which L is positive definite.

Unfortunately, most engineering models are not coercive. This introduces many mathematical difficulties, since the free energy then “controls” only T and Lz :

$$\rho \psi(\varepsilon, z) \geq c(|T|^2 + |Lz|^2)$$

and so no information on ε , ε^p is obtained, only on their difference $\varepsilon - \varepsilon^p = \mathcal{D}^{-1}T$. The famous example is the Prandtl–Reuss model, where T is quite regular (even of class H_{loc}^1 , cf. [6]), but ε and ε^p alone are only Young measures.

If we integrate (1.4) over $x \in \Omega$ and obtain somehow that

$$\mathcal{E}(\varepsilon, z) = \int_{\Omega} \rho \psi(\varepsilon, z) dx$$

is bounded uniformly with respect to t , then from (1.4), regularity of the given boundary data and Korn’s inequality we get that

$$u \in L^\infty((0, T); H^1(\Omega))$$

(see for example [9], inequalities (3.7)–(3.9) for a presentation of this technique).

Using finite differences with respect to t , assuming that f_t and $g_{D,t}$ are sufficiently regular and that g is monotone, one can obtain that in fact $\mathcal{E}(\varepsilon_t, z_t)$ is bounded and so

$$u_t = v \in L^\infty((0, T); H^1(\Omega)).$$

This result will turn out to be very important in our regularity estimates.

For non-coercive models, the class of *linear self-controlling* models will be useful. This class has been introduced by K. Chelmiński and thoroughly studied in his papers (for example [7] and [9]). These models satisfy the estimate

$$|Bg(w)| \leq C(1 + |Lg(w)|), \quad \text{for all } w \in \mathbb{R}^N. \quad (1.5)$$

An important fact is that (1.5) also implies that $u_t \in L^\infty((0, T); H^1(\Omega))$. Moreover, as follows from the paper [9, Theorems 1 and 2] (in particular convergence (5.1) and the estimate (6.8)), we have

$$v_t \in L^\infty(L^2) \quad (1.6)$$

for dynamic coercive or self-controlling models. From this, Eq. (D)₁ and assumptions on the function f it follows that also

$$T_t \in L^\infty(L^2), \quad \operatorname{div}_x T_t \in L^\infty(L^2). \quad (1.7)$$

We will use the (1.6) and (1.7) many times in the regularity proofs for the case of dynamic problems.

We also mention that engineers often like to assume that the process of deformation occurs slowly and so the term ρv_t in (D)₁ is assumed to be small. This leads to *quasistatic equations of viscoplasticity*

$$\begin{aligned} -\operatorname{div}_x \mathcal{D}(\varepsilon(t, x) - Bz(t, x)) &= f(t, x), \quad t > 0, x \in \Omega, \\ \varepsilon(t, x) &= \frac{1}{2}(\nabla u(t, x) + \nabla^T u(t, x)), \quad t > 0, x \in \Omega, \\ z_t(t, x) &= g(-\rho \nabla_z \psi(\varepsilon(t, x), z(t, x))), \quad t > 0, x \in \Omega, \\ u(t, x) &= g_D(t, x), \quad t > 0, x \in \partial\Omega, \\ z(0, x) &= z^0(x), \quad x \in \Omega \end{aligned} \quad (Q)$$

(problem (D) will thus be called *dynamic*).

As it is often done in the literature, we will consider separately both of these problems, even though the general idea of our method is the same.

As for the boundary conditions, (D)₄ or (Q)₄ is the *Dirichlet boundary condition*. It is also possible to study Neumann-type conditions

$$T(t, x) \cdot \nu(x) = g_N(t, x), \quad t > 0, x \in \partial\Omega. \quad (1.8)$$

Also notice that g_D in (D) and in (Q) have, physically, different meanings.

1.2. Function spaces

The Lebesgue space of p -integrable functions $u : \Omega \rightarrow \mathbb{R}^n$ is denoted by $L^p(\Omega; \mathbb{R}^n)$ or just $L^p(\Omega)$ for short, with the norm

$$\begin{aligned} \|u\|_{p, \Omega} &= \left(\int_{\Omega} |u(x)|^p dx \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty, \\ \|u\|_{\infty, \Omega} &= \operatorname{ess\,sup}_{x \in \Omega} |u(x)|. \end{aligned}$$

The Sobolev space $W^{m,p}(\Omega)$ of functions with p -integrable distributional derivatives up to order m has the norm

$$\|u\|_{m,p,\Omega} = \left(\sum_{|\alpha|=0}^m \|D^\alpha u\|_{p,\Omega}^p \right)^{\frac{1}{p}},$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$. In particular $W^{0,p}(\Omega) = L^p(\Omega)$.

Since $W^{m,2}(\Omega)$ is a Hilbert space, we will denote it by $H^m(\Omega)$. The space $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{1,2,\Omega}$. By $H^{-1}(\Omega)$ we denote the space of continuous linear functionals over $H_0^1(\Omega)$. Functions from $H^1(\Omega)$ have a *trace* (i.e. values at the boundary $\partial\Omega$) and it turns out that they are elements of the space $H^{\frac{1}{2}}(\partial\Omega)$. For functions from $H_0^1(\Omega)$ the trace is zero. The space $H^{-\frac{1}{2}}(\partial\Omega)$ is the dual of $H^{\frac{1}{2}}(\partial\Omega)$ and for $\phi \in H^{-\frac{1}{2}}(\partial\Omega)$ we put

$$\|\phi\|_{-\frac{1}{2},2,\partial\Omega} := \sup_{\substack{g \in H^{\frac{1}{2}}(\partial\Omega) \\ \|g\|_{\frac{1}{2},2,\partial\Omega} = 1}} \langle \phi, g \rangle.$$

The symbol $W_{loc}^{m,p}(\Omega)$ will denote the space of functions u such that $u \in W^{m,p}(\Omega')$ for all $\Omega' \Subset \Omega$. Also $H_{loc}^m(\Omega) = W_{loc}^{m,2}(\Omega)$.

The space

$$L^2_{\text{div}}(\Omega; \mathbb{R}^n) = \{\zeta \in L^2(\Omega; \mathbb{R}^n) : \operatorname{div}_x \zeta \in L^2(\Omega)\}$$

is very useful in viscoplasticity theory. This is a Banach space equipped with the norm

$$\|\zeta\|_{2,\text{div},\Omega} = \|\zeta\|_{2,\Omega} + \|\operatorname{div} \zeta\|_{2,\Omega}.$$

For functions from this space one can define their trace on the boundary in the normal direction, which is stated more precisely in the following theorem:

Theorem 1.1. (See [12,19].) Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with boundary of class C^2 . Then there exists a continuous linear operator $\operatorname{tr}_\nu : L^2_{\text{div}}(\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$ such that for every $\zeta \in C^\infty(\bar{\Omega})$ we have $\operatorname{tr}_\nu \zeta = \zeta \cdot \nu|_{\partial\Omega}$, i.e. for smooth functions tr_ν is the restriction of the trace operator in the normal direction. Moreover, Stokes formula holds for all $\zeta \in L^2_{\text{div}}(\Omega)$ and $\xi \in H^1(\Omega)$:

$$\int_{\Omega} (\zeta \cdot \nabla \xi + \operatorname{div} \zeta \cdot \xi) dx = \langle \operatorname{tr}_\nu \zeta, \xi|_{\partial\Omega} \rangle. \quad (1.9)$$

The continuity of tr_ν means that there exists a positive constant C such that for all $\zeta \in L^2_{\text{div}}(\Omega)$

$$\|\operatorname{tr}_\nu \zeta\|_{-\frac{1}{2},2,\partial\Omega} \leq C \|\zeta\|_{2,\text{div},\Omega}. \quad (1.10)$$

We will also use Besov spaces $B^s_{p,\theta}(\Omega)$ for $s > 0$, $1 \leq p \leq \infty$ and $1 \leq \theta \leq \infty$. These are the spaces of p -integrable functions u for which the seminorm

$$\|u\|_{B^s_{p,\theta}(\Omega)} = \begin{cases} \left(\int_{\mathbb{R}^n} \frac{\|\Delta_{\vec{h}}^{([s]+1)} u\|_{p,\Omega_{\vec{h}}}^\theta}{|\vec{h}|^{\theta s}} \frac{d\vec{h}}{|\vec{h}|^n} \right)^{\frac{1}{\theta}} & \text{for } 1 \leq \theta < \infty, \\ \operatorname{ess\,sup}_{\vec{h} \in \mathbb{R}^n} \frac{\|\Delta_{\vec{h}}^{([s]+1)} u\|_{p,\Omega_{\vec{h}}}}{|\vec{h}|^s} & \text{for } \theta = \infty \end{cases}$$

is finite, where

$$\Omega_{\vec{h}} = \{x \in \Omega : x + j\vec{h} \in \Omega \text{ for all } j = 0, 1, \dots, [s] + 1\}$$

and

$$\begin{aligned} \Delta_{\vec{h}} u(x) &= u(x + \vec{h}) - u(x), \\ (\Delta_{\vec{h}})^{(n)} &= \Delta_{\vec{h}} (\Delta_{\vec{h}})^{(n-1)}, \quad n = 2, 3, \dots \end{aligned}$$

Then we take $\|u\|_{B^s_{p,\theta}(\Omega)} = \|u\|_{p,\Omega} + \|u\|_{B^s_{p,\theta}(\Omega)}$ (cf. [15] or [20] for more details). In particular we have $W^{s,p} = B^s_{p,p}$ (for s non-integer, $W^{s,p}$ is the Sobolev–Slobodeckij space). For $\theta = \infty$ the space $B^s_{p,\theta}(\Omega) = \mathcal{N}^s_p(\Omega)$ is called the Nikol'skii space.

The following important imbedding properties hold (cf. [15, p. 232]):

Theorem 1.2.

a) For $1 \leq p \leq \infty$, $s > 0$, $1 \leq \theta \leq \hat{\theta} \leq \infty$ and $\epsilon > 0$ there hold the imbeddings

$$B^{s+\epsilon}_{p,\infty} \hookrightarrow B^s_{p,1} \hookrightarrow B^s_{p,\theta} \hookrightarrow B^s_{p,\hat{\theta}} \hookrightarrow B^s_{p,\infty} \hookrightarrow B^{s-\epsilon}_{p,1}.$$

b) For $1 \leq p < q \leq \infty$ and $1 \leq \theta \leq \infty$, with $\rho = s - n(\frac{1}{p} - \frac{1}{q}) > 0$ there holds the imbedding³

$$B^s_{p,\theta}(\Omega) \hookrightarrow B^\rho_{q,\theta}(\Omega).$$

If X is a Banach space, then $L^p([0, T]; X)$ will denote the space of strongly measurable (cf. Appendix in [10]) functions $u : [0, T] \rightarrow X$ with the norm

$$\|u\|_{L^p([0, T]; X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$ and

³ Also, there is an interesting imbedding $B^s_{p,q} \hookrightarrow L^{p,q}$ for $s = \frac{np}{n-sp}$, with $L^{p,q}$ being the Lorentz space (cf. [18]), but we will not use it here.

$$\|u\|_{L^\infty((0,T);X)} = \operatorname{ess\,sup}_{0 \leq t \leq T} \|u(t)\|_X$$

for $p = \infty$. Often we will write $L^p(X)$ for short, for example $L^2(H^1)$ will denote $L^2((0,T); H^1(\Omega))$.

1.3. Some notations

Let $V \Subset U$. We will be using a cutoff function ϑ satisfying the assumptions

$$\begin{aligned} \vartheta &: \Omega \rightarrow [0, 1], \\ \vartheta &\in C_0^\infty(\mathbb{R}^n), \\ \vartheta &\equiv 1 \quad \text{on } V, \\ \vartheta &\equiv 0 \quad \text{on } \Omega \setminus U. \end{aligned} \tag{A-}\vartheta$$

For any function ϕ defined on Ω , any $h \in \mathbb{R}$ sufficiently small and $k = 1, \dots, n$, let

$$D_k^h \phi(x) = \frac{1}{h} \Delta_k^h \psi(x) = \frac{\phi(x + h\vec{e}_k) - \phi(x)}{h}$$

be the difference quotient in the direction \vec{e}_k . We have

$$\int_U \zeta \cdot D_k^h \phi \, dx = - \int_U D_k^{-h} \zeta \cdot \phi \, dx \tag{1.11}$$

for h sufficiently small, provided that the functions ζ and ϕ vanish outside U .

By $\varepsilon(A)$ we will denote the symmetric part of the matrix A . Sometimes for the *strain tensor*, i.e. $\varepsilon(\nabla u) = \frac{1}{2}(\nabla u + \nabla^T u)$, we will omit the explicit dependence on ∇u and write only ε .

In the whole paper we assume that the problem is of *monotone type*, i.e. the (multi-)function g is monotone.

2. Constant coefficients

In this section we show how the energy method combined with finite differences and monotonicity of g provide us with estimates that give improved local regularity. The proof will be quite short and simple, yet the results we obtain seem to be new.

In this section we assume that $V \Subset U \Subset W \Subset \Omega$ and ϑ is as in (A- ϑ).

2.1. Quasistatic problem

As the first case we will consider the quasistatic problem (Q). In the following we will assume that the model is linear self-controlling. It is not difficult to see that this class also includes coercive models.

Theorem 2.1. Assume that the problem (Q) is either coercive or linear self-controlling with $g_D \in H^1((0,T); H^{\frac{1}{2}}(\partial\Omega))$ and with solutions of regularity

$$(u, \varepsilon, z) \in H^1((0,T); H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)).$$

Let $z^0 \in H_{loc}^1(\Omega)$ and $f \in H^1((0,T); L^2(\Omega)) \cap L^2((0,T); H_{loc}^1(\Omega))$. Then this solution has regularity

$$(T, Lz) \in L^\infty((0,T); H_{loc}^1(\Omega) \times H_{loc}^1(\Omega)). \tag{2.1}$$

Additionally,

$$(u, \varepsilon) \in L^\infty((0,T); H_{loc}^2(\Omega) \times H_{loc}^1(\Omega))$$

for coercive models.

Proof. When $t = 0$, the first equation in (Q) takes the following elliptic form

$$-\operatorname{div} \mathcal{D}\varepsilon(\nabla u^0) = f(0) - \operatorname{div} \mathcal{D}Bz^0.$$

The function on the right-hand side is in L_{loc}^2 , thus from regularity theory for such problems (cf. [21]) we get that $u^0 \in H_{loc}^2$.

Define the energetic scalar product:

$$\langle (\varepsilon, z), (\bar{\varepsilon}, \bar{z}) \rangle_{\mathcal{E}} = \frac{1}{2} \int_{\Omega} \mathcal{D}(\varepsilon - Bz) \cdot (\bar{\varepsilon} - B\bar{z}) \, dx + \frac{1}{2} \int_{\Omega} Lz \cdot \bar{z} \, dx$$

and the energy associated with this product

$$\mathcal{E}(\varepsilon, z) = \frac{1}{2} \int_{\Omega} \mathcal{D}(\varepsilon - Bz) \cdot (\varepsilon - Bz) \, dx + \frac{1}{2} \int_{\Omega} Lz \cdot z \, dx.$$

Take $\bar{\varepsilon} = -D_k^{-h}(\vartheta^2 D_k^h \varepsilon(\nabla u))$ and $\bar{z} = -D_k^{-h}(\vartheta^2 D_k^h z)$. Notice that for $|h| < d(\partial U, \partial \Omega)$, $\bar{\varepsilon}, \bar{z} \in H^1(L^2)$ and thus these are proper test functions. Also, $\bar{\varepsilon}$ and \bar{z} vanish outside $U_h = \{x \in \Omega : d(x, U) < |h|\} \subset \Omega$, so we can use the formula (1.11). Consider the product

$$P = 2 \langle (\varepsilon(\nabla u_t), z_t), (\bar{\varepsilon}, \bar{z}) \rangle_{\mathcal{E}}. \quad (2.2)$$

On one hand, (2.2) is equal to

$$P = 2 \langle \varepsilon(\nabla D_k^h u_t), D_k^h z_t \rangle_{\mathcal{E}} = \frac{d}{dt} \mathcal{E}(\vartheta \varepsilon(\nabla D_k^h u), \vartheta D_k^h z).$$

On the other hand, computation of (2.2) gives

$$P = \int_{\Omega} \vartheta^2 \mathcal{D}(D_k^h \varepsilon(\nabla u) - B D_k^h z) \cdot D_k^h \varepsilon(\nabla u_t) \, dx - \int_{\Omega} \vartheta^2 [B^T \mathcal{D}(D_k^h \varepsilon(\nabla u) - B D_k^h z) - L D_k^h z] \cdot D_k^h z_t \, dx.$$

The second term on the right is ≤ 0 (this follows from monotonicity of g and the fact that when we have a product of two difference quotients then the term $\frac{1}{h^2} > 0$ appears, which has no contribution to the sign in the inequality). Thus

$$P \leq \int_{\Omega} \vartheta^2 \mathcal{D}(D_k^h \varepsilon(\nabla u) - B D_k^h z) \cdot D_k^h \varepsilon(\nabla u_t) \, dx = \int_{\Omega} \vartheta^2 D_k^h T \cdot D_k^h \nabla u_t \, dx$$

(we have used that for the scalar product of matrices with T symmetric it holds $T \cdot \varepsilon(A) = T \cdot A$). Integration by parts gives

$$\begin{aligned} P &\leq - \int_{\Omega} \vartheta^2 D_k^h \operatorname{div} T \cdot D_k^h u_t \, dx - 2 \int_{\Omega} \vartheta D_k^h T \cdot \varepsilon(D_k^h u_t \otimes \nabla \vartheta) \, dx \\ &= \int_{\Omega} \vartheta^2 D_k^h f \cdot D_k^h u_t \, dx - 2 \int_{\Omega} \vartheta D_k^h T \cdot \varepsilon(D_k^h u_t \otimes \nabla \vartheta) \, dx \\ &\leq C(\|f\|_{1,2,W} \|u_t\|_{1,2,W} + \|\vartheta D_k^h T\|_{2,\Omega} \|u_t\|_{1,2,W}) \\ &\leq \mathcal{E}(\vartheta \varepsilon(\nabla D_k^h u), \vartheta D_k^h z) + C(\|f\|_{1,2,W} \|u_t\|_{1,2,W} + \|u_t\|_{1,2,W}^2). \end{aligned}$$

Thus, from Gronwall's inequality:

$$\mathcal{E}(\vartheta \varepsilon(\nabla D_k^h u), \vartheta D_k^h z)(t) \leq C \left(\mathcal{E}(\vartheta \varepsilon(\nabla D_k^h u^0), \vartheta D_k^h z^0) + \int_0^t \|f\|_{1,2,W} \|u_t\|_{1,2,W} + \|u_t\|_{1,2,W}^2 \, dt \right) \leq C$$

for sufficiently smooth data. This means that

$$(T, Lz) \in L^\infty((0, T); H_{loc}^1(\Omega) \times H_{loc}^1(\Omega)).$$

The regularity of (u, ε) in the coercive case now follows from the regularity of T and inequality (1.3). In fact, for coercive problems we have

$$\mathcal{E}(\vartheta D_k^h \varepsilon, \vartheta D_k^h z)(t) \geq c \|\vartheta D_k^h \varepsilon(t)\|_{2,\Omega}^2,$$

from which we deduce that $\varepsilon \in L^\infty(H_{loc}^1)$. To prove the regularity of u notice that

$$\begin{aligned} \vartheta(\partial_k \nabla u) &= \nabla(\vartheta \partial_k u) - \nabla \vartheta \otimes \partial_k \nabla u, \\ \varepsilon(\nabla(\vartheta \partial_k u)) &= \vartheta \partial_k \varepsilon(\nabla u) + \varepsilon(\partial_k u \otimes \nabla \vartheta) \end{aligned}$$

and so, applying Korn's inequality:

$$\begin{aligned}
\|\vartheta \partial_k \nabla u\|_{2,\Omega} &\leq \|\nabla(\vartheta \partial_k u)\|_{2,\Omega} + \|\nabla \vartheta \otimes \partial_k u\|_{2,\Omega} \\
&\leq C_K [\|\varepsilon(\nabla(\vartheta \partial_k u))\|_{2,\Omega} + \|\vartheta \partial_k u\|_{2,\Omega}] + \|\nabla \vartheta\|_{\infty,\Omega} \|\nabla u\|_{2,\Omega} \\
&\leq C [\|\vartheta \partial_k \varepsilon(\nabla u)\|_{2,\Omega} + \|\nabla u\|_{2,\Omega}] \leq C [1 + \|\vartheta \partial_k \varepsilon(\nabla u)\|_{2,\Omega}].
\end{aligned}$$

This proves that $u \in L^\infty(H^1_{loc})$. \square

For the coercive case the above result has been obtained in [4] and [13] for example. In the case of linear self-controlling models the result seems new.

2.2. Dynamic problem

Now consider the dynamic problem (D). Again we assume that the model is either coercive or linear self-controlling. Existence of solutions has been thoroughly studied in [9], where it is proved that $u_t \in L^2(H^1)$.

Theorem 2.2. Assume that the model (D) is either coercive or linear self-controlling with $g_D \in H^1((0, T); H^{\frac{1}{2}}(\partial\Omega))$ and with solutions of regularity

$$(u, \varepsilon, z) \in H^1((0, T); H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)). \quad (2.3)$$

Suppose that $(v^0, \varepsilon^0, z^0) \in H^2_{loc}(\Omega) \times H^1_{loc}(\Omega) \times H^1_{loc}(\Omega)$ and $f \in H^1((0, T); L^2(\Omega)) \cap L^2((0, T); H^1_{loc}(\Omega))$. Then this solution has regularity

$$(T, Lz) \in L^\infty((0, T); H^1_{loc}(\Omega) \times H^1_{loc}(\Omega)). \quad (2.4)$$

Additionally, for the case of coercive models

$$\varepsilon \in L^\infty((0, T); H^1_{loc}(\Omega))$$

and, if additionally $u^0 \in L^2$ is such that $\varepsilon(\nabla u^0) = \varepsilon^0$ then

$$u \in L^\infty((0, T); H^2_{loc}(\Omega)).$$

Remark 2.3. The regularity (2.3) of solutions is a consequence of inequality (3.1) in [9]. In fact, for coercive models, from $\varepsilon \in H^1(L^2)$ we immediately deduce that $v \in L^2(H^1)$.

For linear self-controlling models, the energy inequality (3.1) from [9] gives that $T_t, Lz_t \in L^2(L^2)$ and then the self-controlling property implies $\varepsilon_t^p \in L^2(L^2)$ and thus $\varepsilon_t \in L^2(L^2)$.

Proof of Theorem 2.2. Again the proof is the same for both cases. This time, the energetic scalar product is

$$\langle (v, \varepsilon, z), (\bar{v}, \bar{\varepsilon}, \bar{z}) \rangle_{\mathcal{E}} = \frac{1}{2} \int_{\Omega} \rho v \cdot \bar{v} \, dx + \frac{1}{2} \int_{\Omega} \mathcal{D}(\varepsilon - Bz) \cdot (\bar{\varepsilon} - B\bar{z}) \, dx + \frac{1}{2} \int_{\Omega} Lz \cdot \bar{z} \, dx.$$

Take the test functions

$$\bar{v} = -D_k^{-h}(\vartheta^2 D_k^h v), \quad \bar{\varepsilon} = -D_k^{-h}(\vartheta^2 D_k^h \varepsilon), \quad \bar{z} = -D_k^{-h}(\vartheta^2 D_k^h z)$$

and consider

$$P = 2 \langle (v_t, \varepsilon_t, z_t), (\bar{v}, \bar{\varepsilon}, \bar{z}) \rangle_{\mathcal{E}} \quad (2.5)$$

(notice that the norm $\|v_t\|_{2,\Omega}$ makes sense, from Eq. (1.6)). Again, moving the difference quotients, we get

$$P = 2 \langle \vartheta (D_k^h v_t, D_k^h \varepsilon_t, D_k^h z_t), \vartheta (D_k^h v, D_k^h \varepsilon, D_k^h z) \rangle_{\mathcal{E}} = \frac{d}{dt} \mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z).$$

Writing out (2.5):

$$\begin{aligned}
P &= \int_{\Omega} \vartheta^2 \rho D_k^h v \cdot D_k^h v_t \, dx + \int_{\Omega} \vartheta^2 \mathcal{D}(D_k^h \varepsilon - B D_k^h z) \cdot (D_k^h \varepsilon_t - B D_k^h z_t) \, dx + \int_{\Omega} \vartheta^2 L D_k^h z \cdot D_k^h z_t \, dx \\
&= \int_{\Omega} \vartheta^2 \rho D_k^h v \cdot D_k^h v_t \, dx + \int_{\Omega} \vartheta^2 \mathcal{D}(D_k^h \varepsilon - B D_k^h z) \cdot D_k^h \varepsilon_t \, dx \\
&\quad - \int_{\Omega} \vartheta^2 [B^T \mathcal{D}(D_k^h \varepsilon - B D_k^h z) - L D_k^h z] D_k^h z_t \, dx.
\end{aligned}$$

The last integral on the right is ≤ 0 , thus

$$P \leq \int_{\Omega} \vartheta^2 D_k^h v \cdot [\rho D_k^h v_t dx - \operatorname{div} \mathcal{D}(D_k^h \varepsilon - B D_k^h z)] dx - 2 \int_{\Omega} \vartheta \mathcal{D}(D_k^h \varepsilon - B D_k^h z) \cdot \varepsilon(D_k^h v \otimes \nabla \vartheta) dx \quad (2.6)$$

$$= \int_{\Omega} \vartheta^2 D_k^h f \cdot D_k^h v dx - 2 \int_{\Omega} \vartheta \mathcal{D}(D_k^h \varepsilon - B D_k^h z) \cdot \varepsilon(D_k^h v \otimes \nabla \vartheta) dx \quad (2.7)$$

$$\leq C(\|f\|_{1,2,W} \|v\|_{1,2,W} + \mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z) + \|v\|_{1,2,W}^2).$$

Again, we use Young's and Gronwall's inequalities to get regularity

$$(T, Lz) \in L^\infty(H_{loc}^1 \times H_{loc}^1).$$

Now, similarly as in the case of quasistatic models, we prove regularity of ε and u . \square

Remark 2.4. We could prove regularity in a similar way for a problem with a nonlinearity in the equation of motion dependent on the velocity:

$$\rho v_t - \operatorname{div} \mathcal{D}(\varepsilon - Bz) = f - F(v), \quad (2.8)$$

with the (multi-)function F being maximal monotone (for existence theory to such problems cf. [12]).

If fact, after integration by parts in (2.6) and application of (2.8) yield

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z)(t) \\ & \leq \int_{\Omega} \vartheta^2 D_k^h v (\rho D_k^h v_t - \operatorname{div} \mathcal{D}(D_k^h \varepsilon - B D_k^h z)) dx dt - 2 \int_{\Omega} \vartheta \mathcal{D}(D_k^h \varepsilon - B D_k^h z) \cdot \varepsilon(D_k^h v \otimes \nabla \vartheta) dx dt \\ & = \int_{\Omega} \vartheta^2 D_k^h v \cdot D_k^h f dx dt - \int_{\Omega} \vartheta^2 D_k^h v \cdot D_k^h F(v) dx dt - 2 \int_{\Omega} \vartheta \mathcal{D}(D_k^h \varepsilon - B D_k^h z) \cdot \varepsilon(D_k^h v \otimes \nabla \vartheta) dx dt. \end{aligned}$$

Monotonicity of F gives us again the inequality

$$\frac{d}{dt} \mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z)(t) \leq \int_{\Omega} \vartheta^2 D_k^h v \cdot D_k^h f dx dt - 2 \int_{\Omega} \vartheta \mathcal{D}(D_k^h \varepsilon - B D_k^h z) \cdot \varepsilon(D_k^h v \otimes \nabla \vartheta) dx dt$$

as in (2.7).

2.3. Nemytskij-type monotone operators

The above reasonings work also for evolution problems involving monotone superposition operators (we don't require maximality, only that the solution exists):

$$\begin{aligned} w_t(t) + \mathcal{A}w(t) &\ni f(t), \\ w(0) &= w^0 \end{aligned} \quad (2.9)$$

where $(\mathcal{A}w)(x) = A(w(x))$, with $A: \mathbb{R}^n \supseteq D(A) \rightsquigarrow \mathbb{R}^n$ a monotone multi-function, which implies that $\mathcal{A}: L^2(\Omega) \supset D(\mathcal{A}) \rightarrow L^2(\Omega)$ is a monotone operator.

Theorem 2.5. Suppose that there exists a solution to (2.9) with $w(t) \in D(\mathcal{A})$ for all $t \in (0, T)$ and with regularity

$$w \in L^\infty((0, T); L_{loc}^2(\Omega)).$$

Let $w^0 \in H_{loc}^1(\Omega) \cap D(\mathcal{A})$ and $f \in L^2((0, T); H_{loc}^1(\Omega))$. Then

$$w \in L^\infty((0, T); H_{loc}^1(\Omega)).$$

Proof. It is easy to compute

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_{\Omega} \vartheta^2 |D_k^h w(t)|^2 dx &= \int_{\Omega} \vartheta^2 D_k^h w(t) \cdot D_k^h w_t(t) dx \leq \int_{\Omega} \vartheta^2 D_k^h w(t) \cdot D_k^h f(t) dx \\ &\leq \frac{1}{2} \int_{\Omega} \vartheta^2 |D_k^h w(t)|^2 dx + C \int_{\Omega} \vartheta^2 |D_k^h f(t)|^2 dx. \end{aligned}$$

Gronwall's inequality gives us

$$\frac{1}{2} \int_{\Omega} \vartheta^2 |D_k^h w(t)|^2 dx \leq C \left(\int_{\Omega} \vartheta^2 |D_k^h w^0|^2 dx + \int_0^t \int_{\Omega} \vartheta^2 |D_k^h f(t)|^2 dx dt \right) < c$$

with $c > 0$ independent of h , which proves the result. \square

3. Variable coefficients – coercive problems

In this section we assume that $\mathcal{D} = \mathcal{D}[x]$ and $L = L[x]$. Variable coefficients introduce some complications, because the operator D_k^h doesn't commute with $\mathcal{D}[x]$ and $L[x]$ anymore. Nevertheless we will show how, with some additional assumptions, one can deal with this problem.

We use the notation $T(t, x) = \mathcal{D}[x](\varepsilon(t, x) - Bz(t, x))$ and

$$f_{\{h,k\}}(x) = f(x + h\vec{e}_k).$$

The following equality holds: $D_k^h(fg) = (D_k^h f)g + f_{\{h,k\}}(D_k^h g)$.

Again, let $U \subseteq V \subseteq W \subseteq \Omega$ and ϑ be as in (A- ϑ).

3.1. Dynamic problem

In the dynamic case the energy associated with our system takes the form

$$\mathcal{E}(\varepsilon, z)(t) = \frac{1}{2} \int_{\Omega} \left\{ \rho |v(t, x)|^2 + \mathcal{D}[x](\varepsilon(t, x) - Bz(t, x)) \cdot (\varepsilon(t, x) - Bz(t, x)) L[x] z(t, x) \cdot z(t, x) \right\} dx.$$

Theorem 3.1. Suppose that the model is coercive, assumptions of Theorem 2.2 hold and $\mathcal{D}, L \in W_{loc}^{1,\infty}(\Omega)$. Assume that (D) possesses a solution of regularity (2.3). Then

$$(\varepsilon, z) \in L^\infty((0, T); H_{loc}^1 \times H_{loc}^1).$$

Proof. We start out exactly as before, dealing carefully with $D_k^h \mathcal{D}[x]$, $D_k^h L[x]$:

$$\begin{aligned} &\frac{d}{dt} \mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z)(t) \\ &= \int_{\Omega} \vartheta^2 \rho D_k^h v \cdot D_k^h v_t dx + \int_{\Omega} \vartheta^2 \mathcal{D}[x] (D_k^h \varepsilon - B D_k^h z) \cdot (D_k^h \varepsilon_t - B D_k^h z_t) dx + \int_{\Omega} \vartheta^2 L[x] D_k^h z \cdot D_k^h z_t dx \\ &= \int_{\Omega} \vartheta^2 \rho D_k^h v \cdot D_k^h v_t dx + \int_{\Omega} \vartheta^2 D_k^h T \cdot (D_k^h \varepsilon_t - B D_k^h z_t) dx + \int_{\Omega} \vartheta^2 D_k^h (L[x] z) \cdot D_k^h z_t dx \\ &\quad - \int_{\Omega} \vartheta^2 (D_k^h \mathcal{D})[x] (\varepsilon_{\{h,k\}} - B z_{\{h,k\}}) \cdot (D_k^h \varepsilon_t - B D_k^h z_t) dx - \int_{\Omega} \vartheta^2 (D_k^h L)[x] z_{\{h,k\}} \cdot D_k^h z_t dx \\ &= \int_{\Omega} \vartheta^2 \rho D_k^h v \cdot D_k^h v_t dx + \int_{\Omega} \vartheta^2 D_k^h T \cdot \varepsilon (\nabla D_k^h v) dx - \int_{\Omega} \vartheta^2 [B^T D_k^h T - (D_k^h L[x] z)] D_k^h z_t dx \\ &\quad - \int_{\Omega} \vartheta^2 (D_k^h L)[x] z_{\{h,k\}} \cdot D_k^h z_t dx - \int_{\Omega} \vartheta^2 (D_k^h \mathcal{D})[x] (\varepsilon_{\{h,k\}} - B z_{\{h,k\}}) \cdot (D_k^h \varepsilon_t - B D_k^h z_t) dx \end{aligned}$$

$$\begin{aligned} &\leq \int_{\Omega} \vartheta^2 D_k^h f \cdot D_k^h v \, dx - 2 \int_{\Omega} \vartheta D_k^h T \cdot \varepsilon(D_k^h v \otimes \nabla \vartheta) \, dx - \int_{\Omega} \vartheta^2 (D_k^h L)[x] z_{\{h,k\}} \cdot D_k^h z_t \, dx \\ &\quad - \int_{\Omega} \vartheta^2 (D_k^h \mathcal{D})[x] (\varepsilon_{\{h,k\}} - Bz_{\{h,k\}}) \cdot (D_k^h \varepsilon_t - BD_k^h z_t) \, dx. \end{aligned}$$

Now integrate with respect to time to get

$$\begin{aligned} &\mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z)(t) \\ &\leq \mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z)(0) + \int_0^t \int_{\Omega} \vartheta^2 D_k^h f \cdot D_k^h v \, dx \, dt \\ &\quad - 2 \int_0^t \int_{\Omega} \vartheta D_k^h T \cdot \varepsilon(D_k^h v \otimes \nabla \vartheta) \, dx \, dt - \int_0^t \int_{\Omega} \vartheta^2 (D_k^h L)[x] z_{\{h,k\}} \cdot D_k^h z_t \, dx \, dt \\ &\quad - \int_0^t \int_{\Omega} \vartheta^2 (D_k^h \mathcal{D})[x] (\varepsilon_{\{h,k\}} - Bz_{\{h,k\}}) \cdot (D_k^h \varepsilon_t - BD_k^h z_t) \, dx \, dt. \end{aligned} \quad (3.1)$$

The first two integrals are estimated by

$$C \int_0^t (\|\vartheta D_k^h f\|_{2,U}^2 + \mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z) + \|\vartheta D_k^h v\|_{2,U}^2) \, dt.$$

Thus, as before, we see that the assumption $v \in L^2(H_{loc}^1)$ is necessary. In the remaining two integrals we integrate by parts to get

$$\begin{aligned} &\int_0^t \int_{\Omega} \vartheta^2 (D_k^h L)[x] z_{\{h,k\},t} \cdot D_k^h z \, dx \, dt + \int_0^t \int_{\Omega} \vartheta^2 (D_k^h \mathcal{D})[x] (\varepsilon_{\{h,k\},t} - Bz_{\{h,k\},t}) \cdot (D_k^h \varepsilon - BD_k^h z) \, dx \, dt \\ &\quad - \int_{\Omega} \vartheta^2 (D_k^h L)[x] z_{\{h,k\}}(t) \cdot D_k^h z(t) \, dx + \int_{\Omega} \vartheta^2 (D_k^h L)[x] z_{\{h,k\}}(0) \cdot D_k^h z(0) \, dx \\ &\quad - \int_{\Omega} \vartheta^2 (D_k^h \mathcal{D})[x] (\varepsilon_{\{h,k\}}(t) - Bz_{\{h,k\}}(t)) \cdot (D_k^h \varepsilon(t) - BD_k^h z(t)) \, dx \\ &\quad + \int_{\Omega} \vartheta^2 (D_k^h \mathcal{D})[x] (\varepsilon_{\{h,k\}}(0) - Bz_{\{h,k\}}(0)) \cdot (D_k^h \varepsilon(0) - BD_k^h z(0)) \, dx. \end{aligned} \quad (3.2)$$

One now sees (by applying Hölder's inequality to all of the terms above) that the assumption $\mathcal{D}, L \in W_{loc}^{1,\infty}(\Omega)$ is reasonable. Next, if $z_t, (\varepsilon_t - Bz_t) \in L^2(L_{loc}^2)$, then we can apply Young's inequality to the first two integrals, obtaining

$$\begin{aligned} &\int_0^t \int_{\Omega} \vartheta^2 \{ (D_k^h L)[x] z_{\{h,k\},t} \cdot D_k^h z + (D_k^h \mathcal{D})[x] (\varepsilon_{\{h,k\},t} - Bz_{\{h,k\},t}) \cdot (D_k^h \varepsilon - BD_k^h z) \} \, dx \, dt \\ &\leq C \int_0^t \int_{\Omega} \vartheta^2 (|D_k^h z|^2 + |D_k^h \varepsilon - BD_k^h z|^2) \, dx \, dt \\ &\quad + C \int_0^t \int_{\Omega} \vartheta^2 (|(D_k^h L)[x]|^2 |z_{\{h,k\},t}|^2 + |(D_k^h \mathcal{D})[x]|^2 |\varepsilon_{\{h,k\},t} - Bz_{\{h,k\},t}|^2) \, dx \, dt \\ &\leq \int_0^t \mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z) \, dt + \int_0^t (\|\vartheta z_t\|_{2,\Omega}^2 + \|\vartheta T_t\|_{2,\Omega}^2) \, dt. \end{aligned}$$

The third integral in (3.2) is estimated using similar techniques:

$$\int_{\Omega} \vartheta^2 (D_k^h L)[x] z_{\{h,k\},t}(t) \cdot D_k^h z(t) dx \leq \frac{1}{4} \mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z)(t) + C \|\vartheta z(t)\|_{2,\Omega}^2$$

and analogously for the fifth integral

$$\begin{aligned} & \int_{\Omega} \vartheta^2 (D_k^h \mathcal{D})[x] (\varepsilon_{\{h,k\},t}(t) - Bz_{\{h,k\},t}(t)) \cdot (D_k^h \varepsilon(t) - BD_k^h z(t)) dx \\ & \leq \frac{1}{4} \mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z)(t) + C \|\vartheta(\varepsilon - Bz)\|_{2,\Omega}^2. \end{aligned}$$

Finally, the fourth and sixth terms in (3.2) are estimated by

$$\begin{aligned} & \int_{\Omega} \vartheta^2 \{ (D_k^h L)[x] z_{\{h,k\},t}(0) \cdot D_k^h z(0) + (D_k^h \mathcal{D})[x] (\varepsilon_{\{h,k\}}(0) - Bz_{\{h,k\}}(0)) \cdot (D_k^h \varepsilon(0) - BD_k^h z(0)) \} dx \\ & \leq C (\|z^0\|_{2,W}^2 \|z^0\|_{1,2,W} + \|\varepsilon(0) - Bz^0\|_{2,W}^2 \|\varepsilon(0) - Bz^0\|_{1,2,W}). \end{aligned}$$

Therefore inequality (3.1) takes the form

$$\begin{aligned} & \frac{1}{2} \mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z)(t) \\ & \leq \mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z)(0) + \int_0^t \mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z) dt + \int_0^t (\|f\|_{1,2,W}^2 + \|v\|_{1,2,W}^2 + \|z\|_{2,W}^2) dt \\ & \quad + C (\|z(t)\|_{2,W}^2 + \|\varepsilon - Bz\|_{2,W}^2 + \|z^0\|_{1,2,W}^2 + \|\varepsilon(0) - Bz^0\|_{1,2,W}^2). \end{aligned}$$

Thus, taking into consideration the assumptions on the solution and given data, integral form of Gronwall's inequality gives

$$\mathcal{E}(\vartheta D_k^h v, \vartheta D_k^h \varepsilon, \vartheta D_k^h z)(t) \leq C$$

and this proves our theorem. \square

Remark 3.2. Similar procedure works for the quasistatic problem. In fact, the estimates are almost the same with the exception that we are now lacking the inertial term

$$\int_{\Omega} \rho \vartheta^2 |D_k^h v|^2 dx$$

in the energy.

Remark 3.3. Coercivity assumption is important: in (3.2) $D_k^h z$ in the first and third integrals have to be absorbed by the energy on the left.

Remark 3.4. We could additionally assume that $\rho = \rho[x]$ with $\rho \in W_{loc}^{1,\infty}(\Omega)$ such that ρ is positive, i.e. $(\exists \eta > 0) (\forall x \in \Omega) \rho[x] > \eta$. We would then get the integral

$$\int_0^t \int_{\Omega} \vartheta^2 \rho[x] D_k^h v \cdot D_k^h v_t dx dt = \int_0^t \int_{\Omega} \vartheta^2 D_k^h (\rho[x] v_t) \cdot D_k^h v dx dt - \int_0^t \int_{\Omega} \vartheta^2 (D_k^h \rho)[x] v_{\{h,k\},t} \cdot D_k^h v dx dt.$$

In the first integral on the right we use the equation of motion (D_1) . The second integral is bounded from assumptions and because $v_t \in L^\infty(L^2)$ (this follows from (1.6)).

4. Boundary regularity – tangential derivatives

Assume that either of the boundary conditions: Dirichlet $(D)_4$ (resp. $(Q)_4$) or Neumann (1.8) holds (in particular, we don't have a problem of mixed boundary type). To fix attention, we first consider coercive models. Also, we study only the case of constant coefficients.

4.1. Tangential derivatives – case $\partial\Omega$ is straight (coercive problems)

We assume that the part of $\partial\Omega$ in which regularity is studied is straight, i.e. contained in the set $\partial\mathbb{R}_+^n = \{(x', x_n): x_n = 0\}$. Moreover, without losing generality, we can assume that a neighborhood of $0 \in \mathbb{R}^n$ exists, which is contained in $\partial\Omega \cap \{x_n = 0\}$. The precise formulation is:

Suppose that $\Omega \subset \mathbb{R}_+^n = \{(x', x_n): x_n > 0\}$ and that for some small $\delta > 0$ and

$$U' = B(0, \delta) \cap \mathbb{R}_+^n, \quad V' = B\left(0, \frac{\delta}{2}\right) \cap \mathbb{R}_+^n \quad \text{we have}$$

$$U' \subset \Omega, \quad \partial U' \cap \{x_n = 0\} \subset \partial\Omega,$$

$$0 \in \Gamma = \text{Int } \partial U' \cap \{x_n = 0\} \quad (\text{Int in the relative topology of } \{x_n = 0\}). \quad (\text{A-straight})$$

Let ϑ be as in (A- ϑ) with $U = B(0, \delta)$, $V = B(0, \frac{\delta}{2})$ (i.e. we take ϑ to be nonzero at the part of $\partial\Omega$ in consideration).

It turns out that for studying regularity of tangential derivatives we can repeat the above reasonings with difference quotients. The only exception is that when we integrate by parts in

$$\int_0^t \int_{\Omega} \vartheta^2 D_k^h T \cdot D_k^h \nabla u_t \, dx \, dt$$

we obtain additionally a surface integral (since now ϑ does not vanish on all of $\partial\Omega$):

$$\mathbb{S} = \int_0^t \int_{\Gamma} \vartheta^2 D_k^h T \cdot \nu \cdot D_k^h u_t \, dS_x \, dt. \quad (4.1)$$

We remark that in fact the surface integral in \mathbb{S} is a duality pairing between $D_k^h T \cdot \nu \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\vartheta^2 D_k^h u_t \in H^{\frac{1}{2}}(\partial\Omega)$. We will discuss later how to exactly interpret it and, in particular, the fact that $D_k^h T$ need not be defined on all of $\partial\Omega$ (only on its straight part under consideration).

As we will show, the estimates of this integral depend on the type of problem studied: coercive/non-coercive, quasistatic/dynamic and on the type of boundary conditions: Dirichlet/Neumann.

For coercive problems the precise statement is as follows:

Theorem 4.1. Suppose that the model is coercive. Let U', V', U, V, Γ be as above and $f \in L^2((0, T); H^1(\Omega))$.

a) For the quasistatic problem, let the assumptions of Theorem 2.1 hold and, additionally

$$z^0 \in H^1(\Omega).$$

Assume that

(a) $g_D \in H^1((0, T); H^{\frac{3}{2}}(\partial\Omega))$ in the case of Dirichlet boundary conditions or

(b) $g_N \in H^1((0, T); H^{\frac{1}{2}}(\partial\Omega))$ for the Neumann boundary condition.

b) For the dynamic problem, let the assumptions of Theorem 2.2 be fulfilled and additionally:

$$(v^0, \varepsilon^0, z^0) \in H^2(\Omega) \times H^1(\Omega) \times H^1(\Omega),$$

$$v_t \in L^2((0, T); L^2(\Omega))$$

and

(a) $g_D \in H^1((0, T); H^{\frac{5}{2}}(\partial\Omega))$ for the Dirichlet case, or

(b) $g_N \in H^1((0, T); H^{\frac{1}{2}}(\partial\Omega))$ for the Neumann condition.

Then the solution (for both types of problems) has regularity

$$(\varepsilon_{x_k}, z_{x_k}) \in L^\infty((0, T); L^2(V'), L^2(V')), \quad \text{for } k = 1, \dots, (n-1) \quad (4.2)$$

or, if we denote by

$$\nabla_\Gamma = (\partial_{x_1}, \dots, \partial_{x_{(n-1)}})$$

the tangential derivative, then

$$(\nabla_\Gamma \varepsilon, \nabla_\Gamma z) \in L^\infty((0, T); L^2(V') \times L^2(V')).$$

Remark 4.2. Of course for the Dirichlet problem (4.2) and the regularity of boundary conditions imply that

$$\nabla_{\Gamma} u \in L^{\infty}(H^1(V')).$$

Proof of Theorem 4.1. We again consider a scalar product as in (2.2) or (2.5). For $k = 1, \dots, (n-1)$ it is not difficult to see that $\vartheta D_k^h \zeta$ are proper test functions if $\zeta = \varepsilon, z$ in the quasistatic case or $\zeta = v, \varepsilon, z$ for the dynamic problem (this is also true for $k = n$ and $h > 0$).

As was stated above, the calculations that we perform are the same as in the proofs of Theorem 2.1 or 2.2 with the exception of estimating the surface integral (4.1). Therefore we focus our attention only on this integral.

a) Quasistatic problem

(a) Dirichlet boundary condition

We substitute $u_t = g_{D,t}$ on $\partial\Omega$ and shift the difference quotient away from $D_k^h T$ to get:

$$\mathbb{S} = - \int_0^t \int_{\partial\Omega} T \cdot \nu \cdot D_k^{-h} (\vartheta^2 D_k^h g_{D,t}) dS_x dt.$$

Now the trace theorem for the space $L_{\text{div}}^2(\Omega)$ gives us that

$$\begin{aligned} \mathbb{S} &\leq C \int_0^t \|T\|_{2,\text{div},\Omega} \|D_k^{-h} (\vartheta^2 D_k^h g_{D,t})\|_{\frac{1}{2},2,\partial\Omega} dt \\ &\leq C \left(\int_0^t [\|T\|_{2,\Omega}^2 + \|f\|_{2,\Omega}^2] dx \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_k (\vartheta^2 \partial_k g_{D,t})\|_{\frac{1}{2},2,\partial\Omega}^2 dt \right)^{\frac{1}{2}} \\ &\leq C \left(\int_0^t \|g_{D,t}\|_{\frac{5}{2},2,\Gamma}^2 dt \right)^{\frac{1}{2}}, \end{aligned} \quad (4.3)$$

which would prove our theorem if we assumed $g_D \in H^1((0,T); H^{\frac{5}{2}}(\partial\Omega))$.

Let us show how to deal with the case when $g_D \in H^1(H^{\frac{3}{2}}(\partial\Omega))$. Such lower regularity of the boundary data poses some technical difficulties which we will now try to describe. We cannot shift the difference quotient onto $g_{D,t}$ anymore, since $g_{D,t}$ is not that regular. Therefore we must somehow deal with $D_k^h T$ on the right-hand side. The only way we can do it is to absorb this term by the energy on the left. But in order to perform this the term $D_k^h T$ must be multiplied by the test function ϑ . The problem is that for the estimate as in (4.3) we use the norm of $D_k^h T$ in the space L_{div}^2 . Now, if we took

$$\mathbb{S} = - \int_0^t \int_{\partial\Omega} \vartheta D_k^h T \cdot \nu \cdot \vartheta D_k^h g_{D,t} dS_x dt$$

and estimated it by

$$C \int_0^t \|\vartheta D_k^h T\|_{2,\text{div},\Omega} \|\vartheta g_{D,t}\|_{\frac{1}{2},2,\partial\Omega} dt,$$

then we would have

$$\text{div}(\vartheta D_k^h T) = \vartheta \text{div} D_k^h T + \nabla \vartheta \cdot D_k^h T.$$

The second term causes problems: it is not possible to absorb $\|\nabla \vartheta \cdot D_k^h T\|_{2,\Omega}^2$ by $\|\vartheta D_k^h T\|_{2,\Omega}^2$ on the left. Hence the idea to consider the integrand in (4.1) as a product of $(\vartheta^2 D_k^h T) \cdot \nu$ and $D_k^h g_{D,t}$. Then, computing

$$\text{div}(\vartheta^2 D_k^h T) = \vartheta^2 \text{div} D_k^h T + 2\vartheta \nabla \vartheta \cdot D_k^h T$$

gives us a term which can now be absorbed by the left-hand side (since $\nabla \vartheta \in L^{\infty}(\Omega)$ and so $|2\vartheta \nabla \vartheta \cdot D_k^h T| \leq C\vartheta |D_k^h T|$). But now the function $D_k^h g_{D,t}$ is not localized to Γ and the trace theorem for the space L_{div}^2 contains

the norm $\|D_k^h g_{D,t}\|_{H^{\frac{1}{2}}(\partial\Omega)}$ on the whole boundary $\partial\Omega$. Since Ω is bounded there is some part of $\partial\Omega$ which is not contained in $\{x_n = 0\}$ and on this part the difference quotient need not make sense (since $g_{D,t}$ need not be defined outside of $\partial\Omega$). We will show now how to deal with this problem by localizing the norm of $g_{D,t}$ only to Γ .

Let $\Sigma = \{x_n = 0\}$, $\Gamma_\mu = B(0, 2\mu) \cap \Sigma$ and assume that $\delta > 0$ is so small that $\Gamma_\delta \subset \partial\Omega \cap \{x_n = 0\}$. Take a smooth cutoff function $\hat{\vartheta}$ satisfying

$$\hat{\vartheta} \equiv 1 \quad \text{on } B(0, \delta), \quad \hat{\vartheta} \equiv 0 \quad \text{on } \mathbb{R}^n \setminus \bar{B}\left(0, \frac{3}{2}\delta\right).$$

Let $\mathfrak{E}_{\Gamma_\delta} : H^{\frac{3}{2}}(\Gamma_\delta) \rightarrow H^{\frac{3}{2}}(\Sigma)$ be the extension operator (existence of $\mathfrak{E}_{\Gamma_\delta}$ is proved in Theorem 7.40 of the book [1]) and define $\hat{g}_D = \mathfrak{E}_{\Gamma_\delta}(\hat{\vartheta} g_D)$ so that

$$\hat{g}_D|_{\Gamma_\delta} = \hat{\vartheta} g_D|_{\Gamma_\delta} \quad \text{and} \quad \hat{g}_D|_\Gamma = g_D|_\Gamma. \quad (4.4)$$

$\mathfrak{E}_{\Gamma_\delta}$ is linear and continuous so

$$\|\hat{g}_D\|_{\frac{3}{2}, 2, \Sigma} \leq C \|\hat{\vartheta} g_D\|_{\frac{3}{2}, 2, \Gamma_\delta} \leq C \|g_D\|_{\frac{3}{2}, 2, \Gamma_\delta}. \quad (4.5)$$

Choose $w \in H^2(\mathbb{R}_+^n)$ such that $w = \mathcal{L}_\Sigma \hat{g}_{D,t}$, where \mathcal{L}_Σ is the lifting operator, i.e. $\mathcal{L}_\Sigma : H^{\frac{3}{2}}(\Sigma) \rightarrow H^2(\mathbb{R}_+^n)$ is linear, continuous and $w|_\Sigma = \hat{g}_{D,t}$ (Theorem 7.53 in [1]). In particular, continuity of \mathcal{L}_Σ means that

$$\|w\|_{2, 2, \mathbb{R}_+^n} \leq C \|\hat{g}_{D,t}\|_{\frac{3}{2}, 2, \Sigma}. \quad (4.6)$$

Now we will “localize” the trace theorem for the space L_{div}^2 . First let us recall that the surface integral in \mathbb{S} is in fact a duality pairing between the functional

$$\vartheta^2 D_k^h T \cdot \nu|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$$

and the function $D_k^h g_{D,t}$. Notice that $D_k^h T$, $D_k^h g_{D,t}$ are not necessarily defined for $x \notin \Gamma_{\frac{\delta}{2}}$ but as we are about to observe, the localization property of ϑ provides us with a meaningful definition of \mathbb{S} .

The functional $\vartheta^2 D_k^h T \cdot \nu|_{\partial\Omega}$ is actually defined by the formula

$$X_\xi(\phi) = \int_{\Omega} (\text{div } \xi \cdot \eta + \xi \cdot \nabla \eta) dx, \quad (4.7)$$

where $\xi = \vartheta^2 D_k^h T$ and ϕ, η are such that $\eta|_{\partial\Omega} = \phi$ (compare with (1.9)). Approximating ξ and η by smooth functions it is not difficult to observe that

$$X_\xi(\eta) = X_\xi(\hat{\eta})$$

whenever $\eta|_\Gamma = D_k^h g_{D,t}$ and $\hat{\eta}|_\Gamma = D_k^h \hat{g}_{D,t}$ (since (4.4) holds and the integral in (4.7) is localized to U'). Therefore taking $\eta = D_k^h w$ we can estimate (see also (3.4) in [12])

$$\begin{aligned} |X_\xi(\eta)| &\leq C \|\xi\|_{2, \text{div}, \Omega} \|\eta\|_{1, 2, \Omega} = C \|\vartheta^2 D_k^h T\|_{2, \text{div}, \Omega} \|D_k^h w\|_{1, 2, \Omega} \\ &\leq C \|\vartheta^2 D_k^h T\|_{2, \text{div}, \Omega} \|w\|_{2, 2, \mathbb{R}_+^n}. \end{aligned}$$

Application of (4.5) and (4.6) leads to

$$\begin{aligned} \mathbb{S} &\leq C \int_0^t \|\vartheta^2 D_k^h T\|_{2, \text{div}, \Omega} \|w\|_{2, 2, \mathbb{R}_+^n} dt \leq C \int_0^t \|\vartheta^2 D_k^h T\|_{2, \text{div}, \Omega} \|\hat{g}_{D,t}\|_{\frac{3}{2}, 2, \Sigma} dt \\ &\leq C \int_0^t \|\vartheta^2 D_k^h T\|_{2, \text{div}, \Omega} \|g_{D,t}\|_{\frac{3}{2}, 2, \Gamma_\delta} dt \end{aligned}$$

and so we get

$$\begin{aligned}
\mathbb{S} &\leq C \int_0^t \|\vartheta^2 D_k^h T\|_{2, \text{div}, \Omega} \|g_{D,t}\|_{\frac{3}{2}, 2, \Gamma_\delta} dt \\
&\leq C \left(\int_0^t [\|\vartheta^2 D_k^h T\|_{2, \Omega}^2 + \|\vartheta^2 D_k^h f\|_{2, \Omega}^2 + \|\vartheta \nabla \vartheta \cdot D_k^h T\|_{2, \Omega}^2] dt \right)^{\frac{1}{2}} \left(\int_0^t \|g_{D,t}\|_{\frac{3}{2}, 2, \Gamma_\delta}^2 dt \right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^t [\|\vartheta D_k^h T\|_{2, \Omega}^2 + \|\vartheta^2 D_k^h f\|_{2, \Omega}^2] dt \right)^{\frac{1}{2}} \left(\int_0^t \|g_{D,t}\|_{\frac{3}{2}, 2, \Gamma_\delta}^2 dt \right)^{\frac{1}{2}}
\end{aligned} \tag{4.8}$$

(we have used that $\nabla \vartheta \in L^\infty(\Omega)$ and $\vartheta^2 \leq \vartheta$). This proves the statement of our theorem for the case of Dirichlet boundary condition.

(b) Neumann boundary condition

We substitute $T \cdot \nu = g_N$ on $\partial\Omega$. The time differential is moved away from $D_k^h u_t$ to obtain

$$\begin{aligned}
\mathbb{S} &= - \int_0^t \int_\Gamma \vartheta^2 D_k^h g_{N,t} \cdot D_k^h u \, dS_x \, dt + \int_\Gamma \vartheta^2 D_k^h g_N(t) \cdot D_k^h u(t) \, dS_x - \int_\Gamma \vartheta^2 D_k^h g_N(0) \cdot D_k^h u^0 \, dS_x \\
&\leq \int_0^t \|\vartheta D_k^h g_{N,t}\|_{-\frac{1}{2}, 2, \partial\Omega} \|\vartheta D_k^h u\|_{\frac{1}{2}, 2, \partial\Omega} \, dt + \|\vartheta D_k^h g_N(t)\|_{-\frac{1}{2}, 2, \partial\Omega} \|\vartheta D_k^h u(t)\|_{\frac{1}{2}, 2, \partial\Omega} \\
&\quad + \|\vartheta D_k^h g_N(0)\|_{-\frac{1}{2}, 2, \partial\Omega} \|\vartheta D_k^h u^0\|_{\frac{1}{2}, 2, \partial\Omega}.
\end{aligned} \tag{4.9}$$

Now, using weighted Young's inequality with a small constant ϵ we get

$$\begin{aligned}
\mathbb{S} &\leq \epsilon \left(\int_0^t \|\vartheta D_k^h u\|_{\frac{1}{2}, 2, \Gamma}^2 \, dt + \|\vartheta D_k^h u(t)\|_{\frac{1}{2}, 2, \Gamma} \right) \\
&\quad + C \left(\int_0^t \|\vartheta D_k^h g_{N,t}\|_{-\frac{1}{2}, 2, \Gamma}^2 \, dt + \|\vartheta D_k^h g_N(t)\|_{-\frac{1}{2}, 2, \Gamma}^2 + \|\vartheta D_k^h g_N(0)\|_{-\frac{1}{2}, 2, \Gamma} \|\vartheta D_k^h u^0\|_{\frac{1}{2}, 2, \Gamma} \right).
\end{aligned} \tag{4.10}$$

The trace theorem in the Sobolev space $H^1(\Omega)$ and Korn's inequality give us

$$\begin{aligned}
\|\vartheta D_k^h u\|_{\frac{1}{2}, 2, \partial\Omega} &\leq C \|\vartheta D_k^h u\|_{1, 2, \Omega} \leq C (\|\varepsilon(\nabla(\vartheta D_k^h u))\|_{2, \Omega} + \|\vartheta D_k^h u\|_{2, \Omega}) \\
&\leq C (\|\vartheta D_k^h \varepsilon(\nabla u)\|_{2, \Omega} + \|\varepsilon(D_k^h u \otimes \nabla \vartheta)\|_{2, \Omega} + \|\vartheta D_k^h u\|_{2, \Omega}) \\
&\leq C (\|\vartheta D_k^h \varepsilon(\nabla u)\|_{2, \Omega} + \|\nabla u\|_{2, U'}).
\end{aligned}$$

Plugging the above estimate into (4.10) yields

$$\begin{aligned}
\mathbb{S} &\leq \epsilon \left(\int_0^t \|\vartheta D_k^h \varepsilon(\nabla u)\|_{2, \Omega}^2 \, dt + \|\vartheta D_k^h \varepsilon(\nabla u(t))\|_{2, \Omega}^2 \right) \\
&\quad + C \left(\int_0^t \|g_{N,t}\|_{\frac{1}{2}, 2, \Gamma}^2 \, dt + \|g_{N,t}(t)\|_{\frac{1}{2}, 2, \Gamma}^2 + \|g_{N,t}(0)\|_{\frac{1}{2}, 2, \Gamma} \|\vartheta D_k^h \varepsilon^0\|_{2, \Omega} + \sup_{(0,t)} \|\nabla u\|_{2, U'} \right).
\end{aligned} \tag{4.11}$$

But

$$\int_0^t \|\vartheta D_k^h \varepsilon(\nabla u)\|_{2, \Omega}^2 \, dt \leq t \sup_{(0,t)} \|\vartheta D_k^h \varepsilon(\nabla u)\|_{2, \Omega}^2$$

and so we can absorb the terms with “ ϵ ” in (4.11) by the energy $\mathcal{E}(\vartheta D_k^h \varepsilon, \vartheta D_k^h z)$.

b) **Dynamic problem**(a) Dirichlet boundary condition

We proceed in a similar fashion as in the first part of the proof for the quasistatic case to get that

$$\begin{aligned} \mathbb{S} &\leq C \int_0^t \|T\|_{2,\text{div},\Omega} \|D_k^{-h}(\vartheta^2 D_k^h g_{D,t})\|_{\frac{1}{2},2,\partial\Omega} dt \\ &\leq C \left(\int_0^t [\|\rho v_t\|_{2,\Omega}^2 + \|f\|_{2,\Omega}^2 + \|T\|_{2,\Omega}^2] dt \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_k(\vartheta^2 D_k^h g_{D,t})\|_{\frac{1}{2},2,\partial\Omega}^2 dt \right)^{\frac{1}{2}} \\ &\leq C \left(\int_0^t \|g_{D,t}\|_{\frac{3}{2},2,\Gamma_\delta}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

In fact, the only difference from the quasistatic case is the additional term ρv_t coming from the L^2 -norm of $\text{div } T$. As was observed in Remark 2.3 it is in $L^2(L^2)$.

If we were to proceed as in the second part of the proof for (Q) then inequality (4.8) would take the form

$$\mathbb{S} \leq C \left(\int_0^t [\|\vartheta D_k^h T\|_{2,\Omega}^2 + \|\vartheta^2 D_k^h f\|_{2,\Omega}^2 + \|\vartheta^2 D_k^h v_t\|_{2,\Omega}^2] dt \right)^{\frac{1}{2}} \left(\int_0^t \|g_{D,t}\|_{\frac{3}{2},2,\partial\Gamma_\delta}^2 dt \right)^{\frac{1}{2}}.$$

Unfortunately, we do not have any information about the norm of $D_k^h v_t$ in $L^2(\Omega)$ and we do not know how to deal with weaker assumptions on g_D in this case.

c) Neumann boundary condition

The proof is done in exactly the same way as for the quasistatic case. \square

4.2. *Tangential derivatives – general case (coercive problems)*

Now we will shortly present how the above estimates can be carried out for curved boundaries. We will use the idea from [4]. Assume that $\partial\Omega$ is of class C^2 . Thus, in some neighborhood of $x_0 \in \partial\Omega$ there exists a diffeomorphism $\Psi : U' = \Omega \cap B(x_0, \epsilon) \rightarrow \tilde{U} = \mathbb{R}_+^n \cap B(0, \delta)$ of class C^2 such that $\Psi(x_0) = 0$ and $\Psi(\partial\Omega \cap B(x_0, \epsilon)) \subset \{x_n = 0\}$. Let $\tilde{V} = B(0, \frac{\delta}{2})$, $\Gamma = B(0, \delta) \cap \{x_n = 0\}$ and assume that ϑ is as in (A- ϑ) with $U = B(0, \delta)$ and $V = B(0, \frac{\delta}{2})$.

Denote $y = \Psi(x)$ and $\tilde{\zeta}(t, y) = \zeta(t, x)$ for $\zeta = T, \varepsilon, u$, etc. The quasistatic problem takes the form

$$\begin{aligned} -\text{div}_y^* \tilde{T}(t, y) &= \tilde{f}(t, y), \quad t > 0, y \in \tilde{U}, \\ \tilde{T}(t, y) &= \mathcal{D}(\varepsilon(\nabla_y^* \tilde{u}(t, y)) - B\tilde{z}(t, y)), \quad t > 0, y \in \tilde{U}, \\ \tilde{z}_t(t, y) &\in g(B^T \tilde{T}(t, y) - L\tilde{z}(t, y)), \quad t > 0, y \in \tilde{U}, \\ \tilde{z}(0, y) &= \tilde{z}^0(y), \quad y \in \tilde{U}, \\ \tilde{u}(t, y) &= \begin{cases} \tilde{g}_D(t, y) & \text{for } t > 0, y \in \Gamma, \\ \tilde{u}(t, y) & \text{for } t > 0, y \in \partial\tilde{U} \setminus \Gamma \end{cases} \end{aligned}$$

where

$$\begin{aligned} \text{div}_y^* \tilde{T}(t, y) &= \text{div}_x T(t, \Psi(x))|_{x=\Psi^{-1}(y)}, \\ \nabla_y^* \tilde{u}(t, y) &= \nabla_x u(t, \Psi(x))|_{x=\Psi^{-1}(y)}. \end{aligned}$$

As was observed in [4], the operators $-\text{div}_y^*$ and ∇_y^* are adjoint with respect to the scalar product

$$(\zeta, \eta)^* = \int_{\tilde{U}} \zeta(y) \cdot \eta(y) \cdot |\det \nabla \Psi^{-1}(y)| dy \quad (4.12)$$

(provided that ζ, η vanish on $\partial\tilde{U}$). Moreover, the following sequence of equalities is very useful when transferring estimations from the proof for the case of straight boundary:

Lemma 4.3. Assume that $T \in L^2_{\text{div}}(U)$, $u \in H^1(U)$ and \tilde{T}, \tilde{u} be defined as above. Then

$$\begin{aligned}
 & \int_U (\text{div}_x T(x) \cdot u(x) + T(x) \cdot \nabla_x u(x)) dx \\
 &= \int_{\partial U} T(x) \cdot \nu_U(x) \cdot u(x) dS_x \\
 &= \int_{U'} (\text{div}_y^* \tilde{T}(y) \cdot \tilde{u}(y) + \tilde{T}(y) \cdot \nabla_y^* \tilde{u}(y)) |\det \nabla_y \Psi^{-1}(y)| dy \\
 &= \int_{\partial U'} \tilde{T}(y) \cdot \nu_{U'}(y) \cdot \nabla_y^T \Psi^{-1}(y) \cdot \tilde{u}(y) |\det \nabla_y \Psi(y)| dS_y
 \end{aligned} \tag{4.13}$$

where $\nu_U, \nu_{U'}$ are outward normal vectors to U and U' respectively.

Having (4.12) at hand and estimates from Theorem 4.1 it is not difficult to prove an analogous result for curved boundaries. In fact, considering the energy associated with the scalar product (4.12)

$$\mathcal{E}^*(\tilde{\varepsilon}, \tilde{z}) = \int_{\tilde{U}} [\mathcal{D}(\tilde{\varepsilon}(y) - B\tilde{z}(y)) \cdot (\tilde{\varepsilon}(y) - B\tilde{z}(y)) + L\tilde{z}(y) \cdot \tilde{z}(y)] |\det \nabla \Psi^{-1}(y)| dy \tag{4.14}$$

we get

$$\begin{aligned}
 \frac{d}{dt} \mathcal{E}^*(\vartheta D_k^h \varepsilon(\nabla_y^* \tilde{u}(t, \cdot)), \vartheta D_k^h \tilde{z}(t, \cdot)) &= \int_{\tilde{U}} \vartheta^2(y) D_k^h \tilde{T}(t, y) \cdot D_k^h \varepsilon(\nabla_y^* \tilde{u}_t(t, y)) \cdot |\det \nabla \Psi^{-1}(y)| dy \\
 &\quad - \int_{\tilde{U}} \vartheta^2(y) [B^T D_k^h \tilde{T}(t, y) - L D_k^h \tilde{z}(t, y)] \cdot D_k^h \tilde{z}_t(t, y) \cdot |\det \nabla \Psi^{-1}(y)| dy \\
 &\leq - \int_{\tilde{U}} \vartheta^2(y) D_k^h \text{div}_y^* \tilde{T}(t, y) \cdot D_k^h \tilde{u}_t(t, y) \cdot |\det \nabla \Psi^{-1}(y)| dy \\
 &\quad - 2 \int_{\tilde{U}} \vartheta(y) D_k^h \tilde{T}(t, y) \cdot \varepsilon(D_k^h \tilde{u}_t(t, y) \otimes \nabla_y^* \vartheta(y)) \cdot |\det \nabla \Psi^{-1}(y)| dy \\
 &\quad + \int_{\Gamma} \vartheta^2(y) D_k^h \tilde{T}(t, y) \cdot \nu(y) \cdot D_k^h \tilde{u}_t(t, y) \cdot |\det \nabla \Psi^{-1}(y)| dS_y.
 \end{aligned}$$

The estimates that need to be carried out now are similar to those from the previous section. Along the way we apply the trace theorem in the space $L^2_{\text{div}}(\tilde{U})$, which follows from Theorem 1.1 by a change of variables. In fact, one defines the trace of $\tilde{T} \cdot \nu_{U'}$ using equality (4.13):

$$\langle \tilde{T} \cdot \nu_{U'}, \phi \rangle = \int_{U'} (\text{div}_y^* \tilde{T} \cdot \phi + \tilde{T} \cdot \nabla_y^* \phi) dy.$$

This way Theorem 4.1 holds also for curved boundaries.

4.3. Tangential regularity for non-coercive models

To fix the attention, we again assume that $\partial\Omega$ is straight and use the notation from Section 4.1.

For non-coercive models the Neumann condition causes some difficulties. We cannot carry out estimates as in (4.11), since we do not control $D_k^h \varepsilon(\nabla u)$ anymore. Also, we cannot write

$$D_k^h \varepsilon(\nabla u) = \mathcal{D}^{-1} T + B D_k^h z$$

and use the self-controlling estimate on $D_k^h z$, since this estimate is on z only and not on the difference $\Delta_k^h z$ (for the same reason we needed the coercivity assumption while estimating the first and third integrals in (3.2) with the term $D_k^h z$).

A way to avoid this problem is to shift the difference quotient away from u and assume more regularity on g_N . The precise formulation is as follows:

Theorem 4.4. Let the model be non-coercive and let the assumptions of Theorem 4.1 be fulfilled with a stronger assumption:

(b) $g_N \in H^1((0, T); H^{\frac{3}{2}}(\partial\Omega))$ for the Neumann boundary condition.

Then the solution (to both quasistatic and dynamic problems) has regularity

$$(T_{x_k}, Lz_{x_k}) \in L^\infty((0, T); L^2(V') \times L^2(V')), \quad \text{for } k = 1, \dots, (n-1).$$

Proof. For the Dirichlet boundary the proof is carried out exactly as in Theorem 4.1.

For the Neumann condition we integrate by parts with respect to t in the surface integral \mathbb{S} as in (4.9) and then shift the difference quotients away from u :

$$\mathbb{S} = \int_0^t \int_\Gamma D_k^{-h}(\vartheta^2 D_k^h g_{N,t}) \cdot u \, dS_x - \int_\Gamma D_k^{-h}(\vartheta^2 D_k^h g_N(t)) \cdot u(t) \, dS_x + \int_\Gamma D_k^{-h}(\vartheta^2 D_k^h g_N(0)) \cdot u^0 \, dS_x.$$

Now, for fixed t consider the surface integral

$$\mathbb{I} = \int_\Gamma D_k^{-h}(\vartheta^2 D_k^h \zeta(t)) \cdot u(t) \, dS_x$$

with either $\zeta = g_{N,t}$ or $\zeta = g_N$. We have

$$D_k^{-h}(\vartheta^2 D_k^h \zeta(t)) = (D_k^{-h} \vartheta)(\vartheta D_k^h \zeta(t)) + \vartheta_{\{-h,k\}} D_k^{-h}(\vartheta D_k^h \zeta(t)).$$

Then

$$\begin{aligned} \mathbb{I} &\leq \|\vartheta D_k^h \zeta(t)\|_{-\frac{1}{2}, 2, \partial\Omega} \|(D_k^{-h} \vartheta)u(t)\|_{\frac{1}{2}, 2, \partial\Omega} + \|D_k^{-h}(\vartheta D_k^h \zeta(t))\|_{-\frac{1}{2}, 2, \partial\Omega} \|\vartheta u(t)\|_{\frac{1}{2}, 2, \partial\Omega} \\ &\leq C \|\zeta(t)\|_{\frac{3}{2}, 2, \Gamma} \|u(t)\|_{1, 2, U'}. \end{aligned}$$

Collecting the results we get that \mathbb{S} is bounded, which proves the regularity statement of the theorem. \square

5. Boundary regularity – normal derivatives, quasistatic model

The previous sections give us that for coercive quasistatic problems $\varepsilon, z \in L^\infty(H_{loc}^1)$ and $\nabla_\Gamma \varepsilon, \nabla_\Gamma z \in L^\infty(L^2)$. Moreover, differentiating (Q)₁ with respect to time and assuming some regularity of f we have $T_t \in L^2(L_{div}^2)$.

Let $D = \Gamma \times (-1, 1) \subset \mathbb{R}^n$, where $\Gamma \subset \mathbb{R}^{n-1}$ is some open bounded set and let $D^+ = D \cap \{x_n > 0\} = \Gamma \times (0, 1)$. Recall part a) of Lemma 3.3 from [4]:

Lemma 5.1. There is a constant $C > 0$ such that for all $v \in H^1(D^+)$ with $\nabla_\Gamma v \in H^1(D^+)$ and for all sufficiently small $h > 0$ the following inequality holds

$$\|v_h - v\|_{H^1(\Gamma)} \leq Ch^{\frac{1}{2}} \|v\|_{H_\Gamma^2(D^+)}, \quad (5.1)$$

where $v_h = v(x + h\vec{e}_n)$ and $\|v\|_{H_\Gamma^2(D^+)}^2 = \|v\|_{H^1(D^+)}^2 + \|\nabla_\Gamma v\|_{H^1(D^+)}^2$.

Theorem 5.2. Assume that the problem (Q) is coercive. Let the assumptions from Theorem 2.1 be satisfied with $f_t \in L^2((0, T); H^1)$. Let the boundary data have regularity: $g_N \in L^2((0, T); H^{\frac{3}{2}})$ or $g_D \in L^2((0, T); H^{\frac{3}{2}})$. Then the global regularity

$$(u, \varepsilon, z) \in L^\infty((0, T); B_{2,\infty}^{\frac{5}{4}}(\Omega) \times B_{2,\infty}^{\frac{1}{4}}(\Omega) \times B_{2,\infty}^{\frac{1}{4}}(\Omega))$$

holds.

Proof. Let $\Delta_k^h u(x) = u(x + h\vec{e}_k) - u(x)$ and let $D_k^{h,\alpha} = \frac{1}{h^\alpha} \Delta_k^h$. Let $\alpha \in (0, 1)$. We have

$$\begin{aligned} \mathcal{E}(\vartheta D_n^{h,\alpha} \varepsilon, \vartheta D_n^{h,\alpha} z)(t) &\leq \mathcal{E}(\vartheta D_n^{h,\alpha} \varepsilon, \vartheta D_n^{h,\alpha} z)(0) + \int_0^t \int_\Omega \vartheta^2 D_n^{h,\alpha} f \cdot D_n^{h,\alpha} u_t \\ &\quad - 2 \int_0^t \int_\Omega \vartheta D_n^{h,\alpha} T \cdot \varepsilon(D_n^{h,\alpha} u_t \otimes \nabla \vartheta) + \int_0^t \int_\Gamma \vartheta^2 D_n^{h,\alpha} T \cdot \nu \cdot D_n^{h,\alpha} u_t. \end{aligned}$$

Since $v \in L^2(H^1)$ the two first integrals are finite (in the second one we use Young's inequality to move $D_n^{h,\alpha} T$ to the left, with sup over $(0, t)$). The third integral is equal to

$$\begin{aligned} \int_0^t \int_{\Gamma} \vartheta^2 D_n^{h,\alpha} T \cdot v \cdot D_n^{h,\alpha} u_t &= - \int_0^t \int_{\Gamma} \vartheta^2 D_n^{h,\alpha} T_t \cdot v \cdot D_n^{h,\alpha} u + \int_{\Gamma} \vartheta^2 D_n^{h,\alpha} T(t) \cdot v \cdot D_n^{h,\alpha} u(t) \\ &\quad - \int_{\Gamma} D_n^{h,\alpha} T(0) \cdot v \cdot D_n^{h,\alpha} u(0) \\ &= - \int_0^t \int_{\Gamma} \vartheta^2 \Delta_n^h T_t \cdot v \cdot D_n^{h,2\alpha} u + \int_{\Gamma} \vartheta^2 \Delta_n^h T(t) \cdot v \cdot D_n^{h,2\alpha} u(t) \\ &\quad - \int_{\Gamma} \vartheta^2 D_n^{h,\alpha} T(0) \cdot v \cdot D_n^{h,\alpha} u(0) \\ &\leq C \left(\int_0^t \|\Delta_n^h T_t\|_{2,D^+}^2 + \int_0^t \|\Delta_n^h f_t\|_{2,D^+}^2 \right) \sup_{(0,t)} \|\vartheta D_n^{h,2\alpha} u\|_{\frac{1}{2},2,\Gamma} \\ &\leq Ch^{-2\alpha} \sup_{(0,t)} \|\vartheta \Delta_n^h u\|_{1,2,\Gamma}. \end{aligned} \quad (**)$$

Applying the above lemma, we see that for $\alpha = \frac{1}{4}$ we get boundedness of the energy and so this proves that $D_n^{h,\frac{1}{4}} \varepsilon, D_n^{h,\frac{1}{4}} z$ are bounded independently of h in the space $L^\infty(L^2)$ (notice that from what we have proved about tangential derivatives, the norm $\|u\|_{H_{\Gamma}^2(D^+)}$, which is comparable with $\|\varepsilon\|_{H_{\Gamma}^1(D^+)}$, is finite). Thus $u \in L^\infty(B_{2,\infty}^{\frac{5}{4}})$, $T, z \in L^\infty(B_{2,\infty}^{\frac{1}{4}})$, for coercive problems. \square

We can use a bootstrap argument and with the aid of part b) of Lemma 3.3 from [4] obtain that $u \in L^\infty(H^{\frac{4}{3}-\delta})$ and $T, z \in L^\infty(H^{\frac{1}{3}-\delta})$.

We have derived a different technique for bootstrapping, which uses only first-order finite differences and thus gives a worse regularity result.⁴ The idea is to improve Lemma 5.1 so that it is true for the space $H^{\frac{1}{2}}(\Gamma)$ and then plug this into (**).

Lemma 5.3. *There is a constant C such that for all $v \in H^1(D^+)$ with $v|_{\partial D^+ \setminus \Gamma} = 0$ and $v_{x_n} \in L^p(D^+)$ for $p > 2$ and with $\nabla_{\Gamma} v \in H^1(D^+)$ and all sufficiently small $h > 0$ we have*

$$\|v_h - v\|_{H^{\frac{1}{2}}(\Gamma)}^2 \leq Ch^{1+\frac{p-2}{2p}} \|v_{x_n}\|_{H_{\Gamma}^1(D^+)} \|v_{x_n}\|_{L^p(D^+)},$$

where $v_{x_n} = \frac{\partial v}{\partial x_n}$.

Proof. We will use Fourier's transform. First, take a function $u \in C_0^\infty(D)$. Let $v(x') = u(x', 0)$ and $v_h(x') = u(x', h)$. Let the symbol $\tilde{\cdot}$ denote the Fourier transform with respect to the first $(n-1)$ variables. Then we have

$$(\tilde{v}_h - \tilde{v})(\xi') = \int_0^h \tilde{u}_{x_n}(\xi', x_n) dx_n.$$

Now, by the definition of Sobolev–Slobodeckij spaces

$$\begin{aligned} \|v_h - v\|_{H^{\frac{1}{2}}(\Gamma)}^2 &= \int_{\mathbb{R}^{n-1}} (1 + |\xi'|^2)^{\frac{1}{2}} |\tilde{v}_h(\xi') - \tilde{v}(\xi')|^2 d\xi' = \int_{\mathbb{R}^{n-1}} (1 + |\xi'|^2)^{\frac{1}{2}} \left| \int_0^h \tilde{u}_{x_n}(\xi', x_n) dx_n \right|^2 d\xi' \\ &= \int_{\mathbb{R}^{n-1}} \left| \int_0^h (1 + |\xi'|^2)^{\frac{1}{4}} \tilde{u}_{x_n}(\xi', x_n) dx_n \right|^2 d\xi' \end{aligned}$$

⁴ At its core, the technique we present and that of [4] have a lot in common: we use Fourier's transform, while the authors in [4] use interpolation between H^s spaces, which essentially is the same. The difference is that we apply Besov imbeddings, while in [4] the second-order differences are used. This in part explains why our result is worse.

$$\begin{aligned}
&\leq h \int_{\mathbb{R}^{n-1}} \int_0^h (1 + |\xi'|^2)^{\frac{1}{2}} |\tilde{u}_{x_n}(\xi', x_n)|^2 dx_n d\xi' \\
&\leq h \int_0^h \left(\int_{\mathbb{R}^{n-1}} (1 + |\xi'|^2) |\tilde{u}_{x_n}(\xi', x_n)|^2 d\xi' \right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^{n-1}} |\tilde{u}_{x_n}(\xi', x_n)|^2 d\xi' \right)^{\frac{1}{2}} dx_n \\
&\leq h \left(\int_0^h \int_{\mathbb{R}^{n-1}} (1 + |\xi'|^2) |\tilde{u}_{x_n}(\xi', x_n)|^2 d\xi' dx_n \right)^{\frac{1}{2}} \cdot \left(\int_0^h \int_{\mathbb{R}^{n-1}} |\tilde{u}_{x_n}(\xi', x_n)|^2 d\xi' dx_n \right)^{\frac{1}{2}} \\
&\leq Ch \left(\int_0^h \|u_{x_n}(\cdot, x_n)\|_{H^1_\Gamma(\Gamma)}^2 dx_n \right)^{\frac{1}{2}} \cdot \left(\int_0^h \|u_{x_n}(\cdot, x_n)\|_{L^2(\Gamma)}^2 dx_n \right)^{\frac{1}{2}} \\
&\leq Ch^{1+\frac{p-2}{2p}} \left(\int_0^h \|u_{x_n}(\cdot, x_n)\|_{H^1_\Gamma(\Gamma)}^2 dx_n \right)^{\frac{1}{2}} \cdot \left(\int_0^h \|u_{x_n}(\cdot, x_n)\|_{L^p(\Gamma)}^p dx_n \right)^{\frac{1}{p}} \\
&\leq Ch^{1+\frac{p-2}{2p}} \|u_{x_n}\|_{H^1_\Gamma(D^+)} \|u_{x_n}\|_{L^p(D^+)},
\end{aligned}$$

where $\Gamma \times (0, h) \subset D^+$ (we have repeatedly used Hölder's inequality and isometry of Fourier's transform in the space $L^2(\mathbb{R}^{n-1})$).

Now we use a standard density argument and the inequality is proved. \square

Theorem 5.4. *Let the assumptions of Theorem 5.2 be satisfied. Then for $n = 3$ and all small $\delta > 0$*

$$(u, \varepsilon, z) \in L^\infty((0, T); B_{2,\infty}^{\frac{14}{11}-\delta}(\Omega) \times B_{2,\infty}^{\frac{3}{11}-\delta}(\Omega) \times B_{2,\infty}^{\frac{3}{11}-\delta}(\Omega)).$$

Proof. We will iteratively apply Lemma 5.3 to the function $v = \vartheta \Delta_n^h u$.

Suppose that $\varepsilon \in L^\infty(B_{2,\infty}^{\alpha_k}(D^+))$ (we have proved above that $\alpha_0 = \frac{1}{4}$). From the imbedding theorem for Besov spaces we get $\varepsilon \in L^\infty(L^{p_k}(D^+))$, where $p_k = \frac{6}{3-2\alpha_k+2\delta}$ (from the assumption on the dimension n) for arbitrarily small $\delta > 0$. The right-hand side of inequality (**) with $\alpha = \alpha_{k+1}$ takes the form:

$$C \sup_{(0,t)} \|\vartheta \Delta_n^{h, 2\alpha_{k+1}} u\|_{\frac{1}{2}, 2, \Gamma} = Ch^{-2\alpha_{k+1}} \sup_{(0,t)} \|\vartheta \Delta_n^h u\|_{\frac{1}{2}, 2, \Gamma} \leq Ch^{\frac{1}{2} + \frac{p_k-2}{4p_k} - 2\alpha_{k+1}} \sup_{(0,t)} \|\vartheta u\|_{H_\Gamma^2(D^+)} \|\varepsilon\|_{p_k, D^+}.$$

The exponent of h should be zero, thus

$$\alpha_{k+1} = \frac{1}{4} + \frac{p_k-2}{8p_k} = \frac{1}{4} + \frac{1}{12}\alpha_k - \frac{1}{12}\delta.$$

Therefore, solving the above difference equation gives

$$\alpha_k = \left(\frac{1}{11}\delta - \frac{1}{44} \right) \left(\frac{1}{12} \right)^k + \frac{3}{11} - \frac{1}{11}\delta.$$

Thus $\alpha_k \nearrow \frac{3}{11} - \frac{1}{11}\delta$ for $k \rightarrow \infty$ and so $u \in L^\infty(B_{2,\infty}^{\frac{14}{11}-\delta})$ and $T, z \in L^\infty(B_{2,\infty}^{\frac{3}{11}-\delta})$ for arbitrarily small $\delta > 0$. \square

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Appendix A. Some other local regularity results

The assumption $u_t \in L^2((0, T); H^1(\Omega'))$ can be weakened to $u_t \in L^2((0, T); B_{2,\infty}^s(\Omega'))$, with $\Omega' \Subset \Omega$ and $s \leq 1$, providing a worse local regularity result than the one from the first part of the paper. We are not aware of any existence theorems for

viscoplasticity in Besov spaces, that's why we only assume that the solution exists. Also, this is the reason that the following results aren't theorems but only "observations".

Starting out exactly as in the first sections, but taking energy only on differences (without dividing by h for now), for the quasistatic problem, we arrive at the following inequality

$$\frac{d}{dt} \mathcal{E}(\vartheta \Delta_k^h \varepsilon, \vartheta \Delta_k^h z)(t) \leq \int_{\Omega} \vartheta^2 \Delta_k^h f \cdot \Delta_k^h u_t \, dx - 2 \int_{\Omega} \vartheta \Delta_k^h T \cdot \varepsilon(\Delta_k^h u_t \otimes \nabla \vartheta) \, dx.$$

Thus, after integration with respect to time we obtain

$$\mathcal{E}(\vartheta \Delta_k^h \varepsilon, \vartheta \Delta_k^h z)(t) \leq \mathcal{E}(\vartheta \Delta_k^h \varepsilon^0, \vartheta \Delta_k^h z^0) + \int_0^t \int_{\Omega} \vartheta^2 \Delta_k^h f \cdot \Delta_k^h u_t \, dx \, dt - 2 \int_0^t \int_{\Omega} \vartheta \Delta_k^h T \cdot \varepsilon(\Delta_k^h u_t \otimes \nabla \vartheta) \, dx \, dt.$$

Now divide by h^{2s} to get

$$\begin{aligned} \mathcal{E}(\vartheta D_k^{h,s} \varepsilon, \vartheta D_k^{h,s} z)(t) &\leq \mathcal{E}(\vartheta D_k^{h,s} \varepsilon^0, \vartheta D_k^{h,s} z^0) + \int_0^t \int_{\Omega} \vartheta^2 D_k^{h,s} f \cdot D_k^{h,s} u_t \, dx \, dt \\ &\quad - 2 \int_0^t \int_{\Omega} \vartheta D_k^{h,s} T \cdot \varepsilon(D_k^{h,s} u_t \otimes \nabla \vartheta) \, dx \, dt. \end{aligned} \quad (\text{A.1})$$

Thus the last integral on the right can be split up with the aid of Young's inequality to absorb the difference quotient of the stress by the left-hand side. The remaining terms are bounded independently of h provided

$$f \in L^2((0, T); B_{2,\infty}^s(\Omega')) \quad \text{and} \quad \varepsilon^0, z^0 \in B_{2,\infty}^s(\Omega'). \quad (\text{A.2})$$

The same reasoning can be applied to the dynamic problem.

Observation 1. Assume that $u_t \in L^2((0, T); B_{2,\infty}^s(\Omega'))$ and suppose that (A.2) holds. Then the solution to both the quasistatic and dynamic problems has regularity

$$T, Lz \in L^\infty((0, T); B_{2,\infty}^s(\Omega')).$$

For coercive problems this regularity result also applies to the whole vector z and to the strain tensor: $\varepsilon \in L^\infty((0, T); B_{2,\infty}^s(\Omega'))$.

Also, notice that in the last integral in (A.1) we can shift the time derivative away from $D_k^{h,s} u_t$ and move h^s from $\Delta_k^h T$ to $D_k^{h,s} u_t$, which gives

$$\begin{aligned} \mathcal{E}(\vartheta D_k^{h,s} \varepsilon, \vartheta D_k^{h,s} z)(t) &\leq \mathcal{E}(\vartheta D_k^{h,s} \varepsilon^0, \vartheta D_k^{h,s} z^0) + \int_0^t \int_{\Omega} \vartheta^2 D_k^{h,s} f \cdot D_k^{h,s} u_t \, dx \, dt \\ &\quad + 2 \int_0^t \int_{\Omega} \vartheta \Delta_k^h T_t \cdot \varepsilon(D_k^{h,2s} u \otimes \nabla \vartheta) \, dx \, dt - 2 \int_{\Omega} \vartheta \Delta_k^h T(t) \cdot \varepsilon(D_k^{h,2s} u(t) \otimes \nabla \vartheta) \, dx \\ &\quad + 2 \int_{\Omega} \vartheta D_k^{h,s} T(0) \cdot \varepsilon(D_k^{h,s} u(0) \otimes \nabla \vartheta) \, dx. \end{aligned}$$

The same procedure can be applied to the integral from (A.1) with the force density. Now it is easy to prove the following

Observation 2. Assume that $u \in L^\infty((0, T); B_{2,\infty}^{2s}(\Omega'))$, $T_t \in L^2((0, T); L^2(\Omega))$ and that $f_t \in L^2((0, T); L^2(\Omega))$. Suppose ε^0, z^0 are as in (A.2). Then

$$T, Lz \in L^\infty((0, T); B_{2,\infty}^s(\Omega')).$$

For the dynamic problem also

$$v \in L^\infty((0, T); B_{2,\infty}^s(\Omega')).$$

Thus, locally, the smoothness of T and Lz is the "same" as the smoothness of u_t and "half of" the smoothness of u .

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