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Bounds for the L^q-spectra of self-similar measures without any separation conditions

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ABSTRACT

In this paper we obtain non-trivial bounds for the L^q-spectra of self-similar measures without any separation conditions for $q \leqslant 1$. As an application we obtain non-trivial upper bounds for the multifractal spectra of arbitrary self-similar measure without any separation conditions. Some examples illustrating our results are also discussed.

1. Introduction and statement of results

Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ $(i = 1, 2, \dots, N)$ be contracting similarities with contraction ratios $r_i \in (0, 1)$ and let $(p_1, \dots, p_N)$ be a probability vector (i.e. $0 < p_i < 1$ for all i and $\sum_{i=1}^N p_i = 1$). It follows from [7] that there is a unique non-empty and compact subset K of $\mathbb{R}^d$ and a unique Borel probability measure μ on $\mathbb{R}^d$ such that

$$K = \bigcup_i S_i(K), \tag{1.1}$$

and

$$\mu = \sum_i p_i \mu \circ S_i^{-1}. \tag{1.2}$$

The set K is called the self-similar set associated with the list $(S_1, \dots, S_N)$ and the measure μ is called the self-similar measure associated with the list $(S_1, \dots, S_N, p_1, \dots, p_N)$. It is well known that the support of μ equals K. We say the list $(S_1, \dots, S_N)$ satisfies the open set condition (OSC) if there exists a non-empty, bounded and open set U such that $S_i(U) \subset U$ for all i and $S_i(U) \cap S_j(U) = \emptyset$ for all $i \neq j$.

For $r > 0$ and a real number q, write

$$I_\mu(r; q) = \int_K \mu\big(B(x, r)\big)^{q-1}\, d\mu(x). \tag{1.3}$$

The lower and upper L^q-spectra of μ are now defined by

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$$\underline{\tau}_\mu(q) = \liminf_{r \searrow 0} \frac{\log I_\mu(r;q)}{-\log r},$$
$$\overline{\tau}_\mu(q) = \limsup_{r \searrow 0} \frac{\log I_\mu(r;q)}{-\log r}.$$

The main significance of the L^q-spectra is their relationship with multifractal analysis, cf. [2–4,6,5,10–15] and the references therein. Arbeiter and Patzschke [1] proved a beautiful result providing a formula for the L^q-spectra of self-similar measures satisfying the OSC. Before stating their result we introduce the following definition. Define the function $\beta(q) : \mathbb{R} \to \mathbb{R}$ by

$$\sum_{i=1}^{N} p_i^q r_i^{\beta(q)} = 1. \tag{1.4}$$

We can now state Arbeiter and Patzschke's result.

Theorem A. *(See [1].) Assume that the OSC is satisfied. Then the L^q-spectra $\underline{\tau}_\mu(q)$ and $\overline{\tau}_\mu(q)$ are given by $\underline{\tau}_\mu(q) = \overline{\tau}_\mu(q) = \beta(q)$ for all $q \in \mathbb{R}$.*

Unfortunately, it is very difficult to determine the L^q-spectra of self-similar measures not satisfying the OSC and, as a result of this, previous work analyzing the multifractal structure of self-similar (or self-conformal) measures not satisfying the OSC has almost entirely concentrated on the following two different aspects. Namely, firstly, analyzing general self-similar measures assumed to satisfy separation conditions more general than the OSC. For examples, in [6,8,10] measures satisfying the "weak separation condition" (WSC) (which is weaker than the OSC) are investigated. Secondly, analyzing the multifractal structure of specific self-similar measures or families of self-similar measures not satisfying the OSC, see, for example, [4,5] and the references therein.

However, the scope of this paper is different. Indeed, instead of computing the L^q-spectra of specific families of self-similar measures, we are interested in obtaining non-trivial bounds of the L^q-spectra of *arbitrary* self-similar measures without any separation conditions. In fact, this line of investigation was initiated in [12] where Olsen obtained non-trivial bounds for the L^q-spectra of *arbitrary* self-similar measures without any separation conditions for $q \geqslant 1$, and the main purpose of this paper is to extend the analysis from [12] to $q < 1$. However, before we do this, it is instructive to recall the main result from [12].

We start by introducing some notation. Let

$$\Sigma^n = \{1, \dots, N\}^n, \qquad \Sigma^* = \bigcup_n \Sigma^n, \qquad \Sigma^{\mathbb{N}} = \{1, \dots, N\}^{\mathbb{N}},$$

i.e. Σ^n is the family of all finite strings $\mathbf{i} = i_1 \dots i_n$ of length n with $i_j \in \{1, \dots, N\}$, Σ^* and $\Sigma^{\mathbb{N}}$ denote the family of all finite strings $\mathbf{i} = i_1 \dots i_n$ and the family of all infinite strings $\mathbf{i} = i_1 i_2 \dots$ with $i_j \in \{1, \dots, N\}$, respectively. For $\mathbf{i} = i_1 \dots i_n \in \Sigma^n$, we will write $|\mathbf{i}| = n$ for the length of $\mathbf{i}$. For $\mathbf{i} = i_1 i_2 \dots \in \Sigma^{\mathbb{N}}$ and a positive integer n, let $\mathbf{i}|n = i_1 \dots i_n$ denote the truncation of $\mathbf{i}$ to the nth place. Furthermore, for $\mathbf{i} = i_1 \dots i_n \in \Sigma^n$, we write $S_\mathbf{i} = S_{i_1} \circ \dots \circ S_{i_n}$ and $K_\mathbf{i} = S_\mathbf{i} K$. Also, write $p_\mathbf{i} = p_{i_1} \dots p_{i_n}$ and $r_\mathbf{i} = r_{i_1} \dots r_{i_n}$ for $\mathbf{i} = i_1 \dots i_n \in \Sigma^n$. We can now state the main result from Olsen [12].

Theorem B. *(See [12].) For a positive integer n, let*

$$\mathcal{I}_n = \left\{ I \subseteq \Sigma^n \;\middle|\; \bigcap_{\mathbf{i} \in I} K_\mathbf{i} \neq \emptyset \right\}$$

(*observe that $\mathcal{I}_n$ is non-empty since $\{\mathbf{i}\} \in \mathcal{I}_n$ for all $\mathbf{i} \in \Sigma^n$). There exists a unique $s_n \in \mathbb{R}$ such that*

$$1 = \max_{I \in \mathcal{I}_n} \sum_{\mathbf{i} \in I} p_\mathbf{i} r_\mathbf{i}^{-s_n}.$$

Let $s = \sup_n s_n$. For all $q \in \mathbb{R}$ with $q \geqslant 1$, we have

$$\beta(q) \leqslant \underline{\tau}_\mu(q) \leqslant \overline{\tau}_\mu(q) \leqslant s(1-q).$$

Unfortunately, when $q \leqslant 1$, almost nothing is known about the L^q-spectra of self-similar measures *without any separation conditions*. It is clear that if $q \leqslant 1$, then the L^q-spectra are extremely sensitive to small variations in the distribution of μ. This makes the problem of analyzing the L^q-spectra for $q \leqslant 1$ much more difficult than for $q \geqslant 1$. In particular, we note that the approach from [12] cannot be applied in this case. The purpose of this paper is to provide non-trivial bounds for the L^q-spectra of self-similar measures *without any separation conditions* for $q \leqslant 1$. More precisely, we have the following result.

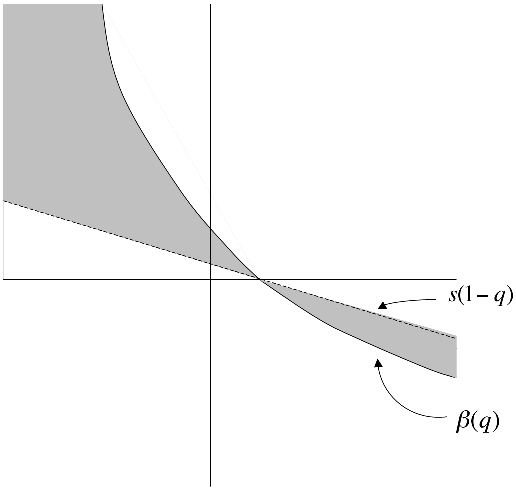

Fig. 1. The graph of the function $q \to \beta(q)$ is shown as a solid curve, and the graph of the function $q \to s(1-q)$ is shown as a dashed line. It follows from Theorem B and Theorem 1.1 that the L^q-spectra $\underline{\tau}_\mu(q)$ and $\overline{\tau}_\mu(q)$ lie in the shaded region bounded by $s(1-q)$ and $\beta(q)$.

Theorem 1.1. *Let $s \in \mathbb{R}$ be defined as in Theorem* B. *For all $q \in \mathbb{R}$ with $q \leqslant 1$, we have*

$$s(1-q) \leqslant \underline{\tau}_\mu(q) \leqslant \overline{\tau}_\mu(q) \leqslant \beta(q).$$

Fig. 1 above illustrates the statements in Theorem B and Theorem 1.1.

It follows from Theorem 1.1 that if K is not a singleton, then $\underline{\tau}_\mu(q) > 0$ for $q < 1$. More precisely, we have Proposition 1.2 below.

Proposition 1.2. *Let s be as in Theorem* B. *If K is not a singleton, then $s > 0$. In particular, if K is not a singleton, then*

$$0 < s(1-q) \leqslant \underline{\tau}_\mu(q)$$

for all $q < 1$.

The proof of Proposition 1.2 is similar to Proposition 2.2 in [12] and is therefore omitted. From Proposition 1.2, we can see that the lower bound for $\underline{\tau}_\mu(q)$ provided by Theorem 1.1 is non-trivial.

If all the contraction ratios $r_1, \dots, r_N$ coincide and equal $r \in (0,1)$, then Theorem 1.1 can be simplified. Indeed, in this case, it is clear that

$$s_n = \frac{1}{n \log r} \log\left(\max_{I \in \mathcal{I}_n} \sum_{\mathbf{i} \in I} p_{\mathbf{i}} \right),$$

and we obtain the following corollary from Theorem 1.1.

Corollary 1.3. *Assume that $r_1 = \cdots = r_N = r$. Write*

$$s = \sup_n \frac{1}{n \log r} \log\left(\max_{I \in \mathcal{I}_n} \sum_{\mathbf{i} \in I} p_{\mathbf{i}} \right).$$

For all $q \leqslant 1$, we have

$$s(1-q) \leqslant \underline{\tau}_\mu(q) \leqslant \overline{\tau}_\mu(q) \leqslant \beta(q).$$

The paper is organized as follows. In Section 2 we illustrate Theorem 1.1 by two examples, namely, we discuss the L^q-spectra of the $(2,3)$-Bernoulli convolution and the λ-Cantor measure. In Section 3, as an application of our results, we obtain a non-trivial upper bound for the multifractal spectra of an *arbitrary self-similar measure*. In Section 4 we prove Theorem 1.1. Finally, in Section 5 we discuss how our results can be extended to the mixed multifractal setting.

2. Examples

In this section we illustrate Theorem 1.1 by two examples.

Example 2.1 *(The* $(2,3)$*-Bernoulli convolution).* The $(2,3)$-Bernoulli convolution is defined as follows. Define $S_1, S_2, S_3 : \mathbb{R} \to \mathbb{R}$ by $S_i(x) = \frac{1}{2}x + \frac{i-1}{4}$ and let $(p_1, p_2, p_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The $(2,3)$-Bernoulli convolution is defined as the self-similar measure μ associated with the probabilistic iterated function system $(S_1, S_2, S_3; p_1, p_2, p_3)$, cf. [5,12]. By Corollary 1.3, we see that

$$\begin{aligned} s = \sup_n s_n &= \sup_n \frac{-1}{n \log 2} \log\Bigl(\max_{I \in \mathcal{I}_n} \sum_{\mathbf{i} \in I} p_{\mathbf{i}}\Bigr) \\ &= \sup_n \frac{-1}{n \log 2} \log\bigl(\max_{I \in \mathcal{I}_n} (\# I) 3^{-n}\bigr) \\ &= \frac{\log 3}{\log 2} - \frac{1}{\log 2} \inf_n \frac{1}{n} \log \max_{I \in \mathcal{I}_n} (\# I), \end{aligned}$$

where $\# A$ denotes the cardinality of A. A simple calculation shows that $\beta(q) = \frac{\log 3}{\log 2}(1-q)$, and we therefore conclude from Theorem 1.1 that if $q \leqslant 1$, then

$$\Bigl(\frac{\log 3}{\log 2} - \frac{1}{\log 2} \inf_n \frac{1}{n} \log \max_{I \in \mathcal{I}_n} (\# I)\Bigr)(1-q) \leqslant \underline{\tau}_\mu(q) \leqslant \overline{\tau}_\mu(q) \leqslant \frac{\log 3}{\log 2}(1-q).$$

In particular, for example, we see that $\max_{I \in \mathcal{I}_2} (\# I) = 5$. Hence, if $q \leqslant 1$, then

$$0 < \Bigl(\frac{\log 3}{\log 2} - \frac{\log 5}{\log 4}\Bigr)(1-q) \leqslant \underline{\tau}_\mu(q) \leqslant \overline{\tau}_\mu(q) \leqslant \frac{\log 3}{\log 2}(1-q).$$

Example 2.2 *(The* λ*-Cantor measure).* For $\lambda \in [0,1]$, the λ-Cantor measure is defined as follows. Define $S_1, S_2, S_3 : \mathbb{R} \to \mathbb{R}$ by $S_1(x) = \frac{1}{3}x$, $S_2(x) = \frac{1}{3}x + \frac{\lambda}{3}$, $S_3(x) = \frac{1}{3}x + \frac{2}{3}$ and let $(p_1, p_2, p_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The λ-Cantor measure is defined as the self-similar measure μ_λ associated with the probabilistic iterated function system $(S_1, S_2, S_3; p_1, p_2, p_3)$. Let us denote the corresponding self-similar set by E_λ.

When $\lambda = 0$, the set E_λ equals the classical Cantor middle-third set. Define $S'_1, S'_2 : \mathbb{R} \to \mathbb{R}$ by $S'_1(x) = \frac{1}{3}x$, $S'_2(x) = \frac{1}{3}x + \frac{2}{3}$. Let $(p'_1, p'_2) = (\frac{2}{3}, \frac{1}{3})$, and let ν denote the self-similar measure corresponding to the probabilistic iterated function system $(S'_1, S'_2; p'_1, p'_2)$. Then μ_λ coincides with ν. Obviously, (S'_1, S'_2) satisfies the OSC and the L^q-spectrum of $\mu_\lambda = \nu$ can therefore be found using Theorem A.

When $\lambda = 1$, the set E_λ equals $[0,1]$ and μ_λ coincides with Lebesgue measure on $[0,1]$. Hence, for $\lambda = 1$ it is not interesting to study the L^q-spectrum of μ_λ since it is trivial.

Now we will focus our attention on the case $\lambda \in (0,1)$. Obviously, in this case, the OSC is not satisfied, and we will now use Theorem 1.1 to obtain non-trivial bounds for the L^q-spectra of the λ-Cantor measure. By Corollary 1.3, we see that

$$\begin{aligned} s = \sup_n s_n &= \sup_n \frac{-1}{n \log 3} \log\Bigl(\max_{I \in \mathcal{I}_n} \sum_{\mathbf{i} \in I} p_{\mathbf{i}}\Bigr) \\ &= \sup_n \frac{-1}{n \log 3} \log\bigl(\max_{I \in \mathcal{I}_n} (\# I) 3^{-n}\bigr) \\ &= 1 - \frac{1}{\log 3} \inf_n \frac{1}{n} \log \max_{I \in \mathcal{I}_n} (\# I). \end{aligned}$$

A simple calculation shows that $\beta(q) = (1-q)$, and we therefore conclude from Theorem 1.1 that if $q \leqslant 1$, then

$$\Bigl(1 - \frac{1}{\log 3} \inf_n \frac{1}{n} \log \max_{I \in \mathcal{I}_n} (\# I)\Bigr)(1-q) \leqslant \underline{\tau}_{\mu_\lambda}(q) \leqslant \overline{\tau}_{\mu_\lambda}(q) \leqslant (1-q).$$

In particular, for example, we see that $\max_{I \in \mathcal{I}_2} (\# I) = 4$. Hence, if $q < 1$, then

$$0 < \Bigl(1 - \frac{\log 2}{\log 3}\Bigr)(1-q) \leqslant \underline{\tau}_{\mu_\lambda}(q) \leqslant \overline{\tau}_{\mu_\lambda}(q) \leqslant (1-q).$$

3. An application: non-trivial upper bounds for the multifractal spectra of arbitrary self-similar measures

As an application of Theorem 1.1, we will now obtain a non-trivial upper bound for the multifractal spectra of an arbitrary self-similar measure. Recall that the main significance of the L^q-spectra is their relationship with the multifractal spectra, which we will define below. For a probability measure μ on $\mathbb{R}^d$, we define the Hausdorff multifractal spectrum function, $f_{H,\mu}$, of μ by

$$f_{H,\mu}(\alpha) = \dim_H \left\{ x \in K \;\middle|\; \lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\},$$

for $\alpha \geqslant 0$, where $\dim_H$ denotes the Hausdorff dimension. We define the packing multifractal spectrum function, $f_{P,\mu}$, of μ similarly, namely, we let

$$f_{P,\mu}(\alpha) = \dim_P \left\{ x \in K \;\middle|\; \lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\},$$

for $\alpha \geqslant 0$, where $\dim_P$ denotes the packing dimension. In the 1980s it was conjectured in the physics literature that for "good" measures the following result, known as the Multifractal Formalism, holds: the multifractal spectra equal the Legendre transform of the L^q-spectra. During the 1990s there has been an enormous interest in the mathematical literature in verifying the Multifractal Formalism and computing the multifractal spectra of measures, and within the last 15 years the multifractal spectra of various classes of measures in Euclidean space $\mathbb{R}^d$ exhibiting some degree of self-similarity have been computed rigorously, cf. [3] and the references therein.

For example, the following result due to Arbeiter and Patzschke [1] provides a formula for the multifractal spectra of a self-similar measure satisfying the OSC. To state this result, we recall that the Legendre transform φ^* of a function $\varphi : \mathbb{R} \to \mathbb{R}$ is defined by

$$\varphi^*(x) = \inf_y \bigl(xy + \varphi(y)\bigr).$$

Also, write

$$\alpha_{\min} = \min_i \frac{\log p_i}{\log r_i}, \qquad \alpha_{\max} = \max_i \frac{\log p_i}{\log r_i}.$$

We can now state Arbeiter and Patzschke's result.

Theorem C. *(See [1].) Let μ be the self-similar measure satisfying* (1.2) *and assume that the OSC is satisfied.*

(1) *We have*

$$f_{H,\mu}(\alpha) = f_{P,\mu}(\alpha) = \underline{\tau}_\mu^*(\alpha) = \overline{\tau}_\mu^*(\alpha) = \beta^*(\alpha)$$

for all $\alpha \in [\alpha_{\min}, \alpha_{\max}]$.

(2) *We have*

$$\left\{ x \in K \;\middle|\; \lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\} = \emptyset$$

for all $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$.

As an application of Theorem 1.1, we now obtain an upper bound for the multifractal spectra of an arbitrary self-similar measure not satisfying any separation condition. In particular, we emphasize that the upper bound in Corollary 3.1 does not require the OSC is satisfied.

Corollary 3.1. *Let μ be the self-similar measure satisfying* (1.2)*. Write $t = \frac{\sum_i p_i \log p_i}{\sum_i r_i \log p_i}$. Then we have*

$$f_{H,\mu}(\alpha) \leqslant f_{P,\mu}(\alpha) \leqslant f(\alpha),$$

where $f : [s, \alpha_{\max}] \to \mathbb{R}$ is defined by

$$f(\alpha) = \begin{cases} \min(\alpha, d) & \text{for } \alpha \in [s,t]; \\ \min(\beta^*(\alpha), d) & \text{for } \alpha \in [t, \alpha_{\max}]. \end{cases}$$

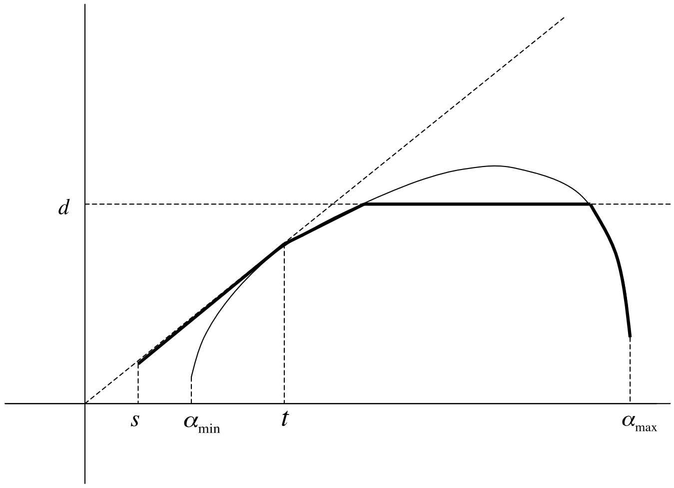

Fig. 2. The graph of the function $\alpha \to \beta^*(\alpha)$ is shown as a solid curve and the geometrical significance of the numbers s, t, $\alpha_{\min}$ and $\alpha_{\max}$ are illustrated. The graph of the function f defined in Corollary 3.1 is shown as a bold solid curve. It follows from Corollary 3.1 that $f(\alpha)$ is an upper bound for the multifractal spectrum $f_{P,\mu}(\alpha)$.

Proof. Define $b : \mathbb{R} \to \mathbb{R}$ by $b(q) = \max(s(1-q), \beta(q))$, i.e., $b(q) = \beta(q)$ for $q \leqslant 1$ and $b(q) = s(1-q)$ for $q > 1$. Since $f_{H,\mu}(\alpha) \leqslant f_{P,\mu}(\alpha) \leqslant \overline{\tau}^*_\mu(\alpha)$ for all α (see [3]) and $\overline{\tau}_\mu(q) \leqslant b(q)$ (by Theorem B and Theorem 1.1), we conclude that $f_{H,\mu}(\alpha) \leqslant f_{P,\mu}(\alpha) \leqslant b^*(\alpha)$. Finally, since $-\beta'(1) = t$, it is not difficult to see that $b^*(\alpha) = \alpha$ for $\alpha \in [s,t]$ and $b^*(\alpha) = \beta^*(\alpha)$ for $\alpha \in [t, \alpha_{\max}]$. This together with the fact that $f_{P,\mu}(\alpha) \leqslant d$, gives the desired result. □

Fig. 2 above illustrates the statement in Corollary 3.1.

We note that the upper bound in Corollary 3.1 can be strictly bigger than $f_{P,\mu}(\alpha)$. Indeed, as an example of this we recall the following result due to Hu and Lau [10]. Namely, for $i = 1,2,3,4$ define $S_i : \mathbb{R} \to \mathbb{R}$ by $S_i = \frac{1}{3}x + \frac{2}{3}(i-1)$ and $p_i = 2^{-3}\binom{m}{i-1}$, and let μ be the corresponding self-similar measure satisfying (1.2). In this case the OSC is not satisfied, and it follows from [10] that

$$\left\{x \in K \;\middle|\; \lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha\right\} = \emptyset$$

for all $\alpha \in (\alpha_0, \alpha_{\max}) \subseteq (s, \alpha_{\max})$, where $\alpha_0 = \frac{3\log 2}{\log 3} - \frac{\log((7+\sqrt{13})/2)}{2\log 3}$. In particular, this shows that $f_{H,\mu}(\alpha) = f_{P,\mu}(\alpha) = 0$ for $\alpha \in (\alpha_0, \alpha_{\max})$, whereas the upper bound in Corollary 3.1 is strictly positive for $\alpha \in (\alpha_0, \alpha_{\max})$.

4. Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. In order to prove Theorem 1.1 we must show that if $q \leqslant 1$, then

$$\overline{\tau}_\mu(q) \leqslant \beta(q) \tag{4.1}$$

and

$$s(1-q) \leqslant \underline{\tau}_\mu(q). \tag{4.2}$$

Proof of inequality (4.1). Firstly, we introduce two quantities that are related to the upper L^q-spectrum, namely, the upper covering Rényi dimension, $\overline{\tau}^c_\mu(q)$, and upper packing Rényi dimension, $\overline{\tau}^p_\mu(q)$. The main reason for introducing the Rényi dimensions is that they are easier to work with.

For a probability measure μ on $\mathbb{R}^d$ with support equal to K, the upper covering Rényi dimension and the upper packing Rényi dimension of μ are defined as follows. Recall that a finite or countable family $(B(x_i,r))_i$ of balls is called a centered cover of K if $K \subset \bigcup_i B(x_i,r)$ and $x_i \in K$ for all i, and that a finite or countable family $(B(x_i,r))_i$ of balls is called a centered packing of K if $B(x_i,r) \cap B(x_j,r) = \emptyset$ for all $i \neq j$ and $x_i \in K$ for all i. For $r > 0$ and $q \in \mathbb{R}$, write

$$M^c_\mu(r;q) = \inf\left\{\sum_i \mu\big(B(x_i,r)\big)^q \;\middle|\; \big(B(x_i,r)\big)_i \text{ is a centered cover of } K\right\}, \tag{4.3}$$

and

$$M_\mu^p(r;q) = \sup\left\{ \sum_i \mu\big(B(x_i,r)\big)^q \;\Big|\; \big(B(x_i,r)\big)_i \text{ is a centered packing of } K \right\}. \tag{4.4}$$

Then, the upper covering Rényi dimension, $\overline{\tau}_\mu^c(q)$, and upper packing Rényi dimension, $\overline{\tau}_\mu^p(q)$ are defined by

$$\overline{\tau}_\mu^c(q) = \limsup_{r \searrow 0} \frac{\log M_\mu^c(r;q)}{-\log r}, \tag{4.5}$$

and

$$\overline{\tau}_\mu^p(q) = \limsup_{r \searrow 0} \frac{\log M_\mu^p(r;q)}{-\log r}, \tag{4.6}$$

respectively. The numbers of $\overline{\tau}_\mu^c(q)$, $\overline{\tau}_\mu^p(q)$ and $\overline{\tau}_\mu(q)$ do not necessarily coincide. However, certain inequalities are always satisfied; this is the content of Lemma 4.1 and Lemma 4.2. These inequalities allow us to work with the more manageable upper packing Rényi dimensions when proving (4.1).

Lemma 4.1. *(See [14, Proposition 2.19 and Proposition 2.20].) Let μ be a Borel probability measure on $\mathbb{R}^d$. Then $\overline{\tau}_\mu^c(q) \leqslant \overline{\tau}_\mu^p(q)$ for all $q \in \mathbb{R}$.*

Lemma 4.2. *Let μ be a Borel probability measure on $\mathbb{R}^d$. Then $\overline{\tau}_\mu(q) \leqslant \overline{\tau}_\mu^c(q)$ for all $q \leqslant 1$.*

Proof. Let $(B(x_i,r))_i$ be a centered cover of $\operatorname{supp}\mu$ (here and below we write $\operatorname{supp}\mu$ for the support of μ). Since for any $x \in B(x_i,r)$, the ball $B(x,2r)$ contains the ball $B(x_i,r)$, we have for $q \leqslant 1$,

$$\begin{aligned} I_\mu(2r;q) &= \int_{\operatorname{supp}\mu} \mu\big(B(x,2r)\big)^{q-1}\, d\mu(x) \leqslant \sum_i \int_{B(x_i,r)} \mu\big(B(x,2r)\big)^{q-1}\, d\mu(x) \\ &\leqslant \sum_i \int_{B(x_i,r)} \mu\big(B(x_i,r)\big)^{q-1}\, d\mu(x) \\ &= \sum_i \mu\big(B(x_i,r)\big)^{q-1} \mu\big(B(x_i,r)\big) \\ &= \sum_i \mu\big(B(x_i,r)\big)^q. \end{aligned}$$

Since this is true for all centered covers of $\operatorname{supp}\mu$, we obtain $I_\mu(2r;q) \leqslant M_\mu^c(r;q)$. This clearly implies that $\overline{\tau}_\mu(q) \leqslant \overline{\tau}_\mu^c(q)$. □

Next, recall the definitions of Σ^n, Σ^* and $\Sigma^{\mathbb{N}}$ from Section 1, namely,

$$\Sigma^n = \{1,\dots,N\}^n, \qquad \Sigma^* = \bigcup_n \Sigma^n, \qquad \Sigma^{\mathbb{N}} = \{1,\dots,N\}^{\mathbb{N}}.$$

It is well known that $\mu(S_{\mathbf{i}}K) = p_{\mathbf{i}}$ for all $\mathbf{i} \in \Sigma^*$ if the OSC is satisfied, cf. [9]. However, we always have the inequality in Lemma 4.3.

Lemma 4.3. *For $\mathbf{i} \in \Sigma^*$, we have $\mu(S_{\mathbf{i}}(K)) \geqslant p_{\mathbf{i}}$.*

Proof. Iterating the equality

$$\mu = \sum_{i=1}^N p_i \mu \circ S_i^{-1},$$

we have

$$\mu = \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} \mu \circ S_{\mathbf{j}}^{-1}$$

for all n. Hence, for $\mathbf{i} \in \Sigma^*$ with $|\mathbf{i}| = n$,

$$\begin{aligned}\mu\big(S_{\mathbf{i}}(K)\big) &= \sum_{\mathbf{i}} p_{\mathbf{i}}\mu\circ S_{\mathbf{i}}^{-1}\big(S_{\mathbf{i}}(K)\big) + \sum_{\mathbf{j}\neq\mathbf{i},|\mathbf{j}|=n} p_{\mathbf{j}}\mu\circ S_{\mathbf{j}}^{-1}\big(S_{\mathbf{i}}(K)\big)\\ &= p_{\mathbf{i}} + \sum_{\mathbf{j}\neq\mathbf{i},|\mathbf{j}|=n} p_{\mathbf{j}}\mu\circ S_{\mathbf{j}}^{-1}\big(S_{\mathbf{i}}(K)\big)\\ &\geqslant p_{\mathbf{i}}.\end{aligned}$$

This completes the proof of Lemma 4.3. □

The next lemma is a standard result.

Lemma 4.4. *Fix $q\in[0,1]$. Let $m\in\mathbb{N}$ and $a_1,\dots,a_m\geqslant 0$. Then $(\sum_i a_i)^q\leqslant\sum_i a_i^q$.*

Now we state our main technical lemma.

Lemma 4.5. *Let μ be the self-similar measure satisfying* (1.2) *and let $\beta(q)$ be as in* (1.4)*. Then*

$$\overline{\tau}_{\mu}^{p}(q)\leqslant\beta(q)$$

for $q\leqslant 1$.

Proof. Let $r_{\min}=\min_i r_i$ and $r_{\max}=\max_i r_i$. The proof of Lemma 4.5 is divided into the following two cases.

Case 1: $q\leqslant 0$. Let $\varepsilon>0$. We will now show that there is a constant $c>0$ such that

$$M_{\mu}^{p}(r;q)\leqslant cr^{-(\beta(q)+\varepsilon)} \tag{4.7}$$

for all r with $0<r<r_{\min}$.

We therefore fix r with $0<r<r_{\min}$ and let $(B(x_i,r))_{i\in I}$ be a centered packing of K. Next, we observe that it is well known that $K=\bigcup_{\mathbf{i}\in\Sigma^{\mathbb{N}}}\bigcap_n K_{\mathbf{i}|n}$. It follows from this that for each $x\in K$ there is an infinite string $\mathbf{i}\in\Sigma^{\mathbb{N}}$ such that $x\in K_{\mathbf{i}|n}$ for all n. In particular, we conclude that for each $i\in I$ there is a (not necessary unique) finite string $\mathbf{i}_i\in\Sigma^*$ such that

$$x_i\in K_{\mathbf{i}_i},$$

and

$$\operatorname{diam}(K_{\mathbf{i}_i})\leqslant r<\operatorname{diam}(K_{\mathbf{i}_i|(|\mathbf{i}_i|-1)}).$$

Fix $i\in I$. Since $x_i\in K_{\mathbf{i}_i}$ and $\operatorname{diam}(K_{\mathbf{i}_i})\leqslant r$, we deduce that $K_{\mathbf{i}_i}\subseteq B(x_i,r)$, and so, using the fact $q\leqslant 0$, we have

$$\mu\big(B(x_i,r)\big)^q\leqslant\mu(K_{\mathbf{i}_i})^q. \tag{4.8}$$

Also, since $r<\operatorname{diam}(K_{\mathbf{i}_i|(|\mathbf{i}_i|-1)})$ and $\beta(q)+\varepsilon>\beta(q)\geqslant 0$ (because $q\leqslant 0$), we deduce that

$$\begin{aligned}r^{\beta(q)+\varepsilon}&\leqslant\operatorname{diam}(K_{\mathbf{i}_i|(|\mathbf{i}_i|-1)})^{\beta(q)+\varepsilon}\\ &\leqslant\frac{1}{r_{\min}^{\beta(q)+\varepsilon}}\operatorname{diam}(K_{\mathbf{i}_i})^{\beta(q)+\varepsilon}.\end{aligned}\tag{4.9}$$

Combining (4.8), (4.9) and Lemma 4.3 gives

$$\begin{aligned}\sum_{i\in I}\mu\big(B(x_i,r)\big)^q&\leqslant\sum_{i\in I}\mu(K_{\mathbf{i}_i})^q\\ &\leqslant\frac{1}{r_{\min}^{\beta(q)+\varepsilon}}r^{-(\beta(q)+\varepsilon)}\sum_{i\in I}\mu(K_{\mathbf{i}_i})^q\operatorname{diam}(K_{\mathbf{i}_i})^{\beta(q)+\varepsilon}\\ &\leqslant\frac{\operatorname{diam}(K)}{r_{\min}^{\beta(q)+\varepsilon}}r^{-(\beta(q)+\varepsilon)}\sum_{i\in I}p_{\mathbf{i}_i}^q r_{\mathbf{i}_i}^{\beta(q)+\varepsilon}\\ &=c_0r^{-(\beta(q)+\varepsilon)}\sum_{k=1}^{\infty}\sum_{i\in I,|\mathbf{i}_i|=k}p_{\mathbf{i}_i}^q r_{\mathbf{i}_i}^{\beta(q)+\varepsilon},\end{aligned}\tag{4.10}$$

where $c_0=\frac{\operatorname{diam}(K)}{r_{\min}^{\beta(q)+\varepsilon}}$.

Next, we prove that if $i_1, i_2 \in I$, then the following holds:

$$i_1 \neq i_2 \quad \Rightarrow \quad \mathbf{i}_{i_1} \neq \mathbf{i}_{i_2}. \tag{4.11}$$

Indeed, otherwise there is $i_1, i_2 \in I$ with $i_1 \neq i_2$ such that $\mathbf{i}_{i_1} = \mathbf{i}_{i_2}$. This implies that $x_{i_1} \in K_{\mathbf{i}_{i_1}} = K_{\mathbf{i}_{i_2}} \subseteq B(x_{i_2}, r)$, and so $B(x_{i_1}, r) \cap B(x_{i_2}, r) \neq \emptyset$, contradicting the fact that $(B(x_i, r))_{i \in I}$ is a packing. This proves (4.11).

It follows immediately from (4.11) that

$$\sum_{i \in I, |\mathbf{i}_i| = k} p_{\mathbf{i}_i}^q r_{\mathbf{i}_i}^{\beta(q)+\varepsilon} \leqslant \sum_{\mathbf{i} \in \Sigma^*, |\mathbf{i}| = k} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)+\varepsilon}. \tag{4.12}$$

Write $u = \sum_{i=1}^N p_i^q r_i^{\beta(q)+\varepsilon}$ and observe that

$$u = \sum_{i=1}^N p_i^q r_i^{\beta(q)+\varepsilon} < \sum_{i=1}^N p_i^q r_i^{\beta(q)} = 1.$$

Combining (4.10), (4.12) and using the fact that $u < 1$, we now conclude that

$$\begin{aligned} \sum_{i \in I} \mu\big(B(x_i, r)\big)^q &\leqslant c_0 r^{-(\beta(q)+\varepsilon)} \sum_{k=1}^{\infty} \sum_{i \in I, |\mathbf{i}_i| = k} p_{\mathbf{i}_i}^q r_{\mathbf{i}_i}^{\beta(q)+\varepsilon} \\ &\leqslant c_0 r^{-(\beta(q)+\varepsilon)} \sum_{k=1}^{\infty} \sum_{\mathbf{i} \in \Sigma^*, |\mathbf{i}| = k} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)+\varepsilon} \\ &\leqslant c_0 r^{-(\beta(q)+\varepsilon)} \sum_{k=1}^{\infty} \left(\sum_{i=1}^N p_i^q r_i^{\beta(q)+\varepsilon} \right)^k \\ &= c_0 r^{-(\beta(q)+\varepsilon)} \sum_{k=1}^{\infty} u^k \\ &= c_0 \frac{u}{1-u} r^{-(\beta(q)+\varepsilon)} \\ &= c r^{-(\beta(q)+\varepsilon)}, \end{aligned} \tag{4.13}$$

where $c = c_0 \frac{u}{1-u}$.

Finally, taking supremum in (4.13) over all packings $(B(x_i, r))_{i \in I}$ gives

$$M_\mu^p(r; q) \leqslant c r^{-(\beta(q)+\varepsilon)}$$

for all r with $0 < r < r_{\min}$. This proves (4.7).

It follows immediately from (4.7) that $\overline{\tau}_\mu^p(q) = \limsup_{r \searrow 0} \frac{\log M_\mu^p(r;q)}{-\log r} \leqslant \beta(q) + \varepsilon$. Letting $\varepsilon \searrow 0$ gives the desired result. This completes the proof of Case 1.

Case 2: $0 \leqslant q \leqslant 1$. Define $V : (0, \infty) \to \mathbb{R}$ by

$$V(r) = M_\mu^p(r; q) r^{\beta(q)}.$$

Let $(B(x_i, r))_{i \in I}$ be a centered packing of K. It follows from (1.2) and Lemma 4.4 that

$$\begin{aligned} \mu\big(B(x_i, r)\big)^q &= \left(\sum_{j=1}^N p_j \mu\big(S_j^{-1}\big(B(x_i, r)\big)\big) \right)^q \\ &= \left(\sum_{j=1}^N p_j \mu\left(B\left(S_j^{-1} x_i, \frac{r}{r_j} \right) \right) \right)^q \\ &\leqslant \sum_{j=1}^N p_j^q \mu\left(B\left(S_j^{-1} x_i, \frac{r}{r_j} \right) \right)^q, \end{aligned}$$

and so

$$\sum_i \mu\big(B(x_i,r)\big)^q r^{\beta(q)} \leqslant \sum_i \sum_{j=1}^N p_j^q \mu\left(B\left(S_j^{-1}x_i, \frac{r}{r_j}\right)\right)^q r^{\beta(q)}$$
$$= \sum_{j=1}^N p_j^q r_j^{\beta(q)} \sum_i \mu\left(B\left(S_j^{-1}x_i, \frac{r}{r_j}\right)\right)^q \left(\frac{r}{r_j}\right)^{\beta(q)}. \tag{4.14}$$

Next we prove that:

$$\left(B\left(S_j^{-1}x_i, \frac{r}{r_j}\right)\right)_{i\in I} \quad \text{is a packing.} \tag{4.15}$$

Indeed, otherwise there are $i_1, i_2 \in I$ with $i_1 \neq i_2$ and $x \in B(S_j x_{i_1}, \frac{r}{r_j}) \cap B(S_j x_{i_2}, \frac{r}{r_j})$. This implies that $|S_j^{-1}x_{i_1} - S_j^{-1}x_{i_2}| \leqslant |S_j^{-1}x_{i_1} - x| + |x - S_j^{-1}x_{i_2}| \leqslant \frac{2r}{r_j}$, whence $|x_{i_1} - x_{i_2}| \leqslant 2r$, contradicting the fact that $(B(x_i,r))_{i\in I}$ is a packing. This proves (4.15). It follows immediately from (4.15) that

$$\sum_i \mu\left(B\left(S_j^{-1}x_i, \frac{r}{r_j}\right)\right)^q \leqslant M_\mu^p\left(\frac{r}{r_j}; q\right). \tag{4.16}$$

Combining (4.14) and (4.16) gives

$$\sum_i \mu\big(B(x_i,r)\big)^q r^{\beta(q)} \leqslant \sum_{j=1}^N p_j^q r_j^{\beta(q)} \sum_i \mu\left(B\left(S_j^{-1}x_i, \frac{r}{r_j}\right)\right)^q \left(\frac{r}{r_j}\right)^{\beta(q)}$$
$$\leqslant \sum_{j=1}^N p_j^q r_j^{\beta(q)} M_\mu^p\left(\frac{r}{r_j}; q\right)\left(\frac{r}{r_j}\right)^{\beta(q)}$$
$$= \sum_{j=1}^N p_j^q r_j^{\beta(q)} V\left(\frac{r}{r_j}\right). \tag{4.17}$$

Finally, taking supremum in (4.17) over all packings $(B(x_i,r))_{i\in I}$, we now obtain

$$V(r) = M_\mu^p(r;q) r^{\beta(q)} \leqslant \sum_{j=1}^N p_j^q r_j^{\beta(q)} V\left(\frac{r}{r_j}\right). \tag{4.18}$$

Next, we note if a and b are real numbers with $a \leqslant b$, then it follows from (4.18) that

$$\sup_{a\leqslant r<b} V(r) \leqslant \sup_{a\leqslant r<b} \sum_{j=1}^N p_j^q r_j^{\beta(q)} V\left(\frac{r}{r_j}\right)$$
$$\leqslant \sum_{j=1}^N p_j^q r_j^{\beta(q)} \sup_{a\leqslant r<b} V\left(\frac{r}{r_j}\right)$$
$$= \sum_{j=1}^N p_j^q r_j^{\beta(q)} \sup_{\frac{a}{r_j}\leqslant r<\frac{b}{r_j}} V(r)$$
$$\leqslant \sum_{j=1}^N p_j^q r_j^{\beta(q)} \sup_{\frac{a}{r_{\max}}\leqslant r<\frac{b}{r_{\min}}} V(r)$$
$$= \sup_{\frac{a}{r_{\max}}\leqslant r<\frac{b}{r_{\min}}} V(r). \tag{4.19}$$

Write k_0 for the unique positive integer such that $r_{\max}^{k_0} \leqslant r_{\min} < r_{\max}^{k_0-1}$. Fix $\lambda > 0$ and a positive integer l with $l > k_0$. Using (4.19) and the fact that $r_{\max}^l\lambda \leqslant r_{\max}^{l-1}\lambda \leqslant r_{\max}^{k_0}\lambda \leqslant r_{\min}\lambda < \lambda$, we now conclude that

$$\begin{aligned}
\sup_{r_{\max}^{l}\lambda\leqslant r\leqslant\lambda} V(r) &= \max\Bigl(\sup_{r_{\max}^{l}\lambda\leqslant r\leqslant r_{\min}\lambda} V(r), \sup_{r_{\min}\lambda\leqslant r\leqslant\lambda} V(r)\Bigr)\\
&\leqslant \max\Bigl(\sup_{r_{\max}^{l-1}\lambda\leqslant r\leqslant\lambda} V(r), \sup_{r_{\min}\lambda\leqslant r\leqslant\lambda} V(r)\Bigr)\\
&= \sup_{r_{\max}^{l-1}\lambda\leqslant r\leqslant\lambda} V(r).
\end{aligned}\tag{4.20}$$

Repeated application of (4.20) shows that if $k > k_0$, then

$$\begin{aligned}
\sup_{r_{\max}^{k}\lambda\leqslant r\leqslant\lambda} V(r) &\leqslant \sup_{r_{\max}^{k-1}\lambda\leqslant r\leqslant\lambda} V(r)\\
&\leqslant \sup_{r_{\max}^{k-2}\lambda\leqslant r\leqslant\lambda} V(r)\\
&\ \ \vdots\\
&\leqslant \sup_{r_{\max}^{k_0}\lambda\leqslant r\leqslant\lambda} V(r),
\end{aligned}$$

and so

$$\begin{aligned}
\sup_{0\leqslant r\leqslant\lambda} V(r) &= \sup_{k>k_0}\sup_{r_{\max}^{k}\lambda\leqslant r\leqslant\lambda} V(r)\\
&\leqslant \sup_{k>k_0}\sup_{r_{\max}^{k_0}\lambda\leqslant r\leqslant\lambda} V(r)\\
&= \sup_{r_{\max}^{k_0}\lambda\leqslant r\leqslant\lambda} V(r).
\end{aligned}\tag{4.21}$$

Writing $\lambda_0 = \frac{\operatorname{diam}(K)}{r_{\max}^{k_0}}$ and putting $\lambda = \lambda_0$ in (4.21) now yields

$$\begin{aligned}
\sup_{0<r\leqslant\lambda_0} V(r) &\leqslant \sup_{r_{\max}^{k_0}\lambda_0\leqslant r\leqslant\lambda_0} V(r)\\
&= \sup_{\operatorname{diam}(K)\leqslant r\leqslant\lambda_0} V(r)\\
&= \sup_{\operatorname{diam}(K)\leqslant r\leqslant\lambda_0} M_\mu^p(r;q)r^{\beta(q)}.
\end{aligned}\tag{4.22}$$

Since $M_\mu^p(r;q) = 1$ for $r > \operatorname{diam}(K)$ and $r^{\beta(q)} \leqslant \lambda_0^{\beta(q)}$ for $r \leqslant \lambda_0$ (because $\beta(q) \geqslant 0$), we deduce from (4.22) that

$$\sup_{0<r\leqslant\lambda_0} V(r) \leqslant c$$

where $c = \lambda_0^{\beta(q)}$, and so $M_\mu^p(r;q)r^{\beta(q)} = V(r) \leqslant c$ for all $0 < r \leqslant \lambda_0$, i.e., $M_\mu^p(r;q) \leqslant cr^{-\beta(q)}$ for all $0 < r \leqslant \lambda_0$. We conclude immediately from this that $\overline{\tau}_\mu^p(q) = \limsup_{r\searrow 0}\frac{\log M_\mu^p(r;q)}{-\log r} \leqslant \beta(q)$. This completes the proof of Case 2. □

We can now prove inequality (4.1).

Proof of (4.1). This follows by combining Lemma 4.1, Lemma 4.2 and Lemma 4.5. □

Proof of inequality (4.2). To prove inequality (4.2), we need the following result from [12]. Define $M : (0,\infty) \to \mathbb{R}$ by $M(r) = \sup_{x\in K}\mu(B(x,r))$.

Lemma 4.6. *(See [12, Proposition 4.3].) Fix a positive integer n. There exists a constant $c > 0$ such that*

$$M(r) \leqslant cr^{s_n}$$

for all $r > 0$.

We can now prove inequality (4.2).

Proof of (4.2). Fix $r > 0$. Using Lemma 4.6 and the fact that $q \leqslant 1$, we have

$$I_\mu(r;q) = \int_K \mu\big(B(x,r)\big)^{q-1}\,d\mu(x) \geqslant \int_K \big(cr^{s_n}\big)^{q-1}\,d\mu(x) = (cr)^{s_n(q-1)}.$$

This clearly implies that $s_n(1-q) \leqslant \underline{\tau}_\mu(q)$. Since this is true for all n, we can conclude that $s(1-q) \leqslant \underline{\tau}_\mu(q)$. This completes the proof of (4.2). $\square$

5. Mixed multifractal setting

In [12], Olsen generalized Theorem B to the mixed multifractal setting. Let us recall the definition of mixed L^q-spectra. Fix a positive integer k, and let $\mathbf{p}_j = (p_{j,i})_{i=1}^N$ be a probability vector for $j = 1,\dots,k$. Let μ_j denote the self-similar measure associated with the probabilistic iterated function system $(S_1,\dots,S_N;\mathbf{p}_j)$, i.e. μ_j satisfies

$$\mu_j = \sum_{i=1}^N p_{j,i}\mu_j \circ S_i^{-1}.$$

Let K denote the common support of the measures $\mu_1,\dots,\mu_k$. Finally, we define $\beta : \mathbb{R}^k \to \mathbb{R}$ by

$$\sum_{i=1}^N p_{1,i}^{q_1}\dots p_{k,i}^{q_k} r_i^{\beta(\mathbf{q})} = 1$$

for $\mathbf{q} = (q_1,\dots,q_k) \in \mathbb{R}^k$. Obviously, this definition reduces to (1.4) for $k = 1$. Now the mixed L^q-spectra of the list $\mu = (\mu_1,\dots,\mu_k)$ are defined as follows. Let $\mathbb{D}_k$ denote the diagonal ray in $\mathbb{R}^k$, i.e.

$$\mathbb{D}_k = \big\{(x,\dots,x) \in \mathbb{R}^k \;\big|\; x \in \mathbb{R}\big\}.$$

If E is a subset of $\mathbb{R}^k$ and $r > 0$, we write $B(E,r)$ for the r neighborhood of E, i.e. $B(E,r) = \{x \in \mathbb{R}^k \mid \operatorname{dist}(x,E) < r\}$. The lower and upper mixed L^q-spectra, denoted $\underline{\tau}_\mu(\mathbf{q})$ and $\overline{\tau}_\mu(\mathbf{q})$, of $\mu = (\mu_1,\dots,\mu_k)$ are now defined by

$$\underline{\tau}_\mu(\mathbf{q}) = \liminf_{r\searrow 0} \frac{\log I_\mu(r;\mathbf{q})}{-\log r},$$
$$\overline{\tau}_\mu(\mathbf{q}) = \limsup_{r\searrow 0} \frac{\log I_\mu(r;\mathbf{q})}{-\log r},$$

where

$$I_\mu(r;\mathbf{q}) = \int_{K^k \cap B(\mathbb{D}_k,r)} \mu_1\big(B(x_1,r)\big)^{q_1-1}\dots\mu_k\big(B(x_k,r)\big)^{q_k-1}\,d(\mu_1\times\dots\times\mu_k)(x_1,\dots,x_k).$$

We write $\mathbf{x} \geqslant \mathbf{y}$ for $\mathbf{x} = (x_1,\dots,x_k), \mathbf{y} = (y_1,\dots,y_k) \in \mathbb{R}^k$ if $x_i \geqslant y_i$ for all i. We also write $p_{i,\mathbf{i}} = p_{i,i_1}\dots p_{i,i_n}$ for all $\mathbf{i} = i_1\dots i_1 \in \Sigma^n$, and put $\mathbf{0} = (0,\dots,0) \in \mathbb{R}^k$ and $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^k$. Let $\langle\cdot|\cdot\rangle$ denote the usual inner product in $\mathbb{R}^k$. With minor modifications of the proof of Theorem 1.1, we obtain the following theorem, which generalizes Theorem 1.1 to the mixed multifractal setting.

Theorem 5.1. *For a positive integer n, let*

$$\mathcal{I}_n = \Big\{I \subseteq \Sigma^n \;\Big|\; \bigcap_{\mathbf{i}\in I} K_{\mathbf{i}} \neq \emptyset\Big\}$$

(*observe that $\mathcal{I}_n$ is non-empty since $\{\mathbf{i}\} \in \mathcal{I}_n$ for all $\mathbf{i} \in \Sigma^n$*). *There exists a unique $s_n \in \mathbb{R}$ such that*

$$1 = \max_{I\in\mathcal{I}_n} \sum_{\mathbf{i}\in I} p_{i,\mathbf{i}} r_{\mathbf{i}}^{-s_n}.$$

Let $s = \sup_n s_n$ and $\mathbf{s} = (s,\dots,s) \in \mathbb{R}^k$. For all $\mathbf{q} \in \mathbb{R}^k$ with $\mathbf{q} \leqslant \mathbf{1}$, we have

$$\langle\mathbf{s}|\mathbf{1}-\mathbf{q}\rangle \leqslant \underline{\tau}_\mu(\mathbf{q}) \leqslant \overline{\tau}_\mu(\mathbf{q}) \leqslant \beta(\mathbf{q}).$$

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References

[1] M. Arbeiter, N. Patzschke, Random self-similar multifractals, Math. Nachr. 181 (1996) 5–42.
[2] J. Barral, I. Bhouri, Multifractal analysis for projections of Gibbs and related measures, Ergodic Theory Dynam. Systems 31 (2011) 673–701.
[3] K.J. Falconer, Techniques in Fractal Geometry, John Wiley and Sons, Chichester, 1997.
[4] D.J. Feng, Gibbs properties of self-conformal measures and the multifractal formalism, Ergodic Theory Dynam. Systems 27 (2007) 787–812.
[5] D.J. Feng, E. Olivier, Multifractal analysis of weak Gibbs measures and phase transition-application to some Bernoulli convolutions, Ergodic Theory Dynam. Systems 23 (2003) 1751–1784.
[6] D.J. Feng, K.S. Lau, Multifractal formalism for self-similar measures with weak separation condition, J. Math. Pures Appl. 92 (2009) 407–428.
[7] J. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (1981) 713–747.
[8] K.S. Lau, S.M. Ngai, Multifractal measures and a weak separation condition, Adv. Math. 141 (1999) 45–96.
[9] S. Graf, On Bandt's tangential distribution for self-similar measures, Monatsh. Math. 120 (1995) 223–246.
[10] T.Y. Hu, K.S. Lau, Multifractal structure of convolution of the Cantor measure, Adv. Math. 27 (2001) 1–16.
[11] S.M. Ngai, Multifractal structure of non-compactly supported infinite measures, Fractals 3 (2008) 209–226.
[12] L. Olsen, Bounds for the L^q-spectra of a self-similar multifractals not satisfying the Open Set Condition, J. Math. Anal. Appl. 355 (2009) 12–21.
[13] L. Olsen, Mixed generalized dimensions of self-similar measures, J. Math. Anal. Appl. 306 (2005) 516–539.
[14] L. Olsen, A multifractal formalism, Adv. Math. 116 (1995) 82–196.
[15] M. Wu, The multifractal spectrum of some Moran measures, Sci. China Ser. A 48 (2005) 1097–1112.