J. Math. Anal. Appl. 390 (2012) 121–125

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Existence of positive stationary solutions for a diffusive variable-territory prey–predator model [☆]

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ARTICLE INFO

Article history:
Received 17 November 2011
Available online 18 January 2012
Submitted by Y. Lou

Keywords:
Prey–predator model
Stationary solution
Existence

ABSTRACT

This paper concerns the existence of positive stationary solutions for a diffusive variable-territory prey–predator model, and completely settles an open problem of Wang and Pang (2009). The main result closes a gap in an earlier result (2011) by the authors.

1. Introduction

The paper studies the existence of positive solutions to the following problem:

$$\begin{cases} -\Delta u = \lambda u - a(x)u^2 - buv & \text{in } \Omega, \\ -\Delta v = v\left(u - 1 - \dfrac{v}{u}\right) & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with appropriately smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, $\partial_\nu = \frac{\partial}{\partial\nu}$, λ and b are positive constants, and $a(x)$ is a nonconstant, continuous function satisfying one of the following conditions:

(H1) $a(x) > 0$ on $\overline{\Omega}$;
(H2) $a(x) = 0$ on $\overline{D} \subset \Omega$ and $a(x) > 0$ on $\overline{\Omega} \setminus \overline{D}$, where D is a simply connected domain with smooth boundary.

The system (1.1) models the steady state behavior of a diffusive variable-territory prey–predator ecosystem in a heterogeneous environment. We refer the reader to [4] and [5] for more detailed biological background of (1.1). Wang and Pang [5] studied the existence, uniqueness and stability of positive solutions of (1.1), and raised an open problem [5, Remark 1] concerning the existence of solutions in the large λ regime. In a recent paper [6], the authors improved the existence

☆ The first author is supported by National University of Singapore Academic Research Fund (grant No. R-146-000-123-112). The second author is supported by NNSFC of China (grant Nos. 10901030, 11071100) and the Department of Education of Liaoning Province (grant No. 2009A152).

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.01.033

results of [5] significantly, and resolved the open problem partially. However, they were unable to settle the existence/non-existence question for the complete range of the parameter λ. The purpose of this short note is to fill in the gap completely, and answer the existence question conclusively. In fact, we prove that there exists a critical value λ_* of λ such that positive solutions exist when $\lambda > \lambda_*$, and do not exist when $0 < \lambda \leqslant \lambda_*$.

We first introduce some notation: For any $\psi \in C(\overline{\Omega})$, denote by $\lambda_1^{\Omega}(\psi)$ and $\lambda_1^{N,\Omega}(\psi)$ the smallest eigenvalue of the operator $-\Delta + \psi$ on Ω with the homogeneous Dirichlet and Neumann boundary conditions, respectively. For simplicity, we write $\lambda_1^{\Omega}(0) = \lambda_1^{\Omega}$. It is well known that $\lambda_1^{\Omega}(\psi)$ and $\lambda_1^{N,\Omega}(\psi)$ exist for any $\psi \in C(\overline{\Omega})$ and are the only eigenvalues whose corresponding eigenfunctions do not change sign. Moreover, $\lambda_1^{\Omega}(\psi)$ and $\lambda_1^{N,\Omega}(\psi)$ are continuous and strictly increasing in $\psi \in C(\overline{\Omega})$.

Let us recall the following results [3]:

(1) Let (H1) hold. Then for any $\lambda > 0$, the problem

$$-\Delta u = \lambda u - a(x)u^2 \quad \text{in } \Omega, \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

has a unique positive solution, denoted by u_λ.

(2) Let (H2) hold. Then (1.2) has a unique positive solution, denoted by u_λ^D, if $\lambda \in (0, \lambda_1^D)$; and has no positive solution if $\lambda \geqslant \lambda_1^D$.

Moreover, by comparison and compactness arguments, one can easily derive that u_λ and u_λ^D are strictly increasing in $\lambda > 0$ and $\lambda \in (0, \lambda_1^D)$, respectively, moreover, the mappings u_λ and u_λ^D, as functions of λ to the function space $C^1(\overline{\Omega})$, are uniformly continuous.

The main result of this paper is as follows:

Theorem 1.1.

(i) *Assume that* (H1) *holds and* $\lambda > 0$. *Then there exists a constant* $\lambda_* \in [\min_{\overline{\Omega}} a, \bar{a})$, *where* $\bar{a} = \frac{1}{|\Omega|}\int_\Omega a(x)\,dx$, *such that* (1.1) *admits at least one positive solution if* $\lambda \in (\lambda_*, \infty)$, *and has no positive solution if* $\lambda \in (0, \lambda_*]$.

(ii) *Assume that* (H2) *holds and* $\lambda > 0$. *Then there exists a constant* $\lambda_* \in (0, \lambda_1^D)$ *such that* (1.1) *admits at least one positive solution if* $\lambda \in (\lambda_*, \infty)$, *and has no positive solution if* $\lambda \in (0, \lambda_*]$.

2. Proof of the theorem

Lemma 2.1. *(See [1].) Let* $a(x), b(x) \in C^1(\overline{\Omega})$ *with* $b(x) > 0$ *on* $\overline{\Omega}$. *Then the problem*

$$-\Delta w = \big(a(x) - b(x)w\big)w \quad \text{in } \Omega, \qquad \partial_\nu w = 0 \quad \text{on } \partial\Omega,$$

has a positive solution if and only if $\lambda_1^{N,\Omega}(-a(x)) < 0$.

Lemma 2.2.

(i) *Assume that* (H1) *holds and* $\lambda > 0$. *Then* (1.1) *admits at least one positive solution if and only if* $\lambda_1^{N,\Omega}(1 - u_\lambda) < 0$.

(ii) *Assume that* (H2) *holds and* $\lambda \in (0, \lambda_1^D)$. *Then* (1.1) *admits at least one positive solution if and only if* $\lambda_1^{N,\Omega}(1 - u_\lambda^D) < 0$.

Proof. We only prove (ii) since the argument for (i) is similar.

We first prove the necessity assertion. Assume that (1.1) has a positive solution (u_0, v_0) for some $\lambda_0 \in (0, \lambda_1^D)$. From Lemma 2.1 and the equation of v_0, it follows that $\lambda_1^{N,\Omega}(1 - u_0) < 0$. By the comparison theorem, we find that $u_0 \leqslant u_{\lambda_0}^D$, and hence $\lambda_1^{N,\Omega}(1 - u_{\lambda_0}^D) < \lambda_1^{N,\Omega}(1 - u_0) < 0$. This establishes the necessity.

Next we prove the sufficiency assertion. Assume that, for some $\lambda \in (0, \lambda_1^D)$,

$$\lambda_1^{N,\Omega}\big(1 - u_\lambda^D\big) < 0. \tag{2.1}$$

As in [5] and [6], we will use the degree theory to prove the existence. Assume that (u, v) is a positive solution of the following problem with parameter $t \in [0, 1]$:

$$\begin{cases} -\Delta u = \lambda u - a(x)u^2 - tbuv & \text{in } \Omega, \\ -\Delta v = v\left(u - 1 - \dfrac{v}{u}\right) & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Without loss of generality, we assume that $a \in C^1(\overline{\Omega})$. By the regularity theory of elliptic equations, $(u, v) \in [C^2(\overline{\Omega})]^2$. Next we estimate the bounds of (u, v).

By the bounds (7) and (10) of [6], there exist positive constants C_1, C_2 independent of $t \in [0,1]$ such that

$$C_1 < u < C_2, \qquad v < C_2, \quad \text{on } \overline{\Omega}. \tag{2.3}$$

Below we estimate the positive lower bound for v, for which the condition (2.1) will play an important role. We shall prove that there exists a positive constant C_0 independent of t such that

$$\min_{\overline{\Omega}} v \geqslant C_0. \tag{2.4}$$

To see this, suppose, on the contrary, that there exist a sequence $\{t_n \in [0,1]\}$ and solutions (u_n, v_n) of (2.2) with $t = t_n$ such that

$$\min_{\overline{\Omega}} v_n \to 0 \quad \text{as } n \to +\infty. \tag{2.5}$$

By (2.3), we have, for all $n \geqslant 1$,

$$C_1 < u_n < C_2, \qquad v_n < C_2, \quad \text{on } \overline{\Omega}. \tag{2.6}$$

Thanks to (2.5) and (2.6), we have, by the same argument as in the proof of the limit (13) of [6],

$$\max_{\overline{\Omega}} v_n \to 0 \quad \text{as } n \to +\infty.$$

Denote $\hat{v}_n = v_n/\|v_n\|_\infty$. Then, $\|\hat{v}_n\|_\infty = 1$. It follows from (2.6) and the equations of u_n and $\hat{v}_n$ that $\{(\Delta u_n, \Delta \hat{v}_n)\}$ and $\{(u_n, \hat{v}_n)\}$ are bounded sets in $[L^\infty(\Omega)]^2$. By the standard elliptic theory, $\{(u_n, \hat{v}_n)\}$ is bounded in $[W^{2,p}(\Omega)]^2$ for any $p > 1$. Hence, there exist a subsequence of $\{(u_n, \hat{v}_n)\}$, denoted by itself, and a pair of positive functions $(w, \hat{v})$ such that $(u_n, \hat{v}_n) \to (w, \hat{v})$ in $[C^1(\overline{\Omega})]^2$, $\hat{v} \neq 0$, and

$$\begin{cases} -\Delta w = \lambda w - a(x) w^2 \quad \text{in } \Omega, \qquad \partial_\nu w = 0 \quad \text{on } \partial\Omega, \\ -\Delta \hat{v} = (w-1)\hat{v} \quad \text{in } \Omega, \qquad \partial_\nu \hat{v} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Clearly, $w = u_\lambda^D$. Since $\hat{v} \neq 0$, by the Harnack inequality ([2]), we have $\hat{v} > 0$ on $\overline{\Omega}$. Thus, $\hat{v}$ is a positive solution of the problem

$$-\Delta w = \big(u_\lambda^D - 1\big) w \quad \text{in } \Omega, \qquad \partial_\nu w = 0 \quad \text{on } \partial\Omega,$$

which implies that $\lambda_1^{N,\Omega}(1 - u_\lambda^D) = 0$. This contradicts (2.1). Therefore, (2.4) holds.

Define

$$\mathcal{O} = \big\{(u, v) \in C(\overline{\Omega} \times \overline{\Omega});\ c < u, v < C_2\big\},$$

where $c = \frac{1}{2}\min\{C_0, C_1\}$. From the above discussion we see that for all $t \in [0,1]$, (2.2) has no solution on $\partial\mathcal{O}$.

Denote

$$A(t; u, v) = \big(Lf(t,u,v), Lg(u,v)\big),$$

where

$$L = (-\Delta + I)^{-1}, \qquad f(t,u,v) = u + u\big(\lambda - a(x)u - tbv\big), \qquad g(u,v) = v + v\left(u - 1 - \frac{v}{u}\right).$$

Then $A : [0,1] \times \overline{\Omega} \to C(\overline{\Omega}) \times C(\overline{\Omega})$ is compact, and for $(u,v) \in \overline{\mathcal{O}}$, it is a solution of (2.2) if and only if it is a fixed point of $A(t;\cdot)$, i.e. $(u,v) = A(t;u,v)$. Thus,

$$(u,v) \neq A(t;u,v), \quad \forall t \in [0,1],\ \forall (u,v) \in \partial\mathcal{O}.$$

Furthermore, the degree $\deg(I - A(t;\cdot), \mathcal{O}, 0)$ is well defined and independent of $t \in [0,1]$.

When $t = 0$, the problem (2.2) becomes

$$\begin{cases} -\Delta u = \lambda u - a(x)u^2 & \text{in } \Omega, \\ -\Delta v = v\left(u - 1 - \dfrac{v}{u}\right) & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Again using (2.1), one derives from Lemma 2.1 that (2.7) has a unique positive solution (u_*, v_*), where $u^* = u_\lambda^D$ and v^* is the unique positive solution of

$$-\Delta v = v\left(u^* - 1 - \frac{v}{u^*}\right) \quad \text{in } \Omega, \qquad \partial_\nu v = 0 \quad \text{on } \partial\Omega.$$

Hence

$$\deg\bigl(I - A(0;\cdot), \mathcal{O}, 0\bigr) = \operatorname{index}\bigl(I - A(0;\cdot), \bigl(u^*, v^*\bigr)\bigr).$$

Moreover, we can prove that (u^*, v^*) as a solution of (2.7) is non-degenerate and linearly stable. In fact, the linearized eigenvalue problem of (2.7) at (u^*, v^*) is

$$\begin{cases} -\Delta h = \lambda h - 2a(x)u^* h + \eta h & \text{in } \Omega, \\ -\Delta k = \left(u^* - 1 - \dfrac{2v^*}{u^*}\right)k + \left[v^* + \left(\dfrac{v^*}{u^*}\right)^2\right]h + \eta k & \text{in } \Omega, \\ \partial_\nu h = \partial_\nu k = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where η denotes the eigenvalue and (h,k) the corresponding eigenfunction. By the first equation of (2.7) we see that $\lambda_1^{\Omega,N}(au^* - \lambda) = 0$. If $h \not\equiv 0$, from the first equation of (2.8), we have

$$\eta \geqslant \lambda_1^{\Omega,N}\bigl(2au^* - \lambda\bigr) > \lambda_1^{\Omega,N}\bigl(au^* - \lambda\bigr) = 0.$$

Furthermore, if $h \equiv 0$ and $k \not\equiv 0$, it follows from the second equation of (2.7) that $\lambda_1^{\Omega,N}(\frac{v^*}{u^*} - u^* + 1) = 0$. By the second equation of (2.8), we obtain

$$\eta \geqslant \lambda_1^{\Omega,N}\left(\frac{2v^*}{u^*} - u^* + 1\right) > \lambda_1^{\Omega,N}\left(\frac{v^*}{u^*} - u^* + 1\right) = 0.$$

In conclusion, we always have $\eta > 0$. Consequently,

$$\operatorname{index}\bigl(I, A(0;\cdot), \bigl(u^*, v^*\bigr)\bigr) = 1.$$

Hence $\deg(I, A(1;\cdot), \mathcal{O}, 0) = 1$, and thus $A(1;\cdot)$ has at least one fixed point in $\mathcal{O}$. In other words, problem (1.1) has at least one positive solution. The proof is complete. □

Proof of Theorem 1.1. We only prove (ii), as (i) can be proved similarly.

We recall [6, Theorem 1.1(ii,a)] that, under the condition (H2), there exists a sufficiently small constant $\epsilon > 0$ such that (1.1) admits at least one positive solution for all $\lambda > \lambda_1^D - \epsilon$. Thus, we only need to consider the case where $\lambda \in (0, \lambda_1^D)$. Let

$$\lambda_* = \inf\bigl\{\lambda \in \bigl(0, \lambda_1^D\bigr);\ \lambda_1^{N,\Omega}\bigl(1 - u_\lambda^D\bigr) < 0\bigr\}. \tag{2.9}$$

By [6, Theorem 1.1(ii)] and Lemma 2.2(ii), we find that $\lambda_* \in (0, \lambda_1^D)$, and (1.1) has no positive solution for any $\lambda \in (0, \lambda_*)$.

Next we prove that (1.1) has a positive solution for all $\lambda \in (\lambda_*, \lambda_1^D)$. Suppose that for some $\lambda_0 \in (\lambda_*, \lambda_1^D)$, (1.1) with $\lambda = \lambda_0$ has no positive solution. Then, by Lemma 2.2(ii), $\lambda_1^{N,\Omega}(1 - u_{\lambda_0}^D) \geqslant 0$. Hence, $\lambda_1^{N,\Omega}(1 - u_\lambda^D) \geqslant 0$ for all $\lambda \in (0, \lambda_0)$, which implies $\lambda_0 \leqslant \lambda_*$, a contradiction.

Finally, we deal with the case $\lambda = \lambda_*$. By the above arguments, we see that

$$\lambda_1^{N,\Omega}\bigl(1 - u_\lambda^D\bigr) \geqslant 0, \quad \forall \lambda \in (0, \lambda_*),$$
$$\lambda_1^{N,\Omega}\bigl(1 - u_\lambda^D\bigr) < 0, \quad \forall \lambda \in \bigl(\lambda_*, \lambda_1^D\bigr).$$

By continuity, we find that $\lambda_1^{N,\Omega}(1 - u_{\lambda_*}^D) = 0$. It follows from Lemma 2.2(ii) that (1.1) with $\lambda = \lambda_*$ has no positive solution. The proof is complete. □

We end this paper with the following discussion: From (2.9), one sees that the critical value λ_* provides a strong link between Eqs. (1.1) and (2.7), which is the limit of (1.1) as $b \to 0^+$. Under this limit, we note that the solution (u_b, v_b) of (1.1) converges to $(u^* = u_\lambda^D, v^*)$ of (2.7) in $[C^1(\overline{\Omega})]^2$. Moreover, as $b \to 0^+$, $\lambda_1^{N,\Omega}(1 - u_b) \to \lambda_1^{N,\Omega}(1 - u_\lambda^D)$. The critical condition $\lambda_1^{N,\Omega}(1 - u_{\lambda_*}^D) = 0$ is thus directly related to the persistent solution of the system (2.7). Furthermore, we recall from [5, Theorem 7] that for b sufficiently small, the positive solution (u_b, v_b) is unique and linearly stable. Thus, no bifurcation occurs in the regime of small b. However, the general bifurcation picture as λ varies from λ_* is unclear.

Acknowledgment

The authors would like to thank the referee for helpful comments.

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