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Sets of filter convergence of sequences of continuous functions

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ABSTRACT

We consider sets of filter convergence and divergence to infinity of sequences of continuous functions for Borel filters. We characterize the sets of filter convergence of sequences of continuous functions on the real line for Borel filters. We also give sufficient and necessary conditions for results involving sets of filter divergence to infinity. In particular, we give the full description of such sets for the statistical convergence for metric spaces.

For a sequence $(f_n)_n$ of continuous real valued functions defined on a space X one can consider the set $C((f_n)) = \{x \in X : \lim_n f_n(x) \text{ exists}\}$. Hahn and Sierpiński proved that the family of such sets on a metric space X coincides with the family $\mathbf{\Pi}^0_3(X)$ $(F_{\sigma\delta}(X))$. Further investigation involved also infinities; the full description of these sets was given by M. A. Lunina in the following theorem (see [1]).

For a sequence of continuous real functions (f_n) we consider seven types of sets of convergence and divergence.

$$\begin{aligned}
&E^1((f_n)) = \{x\colon (f_n(x)) \text{ converges}\},\\
&E^2((f_n)) = \{x\colon \lim f_n(x) = -\infty\},\\
&E^3((f_n)) = \{x\colon \lim f_n(x) = +\infty\},\\
&E^4((f_n)) = \{x\colon -\infty < \underline{\lim}\, f_n(x) < \overline{\lim}\, f_n(x) < +\infty\},\\
&E^5((f_n)) = \{x\colon -\infty = \underline{\lim}\, f_n(x) < \overline{\lim}\, f_n(x) < +\infty\},\\
&E^6((f_n)) = \{x\colon -\infty < \underline{\lim}\, f_n(x) < \overline{\lim}\, f_n(x) = +\infty\},\\
&E^7((f_n)) = \{x\colon -\infty = \underline{\lim}\, f_n(x) \text{ and } \overline{\lim}\, f_n(x) = +\infty\}.
\end{aligned}$$

We call $(E^1, \ldots, E^7)$ a *Lunina's 7-tuple* if there exists a sequence of real-valued continuous functions (f_n) such that $E^i = E^i((f_n))$ for $i = 1, 2, \ldots, 7$.

Theorem 1 (*Lunina [1]*)**.** *Suppose that a metric space X is a union of 7 disjoint sets $E^1, E^2, \ldots, E^7$. Then $(E^1, \ldots, E^7)$ is a Lunina's 7-tuple iff E^1, E^2, E^3 are $F_{\sigma\delta}$ in X and $E^2 \cup E^5 \cup E^7$, $E^3 \cup E^6 \cup E^7$ are G_δ in X.*

In this paper we will investigate analogous relationships for the filter convergence. $\mathcal{F} \subset P(\omega)$ is a *filter* if $\mathcal{F}$ is closed under taking supersets and finite intersections, $\omega \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. We shall use the notation $\mathcal{D}^* = \{\omega \setminus Z : Z \in \mathcal{D}\}$, $\mathrm{FIN} = \{A \subset \omega : A \text{ is finite}\}$ and $\mathrm{FIN} \times \mathrm{FIN} = \{Z \subset \omega \times \omega : \forall^\infty_n \{k : (n,k) \in Z\} \text{ is finite}\}$. The filter FIN^* is called the Fréchet filter. Throughout this paper we only consider filters containing the Fréchet filter.

$C(X)$, $B(X)$, $B_\alpha(X)$ will denote the class of continuous functions, Borel functions and functions of Borel class α mapping the space X into $\mathbb{R}$.

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For a given filter $\mathcal{F}$, $x \in \mathbb{R}$ and $(x_n) \in \mathbb{R}^\omega$ we denote

1. $\mathcal{F} - \lim x_n = x$ if for each $\epsilon > 0$ $\{n \in \omega : |x - x_n| < \epsilon\} \in \mathcal{F}$,
2. $\mathcal{F} - \lim x_n = \infty$ if for each $M \in \mathbb{R}$ $\{n \in \omega : x_n > M\} \in \mathcal{F}$,
3. $\mathcal{F} - \lim x_n = -\infty$ if for each $M \in \mathbb{R}$ $\{n \in \omega : x_n < M\} \in \mathcal{F}$.

Next we denote four types of sets of convergence and divergence.

1. $C_{\mathcal{F}} = \{(x_n) \in \mathbb{R}^\infty : \mathcal{F} - \lim x_n \text{ exists}\}$,
2. $c_{\mathcal{F}} = \{(x_n) \in \mathbb{R}^\infty : \mathcal{F} - \lim x_n = 0\}$,
3. $c_{\mathcal{F}}^{\infty} = \{(x_n) \in \mathbb{R}^\infty : \mathcal{F} - \lim x_n = \infty\}$,
4. $c_{\mathcal{F}}^{-\infty} = \{(x_n) \in \mathbb{R}^\infty : \mathcal{F} - \lim x_n = -\infty\}$.

For a topological space X and $(f_n) \in C(X)^\omega$ we define

1. $C_{\mathcal{F}}((f_n)) = \{x \in X : (f_n(x)) \in C_{\mathcal{F}}\}$,
2. $c_{\mathcal{F}}((f_n)) = \{x \in X : (f_n(x)) \in c_{\mathcal{F}}\}$,
3. $c_{\mathcal{F}}^{\infty}((f_n)) = \{x \in X : (f_n(x)) \in c_{\mathcal{F}}^{\infty}\}$,
4. $c_{\mathcal{F}}^{-\infty}((f_n)) = \{x \in X : (f_n(x)) \in c_{\mathcal{F}}^{-\infty}\}$,

and

1. $C_{\mathcal{F}}(X) = \{C_{\mathcal{F}}((f_n)) : (f_n) \in C(X)^\omega\}$,
2. $c_{\mathcal{F}}(X) = \{c_{\mathcal{F}}((f_n)) : (f_n) \in C(X)^\omega\}$,
3. $c_{\mathcal{F}}^{\infty}(X) = \{c_{\mathcal{F}}^{\infty}((f_n)) : (f_n) \in C(X)^\omega\}$,
4. $c_{\mathcal{F}}^{-\infty}(X) = \{c_{\mathcal{F}}^{-\infty}((f_n)) : (f_n) \in C(X)^\omega\}$.

The standard convergence coincides with the convergence with respect to the Fréchet filter FIN*. One can expect that for every filter $\mathcal{F}$ the sets in $C_{\mathcal{F}}(X)$, $c_{\mathcal{F}}(X)$, $c_{\mathcal{F}}^{\infty}(X)$ and $c_{\mathcal{F}}^{-\infty}(X)$ are Borel. Unfortunately, this is not true. There is a canonical example of a sequence of continuous functions which shows that the sets of convergence do not need to be nice. We identify $P(\omega)$ with the Cantor space and define the continuous functions $f_n : P(\omega) \to \mathbb{R}$ by

$$f_n(Z) = \begin{cases} 0 & \text{if } n \in Z \\ n & \text{otherwise.} \end{cases}$$

Observe that $C_{\mathcal{F}}((f_n)) = \mathcal{F}$ and $c_{\mathcal{F}}^{\infty}((f_n)) = \mathcal{F}^*$, so in the case when F is an ultrafilter, i.e. a maximal filter, these sets are not Borel. In [2] the authors consider F_σ-filters (in fact, ideals but those notions are dual to each other) and proved a Lunina-like theorem. In this paper we will consider general Borel filters.

Theorem 2 (*Dobrowolski et al. [3,4], Solecki [5], Debs and Saint Raymond [6]*)**.** *If $\mathcal{F} \in \mathbf{\Pi}_\alpha^0 \setminus \bigcup_{\beta<\alpha} \mathbf{\Pi}_\beta^0$, then $C_{\mathcal{F}}, c_{\mathcal{F}} \in \mathbf{\Pi}_\alpha^0 \setminus \mathbf{\Sigma}_\alpha^0$.*

Proof. For the case $c_{\mathcal{F}}$ the statement follows from [3, Lemma 4.2]. The result $C_{\mathcal{F}} \in \mathbf{\Pi}_\alpha^0$ was proved in [6, Remark 2.11]. From [4, Proposition 3.9] it follows that $\{(x_n) \in C_{\mathcal{F}} : (x_n) \text{ is bounded}\} \notin \mathbf{\Sigma}_\alpha^0$. Since the set of bounded sequences is an F_σ-set, it follows that $C_{\mathcal{F}}$ is not in $\mathbf{\Sigma}_\alpha^0$. □

Corollary 3. *If $\mathcal{F} \in \mathbf{\Pi}_\alpha^0 \setminus \bigcup_{\beta<\alpha} \mathbf{\Pi}_\beta^0$ then $c_{\mathcal{F}}^{\infty}, c_{\mathcal{F}}^{-\infty} \in \mathbf{\Pi}_\alpha^0 \setminus \mathbf{\Sigma}_\alpha^0$.*

Proof. Observe that the functions $H, G : \mathbb{R}^\omega \to \mathbb{R}^\omega$; $H((x_n)) = (1/(|x_n| + 1/n))$ and $G((x_n)) = (e^{-x_n})$ are continuous. Since $G^{-1}[c_{\mathcal{F}}] = c_{\mathcal{F}}^{\infty}$ and $H^{-1}[c_{\mathcal{F}}^{\infty}] = c_{\mathcal{F}}$, we may use Theorem 2. Similar proof works for $c_{\mathcal{F}}^{-\infty}$. □

Theorem 4. *Let X be a metric space. If $\mathcal{F} \in \mathbf{\Pi}_\alpha^0 \setminus \bigcup_{\beta<\alpha} \mathbf{\Pi}_\beta^0$, then*

1. $C_{\mathcal{F}}(X) \cup c_{\mathcal{F}}^{\infty}(X) \cup c_{\mathcal{F}}^{-\infty}(X) \subset \mathbf{\Pi}_\alpha^0(X)$.
2. *If X is a separable, zero-dimensional metric space, then $C_{\mathcal{F}}(X) = c_{\mathcal{F}}^{\infty}(X) = c_{\mathcal{F}}^{-\infty}(X) = \mathbf{\Pi}_\alpha^0(X)$.*
3. *If $\alpha = 3$ then $C_{\mathcal{F}}(X) = c_{\mathcal{F}}^{\infty}(X) = c_{\mathcal{F}}^{-\infty}(X) = \mathbf{\Pi}_\alpha^0(X)$.*

Proof. 1. The function $H : X \to \mathbb{R}^\omega$ given by $H(x) = (f_n(x))$ is continuous, and thus $C_{\mathcal{F}}((f_n)) = H^{-1}[C_{\mathcal{F}}]$, $c_{\mathcal{F}}^{\infty}((f_n)) = H^{-1}[c_{\mathcal{F}}^{\infty}]$, $c_{\mathcal{F}}^{-\infty}((f_n)) = H^{-1}[c_{\mathcal{F}}^{-\infty}]$.

2. Let $E \in \mathbf{\Pi}_\alpha^0(X)$ be arbitrary. By $\mathbf{\Pi}_\alpha^0$-completeness of $C_{\mathcal{F}}$ (see [7]), there is a continuous function $H : X \to \mathbb{R}^\omega$ with $E = H^{-1}[C_{\mathcal{F}}]$. Then $E = C_{\mathcal{F}}[(H_n)]$, where $H_n = \pi_n \circ H$. The other cases can be handled by the same argument.
3. For $A \in \mathbf{\Pi}_3^0(X)$ we apply [2, Corollary 4] and Lunina's Theorem for the 7-tuple $(A, \emptyset, \emptyset, X \setminus A, \emptyset, \emptyset, \emptyset)$ in the case of $C_{\mathcal{F}}$; $(\emptyset, A, \emptyset, \emptyset, X \setminus A, \emptyset, \emptyset)$ in the case of $c_{\mathcal{F}}^{\infty}$, and $(\emptyset, \emptyset, A, \emptyset, \emptyset, X \setminus A, \emptyset)$ in the case of $c_{\mathcal{F}}^{-\infty}$. □

Theorem 5. *If $\mathcal{F} \in \mathbf{\Pi}_\alpha^0 \setminus \bigcup_{\beta<\alpha} \mathbf{\Pi}_\beta^0$, then $C_{\mathcal{F}}(\mathbb{R}) = \mathbf{\Pi}_\alpha^0(\mathbb{R})$.*

Proof. It is enough to prove that for each $A \in \mathbf{\Pi}^0_\alpha(\mathbb{R})$ there is a sequence of continuous functions $f = (f_n)_n \in (\mathbb{R}^{\mathbb{R}})^\omega$ such that $C_{\mathcal{F}}((f_n)) = A$, because the converse inclusion follows from Theorem 2.

The set A has the Baire property so there are G_δ-sets G_0, G_1 and a meager F_σ-set F such that $\{G_0, G_1, F\}$ is a partition of $\mathbb{R}$, $G_0 \subset A$ and $G_1 \cap A = \emptyset$. Since F is meager on the real line, there are compact pairwise disjoint sets F_n with $\bigcup_n F_n = F$. We know that $C_{\mathcal{F}} \cap [0,1]^\omega \in \mathbf{\Pi}^0_\alpha([0,1]^\omega) \setminus \mathbf{\Sigma}^0_\alpha([0,1]^\omega)$, so by Louveau and Saint-Raymond [8, Theorem 3] there are continuous functions $H_n : F_n \to [0,1]^\omega$ such that $H_n^{-1}[C_{\mathcal{F}}] = F_n \cap A$. The functions $h^n_k = \pi_k \circ H_n$ are continuous on F_n. Let us extend $\bigcup_{i \le k} h^i_k$ from $\bigcup_{i \le k} F_i$ to a continuous function $g_k : \mathbb{R} \to [0,1]$. Observe that $g_k|F_n = h^n_k$ for $k \ge n$, so $C_{\mathcal{F}}((g_k)) \cap F_n = A \cap F_n$. Thus $C_{\mathcal{F}}((g_k)) \cap F = A \cap F$.

Claim. *There is a sequence of continuous functions $b_k : \mathbb{R} \to (0, \infty)$ such that*

1. $\lim_k b_k(x) = 0$, *for $x \in G_0$;*
2. $\lim_k b_k(x)$ *exists and is greater than* 0, *for $x \in F$;*
3. $\lim_k b_k(x) = \infty$, *for $x \in G_1$.*

Proof of the claim. Applying Lunina's Theorem (Theorem 1) for the 7-tuple $(F, G_1, G_0, \emptyset, \emptyset, \emptyset, \emptyset)$ we get a sequence (s_k) of continuous real functions such that $s_k(x)$ is convergent (in the standard sense) on F, convergent to ∞ on G_1 and convergent to $-\infty$ on G_0. Then $b_k(x) = e^{s_k(x)}$ has the required properties. □

Now let $f_k = b_k \cdot (g_k + 1)$.

1. For $x \in G_0$, $\mathcal{F} - \lim f_k(x)$ exists because $\lim_k f_k(x) = 0$ and $g_k(x)$ is bounded.
2. For $x \in G_1$, $\mathcal{F} - \lim f_k(x) = \infty$ (so the $\mathcal{F}$-limit does not exist) because $\lim_k b_k(x) = \infty$ and $(g_k + 1)(x) \ge 1$.
3. For $x \in A \cap F$, $\mathcal{F} - \lim f_k(x)$ exists because $\lim_k b_k(x)$ and $\mathcal{F} - \lim g_k(x)$ exist.
4. For $x \in F \setminus A$, $\mathcal{F} - \lim f_k(x)$ does not exist because $\lim b_k(x)$ exists and is greater than zero and $\mathcal{F} - \lim(g_k + 1)(x)$ does not exist. □

The above proof does not work in general even for the plane because we use the fact that each meager set of the reals is zero-dimensional and that any meager F_σ-set is the countable union of pairwise disjoint compact sets.

For the standard convergence it is not enough to assume that the sets $C_{\mathrm{FIN}^*}((f_n)), c^{-\infty}_{\mathrm{FIN}^*}((f_n)), c^{\infty}_{\mathrm{FIN}^*}((f_n))$ are pairwise disjoint $F_{\sigma\delta}$-sets to get the right example of continuous functions. (See Theorem 1.) We have some additional separation results, i.e. $C_{\mathrm{FIN}^*}((f_n)) \cup c^{-\infty}_{\mathrm{FIN}^*}((f_n))$ can be separated by an F_σ-set from $c^{\infty}_{\mathrm{FIN}^*}((f_n))$. In fact this set is equal to $\{x : \limsup f_n(x) < \infty\}$. In general the set of points of finite filter upper limit is not of small Borel class. However, in some cases we can give an upper bound for this Borel class.

For a given class $\Gamma \subset P(X)$ and pairwise disjoint sets $A, B \subset X$ we say that A can be Γ-separated from B if there exists $E \in \Gamma$ with $A \subset E$ and $E \cap B = \emptyset$. In [6] the authors defined the rank of a filter:

$$rk(\mathcal{F}) = \min\{\xi < \omega_1 : \mathcal{F} \text{ is } \mathbf{\Sigma}^0_{1+\xi}\text{-separated from } \mathcal{F}^*\}.$$

Analytic filters have countable rank. The authors prove that if $\mathcal{F} \in \mathbf{\Pi}^0_\alpha$ then $1 + rk(\mathcal{F}) < \alpha$.

Theorem 6. *Assume that $\mathcal{F}$ is an analytic filter. Then $C_{\mathcal{F}} \cup c^{-\infty}_{\mathcal{F}}$ ($C_{\mathcal{F}} \cup c^{\infty}_{\mathcal{F}}$) is $\mathbf{\Sigma}^0_{1+rk(\mathcal{F})}$-separated but not $\mathbf{\Pi}^0_{1+rk(\mathcal{F})}$-separated from $c^{\infty}_{\mathcal{F}}$ ($c^{-\infty}_{\mathcal{F}}$).*

Proof. Let us consider the sets $B_0 = \{(x_n) \in [0,1]^\omega : \mathcal{F} - \lim x_n = 0\}$, $B_1 = \{(x_n) \in [0,1]^\omega : \mathcal{F} - \lim x_n = 1\}$, $B_a = \{(x_n) \in [0,1]^\omega : 0 < \mathcal{F} - \lim x_n < 1\}$. First we show that $B_a \cup B_1$ can be $\mathbf{\Sigma}_{1+rk(\mathcal{F})}$-separated from B_0. Suppose this is not true, and fix $B \in \mathbf{\Pi}_{1+rk(\mathcal{F})}(2^\omega) \setminus \mathbf{\Sigma}_{1+rk(\mathcal{F})}(2^\omega)$. Then there is a continuous function $s : 2^\omega \to (B_a \cup B_1) \cup B_0$ with $s^{-1}[B_a \cup B_1] = B$ (see [8, Theorem 3]). Let $h = \mathcal{F} - \lim s_n$, where $s_n = \pi_n \circ s$. By Debs and Saint Raymond [6, Theorem 2.6], we have $h \in B_{rk(\mathcal{F})}$ which is impossible because $h^{-1}[(0,1)] = B \notin \mathbf{\Sigma}_{1+rk(\mathcal{F})}(2^\omega)$. Let $h : \mathbb{R} \to (0,1)$ be a decreasing homeomorphism. Then the preimage by (h^∞) of a set separating $B_a \cup B_1$ from B_0 will separate $C_{\mathcal{F}} \cup c^{-\infty}_{\mathcal{F}}$ from $c^{\infty}_{\mathcal{F}}$.

Let $B'_0 = B_0 \cap (0,1)^\omega$, $B'_1 = B_1 \cap (0,1)^\omega$, $B'_a = B_a \cap (0,1)^\omega$. It is enough to show that B'_0 is not $\mathbf{\Sigma}^0_{1+rk(\mathcal{F})}$-separated from $B'_a \cup B'_1$. Let $A \in \mathbf{\Sigma}^0_{1+rk(\mathcal{F})}(2^\omega) \setminus \mathbf{\Pi}^0_{1+rk(\mathcal{F})}(2^\omega)$. We have $A = \bigcup_n A_n$ for some $A_n \in \mathbf{\Pi}^0_{\beta_n}$ with $\beta_n < 1 + rk(\mathcal{F})$. Since $C_{\mathcal{F}} \cup c^{-\infty}_{\mathcal{F}}$ is not $\mathbf{\Sigma}^0_{\beta_n}$-separated from $c^{\infty}_{\mathcal{F}}$, it follows that $B'_a \cup B'_1$ is not $\mathbf{\Sigma}^0_{\beta_n}$-separated from B'_0. So there are continuous functions $\Psi_n : 2^\omega \to B'_0 \cup (B'_a \cup B'_1)$ such that $\Psi_n^{-1}[(B'_a \cup B'_1)] = A_n$. Let $\phi^n_k = \pi_k \circ \Psi_n$, where π_k is the projection onto the k-th coordinate. Then for each x, $x \in A_n$ iff $\mathcal{F} - \lim_k \phi^n_k(x) \neq 0$, and $x \notin A_n$ iff $\mathcal{F} - \lim_k \phi^n_k(x) = 0$. Let us define $h_k = \mathbf{\Sigma}^{\infty}_{i=0} \frac{1}{2^{i+1}} \phi^i_k$. Then $\mathcal{F} - \lim_k h_k(x)$ exists, $0 < \mathcal{F} - \lim_k h_k(x) < 1$, and $\mathcal{F} - \lim_k h_k(x) \neq 0$ iff $x \in A$. Let $\Psi = (h_k)$. Then $\Psi^{-1}[B'_a \cup B'_1] = A$. Now, because A is not $\mathbf{\Pi}^0_{1+rk(\mathcal{F})}$-separated from $2^\omega \setminus A$, $B'_a \cup B'_1$ cannot be $\mathbf{\Pi}^0_{1+rk(\mathcal{F})}$-separated from B'_0. □

As a consequence we get the following result.

Theorem 7. *Let X be a metric space and let $\mathcal{F} \in \mathbf{\Pi}^0_\alpha \setminus \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta$ be a filter. Assume there is $(f_n) \in C(X)^\omega$ such that $C_{\mathcal{F}}((f_n)) = A$, $c^{\infty}_{\mathcal{F}}((f_n)) = B$ and $c^{-\infty}_{\mathcal{F}}((f_n)) = C$. Then $A, B, C \in \mathbf{\Pi}^0_\alpha$, $A \cup B$ can be $\mathbf{\Sigma}_{1+rk(\mathcal{F})}$-separated from C and $A \cup C$ can be $\mathbf{\Sigma}_{1+rk(\mathcal{F})}$-separated from B.*

This is the best possible result of this type because taking $\pi_n : \mathbb{R}^\omega \to \mathbb{R}$, $\pi_n((x_k)) = x_n$, we have $C_\mathcal{F}((\pi_n)) = C_\mathcal{F}$, $c^\infty_\mathcal{F}((\pi_n)) = c^\infty_\mathcal{F}$ and $c^{-\infty}_\mathcal{F}((\pi_n)) = c^{-\infty}_\mathcal{F}$, and Theorems 2 and 6 show that we cannot get better upper bound in general.

For $\alpha = 3$, the $\mathbf{\Sigma}_{1+rk(\mathcal{F})}$-separation simply means F_σ-separation, and the previous result can be reversed.

Theorem 8. *Let X be a metric space and let $\mathcal{F} \in \mathbf{\Pi}^0_3$ be a filter. Let A, B, C be pairwise disjoint $\mathbf{\Pi}^0_3$ subsets of X. Assume that $A \cup B$ can be F_σ-separated from C and $A \cup C$ can be F_σ-separated from B. Then there is $(f_n) \in C(X)^\omega$ such that $C_\mathcal{F}((f_n)) = A$, $c^\infty_\mathcal{F}((f_n)) = B$ and $c^{-\infty}_\mathcal{F}((f_n)) = C$.*

Proof. We follow the idea of the proof of [2, Corollary 4]. Let E, F be F_σ-sets with $(A \cup B \cup C) \cap E = A \cup B$ and $(A \cup B \cup C) \cap F = A \cup C$. Then the 7-tuple $(A, B, C, E \cap F \setminus A, (X \setminus F \setminus B) \cap E, (X \setminus E \setminus C) \cap F, X \setminus (E \cup F))$ gives a partition with the following properties: $A, B, C \in \mathbf{\Pi}^0_3(X)$, and the unions of the second, fifth and seventh sets and of third, sixth and seventh sets are G_δ-sets. By Lunina's Theorem there is a sequence of continuous functions $g_n : X \to \mathbb{R}$ such that $C_{\mathrm{FIN}^*}((g_n)) = A$, $c^\infty_{\mathrm{FIN}^*}((g_n)) = B$ and $c^{-\infty}_{\mathrm{FIN}^*}((g_n)) = C$. By Talagrand's Theorem ([9, Corollary 3.10.2]) there is a function $h : \omega \to \omega$ with $h^{-1}[M] \in \mathcal{F}$ iff $M \in \mathrm{FIN}^*$ for each $M \subset \omega$. Let the sequence of functions (f_n) be defined by $f_n = g_k$ if $h(n) = k$. This sequence has all the required properties. For details see [2]. □

The previous results give in particular a characterization of sets $C_\mathcal{F}((f_n))$ $c^\infty_\mathcal{F}((f_n))$ and $c^{-\infty}_\mathcal{F}((f_n))$ for $\mathcal{F} = \{Z \subset \omega : \lim_n \frac{|Z\cap\{0,\ldots,n-1\}|}{n} = 1\}$ the filter of sets of density 1, i.e. for the statistical convergence.

The following proposition provides some generalization but only for two sets and zero-dimensional spaces.

Proposition 9. *Let X be a separable, zero-dimensional metric space and let $\mathcal{F} \in \mathbf{\Pi}^0_\alpha \setminus \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta$ be a filter. Assume that A, B are disjoint subsets of X such that $A, B \in \mathbf{\Pi}^0_\alpha$ and A can be $\mathbf{\Sigma}_{1+rk(\mathcal{F})}$-separated from B. Then there is $(f_n) \in C(X)^\omega$ with $f_n \geq 0$ such that $C_\mathcal{F}((f_n)) = A$ and $c^\infty_\mathcal{F}((f_n)) = B$.*

Proof. Let $D \in \mathbf{\Pi}^0_{1+rk\mathcal{F}}(2^\omega)$ be such that $D \cap (A \cup B) = B$. Let $Z = \{(x_n) \in (1, 2)^\omega : \mathcal{F} - \lim x_n = 2\}$. Since $c^\infty_\mathcal{F} \notin \mathbf{\Sigma}^0_\alpha$ and $C_\mathcal{F} \cup c^{-\infty}_\mathcal{F} \in \mathbf{\Pi}^0_\alpha$, it follows that $c^\infty_\mathcal{F}$ cannot be $\mathbf{\Sigma}^0_\alpha$-separated from $\mathbb{R}^\omega \setminus (C_\mathcal{F} \cup c^\infty_\mathcal{F} \cup c^{-\infty}_\mathcal{F})$. So Z cannot be $\mathbf{\Sigma}^0_\alpha$-separated from $(1, 2)^\omega \setminus C_\mathcal{F}$ in $(1, 2)^\omega$. Thus there is a continuous function $a : X \to Z \cup ((1, 2)^\omega \setminus C_\mathcal{F})$ such that $a^{-1}[Z] = A \cup B$.

There is also a continuous function $b : X \to C_\mathcal{F} \cap (1, 2)^\omega$ such that $b^{-1}[Z] = D$ because $D \in \mathbf{\Pi}^0_{1+rk\mathcal{F}}$ and Z is not $\mathbf{\Sigma}^0_{1+rk(\mathcal{F})}$-separated from $(C_\mathcal{F} \setminus Z) \cap (1, 2)^\omega$.

Then $\{x : \mathcal{F} - \lim(a_n \cdot b_n)(x) \text{ exists}\} = A \cup B$, $1 < (a_n \cdot b_n)(x) < 4$ and $\{x : \mathcal{F} - \lim(a_n \cdot b_n)(x) = 4\} = B$. Taking an increasing bijection $\phi : (1, 4) \to (0, \infty)$ we can see that the sequence $f_n = \phi \circ (a_n \cdot b_n)$ has the required properties. □

Finally we will present some results for sets of $\mathcal{F}$-convergence and $\mathcal{F}$-divergence to infinities for filters which are not necessarily Borel (for example, for ultrafilters).

Lemma 10. *Assume that $\mathcal{F} - \lim f_n = f$, where f_n, f are Borel real functions defined on a Polish space X. Then there is an analytic filter $\mathcal{F}' \subset \mathcal{F}$ such that $\mathcal{F}' - \lim f_n = f$.*

Proof. Let us first observe that if $H \subset P(\omega)$ is analytic then the filter generated by H is analytic. This is a consequence of two simple facts. First, the set $H' = \{A \subset \omega : (\exists B \in H) A \supset B\}$ is analytic. Second, the filter generated by H is equal to the set $\bigcup_{n\geq 1} F_n[H'^n]$, where $F_n : (P(\omega))^n \to P(\omega)$ and $F_n(A_1, A_2, \ldots, A_n) = A_1 \cap A_2 \cap \cdots \cap A_n$ are continuous.

Now, let $H = \{A : (\exists x \in X)(\exists n) A = \{k : |f_k(x) - f(x)| < 1/n\}\}$. Observe that H is analytic and $H \subset \mathcal{F}$. Then the filter $\mathcal{F}'$ generated by H is analytic, $\mathcal{F}' \subset \mathcal{F}$ and $\mathcal{F}' - \lim f_n = f$. □

The following result is a generalization of [6, Corollary 7.7]. Denote $\mathfrak{C}_\mathcal{F}(X) = \{f \in \mathbb{R}^X : (\exists (f_n) \in C(X)^\omega)(\forall x \in X) \mathcal{F} - \lim f_n(x) = f(x)\}$.

Proposition 11. *Let X be an uncountable Polish space and let $\mathcal{F}$ be a filter. Then $\mathfrak{C}_\mathcal{F}(X) \cap B(X) = B_1(X)$ iff $\mathcal{F}$ does not contain an isomorphic copy of $(\mathrm{FIN} \times \mathrm{FIN})^*$.*

Proof. If $\mathcal{F}$ contains a copy of $(\mathrm{FIN} \times \mathrm{FIN})^*$ then $B_2(X) \subset \mathfrak{C}_\mathcal{F}(X)$ which is strictly larger than $B_1(X)$ on an uncountable Polish space. Assume that $f \in \mathfrak{C}_\mathcal{F}(X) \cap B(X) \setminus B_1(X)$. There is a sequence $(f_n) \in C(X)^\omega$ such that $\mathcal{F} - \lim f_n = f$. By the previous lemma there is an analytic filter $\mathcal{F}' \subset \mathcal{F}$ with $\mathcal{F}' - \lim f_n = f$. By Debs and Saint Raymond [6, Theorem 7.5] and [6, Theorem 2.6], $\mathcal{F}'$ does contain an isomorphic copy of $(\mathrm{FIN} \times \mathrm{FIN})^*$. □

Fact 12 (*Folklore*). *If $\mathcal{F}$ is an ultrafilter then for any sequence $(f_n) \in (\mathbb{R}^X)^\omega$, $C_\mathcal{F}((f_n)) \cup c^\infty_\mathcal{F}((f_n)) \cup c^{-\infty}_\mathcal{F}((f_n)) = X$.*

Proposition 13. *Let X be a Polish space. For each triple of pairwise disjoint sets A, B, C there is a sequence $(f_n) \in C(X)^\omega$ and a filter $\mathcal{F}$ such that $C_\mathcal{F}((f_n)) = A$, $c^\infty_\mathcal{F}((f_n)) = B$ and $c^{-\infty}_\mathcal{F}((f_n)) = C$. Moreover, if $A \cup B \cup C = X$ then we can assume additionally that $\mathcal{F}$ is an ultrafilter.*

Proof. Define the functions $j_1, j_2 : X \to \mathbb{R}$ by the following formulas:

$$j_1(x) = \begin{cases} 1/3 & \text{if } x \in A, \\ 1 & \text{if } x \in B, \\ 0 & \text{if } x \in C, \\ 1/2 & \text{otherwise;} \end{cases}$$

$$j_2(x) = \begin{cases} 1/3 & \text{if } x \in A, \\ 1 & \text{if } x \in B, \\ 0 & \text{if } x \in C, \\ 2/3 & \text{otherwise.} \end{cases}$$

By Katětov's Theorem (see [10]), there are filters $\mathcal{F}_1$, $\mathcal{F}_2$ and sequences of continuous function $(g_n), (h_n) \in C(X)^\omega$ such that $\mathcal{F}_1 - \lim g_n = j_1$ and $\mathcal{F}_2 - \lim h_n = j_2$. We can assume that $0 < g_n, h_n < 1$ for each n. Then we define a filter $\mathcal{F}$ by

$$\mathcal{F} = \{A \subset \{1, 2\} \times \omega : \{n : (1, n) \in A\} \in \mathcal{F}_1 \& \{n : (2, n) \in A\} \in \mathcal{F}_2\}.$$

We put $f'_{(i,n)} = g_n$ for $i = 1$, and $f'_{(i,n)} = h_n$ for $i = 2$. Then $f'_{(i,n)}(x)$ is $\mathcal{F}$-convergent to $1/3$ for $x \in A$, $\mathcal{F}$-convergent to 1 for $x \in B$, $\mathcal{F}$-convergent to 0 for $x \in C$ and $\mathcal{F}$-divergent for $x \notin A \cup B \cup C$. Finally, the functions $f_{(i,n)} = k \circ f'_{(i,n)}$, where k is an increasing bijection from $(0, 1)$ onto $\mathbb{R}$, have all the required properties. Moreover, if $A \cup B \cup C = X$, then any extension of the filter $\mathcal{F}$ to a ultrafilter together with the same functions have the properties required. This is because convergent sequences remain convergent when we extend a filter. □

Proposition 14. *Let X be a Polish space.*

1. *There is a coanalytic filter $\mathcal{F}$ such that for each triple of pairwise disjoint Borel sets A, B, C there is $(f_n) \in C(X)^\omega$ with $C_{\mathcal{F}}((f_n)) = A$, $c^{\infty}_{\mathcal{F}}((f_n)) = B$ and $c^{-\infty}_{\mathcal{F}}((f_n)) = C$.*
2. *There is an ultrafilter $\mathcal{F}$ such that for each triple of pairwise disjoint Borel sets A, B, C with $A \cup B \cup C = X$ there is $(f_n) \in C(X)^\omega$ such that $C_{\mathcal{F}}((f_n)) = A$, $c^{\infty}_{\mathcal{F}}((f_n)) = B$ and $c^{-\infty}_{\mathcal{F}}((f_n)) = C$.*

Proof. By Louveau [11, Theorem 2] there is a coanalytic filter $\mathcal{H}$ such that each Borel function is the $\mathcal{H}$-limit of a sequence of continuous functions. Then we can follow the proof of the previous proposition, since the functions j_1, j_2 are Borel. If we take $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{H}$, then $\mathcal{F}$ will be coanalytic.

To get the second part it is enough to extend $\mathcal{F}$ to an ultrafilter. □

There is no filter such that the set of triples in 1. or 2. of the previous Proposition is exactly the set of all triples of pairwise disjoint Borel sets. Indeed, observe that if $\mathcal{F}$ is Borel then all such sets are of limited class, by Theorem 2. If $\mathcal{F}$ is not Borel, then $C_{\mathcal{F}}((f_n)) = \mathcal{F}$ is not Borel either, where $f_n : P(\omega) \to \mathbb{R}$ is the canonical example mentioned in the beginning of the paper. The following result shows that there are ultrafilters (at least under some additional set theoretic assumptions), for which we cannot get all triples of Borel sets. Recall that it is consistent that there are ultrafilters which do not contain an isomorphic copy of $(\mathrm{FIN} \times \mathrm{FIN})^*$. (All P-points have this property, see [12]).

Proposition 15. *Assume that X is a Polish space and $\mathcal{F}$ does not contain an isomorphic copy of $(\mathrm{FIN} \times \mathrm{FIN})^*$. Then for any sequence $(f_n) \in C(X)^\omega$ and sets $c^{\infty}_{\mathcal{F}}((f_n)) = B$ and $c^{-\infty}_{\mathcal{F}}((f_n)) = C$, if B, C are Borel and $B \cup C = X$ then $B, C \in \mathbf{\Pi}^0_2(X)$.*

Proof. Let $(f_n) \in C(X)^\omega$ be a sequence with $c^{\infty}_{\mathcal{F}}((f_n)) = B$ and $c^{-\infty}_{\mathcal{F}}((f_n)) = C$. Assume that B, C are Borel and $B \cup C = X$. Fix an increasing bijection $h : \mathbb{R} \to (0, 1)$. Then the sequence $h \circ f_n$ $\mathcal{F}$-converges to the characteristic function $\chi_B : X \to [0, 1]$. Since χ_B is Borel, it follows from Proposition 11 that it is of Baire class 1, and thus $B, C \in \mathbf{\Pi}^0_2(X)$. □

Problems

1. Is Theorem 5 true for the plane, or in general for all Polish spaces?
2. Can Theorem 8 be reversed for Polish spaces or at least for zero-dimensional spaces?

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