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Cauchy–Schwarz inequality in semi-inner product C^*-modules via polar decomposition

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ABSTRACT

By virtue of the operator geometric mean and the polar decomposition, we present a new Cauchy–Schwarz inequality in the framework of semi-inner product C^*-modules over unital C^*–algebras and discuss the equality case. We also give several additive and multiplicative type reverses of it. As an application, we present a Kantorovich type inequality on a Hilbert C^*-module.

1. Introduction

The Cauchy–Schwarz inequality $|\langle x|y\rangle| \leq \langle x|x\rangle^{\frac{1}{2}} \langle y|y\rangle^{\frac{1}{2}}$ in a semi-inner product space $(\mathscr{H}, \langle\cdot|\cdot\rangle)$ over the complex number field plays an important role in functional analysis. There are several generalizations and refinements of this classical inequality in various settings for different objects such as integrals and isotone functionals, see the monograph [1] and references therein. The notion of Hilbert C^*-module is a generalization of that of Hilbert space in which the inner product takes its values in a C^*-algebra instead of the complex numbers. A version of the Cauchy–Schwarz inequality in semi-inner product C^*-modules first appeared in [2] by utilizing the operator norm, afterwards in [3] using the assumption of commutativity, and in [4] using the assumption of invertibility.

On the other hand, spreading out an idea of Kantorovich inequality, Dragomir [5] proposed several additive and multiplicative type reverses of the Cauchy–Schwarz inequality in a pre-inner product space. Afterwards some reverse Cauchy–Schwarz type inequalities in other settings have been investigated: An application of the covariance-variance inequality to the Cauchy–Schwarz inequality was obtained by Fujii–Izumino–Nakamoto–Seo [6]. A refinement of the Cauchy–Schwarz inequality involving connections is investigated by Wada [7]. Niculescu [8], Joiţa [9], Moslehian–Persson [10] and Arambasić–Bakić–Moslehian [11] have investigated the Cauchy–Schwarz inequality and its various reverses in the framework of C^*-algebras and Hilbert C^*-modules. Some operator versions of the Cauchy–Schwarz inequality with simple conditions for the case of equality are presented by Fujii [12]. The authors of [4] gave some reverse Cauchy–Schwarz

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inequalities and presented some Klamkin–Mclenaghan, Shisha–Mond, Cassels and Grüss type inequalities in Hilbert C^*-modules. Other related results may be found in [13,14].

In this paper, by virtue of the operator geometric mean and the polar decomposition, we present a new Cauchy–Schwarz inequality in the framework of semi-inner product C^*-modules over unital C^*-algebras and discuss the equality case. We also give several additive and multiplicative type reverses of it, see also [10]. As an application, we present a Kantorovich type inequality on a Hilbert C^*-module.

2. Preliminaries

Let us fix our notation and terminology. Let $\mathscr{A}$ be a unital C^*-algebra with the unit element e and the center $\mathcal{Z}(\mathscr{A})$. For $a \in \mathscr{A}$, we denote the real part of a by $\mathrm{Re}\, a = \frac{1}{2}(a + a^*)$. An element $a \in \mathscr{A}$ is called positive if it is selfadjoint and its spectrum is contained in $[0, \infty)$. For a positive element $a \in \mathscr{A}$, $a^{\frac{1}{2}}$ denotes the unique positive element $b \in \mathscr{A}$ such that $b^2 = a$. For $a \in \mathscr{A}$, we denote the absolute value of a by $|a| = (a^*a)^{\frac{1}{2}}$. If $a \in \mathcal{Z}(\mathscr{A})$ is positive, then $a^{\frac{1}{2}} \in \mathcal{Z}(\mathscr{A})$. If $a, b \in \mathcal{Z}(\mathscr{A})$ are positive, then ab is positive and $(ab)^{\frac{1}{2}} = a^{\frac{1}{2}} b^{\frac{1}{2}}$.

A complex linear space $\mathscr{X}$ is said to be an inner product $\mathscr{A}$-module (or a pre-Hilbert $\mathscr{A}$-module) if $\mathscr{X}$ is a right $\mathscr{A}$-module together with a C^*-valued map $(x, y) \mapsto \langle x, y\rangle : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ such that

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ $(x, y, x \in \mathscr{X}, \alpha, \beta \in \mathbb{C})$,
(ii) $\langle x, ya\rangle = \langle x, y\rangle a$ $(x, y \in \mathscr{X}, a \in \mathscr{A})$,
(iii) $\langle y, x\rangle = \langle x, y\rangle^*$ $(x, y \in \mathscr{X})$,
(iv) $\langle x, x\rangle \geq 0$ $(x \in \mathscr{X})$ and if $\langle x, x\rangle = 0$, then $x = 0$.

We always assume that the linear structures of $\mathscr{A}$ and $\mathscr{X}$ are compatible. Notice that (ii) and (iii) imply $\langle xa, y\rangle = a^*\langle x, y\rangle$ for all $x, y \in \mathscr{X}$, $a \in \mathscr{A}$. If $\mathscr{X}$ satisfies all conditions for an inner-product $\mathscr{A}$-module except for the second part of (iv), then we call $\mathscr{X}$ a semi-inner product $\mathscr{A}$-module.

In this case, we write $\| \; x \; \| := \sqrt{\| \langle x, x\rangle \|}$, where the latter norm denotes the C^*-norm of $\mathscr{A}$. If an inner-product $\mathscr{A}$-module $\mathscr{X}$ is complete with respect to its norm, then $\mathscr{X}$ is called a *Hilbert $\mathscr{A}$-module*. Three typical examples of Hilbert C^*-modules are as follows:

- Every Hilbert space is a Hilbert $\mathbb{C}$-module via $\langle x, y\rangle = \langle y|x\rangle$.
- Let $\mathscr{A}$ be a C^*-algebra. Then $\mathscr{A}$ is a Hilbert C^*-module over itself via $\langle x, y\rangle = x^*y$ $(x, y \in \mathscr{A})$.
- Let $\ell_2(\mathscr{A}) = \{(a_i)_{i\in\mathbb{N}} : \sum_{i=1}^{\infty} a_i^* a_i$ norm-converges in $\mathscr{A}, \; a_i \in \mathscr{A}, i = 1, 2, \ldots\}$. Then $\ell_2(\mathscr{A})$ is a Hilbert $\mathscr{A}$-module under the natural operations $\lambda(a_i) + \mu(b_i) = (\lambda a_i + \mu b_i)$, $(a_i)a = (a_i a)$ and $\langle (a_i), (b_i)\rangle = \sum_{i=1}^{\infty} a_i^* b_i$.

The theory of Hilbert C^*-modules is, however, not trivial, since some fundamental properties of Hilbert spaces such as adjointability of any bounded linear operator, the Pythagoras equality, triangle inequality for C^*-valued norm $|x| = \langle x, x\rangle^{1/2}$ $(x \in \mathscr{X})$ and decomposition into orthogonal complements are not true in the context of Hilbert C^*-modules in general. For more details on Hilbert C^*-modules, see [2].

3. Cauchy–Schwarz inequality and its reverses

Let a and b be positive elements of a C^*-algebra $\mathscr{A}$. Then the operator geometric mean $a\sharp b$ is defined by

$$a\sharp b = a^{\frac{1}{2}}(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})^{\frac{1}{2}} a^{\frac{1}{2}}$$

if a is invertible, see [15–17]. The operator geometric mean has the symmetric property: $a\sharp b = b\sharp a$, see [16]. If a commutes with b, then $a \sharp b = a^{\frac{1}{2}} b^{\frac{1}{2}}$. From this viewpoint, we would expect that the following Cauchy–Schwarz inequality in a semi-inner product C^*-module holds:

$$|\langle x, y\rangle| \leq \langle x, x\rangle \sharp \langle y, y\rangle \quad (x, y \in \mathscr{X}).$$

Unfortunately we have a counterexample. If $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ in the Hilbert $M_2(\mathbb{C})$-module $M_2(\mathbb{C})$, then we have $|\langle x, y\rangle| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\langle x, x\rangle \sharp \langle y, y\rangle = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, we have $|\langle x, y\rangle| \not\leq \langle x, x\rangle \sharp \langle y, y\rangle$.

By virtue of the polar decomposition, we have the following Cauchy–Schwarz inequality in a semi-inner product C^*-module.

Theorem 3.1. *Let $\mathscr{X}$ be a semi-inner product $\mathscr{A}$-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x, y \in \mathscr{X}$ such that $\langle x, y\rangle = u|\langle x, y\rangle|$ is a polar decomposition in $\mathscr{A}$, i.e., $u \in \mathscr{A}$ is a partial isometry. Then*

$$|\langle x, y\rangle| \leq u^*\langle x, x\rangle u \sharp \langle y, y\rangle. \tag{3.1}$$

Under the assumption that $\mathscr{X}$ is an inner product $\mathscr{A}$-module and $\langle y, y\rangle$ is invertible, the equality in (3.1) holds if and only if $xu = yb$ for some $b \in \mathscr{A}$.

Proof. We may assume that $\langle y, y\rangle$ is invertible. Then

$$\begin{aligned} 0 &\le \langle xu - y\langle y, y\rangle^{-1}\langle y, x\rangle u, xu - y\langle y, y\rangle^{-1}\langle y, x\rangle u\rangle \\ &= u^*\langle x, x\rangle u - u^*\langle x, y\rangle\langle y, y\rangle^{-1}\langle y, x\rangle u \\ &= u^*\langle x, x\rangle u - |\langle x, y\rangle|\langle y, y\rangle^{-1}|\langle x, y\rangle| \end{aligned} \tag{3.2}$$

and we have $|\langle x, y\rangle|\langle y, y\rangle^{-1}|\langle x, y\rangle| \le u^*\langle x, x\rangle u$. Therefore, it follows that

$$|\langle x, y\rangle| = \langle y, y\rangle\sharp|\langle x, y\rangle|\langle y, y\rangle^{-1}|\langle x, y\rangle| \le \langle y, y\rangle\sharp u^*\langle x, x\rangle u$$

and hence we obtain the desired inequality (3.1) by the symmetric property of the operator geometric mean.

Under the assumption that $\mathscr{X}$ is an inner-product $\mathscr{A}$-module and $\langle y, y\rangle$ is invertible, suppose that the equality in (3.1) holds. Then it follows that $u^*\langle x, x\rangle u - |\langle x, y\rangle|\langle y, y\rangle^{-1}|\langle x, y\rangle| = 0$. Since $\mathscr{X}$ is an inner-product $\mathscr{A}$-module, it follows from (3.2) that $xu = y\langle y, y\rangle^{-1}\langle y, x\rangle u$.

Conversely, suppose that $xu = yb$ for some $b \in \mathscr{A}$. Then

$$u^*\langle x, y\rangle\langle y, y\rangle^{-1}\langle y, x\rangle u = b^*\langle y, y\rangle b = u^*\langle x, x\rangle u,$$

whence

$$\langle y, y\rangle\sharp u^*\langle x, x\rangle u = \langle y, y\rangle\sharp|\langle x, y\rangle|\langle y, y\rangle^{-1}|\langle x, y\rangle| = |\langle x, y\rangle|. \quad \square$$

Next, we show several additive and multiplicative reverses of (3.1) by using an idea of [10,4]. For this, we need the following lemma.

Lemma 3.2. *Let $\mathscr{X}$ be a semi-inner product $\mathscr{A}$-module over a unital C^*-algebra $\mathscr{A}$ and $x, y \in \mathscr{X}$. Suppose that there exists a partial isometry $u \in \mathscr{A}$ such that $\langle x, y\rangle = u|\langle x, y\rangle|$ as a polar decomposition of is a polar $\langle x, y\rangle$. Then the following inequalities are mutually equivalent for some $a, b \in \mathcal{Z}(\mathscr{A})$:*

(i) $\mathrm{Re}\langle yb - xu, xu - ya\rangle \ge 0$;
(ii) $u^*\langle x, x\rangle u + \mathrm{Re}(a^*b)\langle y, y\rangle \le \mathrm{Re}(a + b)|\langle x, y\rangle|$;
(iii) $\langle xu - y\frac{a+b}{2}, xu - y\frac{a+b}{2}\rangle \le \frac{|a-b|^2}{4}\langle y, y\rangle$.

Proof. Since $|\langle x, y\rangle| = u^*\langle x, y\rangle = \langle y, x\rangle u$ and $a, b \in \mathcal{Z}(\mathscr{A})$, it follows that $\mathrm{Re}\langle yb - xu, xu - ya\rangle = \mathrm{Re}(a + b)|\langle x, y\rangle| - u^*\langle x, x\rangle u - \mathrm{Re}(a^*b)\langle y, y\rangle = \frac{|a-b|^2}{4}\langle y, y\rangle - \langle xu - y\frac{a+b}{2}, xu - y\frac{a+b}{2}\rangle$. $\square$

Theorem 3.3. *Let $\mathscr{X}$ be a semi-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$ and $x, y \in \mathscr{X}$. Suppose that $\langle x, y\rangle = u|\langle x, y\rangle|$ is a polar decomposition with a partial isometry $u \in \mathscr{A}$, and that there exist $a, b \in \mathcal{Z}(\mathscr{A})$ such that* (i)*of* Lemma 3.2 *is valid for $x, y \in \mathscr{X}$ and $u, a, b \in \mathscr{A}$.*

If $\mathrm{Re}(a^*b)$ is positive invertible, then

(i) $u^*\langle x, x\rangle u\sharp\langle y, y\rangle \le \frac{1}{2}\mathrm{Re}(a + b)\ (\mathrm{Re}(a^*b))^{-1/2}|\langle x, y\rangle|$.

If $\mathrm{Re}(a^*b)$ is positive invertible and $\mathrm{Re}(a + b)$ is invertible, then

(ii) $u^*\langle x, x\rangle u\sharp\langle y, y\rangle - |\langle x, y\rangle| \le \frac{1}{4}((\mathrm{Re}(a + b))^2 - 4\mathrm{Re}(a^*b))(\mathrm{Re}(a + b))^{-1}\langle y, y\rangle$;
(iii) $u^*\langle x, x\rangle u\sharp\langle y, y\rangle - |\langle x, y\rangle| \le \frac{1}{4}((\mathrm{Re}(a + b))^2 - 4\mathrm{Re}(a^*b))\ (\mathrm{Re}(a^*b)\mathrm{Re}(a + b))^{-1}u^*\langle x, x\rangle u$.

Proof. (i): Since the arithmetic-geometric mean inequality $a\sharp b \le \frac{a+b}{2}$ holds for all positive elements $a, b \in \mathscr{A}$ in [18, Theorem 1.27], It follows from (ii) of Lemma 3.2 that

$$\begin{aligned} (\mathrm{Re}(a^*b))^{1/2}\, u^*\langle x, x\rangle u\sharp\langle y, y\rangle &= u^*\langle x, x\rangle u\sharp\mathrm{Re}(a^*b)\langle y, y\rangle \le \frac{u^*\langle x, x\rangle u + \mathrm{Re}(a^*b)\langle y, y\rangle}{2} \\ &\le \frac{\mathrm{Re}(a + b)}{2}|\langle x, y\rangle| \end{aligned}$$

and so we have the desired inequality (i).

(ii): We may assume that $\langle y, y\rangle$ is invertible. Put $X = \langle y, y\rangle^{-\frac{1}{2}}u^*\langle x, x\rangle u\langle y, y\rangle^{-\frac{1}{2}}$. Then (ii) of Lemma 3.2 implies that

$$\begin{aligned} \langle y, y\rangle\sharp u^*\langle x, x\rangle u - |\langle x, y\rangle| &\le \langle y, y\rangle^{\frac{1}{2}}X^{\frac{1}{2}}\langle y, y\rangle^{\frac{1}{2}} - (\mathrm{Re}(a + b))^{-1}u^*\langle x, x\rangle u - \mathrm{Re}(a^*b)\ (\mathrm{Re}(a + b))^{-1}\langle y, y\rangle \\ &= \langle y, y\rangle^{\frac{1}{2}}\left(X^{\frac{1}{2}} - (\mathrm{Re}(a + b))^{-1}X\right)\langle y, y\rangle^{\frac{1}{2}} - \mathrm{Re}(a^*b)\ (\mathrm{Re}(a + b))^{-1}\langle y, y\rangle \\ &= -\langle y, y\rangle^{\frac{1}{2}}(\mathrm{Re}(a + b))^{-1}\left(\left(X^{\frac{1}{2}} - \frac{\mathrm{Re}(a + b)}{2}\right)^2 - \frac{(\mathrm{Re}(a + b))^2}{4}\right)\langle y, y\rangle^{\frac{1}{2}} \\ &\quad - \mathrm{Re}(a^*b)\ (\mathrm{Re}(a + b))^{-1}\langle y, y\rangle \\ &\le \frac{1}{4}\left((\mathrm{Re}(a + b))^2 - 4\mathrm{Re}(a^*b)\right)\ (\mathrm{Re}(a + b))^{-1}\langle y, y\rangle. \end{aligned}$$

(iii): We may assume that $u^*\langle x,x\rangle u$ is invertible. Put $Y = (u^*\langle x,x\rangle u)^{-\frac{1}{2}}\langle y,y\rangle(u^*\langle x,x\rangle u)^{-\frac{1}{2}}$. Then it follows from (ii) of Lemma 3.2 that

$$\begin{aligned} u^*\langle x,x\rangle u\sharp\langle y,y\rangle - |\langle x,y\rangle| &\le (u^*\langle x,x\rangle u)^{\frac{1}{2}} Y^{\frac{1}{2}} (u^*\langle x,x\rangle u)^{\frac{1}{2}} - (\mathrm{Re}(a+b))^{-1} u^*\langle x,x\rangle u \\ &\quad - \mathrm{Re}(a^*b)\ (\mathrm{Re}(a+b))^{-1}\langle y,y\rangle \\ &= -(u^*\langle x,x\rangle u)^{\frac{1}{2}}\mathrm{Re}(a^*b)\ (\mathrm{Re}(a+b))^{-1}\left(\left(Y^{\frac{1}{2}} - \frac{1}{2}\mathrm{Re}(a+b)\ (\mathrm{Re}(a^*b))^{-1}\right)^2\right. \\ &\quad \left. - \frac{1}{4}\mathrm{Re}(a+b)^2(\mathrm{Re}(a^*b))^{-2}\right)(u^*\langle x,x\rangle u)^{\frac{1}{2}} - (\mathrm{Re}(a+b))^{-1}u^*\langle x,x\rangle u \\ &\le \frac{1}{4}((\mathrm{Re}(a+b))^2 - 4\mathrm{Re}(a^*b))\ (\mathrm{Re}(a^*b)\mathrm{Re}(a+b))^{-1}u^*\langle x,x\rangle u. \quad \Box \end{aligned}$$

Remark 3.4. We point out that there are some cases where the equalities hold for (i)–(iii) of Theorem 3.3. As a matter of fact, suppose that there exist $x, z \in \mathscr{X}$ such that $\langle x,x\rangle = \langle z,z\rangle$ and $\langle x,z\rangle = 0$. For example, let $\mathscr{A}$ be the unital C^*-algebra $M_2(\mathbb{C})$ and let $\mathscr{X} = \mathscr{A}$ as a Hilbert C^*-module under $\langle x,y\rangle = x^*y$ for $x, y \in \mathscr{X}$. Then there exist $x, z \in \mathscr{X}$ such that $\langle x,x\rangle = \langle z,z\rangle$ and $\langle x,z\rangle = 0$. For each positive invertible $a, b \in \mathcal{Z}(\mathscr{A})$ with $a+b = e$, we put $y = 2x + z\cdot(b-a)\ (ab)^{-1/2}$. Then it follows that $\langle x,y\rangle = 2\langle x,x\rangle = |\langle x,y\rangle|$ and moreover $\langle yb - xu, xu - ya\rangle = 0$. Since $\langle y,y\rangle = \langle x,x\rangle(ab)^{-1}$, we have

$$u^*\langle x,x\rangle u\sharp\langle y,y\rangle = \langle x,x\rangle\sharp(\langle x,x\rangle(ab)^{-1}) = \langle x,x\rangle(ab)^{-1/2} = \frac{1}{2}(ab)^{-1/2}|\langle x,y\rangle|$$

and hence the equality in (i) holds.

To show that the equality in (ii) holds, suppose that there exist $y, z \in \mathscr{X}$ such that $\langle y,y\rangle = \langle z,z\rangle$ and $\langle y,z\rangle = 0$. For each positive invertible $a, b \in \mathcal{Z}(\mathscr{A})$ with $a+b = e$, we put

$$x = y\cdot\frac{1+4ab}{4} + z\cdot\frac{(b-a)[(3a+b)\ (a+3b)]^{1/2}}{4}.$$

Then it follows that $\langle x,y\rangle = \langle y,y\rangle\cdot\frac{1+4ab}{4} = |\langle x,y\rangle|$ and $\langle yb - xu, xu - ya\rangle = 0$. Since $\langle x,x\rangle = \frac{1}{4}\langle y,y\rangle$, we have

$$u^*\langle x,x\rangle u\sharp\langle y,y\rangle - |\langle x,y\rangle| = \frac{1}{4}\langle y,y\rangle\sharp\langle y,y\rangle - \langle y,y\rangle\cdot\frac{1+4ab}{4} = \frac{1-4ab}{4}\langle y,y\rangle$$

and hence the equality in (ii) holds.

If we put $y = x\cdot(e+4ab)\ (4ab)^{-1} + z\cdot(b-a)[(3a+b)\ (a+3b)]^{1/2}(4ab)^{-1}$ for $x, z \in \mathscr{X}$ such that $\langle x,x\rangle = \langle z,z\rangle$ and $\langle x,z\rangle = 0$, and each positive invertible $a, b \in \mathcal{Z}(\mathscr{A})$ with $a+b = e$, then similarly it follows that the equality in (iii) holds.

4. Applications to the Kantorovich inequality

Throughout the final section, let $\mathscr{X}$ be a Hilbert $\mathscr{A}$-module. We define $\mathcal{L}(\mathscr{X})$ to be the set of all maps $T : \mathscr{X} \mapsto \mathscr{X}$ for which there is a map $T^* : \mathscr{X} \mapsto \mathscr{X}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for $x, y \in \mathscr{X}$. Then $\mathcal{L}(\mathscr{X})$ is a C^*-algebra. For $T \in \mathcal{L}(\mathscr{X})$, using the closed graph theorem, it is easy to see that T is $\mathscr{A}$-linear and bounded. In addition, T is positive if and only if $\langle x, Tx\rangle \ge 0$ for all $x \in \mathscr{X}$.

First of all, by virtue of Theorem 3.1, we obtain the following generalized Cauchy–Schwarz inequality on a Hilbert C^*-module.

Theorem 4.1. *Let T be positive in $\mathcal{L}(\mathscr{X})$. Suppose that $x, y \in \mathscr{X}$ such that $\langle x, Ty\rangle = u|\langle x, Ty\rangle|$ is a polar decomposition with a partial isometry $u \in \mathscr{A}$. Then*

$$|\langle x, Ty\rangle| \le u^*\langle x, Tx\rangle u\sharp\langle y, Ty\rangle. \tag{4.1}$$

Proof. By Theorem 3.1, we have

$$\begin{aligned} |\langle x, Ty\rangle| &= |\langle T^{1/2}x, T^{1/2}y\rangle| \le u^*\langle T^{1/2}x, T^{1/2}x\rangle u\sharp\langle T^{1/2}y, T^{1/2}y\rangle \\ &= u^*\langle x, Tx\rangle u\sharp\langle y, Ty\rangle. \quad \Box \end{aligned}$$

By virtue of Theorem 3.3, we similarly have the following multiplicative type reverse of (4.1) in Theorem 4.1.

Theorem 4.2. *Let T be positive in $\mathcal{L}(\mathscr{X})$. Suppose that $x, y \in \mathscr{X}$ such that $\langle x, Ty\rangle = u|\langle x, Ty\rangle|$ is a polar decomposition with a partial isometry $u \in \mathscr{A}$, and there exist $a, b \in \mathcal{Z}(\mathscr{A})$ such that $\mathrm{Re}(a^*b) > 0$ and $\mathrm{Re}\langle yb - xu, T(xu - ya)\rangle \geq 0$. Then*

$$u^*\langle x, Tx\rangle u \sharp \langle y, Ty\rangle \leq \frac{1}{2}\mathrm{Re}(a+b)(\mathrm{Re}(a^*b))^{-1/2}|\langle x, Ty\rangle|.$$

In 1948, Leonid Vital'evich Kantorovich [19] introduced the following inequality

$$\langle Hx, x\rangle\langle H^{-1}x, x\rangle \leq (\lambda_1 + \lambda_n)^2/4\lambda_1\lambda_n$$

where $x = (x_1, \ldots, x_n)$ is a unit vector in $\mathbf{C}^n$ and H is an $n \times n$ positive-definite matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n > 0$. There are many refinements and extensions of this inequality in the literature, see [20,21] and references therein.

We have the following result by Theorem 3.1.

Theorem 4.3. *Let T be positive invertible in $\mathcal{L}(\mathscr{X})$. Then*

$$\langle x, x\rangle \leq \langle x, Tx\rangle \sharp \langle x, T^{-1}x\rangle$$

for all $x \in \mathscr{X}$.

Proof. For each $x \in \mathscr{X}$, if we replace x and y with $T^{-\frac{1}{2}}x$ and $T^{\frac{1}{2}}x$ in Theorem 3.1 respectively, then we have

$$\langle x, x\rangle = \langle T^{\frac{1}{2}}x, T^{-\frac{1}{2}}x\rangle \leq \langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x\rangle \sharp \langle T^{-\frac{1}{2}}x, T^{-\frac{1}{2}}x\rangle = \langle x, Tx\rangle \sharp \langle x, T^{-1}x\rangle. \quad \square$$

As an application of Theorem 3.3, we obtain the following Kantorovich inequality on a Hilbert C^*-module:

Theorem 4.4. *Let T be positive invertible in $\mathcal{L}(\mathscr{X})$ such that $aI \leq T \leq bI$ for some positive invertible elements $a, b \in \mathcal{Z}(\mathscr{A})$. Then*

$$\langle x, Tx\rangle \sharp \langle x, T^{-1}x\rangle \leq \frac{1}{2}(a+b)\,(ab)^{-1/2}\langle x, x\rangle \tag{4.2}$$

for all $x \in \mathscr{X}$.

Proof. The assumption $aI \leq T \leq bI$ implies $(T - aI)\,(bI - T)T^{-1} \geq 0$ and hence $\langle T^{-\frac{1}{2}}xb - T^{\frac{1}{2}}x, T^{\frac{1}{2}}x - T^{-\frac{1}{2}}xa\rangle \geq 0$ for all $x \in \mathscr{X}$. It follows from (i) of Theorem 3.3 that

$$\begin{aligned}\langle x, Tx\rangle \sharp \langle x, T^{-1}x\rangle &= \langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x\rangle \sharp \langle T^{-\frac{1}{2}}x, T^{-\frac{1}{2}}x\rangle \\ &\leq \frac{1}{2}(a+b)\,(ab)^{-1/2}\,|\langle T^{\frac{1}{2}}x, T^{-\frac{1}{2}}x\rangle| = \frac{1}{2}(a+b)\,(ab)^{-1/2}\langle x, x\rangle. \quad \square\end{aligned}$$

Remark 4.5. We point out that there are some cases where the equality holds in (4.2) of Theorem 4.4. As a matter of fact, let $\mathscr{X} = \mathscr{A} = M_2(\mathbb{C})$ be a Hilbert C^*-module over itself via $\langle x, y\rangle = x^*y\ (x, y \in M_2(\mathbb{C}))$. Since $\mathcal{Z}(\mathscr{A}) = \mathbb{C}I$, let $0 < a < b$ be positive scalars. Put $T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then it follows that T is positive invertible such as $aI \leq T \leq bI$, and we have $Ty = ya$ and $Tz = zb$, and $\langle y, y\rangle = \langle z, z\rangle$ and $\langle y, z\rangle = 0$. Put $x = \frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z$. Then we have $\langle x, Tx\rangle = \frac{a+b}{2}\langle y, y\rangle$, $\langle x, T^{-1}x\rangle = \frac{a^{-1}+b^{-1}}{2}\langle y, y\rangle$ and $\langle x, x\rangle = \langle y, y\rangle$. Hence it follows that

$$\langle x, Tx\rangle \sharp \langle x, T^{-1}x\rangle = (a+b)\,(4ab)^{-\frac{1}{2}}\langle y, y\rangle = (a+b)\,(4ab)^{-\frac{1}{2}}\langle x, x\rangle.$$

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