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Hardy spaces related to Schrödinger operators with potentials which are sums of L^p-functions[☆]

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ABSTRACT

We investigate the Hardy space H^1_L associated with the Schrödinger operator $L = -\Delta + V$ on $\mathbb{R}^n$, where $V = \sum_{j=1}^d V_j$. We assume that each V_j depends on variables from a linear subspace $\mathbb{V}_j$ of $\mathbb{R}^n$, $\dim \mathbb{V}_j \geq 3$, and V_j belongs to $L^q(\mathbb{V}_j)$ for certain q. We prove that there exist two distinct isomorphisms of H^1_L with the classical Hardy space. We deduce as a corollary a specific atomic characterization of H^1_L. We also prove that the space H^1_L can be described by means of the Riesz transforms $\mathcal{R}_{L,i} = \partial_i L^{-1/2}$.

1. Introduction and the main results

In the paper we consider a Schrödinger operator on $\mathbb{R}^n$ given by

$$Lf(x) = -\Delta f(x) + V(x)f(x),$$

where Δ denotes the Laplace operator. Throughout the whole paper we assume that the potential V satisfies:

(A_1) there exist $V_j \geq 0$, $V_j \not\equiv 0$ such that

$$V(x) = \sum_{j=1}^d V_j(x),$$

(A_2) for every $j \in \{1, \dots, d\}$ there exists a linear subspace $\mathbb{V}_j$ of $\mathbb{R}^n$ of dimension $n_j \geq 3$ such that if $\Pi_{\mathbb{V}_j}$ denotes the orthogonal projection on $\mathbb{V}_j$ then

$$V_j(x) = V_j(\Pi_{\mathbb{V}_j} x),$$

(A_3) there exists $\kappa > 0$ such that for $j = 1, \dots, d$ we have

$$V_j \in L^r(\mathbb{V}_j)$$

for all r satisfying $|r - n_j/2| \leq \kappa$.

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Denote by $K_t = \exp(-tL)$ and $P_t = \exp(t\Delta)$ the semigroups of linear operators associated with L and Δ respectively. Let $K_t(x, y)$ and $P_t(x-y)$ denote the integral kernels of these semigroups. The Feynman–Kac formula implies that

$$0 \le K_t(x, y) \le P_t(x-y) = (4\pi t)^{-n/2} \exp\left(-|x-y|^2/4t\right). \tag{1.1}$$

Let M_L and M_Δ be the associated maximal operators, i.e.,

$$M_L f(x) = \sup_{t>0} |K_t f(x)| \qquad M_\Delta f(x) = \sup_{t>0} |P_t f(x)|.$$

The Hardy spaces $H^1_L(\mathbb{R}^n)$ and $H^1_\Delta(\mathbb{R}^n)$ are the subspaces of $L^1(\mathbb{R}^n)$ defined by

$$f \in H^1_L(\mathbb{R}^n) \iff M_L f \in L^1(\mathbb{R}^n), \qquad f \in H^1_\Delta(\mathbb{R}^n) \iff M_\Delta f \in L^1(\mathbb{R}^n)$$

with the norms

$$\|f\|_{H^1_L(\mathbb{R}^n)} = \|M_L f\|_{L^1(\mathbb{R}^n)}, \qquad \|f\|_{H^1_\Delta(\mathbb{R}^n)} = \|M_\Delta f\|_{L^1(\mathbb{R}^n)}.$$

Clearly the space $H^1_\Delta(\mathbb{R}^n)$ is the classical Hardy space $H^1(\mathbb{R}^n)$ (see [1]). The goal of the paper is to prove some characterizations of the space $H^1_L(\mathbb{R}^n)$.

Denote by L^{-1} and $(-\Delta)^{-1}$ the operators with the kernels $\Gamma(x, y) = \int_0^\infty K_t(x, y)\, dt$ and $\Gamma_0(x-y) = \int_0^\infty P_t(x-y)\, dt$. Clearly,

$$0 \le \int_0^t K_s(z, y)\, ds \le \Gamma(z, y) \le \Gamma_0(z-y) = C|z-y|^{2-n}. \tag{1.2}$$

We shall see that operators $I - VL^{-1}$ and $I - V\Delta^{-1}$ are bounded on $L^1(\mathbb{R}^n)$ and give the following characterization of the Hardy space $H^1_L(\mathbb{R}^n)$.

Theorem 1.3. *Assume $f \in L^1(\mathbb{R}^n)$. Then f belongs to $H^1_L(\mathbb{R}^n)$ if and only if $(I - VL^{-1})f$ belongs to the classical Hardy space $H^1_\Delta(\mathbb{R}^n)$. Moreover,*

$$\|f\|_{H^1_L(\mathbb{R}^n)} \sim \|(I - VL^{-1})f\|_{H^1_\Delta(\mathbb{R}^n)}.$$

We define the weight function ω by

$$\omega(x) = \lim_{t\to\infty} \int_{\mathbb{R}^n} K_t(x, y)\, dy. \tag{1.4}$$

The above limit exists because, by (1.1) and the semigroup property, the function $t \mapsto K_t\mathbf{1}(x)$ is non-increasing and takes values in $[0, 1]$. Clearly, the function ω is L-harmonic, because by (1.4) for every $t > 0$,

$$\omega(x) = K_t\omega(x) = \int_{\mathbb{R}^n} K_t(x, y)\omega(y)\, dy. \tag{1.5}$$

We shall prove that there exists $\delta > 0$ such that $\delta \le \omega(x) \le 1$ (see Proposition 2.14). Moreover, ω is the unique (up to a multiplicative constant) bounded L-harmonic function. To see this we can briefly argue as follows. Let w be any bounded L-harmonic function. It follows from Corollary 2.7 that $w = (I - L^{-1}V)(I - \Delta^{-1}V)w$ and $h = (I - \Delta^{-1}V)w$ is bounded and Δ-harmonic. Thus, $h(x) = c_0\mathbf{1}(x)$ and, consequently, $w(x) = c_0(I - L^{-1}V)\mathbf{1}(x)$. So we see that the space of bounded L-harmonic functions is one-dimensional.

We are now in a position to state our second main result.

Theorem 1.6. *Let $f \in L^1(\mathbb{R}^n)$. Then f belongs to $H^1_L(\mathbb{R}^n)$ if and only if ωf belongs to $H^1_\Delta(\mathbb{R}^n)$. Additionally,*

$$\|f\|_{H^1_L(\mathbb{R}^n)} \sim \|\omega f\|_{H^1_\Delta(\mathbb{R}^n)}.$$

From Theorem 1.6 we get atomic characterizations of the elements of $H^1_L(\mathbb{R}^n)$. We call a function a an ω-atom if it satisfies:

- there exists a ball $B = B(y, r)$ such that $\operatorname{supp} a \subseteq B$,
- $\|a\|_\infty \le |B|^{-1}$,
- $\int_{\mathbb{R}^n} a(x)\omega(x)\, dx = 0$.

Corollary 1.7. *If a function f belongs to $H^1_L(\mathbb{R}^n)$ then there exist a sequence a_k of ω-atoms and a sequence $\lambda_k \in \mathbb{C}$ such that $\sum_{k=1}^\infty |\lambda_k| < \infty$, $f = \sum_{k=1}^\infty \lambda_k a_k$, and*

$$\|f\|_{H^1_L(\mathbb{R}^n)} \sim \sum_{k=1}^\infty |\lambda_k|.$$

For $i = 1, \dots, n$ denote by ∂_i the derivative in the direction of the ith canonical coordinate of $\mathbb{R}^n$. For $f \in L^1(\mathbb{R}^n)$ the classical Riesz transforms $\mathcal{R}_{\Delta,i}$ are given by

$$\mathcal{R}_{\Delta,i}f = \lim_{\varepsilon\to 0}\int_\varepsilon^{\varepsilon^{-1}} \partial_i P_t f \frac{dt}{\sqrt{t}}.$$

Similarly we define the Riesz transforms $\mathcal{R}_{L,i}$ associated with L by setting

$$\mathcal{R}_{L,i}f = \lim_{\varepsilon\to 0}\int_\varepsilon^{\varepsilon^{-1}} \partial_i K_t f \frac{dt}{\sqrt{t}}.$$

We shall see that the last limits are well-defined in the sense of distributions and they characterize $H^1_L(\mathbb{R}^n)$ in the following sense.

Theorem 1.8. *An $L^1(\mathbb{R}^n)$-function f belongs to $H^1_L(\mathbb{R}^n)$ if and only if $\mathcal{R}_{L,i}f$ belong to $L^1(\mathbb{R}^n)$ for $i = 1, \dots, n$. Additionally,*

$$\|f\|_{H^1_L(\mathbb{R}^n)} \sim \|f\|_{L^1(\mathbb{R}^n)} + \sum_{i=1}^n \|\mathcal{R}_{L,i}f\|_{L^1(\mathbb{R}^n)}.$$

Hardy spaces associated with semigroups of linear operators and in particular Schrödinger semigroups have attracted the attention of many authors; see, e.g., [2–10] and references therein. The present paper generalizes the results of [11,12], where the spaces $H^1_L(\mathbb{R}^n)$ were studied under certain assumptions: $V \geq 0$, $\operatorname{supp} V$ is compact, $V \in L^r(\mathbb{R}^n)$ for some $r > n/2$. Obviously such potentials V satisfy the conditions (A_1)–(A_3). To prove Theorems 1.3, 1.6 and 1.8 we develop methods of [11,12].

Let us finally mention some differences that occur in atomic decompositions of Hardy spaces for Schrödinger operators for various classes of potentials in $\mathbb{R}^d$, $d \geq 1$. In the case considered here, each atom a satisfies the cancelation condition $\int a\,\omega = 0$. On the other hand, the one-dimensional situation is different and was studied in [4]. The authors considered there any non-negative locally integrable potential V and defined a special family of intervals $\{I_j\}_j$ that cover $\mathbb{R}$. Then, the atoms are either classical atoms for $H^1(\mathbb{R})$ supported in $(1+\delta)I_j$ or $a = |I_j|^{-1}\chi_{I_j}$ for some j. A similar situation arises, e.g., for non-negative polynomial potentials or, more generally, for potentials from some non-negative reverse Hölder classes in higher dimensions. In these cases, the atoms are properly scaled local atoms in the sense of Goldberg [13], which means that some of them do not need to satisfy cancelation conditions; see [5–8] for details and more examples.

In [10] the authors provide a very general theory of Hardy spaces for Schrödinger operators. They proved the special atomic decomposition; however their atoms are of a different nature than those considered in [5–8,11,12]. The atoms in [10] are of the form $a = L^N b$, where b is appropriately localized L-regular function; see Definition 2.1 of [10].

Finally, the reader interested in boundedness of spectral multipliers on Hardy spaces associated with Schrödinger operators is referred to [14,9], and references therein.

2. Auxiliary lemmas

In the paper we shall use the following notation. For $z \in \mathbb{R}^n$ and a subspace $\mathbb{V}_j$ of $\mathbb{R}^n$ we write

$$z = z_j + \widetilde{z}_j, \quad z_j = \Pi_{\mathbb{V}_j}(z),\ \widetilde{z}_j = \Pi_{\mathbb{V}_j^\perp}(z), \qquad \widetilde{n}_j = \dim \mathbb{V}_j^\perp = n - n_j.$$

Notice that if $\mathbb{V}_j = \mathbb{R}^n$, then, in fact, there is no $\mathbb{V}_j^\perp$.

The relation between P_t and K_t is given by the perturbation formula:

$$P_t = K_t + \int_0^t P_{t-s} V K_s\, ds. \tag{2.1}$$

The following two lemmas state crucial estimates that will be used in many proofs of this paper.

Lemma 2.2. *There exists $\lambda > 0$ such that*

$$\sup_{y\in\mathbb{R}^n} \|V(\cdot)|\cdot - y|^{2-n+\mu}\|_{L^r(\mathbb{R}^n)} \leq C \quad \textit{for } r \in [1, 1+\lambda] \textit{ and } \mu \in [-\lambda, \lambda]. \tag{2.3}$$

Proof. It suffices to prove (2.3) for $V = V_1$. For fixed $y \in \mathbb{R}^n$ we have

$$\|V_1(\cdot)|\cdot - y|^{2-n+\mu}\|^r_{L^r(\mathbb{R}^n)} \leq C \int_{\mathbb{V}_1}\int_{\mathbb{V}_1^\perp} \frac{V_1(z_1)^r}{|z_1 - y_1|^{-r(2-n+\mu)} + |\widetilde{z}_1 - \widetilde{y}_1|^{-r(2-n+\mu)}}\, d\widetilde{z}_1\, dz_1. \tag{2.4}$$

Observe that if $\lambda > 0$ is sufficiently small, $r \in [1, 1+\lambda]$, and $\mu \in [-\lambda, \lambda]$ then

$$\begin{aligned}\int_{\mathbb{V}_1^\perp} &\left(|z_1 - y_1|^{-r(2-n+\mu)} + |\widetilde{z}_1 - \widetilde{y}_1|^{-r(2-n+\mu)}\right)^{-1} d\widetilde{z}_1 \\ &\le C \int_{|z_1-y_1|>|\widetilde{z}_1-\widetilde{y}_1|} |z_1 - y_1|^{r(2-n+\mu)}\, d\widetilde{z}_1 + C \int_{|z_1-y_1|\le|\widetilde{z}_1-\widetilde{y}_1|} |\widetilde{z}_1 - \widetilde{y}_1|^{r(2-n+\mu)}\, d\widetilde{z}_1 \\ &\le C|z_1 - y_1|^{r(2-n+\mu)+\widetilde{n}_1}. \end{aligned} \tag{2.5}$$

Thus, by (2.5),

$$\begin{aligned}\|V_1(\cdot)|\cdot - y|^{2-n+\mu}\|^r_{L^r(\mathbb{R}^n)} \le C &\int_{|z_1-y_1|\le 1} V_1(z_1)^r |z_1 - y_1|^{r(2-n+\mu)+\widetilde{n}_1}\, dz_1 \\ &+ C\int_{|z_1-y_1|>1} V_1(z_1)^r |z_1 - y_1|^{r(2-n+\mu)+\widetilde{n}_1}\, dz_1. \end{aligned} \tag{2.6}$$

Note that by (A_3) there exist $t, s > 1$ such that $V_1^r \in L^t(\mathbb{V}_1) \cap L^s(\mathbb{V}_1)$ and

$$\chi_{\{|z_1|\le 1\}}(z_1)|z_1|^{r(2-n+\mu)+\widetilde{n}_1} \in L^{t'}(\mathbb{V}_1), \qquad \chi_{\{|z_1|>1\}}(z_1)|z_1|^{r(2-n+\mu)+\widetilde{n}_1} \in L^{s'}(\mathbb{V}_1)$$

for $r \in [1, 1+\lambda]$ and $\mu \in [-\lambda, \lambda]$ provided $\lambda > 0$ is small enough. Thus (2.3) follows from the Hölder inequality. $\square$

Corollary 2.7. *The operators $I - V\Delta^{-1}$ and $I - VL^{-1}$ are bounded on $L^1(\mathbb{R}^n)$ and*

$$(I - VL^{-1})(I - V\Delta^{-1})f = (I - V\Delta^{-1})(I - VL^{-1})f = f \quad \textit{for } f \in L^1(\mathbb{R}^n). \tag{2.8}$$

Lemma 2.9. *There exist $\sigma, \varepsilon > 0$ such that for $s \in [1, 1+\varepsilon]$ and $R \ge 1$ we have*

$$\sup_{y\in\mathbb{R}^n} \int_{|z-y|>R} V(z)^s |z - y|^{s(2-n)}\, dz \le CR^{-\sigma}. \tag{2.10}$$

Proof. It is enough to prove (2.10) for $V = V_1$. Fix $q > 1$ and $\varepsilon > 0$ such that $n_1/q(1+\varepsilon) - 2 > 0$ and $V_1 \in L^{q(1+\varepsilon)}(\mathbb{V}_1) \cap L^q(\mathbb{V}_1)$ (see (A_3)). Set $\sigma = n_1/q - 2$. For $s \in [1, 1+\varepsilon]$ we have

$$\begin{aligned}\int_{|z-y|>R} V_1(z)^s |z-y|^{s(2-n)}\, dz \le &\int_{|z_1-y_1|\ge|\widetilde{z}_1-\widetilde{y}_1|} \chi_{\{|z-y|>R\}}(z) V_1(z)^s |z_1 - y_1|^{s(2-n)}\, dz \\ &+ \int_{|z_1-y_1|<|\widetilde{z}_1-\widetilde{y}_1|} \chi_{|z-y|>R}(z) V_1(z)^s |\widetilde{z}_1 - \widetilde{y}_1|^{s(2-n)}\, dz \\ = &\ T(R) + S(R). \end{aligned} \tag{2.11}$$

If $|z_1 - y_1| \ge |\widetilde{z}_1 - \widetilde{y}_1|$ and $|z - y| > R \ge 1$, then $|z_1 - y_1| > R/2 \ge 1/2$. Thus,

$$\begin{aligned} T(R) &\le C\int_{|z_1-y_1|>R/2} |z_1 - y_1|^{n-n_1} V_1(z_1)^s |z_1 - y_1|^{s(2-n)}\, dz_1 \\ &\le C\|V_1\|^s_{L^{qs}(\mathbb{V}_1)} \left(\int_{|z_1-y_1|>R/2} |z_1 - y_1|^{(s(2-n)+n-n_1)q'}\, dz_1\right)^{1/q'} = CR^{-\sigma}. \end{aligned} \tag{2.12}$$

Similarly, if $|z_1 - y_1| < |\widetilde{z}_1 - \widetilde{y}_1|$ and $|z - y| > R \ge 1$, then $|\widetilde{z}_1 - \widetilde{y}_1| > R/2 \ge 1/2$ and

$$\begin{aligned} S(R) &\le C\int_{|\widetilde{z}_1-\widetilde{y}_1|>R/2} \|V_1\|^s_{L^{sq}(\mathbb{V}_1)} \left(\int_{|z_1-y_1|<|\widetilde{z}_1-\widetilde{y}_1|} dz_1\right)^{1/q'} |\widetilde{z}_1 - \widetilde{y}_1|^{s(2-n)}\, d\widetilde{z}_1 \\ &\le C\int_{|\widetilde{z}_1-\widetilde{y}_1|>R/2} |\widetilde{z}_1 - \widetilde{y}_1|^{s(2-n)+n_1/q'}\, d\widetilde{z}_1 = CR^{-\sigma}. \quad \square \end{aligned} \tag{2.13}$$

We shall need the following properties of the function ω, similar to those that hold in the case of compactly supported potentials (cf. [11, Lemma 2.4]).

Proposition 2.14. *There exist $\gamma, \delta > 0$ such that for $x, y \in \mathbb{R}^n$ we have*

(a) $|\omega(x) - \omega(y)| \le C_\gamma |x - y|^\gamma$,
(b) $\delta \le \omega(x) \le 1$.

Proof. The property (a) can be proved by a slight modification of the proof of (2.6) in [11]. Indeed, thanks to (1.5) and $0 \le \omega(x) \le 1$, it suffices to show that there are $C, \gamma > 0$ such that for $|h| < 1$ we have

$$\int_{\mathbb{R}^n} |K_1(x+h,y) - K_1(x,y)|\, dy \le C|h|^{\gamma}. \tag{2.15}$$

To this purpose, by using (2.1), it is enough to establish that

$$\sum_{j=1}^{d} \int_{\mathbb{R}^n} \left| \int_0^1 \int_{\mathbb{R}^n} (P_s(x+h-z) - P_s(x-z))V_j(z)K_{1-s}(z,y)\, dz\, ds \right| dy \le C|h|^{\gamma}.$$

Consider one summand that contains V_1. Utilizing the fact that $P_s(x) = P_s(x_1)P_s(\tilde{x}_1)$, where $P_s(x_1)$ and $P_s(\tilde{x}_1)$ are the heat kernels on $\mathbb{V}_1$ and $\mathbb{V}_1^{\perp}$ respectively, we have

$$\begin{aligned} I &= \int_{\mathbb{R}^n} \left| \int_0^1 \int_{\mathbb{R}^n} (P_s(x+h-z) - P_s(x-z))V_1(z)K_{1-s}(z,y)\, dz\, ds \right| dy \\ &\le \int_0^1 \int_{\mathbb{R}^n} |P_s(x+h-z) - P_s(x-z)| V_1(z)\, dz\, ds \\ &\le \int_0^1 \int_{\mathbb{R}^n} P_s(x_1+h_1-z_1)|P_s(\tilde{x}_1+\tilde{h}_1-\tilde{z}_1) - P_s(\tilde{x}_1-\tilde{z}_1)|V_1(z_1)\, dz\, ds \\ &\quad + \int_0^1 \int_{\mathbb{R}^n} P_s(\tilde{x}_1-\tilde{z}_1)\, |P_s(x_1+h_1-z_1) - P_s(x_1-z_1)|\, V_1(z_1)\, dz\, ds. \end{aligned} \tag{2.16}$$

By taking $q > n_1/2$ such that $V_1 \in L^q(\mathbb{V}_1)$ and using the Hölder inequality we obtain

$$\begin{aligned} I &\le \int_0^1 \|P_s(x_1)\|_{L^{q'}(\mathbb{V}_1)} \|V_1(z_1)\|_{L^q(\mathbb{V}_1)} \int_{\mathbb{V}_1^{\perp}} |P_s(\tilde{x}_1+\tilde{h}_1-\tilde{z}_1) - P_s(\tilde{x}_1-\tilde{z}_1)|\, d\tilde{z}_1\, ds \\ &\quad + \int_0^1 \left(\int_{\mathbb{V}_1} \left| P_s(x_1+h_1-z_1) - P_s(x_1-z_1) \right|^{q'} dz_1 \right)^{1/q'} \|V_1(z_1)\|_{L^q(\mathbb{V}_1)}\, ds \\ &\le C(|\tilde{h}_1|^{\gamma} + |h_1|^{\gamma}), \end{aligned} \tag{2.17}$$

which finishes the proof of (a).

Next we note that

$$K_t(x,y) > 0 \quad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^n. \tag{2.18}$$

The proof of (2.18) is a straightforward adaptation of the proof of [11, Lemma 2.12]. We omit the details.

Our next task is to establish that there exists $\delta > 0$ such that

$$\omega(x) \ge \delta. \tag{2.19}$$

The proof of (2.19) goes by induction on d. Assume first that we have only one potential V_1, that is, $d = 1$. Then, $K_t(x,y) = K_t^{\{1\}}(x_1,y_1)P_t(\tilde{x}_1 - \tilde{y}_1)$, where $K_t^{\{1\}}(x_1,y_1)$ is the kernel of the semigroup generated by $\Delta - V_1(x_1)$ on $\mathbb{V}_1$ and $P_t(\tilde{x}_1)$ is the classical heat semigroup on $\mathbb{V}_1^{\perp}$. Hence $\omega(x) = \omega_0(x_1)$, where $\omega_0(x_1) = \lim_{t\to\infty} \int_{\mathbb{V}_1} K_t^{\{1\}}(x_1,y_1)\, dy_1$. Therefore, there is no loss of generality in proving (2.19) if we assume that $\mathbb{V}_1 = \mathbb{R}^n$. If we integrate (2.1) over $\mathbb{R}^n$ and take the limit as $t \to \infty$, then we get

$$1 - \omega(x) = \int_{\mathbb{R}^n} V(y)\Gamma(x,y)\, dy, \quad \text{where } \Gamma(x,y) \le C|x-y|^{2-n}. \tag{2.20}$$

By (A_3) and the Hölder inequality we can find $t, s > 1$ such that $V \in L^t(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$, $\chi_{\{|x|\le 1\}}(x)|x|^{2-n} \in L^{t'}(\mathbb{R}^n)$, and $\chi_{\{|x|>1\}}(x)|x|^{2-n} \in L^{s'}(\mathbb{R}^n)$. Thus (2.20) leads to

$$\lim_{|x|\to\infty} \int_{\mathbb{R}^n} V(y)|x-y|^{2-n}\, dy = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \omega(x) = 1. \tag{2.21}$$

Eq. (1.5) combined with (2.18) and (2.21) implies that $w(x) > 0$ for every $x \in \mathbb{R}^n$. Since ω is continuous (see (a)) and $\lim_{|x|\to\infty} \omega(x) = 1$, we get (2.19).

Using induction, we assume that (2.19) is true for V being a sum of $d-1$ potentials. Take $V = V_1 + \cdots + V_d$. As in the case of $d = 1$, we can assume that $\mathrm{lin}\{\mathbb{V}_1, \ldots, \mathbb{V}_d\} = \mathbb{R}^n$. Consider the semigroup $\{S_t\}_{t>0}$ generated by $-\Delta + V_2 + \cdots + V_d$.

Let $\omega_1(x) = \lim_{t\to\infty} \int_{\mathbb{R}^n} S_t(x,y)\,dy$. By the inductive assumption $\omega_1(x) \geq \delta_1$. Similarly to (2.20), the perturbation formula

$$S_t = K_t + \int_0^t S_{t-s} V_1 K_s\,ds$$

implies

$$\delta_1 \leq \omega_1(y) \leq \omega(y) + C\int_{\mathbb{R}^n} V_1(z)|z-y|^{2-n}\,dz \leq \omega(y) + C\int_{\mathbb{V}_1} V_1(z_1)|z_1-y_1|^{2-n_1}\,dz_1, \tag{2.22}$$

where the last inequality is proved in (2.5). If $y_1 \to \infty$ then the integral on the right hand side of (2.22) goes to zero. Hence, $\omega(y) > \delta_1/2$ provided $|y_1| > R_1$. We repeat the argument for each $V_2, \dots, V_d$ instead of V_1 and deduce that there exist $R, \delta > 0$ such that $\omega(x) > \delta$ for $|x| > R$. Consequently, by using (1.5) and (2.18) and continuity of ω we obtain (2.19). □

3. Proof of Theorem 1.3

By (2.1) we get

$$K_t - P_t(I - VL^{-1}) = Q_t - W_t, \tag{3.1}$$

where

$$W_t = \int_0^t (P_{t-s} - P_t)\, V\, K_s\,ds, \qquad Q_t = \int_t^\infty P_t\, V\, K_s\,ds.$$

Let

$$W_t(x,y) = \sum_{j=1}^d W_t^{\langle j\rangle}(x,y) = \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^n} (P_{t-s}(x-z) - P_t(x-z)) V_j(z) K_s(z,y)\,dz\,ds,$$

$$Q_t(x,y) = \sum_{j=1}^d Q_t^{\langle j\rangle}(x,y) = \sum_{j=1}^d \int_{\mathbb{R}^n} P_t(x,z) \int_t^\infty V_j(z) K_s(z,y)\,ds\,dz$$

be the integral kernels of W_t and Q_t respectively. In order to prove Theorem 1.3 it is sufficient to establish that the maximal operators $f \mapsto \sup_{t>0}|W_t f|$ and $f \mapsto \sup_{t>0}|Q_t f|$ are bounded on $L^1(\mathbb{R}^n)$. The proofs of these facts are presented in the following four lemmas.

Lemma 3.2. *The operator $f \mapsto \sup_{t>2}|W_t f|$ is bounded on $L^1(\mathbb{R}^n)$.*

Proof. It suffices to prove that

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{t>2} |W_t(x,y)|\,dx < \infty.$$

Without loss of generality we can consider only $W_t^{\langle j\rangle}(x,y)$. For $0 < \beta < 1$, which will be fixed later on, we write

$$\begin{aligned} W_t^{\langle 1\rangle}(x,y) &= \int_0^t \int_{\mathbb{R}^n} (P_{t-s}(x-z) - P_t(x-z)) V_1(z) K_s(z,y)\,dz\,ds \\ &= \int_0^{t^\beta} \cdots + \int_{t^\beta}^t \cdots = F_1(x,y;t) + F_2(x,y;t). \end{aligned}$$

To estimate F_1 observe that for $t > 2$ and $s \leq t^\beta < t$ there exists $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$|P_{t-s}(x-z) - P_t(x-z)| \leq C\frac{s}{t}\phi_t(x-z). \tag{3.3}$$

Here and subsequently, $f_t(x) = t^{-n/2} f(x/\sqrt{t})$ and $\mathcal{S}$ denotes the Schwartz class of functions. From (1.2) and (3.3), we get

$$|F_1(x,y;t)| \leq Ct^{-1+\beta} \int_{\mathbb{R}^n} \phi_t(x-z) V_1(z)|z-y|^{2-n} dz.$$

Since $\sup_{t>2} t^{-1+\beta}\phi_t(x-z) \leq C(1+|x-z|)^{-n-2+2\beta}$, we have that

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{t>2} |F_1(x,y;t)|dx \leq C\int_{\mathbb{R}^n} V_1(z)|z-y|^{2-n}\,dz \leq C,$$

where the last inequality comes from Lemma 2.2.

To deal with F_2 we write

$$\begin{aligned} F_2(x,y;t) &= \int_{t^\beta}^t \int_{\mathbb{R}^n} P_{t-s}(x-z)V_1(z)K_s(z,y)\,dz\,ds - \int_{t^\beta}^t \int_{\mathbb{R}^n} P_t(x-z)V_1(z)K_s(z,y)\,dz\,ds \\ &= F_2'(x,y;t) - F_2''(x,y;t). \end{aligned}$$

Observe that for $s \in [t^\beta, t]$ we have

$$K_s(z,y) \le Ct^{-\beta n/2} \exp\left(-|z-y|^2/4t\right). \tag{3.4}$$

Also

$$\int_0^t P_{t-s}(x-z)\,ds = \int_0^t P_s(x-z)\,ds \le C|x-z|^{2-n} \exp\left(-|x-z|^2/ct\right). \tag{3.5}$$

As a consequence of (3.4) and (3.5) we obtain

$$F_2'(x,y;t) \le C \int_{\mathbb{R}^n} t^{-\beta n/2}|x-z|^{2-n} \exp\left(-|x-z|^2/ct\right) V_1(z_1) \exp\left(-|z-y|^2/4t\right)\,dz.$$

Then, for $\varepsilon > 0$,

$$\begin{aligned} &\sup_{t>2} t^{-\beta n/2} \exp\left(-|x-z|^2/ct\right) \exp\left(-|z-y|^2/4t\right) \\ &\quad \le C \sup_{t>2} t^{-1-\varepsilon} \exp\left(-|x-z|^2/ct\right) \cdot \sup_{t>2} t^{-\beta n/2+1+\varepsilon} \exp\left(-|z-y|^2/4t\right) \\ &\quad \le C(1+|x-z|)^{-2-2\varepsilon}|z-y|^{2+2\varepsilon-\beta n}. \end{aligned}$$

Consequently,

$$\begin{aligned} \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{t>2} F_2'(x,y;t)\,dx &\le C \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|x-z|^{2-n}}{(1+|x-z|)^{2+2\varepsilon}} |z-y|^{2+2\varepsilon-\beta n} V_1(z)\,dx\,dz \\ &\le C \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} |z-y|^{2+2\varepsilon-\beta n} V_1(z_1)\,dz. \end{aligned}$$

If we choose $\beta < 1$ close to 1 and ε small, then we can apply Lemma 2.2 and get

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{t>2} F_2'(x,y;t)\,dx \le C.$$

We now turn to estimating $F_2''(x,y;t)$. Observe that for $\varepsilon > 0$ we have

$$\int_{t^\beta}^t K_s(z,y)\,ds \le C \int_{t^\beta}^\infty t^{-\beta\varepsilon} s^{-n/2+\varepsilon} \exp\left(-|z-y|^2/(4s)\right)\,ds \le Ct^{-\beta\varepsilon}|z-y|^{2-n+2\varepsilon}.$$

Then from Lemma 2.2 we conclude that

$$\begin{aligned} \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{t>2} F_2''(x,y;t)\,dx &\le C \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sup_{t>2} t^{-\beta\varepsilon} P_t(x-z)V_1(z)|z-y|^{2-n+2\varepsilon}\,dx\,dz \\ &\le C \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+|x-z|)^{-n-2\beta\varepsilon} V_1(z)|z-y|^{2-n+2\varepsilon}\,dx\,dz \\ &\le C \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} V_1(z)|z-y|^{2-n+2\varepsilon}\,dz \le C, \end{aligned}$$

provided $\varepsilon > 0$ is small enough. □

Lemma 3.6. *The operator $f \mapsto \sup_{t\le 2} |W_t f|$ is bounded on $L^1(\mathbb{R}^n)$.*

Proof. It is enough to prove that

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{t\le 2} |W_t^{\langle 1\rangle}(x,y)|\,dx < \infty.$$

We have

$$\int_0^t \int_{\mathbb{R}^n} (P_{t-s}(x-z) - P_t(x-z))V_1(z)K_s(z,y)\,dz\,ds = \int_0^{t/2} \cdots + \int_{t/2}^t \cdots = F_3(x,y;t) + F_4(x,y;t).$$

To deal with F_3 observe that for $t \le 2, s \le t/2$ we have

$$|P_{t-s}(x-z) - P_t(x-z)| \le C\phi_t(x-z),$$

where $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\phi \ge 0$. Therefore

$$\sup_{t\le 2} |F_3(x,y;t)| \le C \sup_{t\le 2} \int_{\mathbb{R}^n} \phi_t(x-z)V_1(z)|z-y|^{2-n}dz.$$

Denote by M_ϕ^0 the classical local maximal operator associated with ϕ, that is,

$$M_\phi^0 f(x) = \sup_{t\le 2} |\phi_t * f(x)|.$$

Then

$$\sup_{t\le 2} |F_3(x,y;t)| \le CM_\phi^0(\xi_y)(x),$$

where $\xi_y(z) = V_1(z)|z-y|^{2-n}$. We claim that

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{t\le 2} |F_3(x,y)|dx \le C \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} M_\phi^0(\xi_y)(x)\,dx \le C. \tag{3.7}$$

To obtain (3.7) we write

$$\xi_y(z) = \sum_{k=1}^{\infty} \xi_{y,k}(z),$$

where

$$\xi_{y,1}(z) = V_1(z)|z-y|^{2-n}\chi_{B(y,2)}(z), \qquad \xi_{y,k}(z) = V_1(z)|z-y|^{2-n}\chi_{B(y,2^k)\setminus B(y,2^{k-1})}(z), \quad k>1.$$

From Lemma 2.2 it follows that there exists $s > 1$ such that

$$\operatorname{supp}\xi_{y,1} \subseteq B(y,2) \quad \text{and} \quad \|\xi_{y,1}\|_{L^s(\mathbb{R}^n)} \le C \le C|B(y,2)|^{-1+1/s}. \tag{3.8}$$

Consider $\xi_{y,k}$ for $k > 1$. Set $q < n_1/2$ such that $V_1 \in L^q(\mathbb{V}_1)$. Then

$$\begin{aligned} &\operatorname{supp}\xi_{y,k} \subseteq B(y,2^k) \\ &\|\xi_{y,k}\|_{L^q(\mathbb{R}^n)} \le C2^{k(2-n)}\|V_1\|_{L^q(\mathbb{V}_1)}2^{k(n-n_1)/q} \le C|B(y,2^k)|^{-1+1/q}2^{-\rho k}, \end{aligned} \tag{3.9}$$

where $\rho = n_1/q - 2$. Now, our claim (3.7) follows from (3.8) and (3.9), and the classical theory of local maximal operators.

It remains to analyze $F_4 = F_5 - F_6$, where

$$F_5(x,y;t) = \int_{t/2}^{t} \int_{\mathbb{R}^n} P_{t-s}(x-z)V_1(z)K_s(z,y)\,dz\,ds,$$

$$F_6(x,y;t) = \int_{t/2}^{t} \int_{\mathbb{R}^n} P_t(x-z)V_1(z)K_s(z,y)\,dz\,ds.$$

Clearly,

$$\sup_{s\in[t/2,t]} K_s(z,y) \le Ct^{-n/2}\exp\left(-|z-y|^2/ct\right).$$

Therefore, for $0 < t \le 2$ and $0 < \gamma < 1$ close to 1 we get

$$\begin{aligned} F_5(x,y;t) &\le C\int_0^{t/2}\int_{\mathbb{R}^n} t^{-\gamma}P_s(x-z)V_1(z)t^{-n/2+\gamma}\exp\left(-|z-y|^2/ct\right)\,dz\,ds \\ &\le C\int_{\mathbb{R}^n} |x-z|^{2-n}t^{-\gamma}\exp\left(-|x-z|^2/ct\right)V_1(z)|z-y|^{-n+2\gamma}\,dz \\ &\le C\int_{\mathbb{R}^n} |x-z|^{2-n-2\gamma}\exp\left(-|x-z|^2/c'\right)V_1(z)|z-y|^{-n+2\gamma}\,dz. \end{aligned}$$

Thus, by using Lemma 2.2, we get

$$\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}\sup_{0<t\le 2} F_5(x,y;t)dx \le C.$$

To deal with F_6 we observe that for $0 < t \le 2$ and $0 < \gamma < 1$ close to 1 we have

$$\begin{aligned} F_6(x, y; t) &\le C \int_{\mathbb{R}^n} tP_t(x-z)V_1(z_1)t^{-n/2} \exp\left(-|z-y|^2/ct\right) dz \\ &\le \int_{\mathbb{R}^n} |x-z|^{2-n-2\gamma} \exp\left(-|x-z|^2/c'\right) V_1(z)|z-y|^{-n+2\gamma} dz \end{aligned}$$

and, consequently,

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{t<2} F_6(x, y; t)\, dx \le C. \quad \Box$$

Lemma 3.10. *The operator $f \mapsto \sup_{t>2} |Q_t f|$ is bounded on $L^1(\mathbb{R}^n)$.*

Proof. Notice that for $\varepsilon > 0$ and $t > 2$ we have

$$\int_t^\infty K_s(z, y)\, ds \le C \int_t^\infty s^{-\varepsilon} s^{-n/2+\varepsilon} \exp\left(-\frac{|y-z|^2}{4s}\right) ds \le Ct^{-\varepsilon}|y-z|^{2-n+2\varepsilon}. \tag{3.11}$$

It causes no loss of generality to consider only $Q_t^{\langle 1\rangle}(x, y)$. If $t > 2$, then

$$0 \le Q_t^{\langle 1\rangle}(x, y) \le C \int_{\mathbb{R}^n} P_t(x-z)V_1(z)t^{-\varepsilon}|y-z|^{2-n+2\varepsilon} dz.$$

Since $\sup_{t>2} t^{-\varepsilon}P_t(x-z) \le C(1+|x-z|)^{-n-2\varepsilon}$, we find that

$$\begin{aligned} \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{t>2} Q_t^{\langle 1\rangle}(x, y)\, dx &\le C \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1+|x-z|)^{-n-2\varepsilon} V_1(z)|y-z|^{2-n+2\varepsilon} dz\, dx \\ &\le C \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} V_1(z)|y-z|^{2-n+2\varepsilon} dz \le C. \end{aligned} \tag{3.12}$$

The last inequality follows from Lemma 2.2. $\Box$

Lemma 3.13. *The operator $f \mapsto \sup_{t\le 2} |Q_t f|$ is bounded on $L^1(\mathbb{R}^n)$.*

Proof. The estimate $\int_t^\infty K_s(z, y)\, ds \le C|z-y|^{2-n}$ implies

$$\sup_{t\le 2} Q_t(x, y) \le C \sup_{t\le 2} \int_{\mathbb{R}^n} P_t(x-z)V(z)|z-y|^{2-n} dz.$$

We claim that for fixed $y \in \mathbb{R}^n$ the foregoing function (of variable x) belongs to $L^1(\mathbb{R}^n)$ and

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{t\le 2} Q_t(x, y)\, dx < \infty.$$

The claim follows by arguments identical to the one that we used to prove (3.7). $\Box$

Now, Theorem 1.3 follows directly from Lemmas 3.2, 3.6, 3.10 and 3.13.

4. Proof of Theorem 1.6

Proof. Thanks to (2.20) and Proposition 2.14, for $g \in L^1(\mathbb{R}^n)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} (I - VL^{-1})(g/\omega)(x)\, dx &= \int_{\mathbb{R}^n} \frac{g(x)}{\omega(x)} dx - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} V(x)\Gamma(x, y)\frac{g(y)}{\omega(y)} dy\, dx \\ &= \int_{\mathbb{R}^n} \frac{g(x)}{\omega(x)} dx - \left(\int_{\mathbb{R}^n} \frac{g(y)}{\omega(y)} dy - w(y)\frac{g(y)}{\omega(y)} dy\right) \\ &= \int_{\mathbb{R}^n} g(y)\, dy. \end{aligned} \tag{4.1}$$

First, we are going to prove that

$$\|\omega f\|_{H^1_\Delta(\mathbb{R}^n)} \le \|f\|_{H^1_L(\mathbb{R}^n)}. \tag{4.2}$$

Theorem 1.3 combined with (2.8) implies that (4.2) is equivalent to

$$\|\omega(I - V\Delta^{-1})f\|_{H^1_\Delta(\mathbb{R}^n)} \le C\|f\|_{H^1_\Delta(\mathbb{R}^n)}. \tag{4.3}$$

Assume that a is a classical $(1, \infty)$-atom associated with $B = B(y_0, r)$, i.e.,

$$\operatorname{supp} a \subseteq B, \qquad \|a\|_\infty \le |B|^{-1}, \qquad \int_B a(x)\,dx = 0. \tag{4.4}$$

By the atomic characterization of $H^1_\Delta(\mathbb{R}^n)$ the inequality (4.3) will be obtained if we have established that $b = \omega(I - V\Delta^{-1})a \in H^1_\Delta(\mathbb{R}^n)$ and

$$\|b\|_{H^1_\Delta(\mathbb{R}^n)} \le C \tag{4.5}$$

with a constant $C > 0$ independent of a.

By (2.8), $a = (I - VL^{-1})(b/\omega)$. Hence, using (4.1) we get

$$\int_{\mathbb{R}^n} b(x)\,dx = 0. \tag{4.6}$$

The proof of (4.5) is divided into two cases.

Case 1: $r \ge 1$. Set

$$b(x) = (b(x) - c_1)\chi_{2B}(x) + \sum_{k=2}^{\infty} \left(b(x)\chi_{2^kB\setminus 2^{k-1}B}(x) + c_{k-1}\chi_{2^{k-1}B}(x) - c_k\chi_{2^kB}(x)\right) = \sum_{k=1}^{\infty} b_k(x),$$

where

$$c_k = -|2^kB|^{-1}\int_{(2^kB)^c} b(x)\,dx, \quad k = 1, 2, \ldots.$$

Here and throughout, $\rho B = B(y_0, \rho r)$ for $B = B(y_0, r)$.

We claim that

$$\sum_{k=1}^{\infty} \|b_k\|_{H^1_\Delta(\mathbb{R}^n)} \le C. \tag{4.7}$$

From Lemma 2.9 and Proposition 2.14 we conclude that there exists $\sigma > 0$ such that

$$\begin{aligned} |c_k| &\le |2^kB|^{-1}\int_{(2^kB)^c} V(x)|\Delta^{-1}a(x)|\,dx \le C|2^kB|^{-1}\int_{(2^kB)^c}\int_B V(x)\,|x-y|^{2-n}|a(y)|\,dy\,dx \\ &\le C|2^kB|^{-1}\int_B |a(y)|\int_{(2^kB)^c} V(x)\,|x-y_0|^{2-n}\,dx\,dy \le C|2^kB|^{-1}(2^kr)^{-\sigma}. \end{aligned} \tag{4.8}$$

Note that $\operatorname{supp} b_k \subseteq 2^kB$ and $\int_{\mathbb{R}^n} b_k(x)\,dx = 0$. Therefore (4.7) follows if we have verified that there exists $q > 1$ such that

$$\sum_{k=1}^{\infty} \|b_k\|_{L^q(\mathbb{R}^n)}|2^kB|^{1-1/q} \le C, \tag{4.9}$$

where C does not depend on a.

If $k = 1$, then

$$|b_1(x)| \le |c_1|\chi_{2B}(x) + |a(x)| + V(x)|\Delta^{-1}a(x)|\chi_{2B}(x)$$

and

$$\|b_1\|_{L^q(\mathbb{R}^n)} \le C|2B|^{-1+1/q} + \left(\int_{2B} V(x)^q|\Delta^{-1}a(x)|^q dx\right)^{1/q}.$$

Notice that

$$\left(\int_{2B} V(x)^q|\Delta^{-1}a(x)|^q dx\right)^{1/q} \le Cr^2|B|^{-1}\sum_{j=1}^{d}\left(\int_{2B} V_j(x)^q dx\right)^{1/q}.$$

We can consider only the summand with V_1. By the Hölder inequality,

$$r^2|B|^{-1}\left(\int_{2B}V_1(x)^q dx\right)^{1/q} \le Cr^2|B|^{-1}r^{\widetilde{n}_1/q}\|V_1\|_{L^{qs}(\mathbb{V}_1)}r^{n_1(1-1/s)/q}$$
$$= C|B|^{-1+1/q}r^{2-n_1/(sq)}.$$

Choosing $q, s > 1$ such that $V_1 \in L^{qs}(\mathbb{V}_1)$ and $2 - n_1/(qs) < 0$ we get

$$\|b_1\|_{L^q(\mathbb{R}^n)} \le C|2B|^{-1+1/q}. \tag{4.10}$$

For $k > 1$, by the definition of b_k, we get that

$$\|b_k\|_{L^q(\mathbb{R}^n)} \le |c_{k-1}||2^{k-1}B|^{1/q} + |c_k||2^kB|^{1/q} + \|b\|_{L^q(2^kB\setminus 2^{k-1}B)}.$$

From (4.8) we see that first two summands can be estimated by $C|2^kB|^{-1+1/q}2^{-k\sigma}$. Then it remains to deal with the last summand. By using Lemma 2.9 there exists $\sigma' > 0$ such that for $q \in (1, 1+\varepsilon]$ we have

$$\begin{aligned}\|b\|_{L^q(2^kB\setminus 2^{k-1}B)} &\le C\left(\int_{2^kB\setminus 2^{k-1}B}\left(\int_B V(x)|x-y|^{2-n}|a(y)|\,dy\right)^q dx\right)^{1/q}\\ &\le C\left(\int_{(2^{k-1}B)^c}V(x)^q|x-y_0|^{q(2-n)}dx\right)^{1/q} \le C(2^kr)^{-\sigma'}\\ &= C|2^kB|^{-1+1/q}(2^kr)^{-\sigma'+n-n/q} \le C|2^kB|^{-1+1/q}2^{-k\delta}\end{aligned} \tag{4.11}$$

provided that $\delta = -\sigma' + n - n/q > 0$.

The estimate (4.9) follows from (4.10) and (4.11). This ends Case 1.

Case 2: $r < 1$. Fix $N \in \mathbb{N}\cup\{0\}$ such that $1/2 < 2^N r \le 1$. Then

$$b(x) = (a(x)\omega(x) - c_0\chi_B(x)) + \sum_{i=1}^{N} c_0|B|\left(|2^{i-1}B|^{-1}\chi_{2^{i-1}B}(x) - |2^iB|^{-1}\chi_{2^iB}(x)\right)$$
$$+\left(b(x) - a(x)\omega(x) + c_0|B||2^NB|^{-1}\chi_{2^NB}(x)\right) = d_0(x) + \sum_{i=1}^{N} d_i(x) + b'(x),$$

where

$$c_0 = |B|^{-1}\int_B a(x)\omega(x)\,dx.$$

By using $\int_B a = 0$ and property (a) from Proposition 2.14, we obtain

$$|c_0| \le |B|^{-1}\int_B |a(x)||\omega(x) - \omega(y_0)|dx \le r^\delta|B|^{-1}. \tag{4.12}$$

Observe that $\operatorname{supp} d_0 \subseteq B$, $\int_B d_0 = 0$, and $\|d_0\|_\infty \le C|B|^{-1}$. Similarly, for $i = 1, \dots, N$, $\operatorname{supp} d_i \subseteq 2^iB$, $\int d_i = 0$ and $\|d_i\|_\infty \le Cr^\delta|2^iB|^{-1}$. Therefore

$$\sum_{i=0}^{N}\|d_i\|_{H^1_\Delta(\mathbb{R}^n)} \le C + CNr^\delta \le C - Cr^\delta\log_2 r \le C.$$

Define $B' = 2^NB$. Obviously $|B'| \sim 1$. To deal with $b'(x)$ we apply the method from Case 1 with respect to B', i.e.,

$$b' = (b'(x) - c_1')\chi_{2B'}(x) + \sum_{k=2}^{\infty}\left(b'(x)\chi_{2^kB'\setminus 2^{k-1}B'}(x) + c_{k-1}'\chi_{2^{k-1}B'}(x) - c_k'\chi_{2^kB'}(x)\right) = \sum_{k=1}^{\infty} b_k',$$

where

$$c_k' = -|2^kB'|^{-1}\int_{(2^kB')^c} b'(x)\,dx.$$

The arguments that we used in Case 1 also give

$$|c_k'| \le C|2^kB'|^{-1}2^{-k\sigma} \quad \text{for } k = 1, 2, \dots \quad \text{and} \quad \sum_{k=2}^{\infty}\|b_k'\|_{H^1_\Delta(\mathbb{R}^n)} \le C. \tag{4.13}$$

It remains to obtain that

$$\|b_1'\|_{H^1_\Delta(\mathbb{R}^n)} \le C. \tag{4.14}$$

It is immediate that $\operatorname{supp} b_1' \subseteq 2B'$ and $\int_{2B'} b_1' = 0$. Also,

$$\|b_1'\|_{L^q(\mathbb{R}^n)} \le \left(\int_{2B'} V(x)^q |\Delta^{-1}a(x)|^q\right)^{1/q} + C|c_0|\|B\|2B'|^{-1+1/q} + C|c_1'| \, \| \, 2B'|^{1/q}. \tag{4.15}$$

By (4.12) and (4.13) only the first summand needs to be estimated. Observe that

$$|\Delta^{-1}a(x)| \le \int_B |x-y|^{2-n}|a(y)|\,dy \le \left\{\begin{matrix} Cr^{2-n} & \text{if } |x-y_0| < 2r \\ C|x-y_0|^{2-n} & \text{if } |x-y_0| > 2r \end{matrix}\right\} \le C|x-y_0|^{2-n}.$$

Therefore, by using Lemma 2.2, we get

$$\|b_1'\|_{L^q(\mathbb{R}^n)} \le C$$

and (4.14) follows, which finishes Case 2 and the proof of (4.2).

In order to finish the proof of Theorem 1.6 it remains to prove that

$$\|f\|_{H^1_L(\mathbb{R}^n)} \le C\|\omega f\|_{H^1_\Delta(\mathbb{R}^n)}. \tag{4.16}$$

By virtue of Theorem 1.3, the inequality (4.16) is equivalent to

$$\left\|(I - VL^{-1})\,(g/\omega)\,\right\|_{H^1_\Delta(\mathbb{R}^n)} \le C\|g\|_{H^1_\Delta(\mathbb{R}^n)}. \tag{4.17}$$

Assume that a is an $H^1_\Delta(\mathbb{R}^n)$-atom (see (4.4)). Set $b = (I - VL^{-1})(a/\omega)$. The proof will be finished if we have obtained that

$$\|b\|_{H^1_\Delta(\mathbb{R}^n)} \le C \tag{4.18}$$

with C independent of atom a. By (4.1), we have

$$\int_{\mathbb{R}^n} b(x)\,dx = \int_{\mathbb{R}^n} a(x)\,dx = 0.$$

Note that the proof of (4.5) only relies on estimates of $\Gamma_0(x,y)$ from above by $C|x-y|^{2-n}$. The same estimates hold for $\Gamma(x,y)$. Moreover, the weight $1/\omega$ has the same properties as ω, that is, boundedness from above and below by positive constants and the Hölder condition. Therefore the proof of (4.18) follows by the same arguments. Details are omitted. □

5. Proof of Theorem 1.8

By (2.1) we get a formula similar to (3.1):

$$K_t - P_t(I - VL^{-1}) = Q_t' - W_t', \tag{5.1}$$

where

$$W_t' = \int_0^t P_{t-s}\,V\,K_s\,ds, \qquad Q_t' = \int_0^\infty P_t\,V\,K_s\,ds.$$

Recall that for $i = 1, \dots, n$ we denote by ∂_i the derivative in the direction of ith standard coordinate. For $f \in L^1(\mathbb{R}^n)$, from (3.1) and (5.1) we get

$$\int_\varepsilon^{\varepsilon^{-1}} \partial_i K_t f \frac{dt}{\sqrt{t}} - \int_\varepsilon^{\varepsilon^{-1}} \partial_i P_t(I - VL^{-1}) f \frac{dt}{\sqrt{t}} = \mathcal{W}'_{i,\varepsilon} f + \mathcal{Q}'_{i,\varepsilon} f + \mathcal{W}_{i,\varepsilon} f + \mathcal{Q}_{i,\varepsilon} f, \tag{5.2}$$

$$\mathcal{Q}_{i,\varepsilon} = \int_2^{\varepsilon^{-1}} \partial_i Q_t \frac{dt}{\sqrt{t}}, \qquad \mathcal{Q}'_{i,\varepsilon} = \int_\varepsilon^2 \partial_i Q_t' \frac{dt}{\sqrt{t}},$$

$$\mathcal{W}_{i,\varepsilon} = -\int_2^{\varepsilon^{-1}} \partial_i W_t \frac{dt}{\sqrt{t}}, \qquad \mathcal{W}'_{i,\varepsilon} = -\int_\varepsilon^2 \partial_i W_t' \frac{dt}{\sqrt{t}}.$$

All the operators above are well-defined and bounded on $L^1(\mathbb{R}^n)$. By the theory of the classical Hardy spaces, $\mathcal{R}_{\Delta,i} f = \lim_{\varepsilon \to 0} \int_\varepsilon^{\varepsilon^{-1}} \partial_i P_t f \frac{dt}{\sqrt{t}} \in L^1(\mathbb{R}^n)$ for every $i = 1, \dots, n$, exactly when $f \in H^1_\Delta(\mathbb{R}^n)$. Moreover,

$$\|f\|_{H^1_\Delta(\mathbb{R}^n)} \sim \|f\|_{L^1(\mathbb{R}^n)} + \sum_{i=1}^n \|\mathcal{R}_{\Delta,i} f\|_{L^1(\mathbb{R}^n)}. \tag{5.3}$$

The subsequent four lemmas prove that the operators $\mathcal{Q}_{i,\varepsilon}$, $\mathcal{Q}'_{i,\varepsilon}$, $\mathcal{W}_{i,\varepsilon}$, $\mathcal{W}'_{i,\varepsilon}$ converge strongly as $\varepsilon \to 0$ in the space of $L^1(\mathbb{R}^n)$-bounded operators.

Lemma 5.4. *For every $i = 1, \dots, n$ the operators $\mathcal{Q}_{i,\varepsilon}$ converge as $\varepsilon \to 0$ in norm-operator topology.*

Proof. The operators $\mathcal{Q}_{i,\varepsilon}$ have the integral kernels

$$\mathcal{Q}_{i,\varepsilon}(x,y) = \int_2^{\varepsilon^{-1}} \int_t^{\infty} \int_{\mathbb{R}^n} \partial_i P_t(x-z)V(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}.$$

The lemma will be proved when we have obtained

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbb{Q}_i^{(j)}(x,y)dx \le C,$$

where

$$\mathbb{Q}_i^{(j)}(x,y) = \int_2^{\infty} \int_t^{\infty} \int_{\mathbb{R}^n} |\partial_i P_t(x-z)| V_j(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}.$$

Since $|\partial_i P_t(x-z)| \le Ct^{-1/2}\phi_t(x-z)$ for some $\phi \in \mathcal{S}(\mathbb{R}^n)$ we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathbb{Q}_i^{(j)}(x,y)dx &\le C \int_{\mathbb{R}^n} \int_2^{\infty} \int_t^{\infty} \int_{\mathbb{R}^n} t^{-1/2}\phi_t(x-z)V_j(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\,dx \\
&\le C \int_2^{\infty} \int_t^{\infty} \int_{\mathbb{R}^n} t^{-1}V_j(z)K_s(z,y)\,dz\,ds\,dt \\
&\le C \int_2^{\infty} \int_t^{\infty} \int_{\mathbb{R}^n} t^{-1-\varepsilon}V_j(z)s^{-n/2+\varepsilon}\exp(-|z-y|^2/4s)\,dz\,ds\,dt \\
&\le C \left(\int_2^{\infty} t^{-1-\varepsilon}dt\right) \cdot \left(\int_{\mathbb{R}^n} V_j(z)|z-y|^{2-n+2\varepsilon}\,dz\right) \le C,
\end{aligned} \tag{5.5}$$

where in the last inequality we used Lemma 2.2, and C does not depend on $y \in \mathbb{R}^n$. □

Lemma 5.6. *For every $i = 1, \dots, n$ the operators $\mathcal{W}_{i,\varepsilon}$ converge as $\varepsilon \to 0$ in norm-operator topology.*

Proof. The operators $\mathcal{W}_{i,\varepsilon}$ have the integral kernels

$$\mathcal{W}_{i,\varepsilon}(x,y) = \int_2^{\varepsilon^{-1}} \int_0^{t} \int_{\mathbb{R}^n} \partial_i\left(P_{t-s}(x-z) - P_t(x-z)\right)V(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}.$$

Set

$$\mathbb{W}_i^{(j)}(x,y) = \int_2^{\infty} \int_0^{t} \int_{\mathbb{R}^n} |\partial_i\left(P_{t-s}(x-z) - P_t(x-z)\right)| V_j(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}.$$

The lemma will be proved when we have obtained that

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbb{W}_i^{(j)}(x,y)dx \le C. \tag{5.7}$$

For fixed $y \in \mathbb{R}^n$ and $0 < \beta < 1$, β will be determined later on; we write

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathbb{W}_i^{(j)}(x,y)dx &\le \int_{\mathbb{R}^n} \int_2^{\infty} \int_0^{t} \int_{\mathbb{R}^n} \left|\partial_i\left(P_{t-s}(x-z) - P_t(x-z)\right)\right| V_j(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\,dx \\
&\le \int_0^{t^\beta} \cdots\,ds + \int_{t^\beta}^{t} \cdots\,ds = J_1 + J_2.
\end{aligned}$$

Observe that there exist $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\psi \geq 0$ such that for $s \in (0, t^\beta)$ and $t > 2$ we have

$$|\partial_i (P_{t-s}(x) - P_t(x))| \leq st^{-3/2}\psi_t(x).$$

Thus by using Lemma 2.2 we get

$$\begin{aligned} J_1 &\leq \int_{\mathbb{R}^n}\int_2^\infty\int_0^{t^\beta}\int_{\mathbb{R}^n} st^{-2}\psi_t(x-z)V_j(z)K_s(z,y)\,dz\,ds\,dt\,dx \\ &\leq C\int_2^\infty t^{-2+\beta}\,dt\cdot\int_{\mathbb{R}^n} V_j(z)|z-y|^{2-n}dz \leq C_1. \end{aligned} \tag{5.8}$$

Note that if $t > 2$ and $s \in [t^\beta, t]$ then $K_s(z) \leq Ct^{-\beta n/2}\exp(-|z|^2/ct)$. Choosing $0 < \beta < 1$, β close to 1, and applying Lemma 2.2 we obtain

$$\begin{aligned} J_2 &\leq \int_{\mathbb{R}^n}\int_2^\infty\int_{t^\beta}^t\int_{\mathbb{R}^n}\left(\frac{\psi_{t-s}(x-z)}{\sqrt{t-s}}+\frac{\psi_t(x-z)}{\sqrt{t}}\right)V_j(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\,dx \\ &\leq C\int_2^\infty\int_0^t\int_{\mathbb{R}^n}\left(((t-s)t)^{-1/2}+t^{-1}\right)V_j(z)t^{-\beta n/2}\exp(-|z-y|^2/ct)\,dz\,ds\,dt \\ &\leq C\int_2^\infty\int_{\mathbb{R}^n} V_j(z)t^{-\beta n/2}\exp(-|z-y|^2/ct)\,dz\,dt \leq C\int_{\mathbb{R}^n} V_j(z)|z-y|^{2-\beta n}dz \leq C_2. \end{aligned} \tag{5.9}$$

Notice that the constants C_1 and C_2 in (5.8) and (5.9) respectively do not depend on $y \in \mathbb{R}^n$. Thus (5.7) follows. $\square$

Lemma 5.10. *For $i = 1, \dots, n$ the operators $\mathcal{W}'_{i,\varepsilon}$ converge as $\varepsilon \to 0$ in norm-operator topology.*

Proof. The operators $\mathcal{W}'_{i,\varepsilon}$ have the integral kernel

$$\mathcal{W}'_{i,\varepsilon}(x,y) = \int_\varepsilon^2\int_0^t\int_{\mathbb{R}^n}\partial_i P_{t-s}(x-z)V(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}.$$

The lemma will be proved if we have shown that

$$\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}\mathbb{W}_i^{(j)'}(x,y)\,dx \leq C, \tag{5.11}$$

where

$$\mathbb{W}_i^{(j)'}(x,y) = \int_0^2\int_0^t\int_{\mathbb{R}^n}\left|\partial_i P_{t-s}(x-z)\right|V_j(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}.$$

Fix $y \in \mathbb{R}^n$. Observe that

$$\begin{aligned} \int_{\mathbb{R}^n}\mathbb{W}_i^{(j)'}(x,y)dx &\leq \int_{\mathbb{R}^n}\int_0^2\int_0^t\int_{\mathbb{R}^n}\left|\partial_i P_{t-s}(x-z)\right|V_j(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\,dx \\ &\leq \int_0^{t/2}\cdots\,ds + \int_{t/2}^t\cdots\,ds = J_3 + J_4. \end{aligned}$$

There exist $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\psi \geq 0$, such that

$$\begin{aligned} J_3 &\leq \int_{\mathbb{R}^n}\int_0^2\int_0^{t/2}\int_{\mathbb{R}^n}(t(t-s))^{-1/2}\psi_{t-s}(x-z)V_j(z)K_s(z,y)\,dz\,ds\,dt\,dx \\ &\leq C\int_0^2\int_0^t\int_{\mathbb{R}^n} t^{-1}V_j(z)K_s(z,y)\,dz\,ds\,dt \\ &\leq C\int_0^2\int_{\mathbb{R}^n} t^{-1}V_j(z)|z-y|^{2-n}\exp\left(-|z-y|^2/ct\right)dz\,dt \\ &\leq C\int_{|z-y|>1/2} V_j(z)|z-y|^{2-n}dz + \int_{|z-y|\leq 1/2} V_j(z)|z-y|^{2-n}|\log|z-y\,\|\,dz \leq C_3 \end{aligned}$$

and

$$J_4 \leq C\int_{\mathbb{R}^n}\int_0^2\int_{t/2}^t\int_{\mathbb{R}^n}(t(t-s))^{-1/2}\psi_{t-s}(x-z)V_j(z)t^{-n/2}\exp\left(-\frac{|z-y|^2}{ct}\right)dz\,ds\,dt\,dx$$

$$\leq C\int_0^2\int_0^{t/2}\int_{\mathbb{R}^n}(ts)^{-1/2}V_j(z)t^{-n/2}\exp\left(-\frac{|z-y|^2}{ct}\right)dz\,ds\,dt$$
$$\leq C\int_{\mathbb{R}^n}V_j(z)\int_0^\infty t^{-n/2}\exp\left(-\frac{|z-y|^2}{ct}\right)dt\,dz\leq C\int_{\mathbb{R}^n}V_j(z)|z-y|^{2-n}dz\leq C_4$$

with constants C_3 and C_4 independent of $y\in\mathbb{R}^n$. So we have obtained (5.11). □

Lemma 5.12. *For $i=1,\dots,n$ the operators $\mathcal{Q}'_{i,\varepsilon}$ converge strongly as $\varepsilon\to 0$.*

Proof. The kernels of $\mathcal{Q}'_{i,\varepsilon}$ are given by

$$\mathcal{Q}'_{i,\varepsilon}(x,y)=\int_\varepsilon^2\int_0^\infty\int_{\mathbb{R}^n}\partial_i P_t(x-z)V(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}.$$

For $f\in L^1(\mathbb{R}^n)$ we have

$$\mathcal{Q}'_{i,\varepsilon}f(x)=\int_{\mathbb{R}^n}\mathcal{Q}'_{i,\varepsilon}(x,y)f(y)\,dy.$$

Note that $\mathcal{Q}'_{i,\varepsilon}(x,y)=H_{i,\varepsilon}*\phi_y(x)$, where $\phi_y(z)=V(z)\Gamma(z,y)$ and $H_{i,\varepsilon}(x)=\int_\varepsilon^2\partial_i P_t(x)\frac{dt}{\sqrt{t}}$.

It follows from the theory of singular integrals operators that for $g\in L^r(\mathbb{R}^n)$, $r>1$, the limits $\lim_{\varepsilon\to 0}H_{i,\varepsilon}*g(x)=H_i g(x)$ exist for a.e. x and in $L^r(\mathbb{R}^n)$ norm. Obviously, H_i are $L^r(\mathbb{R}^n)$-bounded operators. Moreover,

$$\left\|\sup_{0<\varepsilon<2}|H_{i,\varepsilon}*g|\right\|_{L^r(\mathbb{R}^n)}\leq C\|g\|_{L^r(\mathbb{R}^n)}. \tag{5.13}$$

Notice that for $|z|>1/2$ we have

$$\sup_{0<\varepsilon<2}|H_{i,\varepsilon}(z)|\leq C_N|z|^{-N}. \tag{5.14}$$

From (5.13) and (5.14) we deduce that if a is a function supported in a ball $B(y_0,R)$, $R>1/2$, and $\|a\|_{L^r(\mathbb{R}^n)}\leq\tau|B|^{-1+1/r}$, $r>1$, then

$$\left\|\sup_{0<\varepsilon<2}|H_{i,\varepsilon}*a|\right\|_{L^1(\mathbb{R}^n)}\leq C\tau. \tag{5.15}$$

Using Lemma 2.2 we get that for every $y\in\mathbb{R}^n$ the limit $\lim_{\varepsilon\to 0}Q'_{i,\varepsilon}(x,y)=Q'_i(x,y)$ exists for a.e. $x\in\mathbb{R}^n$. The lemma will be proved by using Lebesgue's dominated convergence theorem if we have established that

$$\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}\sup_{0<\varepsilon<2}|\mathcal{Q}'_{i,\varepsilon}(x,y)|\,dx\leq C\quad\text{and} \tag{5.16}$$

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}|\mathcal{Q}'_{i,\varepsilon}(x,y)-\mathcal{Q}'_i(x,y)|\,dx=0\quad\text{for every } y. \tag{5.17}$$

For fixed $y\in\mathbb{R}^n$ let

$$\phi_1(z)=\phi_y(z)\chi_{B(y,2)}(z),\qquad \phi_k(z)=\phi_y(z)\chi_{B(y,2^k)\setminus B(y,2^{k-1})}(z),\quad k\geq 2.$$

Then $\phi_y=\sum_{k=1}^\infty\phi_k$, where the series converges in $L^1(\mathbb{R}^n)$ and $L^r(\mathbb{R}^n)$ norm for r slightly bigger than 1. Notice that $\operatorname{supp}\phi_k\subseteq B(y,2^k)$, $\|\phi_1\|_{L^r(\mathbb{R}^n)}\leq C$, and

$$\begin{aligned}\|\phi_k\|^r_{L^r(\mathbb{R}^n)}&=\int_{B(y,2^k)\setminus B(y,2^{k-1})}V_1(z)^r|z-y|^{(2-n)r}\,dz\leq 2^{k(2-n)r}\int_{B(y,2^k)}V_1(z)^r\,dz\\&\leq C2^{k(2-n)r}2^{k(n-n_1)}\|V_1\|^r_{L^{rq}(\mathbb{V}_1)}2^{kn_1/q'}=C(2^k)^{-nr+n+2r-n_1/q}.\end{aligned} \tag{5.18}$$

Therefore, for $q<n_1/2r$ such that $V_1\in L^{rq}(\mathbb{V}_1)$, we get

$$\|\phi_k\|_{L^r(\mathbb{R}^n)}\leq C|B(y,2^k)|^{-1+1/r}2^{-\sigma k}, \tag{5.19}$$

where $\sigma = n_1/(qr) - 2 > 0$. By using (5.15) combined with (5.19) we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \sup_{0<\varepsilon<2} |\mathcal{Q}'_{i,\varepsilon}(x,y)|\,dx &= \int_{\mathbb{R}^n} \sup_{0<\varepsilon<2} |H_{i,\varepsilon}\phi_y(x)|\,dx \\ &\le \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} \sup_{0<\varepsilon<2} |H_{i,\varepsilon}\phi_k(x)|\,dx \\ &\le C \sum_{k=1}^{\infty} 2^{-\sigma k} \le C, \end{aligned} \tag{5.20}$$

which implies (5.16), since the last constant C does not depend on y. Additionally (5.17) is a consequence of (5.16) and Lebesgue's dominated convergence theorem. □

Now, Theorem 1.8 follows directly by applying (5.2) and (5.3), and Theorem 1.3.

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