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Robust uniform persistence in discrete and continuous nonautonomous systems

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ABSTRACT

This is an extension of the work in Salceanu (2011) [14] to nonautonomous systems of difference and differential equations on the positive cone of $\mathbb{R}^m$ that exhibit a positively invariant boundary hyperplane X. It is shown that when a compact subset of X, which attracts all orbits in X, is a robust uniform weak repeller, robust uniform persistence for the complementary dynamics (*i.e.*, the dynamics in $\mathbb{R}^m \setminus X$) is obtained. Additional assumptions are made, to deal with the nonautonomous nature of the systems. Some particular cases that often occur in applications are discussed and then sufficient conditions for the robust uniform persistence of the disease in two epidemic models from the literature are given.

1. Introduction

Since the first papers that addressed the idea of persistence appeared in the late seventies [1,2], the dynamical systems approach to persistence has provided quite useful tools for dealing with this important, biologically motivated mathematical problem, such as average Lyapunov functions [3], normal or external Lyapunov exponents [3–5], invariant probability measures [3,5–7], or chain recurrence, used in combination with Morse decompositions or acyclicity theory [6,8,9]. One of the first comprehensive texts on this subject was only recently put together by Smith and Thieme [10]. Thieme [11,12] gives sufficient conditions for weakly persistent nonautonomous semiflows to be strongly persistent semiflows. Mierczynski et al. [13] use the idea of unsaturated Morse decomposition of the restriction of the attractor for the skew product semiflow to the boundary of the state space to obtain persistence results for nonautonomous and random Kolmogorov parabolic systems. However, as Thieme [12] points out, the persistence theory for nonautonomous systems is still an underdeveloped area.

In this paper we build on the previous work in [14], where the author used Lyapunov exponents to obtain sufficient conditions for compact subsets of the boundary of the positive cone $\mathbb{R}^m_+$ to be *robust uniform weak repellers* and then used this to obtain robust persistence results for systems of difference and differential equations of the form

$$\begin{cases} x_{n+1} = f(z_n, \xi) \\ y_{n+1} = A(z_n, \xi) y_n \end{cases} \tag{1.1}$$

for discrete time, and respectively

$$\begin{cases} x' = f(z, \xi) \\ y' = A(z, \xi) y \end{cases} \tag{1.2}$$

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for continuous time, where $\xi \in \mathbb{R}^l$ (for some fixed positive integer l) represents a vector of parameters. We mention that similar results, but addressing only the discrete time case and uniform persistence (as opposed to *robust* uniform persistence), were obtained in [15]. Let $\phi(t, z)$ be the solution semiflow generated by (1.1), for $t = n \in \mathbb{Z}_+$ ($\mathbb{Z}_+$ denotes the set of non-negative integers), or by (1.2), for $t \in \mathbb{R}_+$ ($\mathbb{R}_+$ denotes the set of non-negative real numbers). It was shown in [14] that, if there exists a closed set B that absorbs all trajectories (corresponding to a fixed $\xi = \xi_0$) and whose restriction to an arbitrary neighborhood of the set $X = \{z = (x, y) \in \mathbb{R}^p_+ \times \mathbb{R}^q_+ \mid y = 0\}$ is compact, and if all Lyapunov exponents $\lambda(z, \eta)$ corresponding to z in $\Omega(M)$ (the union of omega limit sets of points in $M = B \cap X$) and to nonnegative unit vectors η in $\mathbb{R}^q_+$, are positive, then M is a *robust uniform weak repeller* ($\exists\, \varepsilon > 0$ and Ξ a neighborhood of ξ_0 such that $\limsup_{t\to\infty} d(\phi(t, z, \xi), M) > \varepsilon, \forall z \in \mathbb{R}^{p+q}_+ \setminus X, \xi \in \Xi$) and the system is *robustly uniformly persistent* ($\exists\, \varepsilon > 0$ and Ξ a neighborhood of ξ_0 such that $\liminf_{t\to\infty} d(\phi(t, z, \xi), X) > \varepsilon, z \in \mathbb{R}^{p+q}_+ \setminus X, \xi \in \Xi$).

Here we take a similar approach and extend the results in [14] to the nonautonomous counterparts of (1.1) and (1.2), which are

$$\begin{cases} x_{n+1} = f(n, z_n, \xi) \\ y_{n+1} = A(n, z_n, \xi)y_n \end{cases} \tag{1.3}$$

and

$$\begin{cases} x' = f(t, z, \xi) \\ y' = A(t, z, \xi)y \end{cases} \tag{1.4}$$

respectively. Here the $q \times q$ matrix function A, as well as f, are continuous in t, z and ξ, for $t \in \mathbb{R}_+$, respectively continuous in z and ξ, for $t \in \mathbb{Z}_+$. As before, $\xi \in \mathbb{R}^l$ denotes a parameter vector. Also, $z = (x, y)$ denotes an element of $\mathbb{R}^p_+ \times \mathbb{R}^q_+$ (*i.e.*, $x \in \mathbb{R}^p_+, y \in \mathbb{R}^q_+$). In the context of models (1.3) and (1.4), the concepts of *uniform weak repeller* and *robust uniform persistence* are defined in Section 3, similarly as for (1.1) and (1.2).

We point out that (as we also did in [14]) we do not necessarily assume that the absorbing set B is compact. If B is compact, the system is said to be *dissipative*, which is, more or less, a standard assumption in the literature.

The paper is organized as follows. In Section 2 we introduce the notation and some basic preliminary results. Section 3 contains our results on robust uniform persistence for the type of models presented in Section 2, together with the main assumptions. In Section 3.1 we give a result that applies particulary to periodic systems. Section 4 contains two applications: one discrete time model with periodic coefficients, and one continuous time model. For both of these modes we give sufficient conditions for the persistence of the disease in the host population, as well as for other forms of persistence.

2. Preliminaries

Let $\mathbb{T}$ denote either $\mathbb{Z}_+$, or $\mathbb{R}_+$. When $t \in \mathbb{Z}_+$, we refer to (1.3), while when $t \in \mathbb{R}_+$, we refer to (1.4), in which case we tacitly assume existence and uniqueness of solutions for all t. When $t \in \mathbb{T}$, that means we consider both discrete and continuous cases. Let $F(t, z, \xi)$ denote the right hand side in (1.3) or in (1.4):

$$F(t, z, \xi) = (f(t, z, \xi), A(t, z, \xi)y).$$

Let $\phi(t + s, s, z, \xi)$ be the (non-autonomous) *solution semiflow* (or trajectory through (s, z)) generated by (1.3) (for $t \in \mathbb{Z}_+$) or by (1.4) (for $t \in \mathbb{R}_+$). Thus, for all $r, s, t \in \mathbb{Z}_+, z \in \mathbb{R}^{p+q}_+$ and $\xi \in \mathbb{R}^l$, we have

$$\begin{aligned} &\phi(s, s, z, \xi) = z, \quad \text{and} \\ &\phi(t + s + r, r, z, \xi) = \phi(t + s + r, s + r, \phi(s + r, r, z, \xi), \xi). \end{aligned} \tag{2.1}$$

We say that a certain set S contained in $\mathbb{R}^{p+q}_+$ is *positively invariant* if any solution $\phi(t, s, z, \xi)$ is contained in S for all $s \geq 0, t \geq s$, whenever $z \in S$. Hereafter we assume that $\mathbb{R}^{p+q}_+$ is positively invariant (for all ξ). We consider our state space to be a certain positively invariant set $Z \subseteq \mathbb{R}^{p+q}_+$.

Let

$$X = \{z = (x, y) \in \mathbb{R}^p_+ \times \mathbb{R}^q_+ \mid y = 0\}. \tag{2.2}$$

Hence X is positively invariant. In what follows we are also tacitly assuming that

$Z \setminus X$ is positively invariant.

Our primary motivation for considering systems of the form (1.3) or (1.4), is to investigate robust uniform persistence of the disease/infection in biological populations, in which case the set X consists of all disease-free states, and so it is positively invariant due to biological reasons (assuming that infection cannot invade the ecosystem). Also in the view of this interpretation for X, the assumption that $Z \setminus X$ is positively invariant means that, once present, the infection cannot be completely eradicated from the host population (in finite time).

Let $P(t, s, z, \xi)$ be the solution matrix for

$$u'(t) = A(t, \phi(t, s, z, \xi), \xi)u(t), \quad t \in \mathbb{T},\ t \geq s, \tag{2.3}$$

satisfying $P(s, s, z, \xi) = I$ (for $t \in \mathbb{Z}_+$, $u'(t)$ meaning $u(t+1)$). Then one can easily check that P has the following (cocycle) property:

$$P(s+r+p, r+p, \phi(r+p, p, z, \xi), \xi)P(r+p, p, z, \xi) = P(s+r+p, p, z, \xi), \tag{2.4}$$

for all $z \in \mathbb{R}^{p+q}$, $\xi \in \mathbb{R}^l$ and all $s, r, p \in \mathbb{T}$.

The following lemma, whose proof can be found in the Appendix, gives some useful properties of matrices A and P.

Lemma 2.1. *The following hold:*

(a) $A(n, z, \xi) \geq 0$, $\forall n \in \mathbb{Z}_+, z \in X, \xi \in \mathbb{R}^l$.
(b) $a_{ij}(t, z, \xi) \geq 0$, $t \in \mathbb{R}, z \in X, \xi \in \mathbb{R}^l$, *whenever* $i \neq j$.
(c) $P(t+s, s, z, \xi) \geq 0$, $\forall z \in X, \forall t, s \in \mathbb{T}, \forall \xi \in \mathbb{R}^l$.

Vector norms are denoted by $|\cdot|$ and matrix norms by $\|\cdot\|$. $d(z_1, z_2)$ is the distance (induced by the norm $|\cdot|$) between points z_1 and z_2, while $d(z, S)$ is the distance from point z to set S (with the usual definition). By a neighborhood of a set S we mean an open set containing S.

3. Robust uniform persistence

In this section we fix a parameter $\xi_0 \in \mathbb{R}^l$ and give a sufficient condition for a compact subset of X to be a robust uniform weak repeller and then use this to obtain robust uniform persistence (both defined below with respect to ξ_0). For simplicity we sometimes write $\phi(t, s, z)$ for $\phi(t, s, z, \xi_0)$ and similarly for other quantities (P, A, etc.).

Definition 3.1. We call $S \subset X$ a *robust uniform weak repeller* if there exists $\varepsilon > 0$ and a neighborhood Ξ of ξ_0 such that

$$\limsup_{t\to\infty} d(\phi(t, s, z, \xi), S) > \varepsilon, \quad \forall s \in \mathbb{T},\ z \in Z \setminus X,\ \xi \in \Xi.$$

We occasionally make use of the following notation: if $z(t) = \phi(t, s, z, \xi)$, $t \geq s$ is a solution of (1.3) or (1.4) then let $\phi^{(2)}(t, s, z, \xi) := y(t)$. Note that $|\phi^{(2)}(t, s, z, \xi)| = d(\phi(t, s, z, \xi), X)$.

Definition 3.2. We say that $\phi(t, s, z)$ (or the equation that generates ϕ) is *robustly uniformly persistent* if there exists $\varepsilon > 0$ and a neighborhood Ξ of ξ_0 such that

$$\liminf_{t\to\infty} |\phi(t, s, z, \xi)^{(2)}| > \varepsilon, \quad \forall s \in \mathbb{T},\ z \in Z \setminus X,\ \xi \in \Xi.$$

Further, we make the following assumptions.

(A1) There exists a closed set B that absorbs all trajectories corresponding to ξ_0 (*i.e.*, $\forall s \in \mathbb{T}, z \in Z, \exists\, t(s, z)$ such that $\phi(t+s, s, z, \xi_0) \in B$, $\forall t \geq t(s, z)$).
(A2) Let

$$U = \{\eta \in \mathbb{R}^q_+ \mid |\eta| = 1\} \quad \text{and}$$
$$M = B \cap X.$$

Then $\forall z \in M, \eta \in U, \exists\, T > 0, c > 1, s_0 \in \mathbb{T}$ such that $\forall s \geq s_0, \exists\, T(s) \in (0, T]$ satisfying $|P(T(s)+s, s, z)\eta| > c$.
(A3) $\forall t_0 \in \mathbb{T}, z \in M, \exists\, K > 0$ such that $|\phi(t+s, s, z, \xi_0)| \leq K$, $\forall s \in \mathbb{T}, t \in [0, t_0]$.
(A4) There are $\widetilde{V}_B$ a neighborhood of B and $\widetilde{\Xi}$ a bounded neighborhood of ξ_0 such that, for any compact set $\widetilde{M} \subset X$, there exists a constant $K > 0$ such that:

$$\|A(t, z, \xi_0)\| \leq K \tag{3.1}$$
$$|F(t, z_2, \xi) - F(t, z_1, \xi_0)| \leq K(|z_2 - z_1| + |\xi - \xi_0|) \tag{3.2}$$
$$\|A(t, z_2, \xi) - A(t, z_1, \xi_0)\| \leq K(|z_2 - z_1| + |\xi - \xi_0|), \tag{3.3}$$

$\forall t \in \mathbb{T}, z, z_1 \in \widetilde{M}, z_2 \in \widetilde{V}_B, \xi \in \widetilde{\Xi}$.

Moreover, for every neighborhood V_B of B, $V_B \subseteq \widetilde{V}_B$, there exists Ξ a neighborhood of ξ_0, $\Xi \subseteq \widetilde{\Xi}$, satisfying the following: $\forall z \in Z, s \in \mathbb{T}, \xi \in \Xi, \exists\, t(z, s, \xi) \in \mathbb{T}$ such that

$$\phi(t+s, s, z, \xi) \in V_B, \quad \forall\, t \geq t(z, s, \xi). \tag{3.4}$$

(A5) $\{z = (x, y) \in B \mid |y| \leq \delta\}$ is bounded (hence compact), for some $\delta > 0$.

Note that (A5) implies that M (as defined in (A2)) is compact. We emphasize that the set B does not have to be compact, but only its restriction to a certain neighborhood of X has to be so. Assumption (A2) is our "key assumption" that, together with (A3) and (A4), make M a robust uniform weak repeller. From this, using (A5), we obtain robust uniform persistence (Theorem 3.5). Assumptions (3.1)–(3.3) in (A4) are primarily used (in Lemma 3.3) to ensure that continuity of $\phi(t+s, s, z, \xi)$

at (z, ξ_0), where $z \in M$, is uniform in $s \in \mathbb{T}$ and $t \in [0, T]$, for some $T > 0$. In the case when (3.1)–(3.3) hold with $\widetilde{M}$ replaced by X (that is, when they hold for all $t \in \mathbb{T}$, $z, z_1 \in X$, $z_2 \in \widetilde{V}_B$ and $\xi \in \widetilde{\Xi}$), for example when F and A are Lipschitz in z and (locally Lipschitz in) ξ, assumption (A3) is not needed. However, (A3) holds in most applications (including the ones presented in Section 4), in which case (A4) is "more likely" to hold (especially in the case when B is compact and F, A are C^1 in z and ξ). Condition (3.4) prevents B from being "too sensitive" to changes in parameters. Also note that only (A4) "depends" on ξ, while the other assumptions made above are only for the fixed parameter ξ_0.

One of our main goals is to get M to be a robust uniform weak repeller. Thus, using assumption (3.4), the semiflow property (2.1) and that $Z \setminus X$ is positively invariant, we can assume that, for every V_B and Ξ as in (A4), $V_B \setminus X$ is positively invariant, namely $\phi(t+s, s, z, \xi)$ remains in $V_B \setminus X$, for all $z \in V_B \setminus X$, $s, t \in \mathbb{T}$ and $\xi \in \Xi$.

Lemma 3.3. *Assume* (A1)–(A4) *hold. Then for every* $(\hat{z}, \hat{\eta}) \in M \times U$ *there exist* $\hat{c} > 1$, $\widehat{T} > 0$, $\hat{s} \in \mathbb{T}$ *and bounded neighborhoods* $\widehat{W}$ *and* $\widehat{\Xi}$ *of* $(\hat{z}, \hat{\eta})$ *and* ξ_0 *respectively, such that for all* $s \geq \hat{s}$, *there is a* $T(s) \in (0, \widehat{T}]$ *and*

$$|P(T(s)+s, s, z, \xi)\eta| > \hat{c}, \quad \forall (z, \eta) \in \widehat{W} \text{ with } z \notin X, \ \forall \xi \in \widehat{\Xi}. \tag{3.5}$$

Proof. Let $(\hat{z}, \hat{\eta}) \in M \times U$. From (A2), there exist $\hat{c} > 1$, $\widehat{T} > 0$ and $s_0 \in \mathbb{T}$ such that $\forall s \geq s_0$, $\exists\, 0 < T(s) \leq \widehat{T}$ and

$$|P(T(s)+s, s, \hat{z}, \xi_0)\hat{\eta}| > \hat{c}. \tag{3.6}$$

Let $\hat{a} \in (1, \hat{c})$. Let V_B and Ξ as in (A4). Now we will argue by contradiction and suppose that the conclusion of the lemma does not hold. Then it follows that there exist sequences $(s_k)_k \subseteq \mathbb{T}$, with $0 < T(s_k) \leq \widehat{T}$, and $(z^k, \eta_k)_k \subset (\mathbb{R}^m_+ \setminus X) \times U$, $(\xi_k)_{k\geq 1} \subset \Xi$, with $(z^k, \eta_k) \to (\hat{z}, \hat{\eta})$ and $\xi_k \to \xi_0$ as $k \to \infty$ (hence $z^k \to \hat{z}$ and $\eta_k \to \hat{\eta}$), such that

$$|P(T(s_k)+s_k, s_k, z^k, \xi_k)\eta_k| \leq \hat{a}. \tag{3.7}$$

On the other hand, from (3.6) we have

$$|P(T(s_k)+s_k, s_k, \hat{z}, \xi_0)\hat{\eta}| > \hat{c}. \tag{3.8}$$

Thus,

$$\begin{aligned} \Delta_k &:= |P(T(s_k)+s_k, s_k, z^k, \xi_k)\eta_k - P(T(s_k)+s_k, s_k, \hat{z}, \xi_0)\hat{\eta}| \\ &\geq |P(T(s_k)+s_k, s_k, \hat{z}, \xi_0)\hat{\eta}| - |P(T(s_k)+s_k, s_k, z^k, \xi_k)\eta_k| \\ &> \hat{c} - \hat{a} > 0, \quad \forall k \in \mathbb{Z}_+. \end{aligned} \tag{3.9}$$

$P(t, s, z, \xi)\eta$ is the solution of $u' = A(t, \phi(t, s, z, \xi), \xi)u$, that is equal to η at $t = s$. Let $u_k(t) := P(t, s_k, z^k, \xi_k)\eta_k$ and $\hat{u}_k(t) := P(t, s_k, \hat{z}, \xi_0)\hat{\eta}$.

Assumption (A3) implies that there exists $\widetilde{M}$ a compact set, $M \subseteq \widetilde{M} \subset X$, such that $\phi(t+s_k, s_k, \hat{z}, \xi_0) \in \widetilde{M}$ for all k and all $t \in (0, \widehat{T}]$. It follows then from (3.2) and Gronwall's inequality, for $t \in \mathbb{R}_+ \cap (0, \widehat{T}]$, or from a simple iteration, for $t \in \mathbb{Z}_+ \cap (0, \widehat{T}]$, that we can make $|\phi(t+s_k, s_k, z^k, \xi_k) - \phi(t+s_k, s_k, \hat{z}, \xi_0)|$ arbitrarily small, starting with a certain k. Then, using (3.3) we obtain that, for any $\delta > 0$ there exists a $N \in \mathbb{Z}_+$ such that

$$\|A(t+s_k, \phi(t+s_k, s_k, z^k, \xi_k), \xi_k) - A(t+s_k, \phi(t+s_k, s_k, \hat{z}, \xi_0), \xi_0)\| \leq \delta, \quad \forall k \geq N, \ t \in [0, \widehat{T}]. \tag{3.10}$$

This implies, using (3.1) and the triangle inequality, that there exists $K_1 > 0$ such that

$$\|A(t+s_k, \phi(t+s_k, s_k, z^k, \xi_k), \xi_k)\| \leq K_1, \quad \forall k \geq N, \ t \in [0, \widehat{T}]. \tag{3.11}$$

From (3.11) we obtain (using Gronwall's inequality in the continuous case) that there exists $K_2 > 0$ such that

$$|u_k(t+s_k)| \leq K_2, \quad \forall k \geq N, \ t \in [0, \widehat{T}]. \tag{3.12}$$

First we consider the continuous case. Thus, let $N \in \mathbb{Z}_+$, $\delta, K_1, K_2 > 0$ for which (3.10)–(3.12) hold. We also assume that $|\eta_k - \hat{\eta}| \leq \delta$, for all $k \geq N$. Then

$$\begin{aligned} \Delta_k &= |u_k(T(s_k)+s_k) - \hat{u}_k(T(s_k)+s_k)| \\ &\leq \delta + \int_{s_k}^{T(s_k)+s_k} |A(\tau, \phi(\tau, s_k, z^k, \xi_k), \xi_k)u_k(\tau) - A(\tau, \phi(\tau, s_k, \hat{z}, \xi_0), \xi_0)\hat{u}_k(\tau)| d\tau \\ &\leq \delta + \int_{s_k}^{T(s_k)+s_k} |A(\tau, \phi(\tau, s_k, \hat{z}, \xi_0), \xi_0)\hat{u}_k(\tau) - A(\tau, \phi(\tau, s_k, \hat{z}, \xi_0), \xi_0)u_k(\tau)| d\tau \\ &\quad + \int_{s_k}^{T(s_k)+s_k} |A(\tau, \phi(\tau, s_k, \hat{z}, \xi_0), \xi_0)u_k(\tau) - A(\tau, \phi(\tau, s_k, z^k, \xi_k), \xi_k)u_k(\tau)| d\tau \end{aligned}$$

$$\le \delta + \int_{s_k}^{T(s_k)+s_k} \|A(\tau, \phi(\tau, s_k, \hat{z}, \xi_0), \xi_0)\| |\hat{u}_k(\tau) - u_k(\tau)| d\tau$$
$$+ \int_{s_k}^{T(s_k)+s_k} \|A(\tau, \phi(\tau, s_k, \hat{z}, \xi_0), \xi_0) - A(\tau, \phi(\tau, s_k, z^k, \xi_k), \xi_k)\| |u_k(\tau)| d\tau$$
$$\le \delta + K_1 \int_{s_k}^{T(s_k)+s_k} |\hat{u}_k(\tau) - u_k(\tau)| d\tau + \delta \int_{s_k}^{T(s_k)+s_k} |u_k(\tau)| d\tau$$
$$\le \delta + K_1 \int_{s_k}^{T(s_k)+s_k} |\hat{u}_k(\tau) - u_k(\tau)| d\tau + \delta K_2, \quad \forall k \ge N.$$

Thus, from Gronwall's inequality we have

$$|u_k(T(s_k) + s_k) - \hat{u}_k(T(s_k) + s_k)| \le \delta(1 + K_2) e^{K_1 \widehat{T}}, \quad \forall k \ge N. \tag{3.13}$$

We can choose δ sufficiently small such that (3.13) is in contradiction to (3.9).

Now consider the discrete case. We have

$$P(t + s, s, z) = A(s + t - 1, \phi(s + t - 1, s, z)) \cdot \cdots \cdot A(s, z). \tag{3.14}$$

Using (3.10), (3.11) and (3.1), in combination with the following inequality (which holds for any square matrices A_1, A_2, B_1, B_2)

$$\|A_2 B_2 - A_1 B_1\| \le \|A_2\| \|B_2 - B_1\| + \|A_2 - A_1\| \|B_1\|,$$

it is a simple exercise to show that Δ_k can be made arbitrarily small, beginning with some k, which again contradicts (3.9). This completes our proof. $\square$

With the help of the previous lemma, we give now a sufficient condition for M to be a robust uniform weak repeller, which will be used later (in Theorem 3.5) to obtain robust uniform persistence, in the sense of Definition 3.2.

Theorem 3.4. *Assume* (A1)–(A4) *hold. Then there exist bounded neighborhoods V_M and Ξ_0 respectively of M and ξ_0, and $c > 1, T > 0, s_0 \in \mathbb{T}$ such that for all $s \ge s_0$, $z \in V_M \setminus X$ and $\xi \in \Xi_0$ having the property that $\phi(t+s, s, z, \xi) \in V_M$, $\forall t \in [0, t_0]$, for some $t_0 > 0$, there exist numbers $s = \nu_0, \nu_1, \ldots, \nu_n$ satisfying:*

(1^0) $0 < \nu_i - \nu_{i-1} \le T$, *for all* $i = 1, \ldots, n$
(2^0) $\nu_{n-1} \le t_0 + s < \nu_n$ *and*
(3^0) $|P(\nu_i, s, z, \xi)y| \ge c^i |y|, \forall i \in \{1, \ldots, n\}$.

In particular, every solution $\phi(t, s, z, \xi)$, with $z \in V_M \setminus X$ and $\xi \in \Xi_0$, leaves V_M.

Proof. We apply Lemma 3.3 to each $(\hat{z}, \hat{\eta}) \in M \times U$ and obtain the corresponding $\hat{c}, \widehat{T}, \hat{s}$ and neighborhoods $W(\hat{z}, \hat{\eta})$ and $\widehat{\Xi}$. Since $M \times U$ is compact, the open cover $\cup_{(\hat{z},\hat{\eta}) \in M \times U} W(\hat{z}, \hat{\eta})$ of $M \times U$ has a finite subcover. Thus, there exists a positive integer k such that $M \times U \subset W := \cup_{i=1}^k W_i$ and for each $i \in \{1, \ldots, k\}$ we have: $\forall s \ge s_i$, $\exists\, T(s) \in (0, T_i]$ such that

$$|P(T(s) + s, s, z, \xi)\eta| > c_i, \quad \forall (z, \eta) \in W_i \text{ with } z \notin X,\ \forall \xi \in \Xi_i,$$

where $c_i > 1$, $T_i > 0$, $s_i \in \mathbb{T}$, W_i is a neighborhood of $(z^i, \eta_i) \in M \times U$ and Ξ_i is a neighborhood of ξ_0.

There exists an open, bounded neighborhood V_M of M, such that $V_M \times U \subseteq W$. Let $\Xi_0 \subseteq \Xi_i$, for all $i \in \{1, \ldots, k\}$ and define $c = \min\{c_1, \ldots, c_k\}$, $T = \max\{T_1, \ldots, T_k\}$ (hence $c > 1$, $T > 0$) and $s_0 = \max\{s_1, \ldots, s_k\}$. Hence we have that

$$|P(T(s) + s, s, z, \xi)\eta| > c, \quad \forall z \in V_M \setminus X,\ \eta \in U,\ \xi \in \Xi_0,\ s \ge s_0, \tag{3.15}$$

for some $T(s) \in (0, T]$.

Now let $z = (x, y) \in V_M \setminus X$, $\xi \in \Xi_0$ and $s \ge s_0$, and assume that $\phi(t + s, s, z, \xi) \in V_M$, $\forall t \in [0, t_0]$, for some $t_0 > 0$. Let $\eta = y/|y|$. Then, from (3.15), we have

$$|P(\nu_1, s, z, \xi)y| > c|y|, \quad \text{where } \nu_1 := T(s) + s.$$

If $\nu_1 > t_0 + s$ we stop here, since (1^0)–(3^0) hold with $n = 1$. Otherwise, assuming that we have already obtained $\nu_0, \ldots, \nu_r (\nu_r \le t_0 + s)$, satisfying ($1^0$) and ($3^0$) (with n replaced by r), we define ν_{r+1} in a similar manner as ν_1. Thus, let $z^r = \phi(\nu_r, s, z, \xi)$ and $\eta_r = P(\nu_r, s, z, \xi)y/|P(\nu_r, s, z, \xi)y|$. Note that η_r is well defined, because $Z \setminus X$ is positively invariant. Also, $P(\nu_r, s, z, \xi)y = \phi^{(2)}(\nu_r, s, z, \xi) \ge 0$, hence $\eta_r \in U$. So again, from (3.15),

$$|P(T(\nu_r) + \nu_r, \nu_r, z^r, \xi)\eta_r| > c, \quad \text{where } T(\nu_r) \in (0, T]. \tag{3.16}$$

Define $\nu_{r+1} = T(\nu_r) + \nu_r$. Then (3.16) implies, using (2.4), that

$$|P(\nu_{r+1}, s, z, \xi)y| > c^{r+1}|y|. \tag{3.17}$$

Note that, by construction, $0 < \nu_r - \nu_{r-1} \le T$. Also, for fixed s, $(\nu_r - s) \to \infty$ as $r \to \infty$ since, otherwise, it would follow that $\phi^{(2)}(\nu_r, s, z, \xi)$ is bounded, which would contradict (3.17). Hence we obtain (1^0)–(3^0) by defining $n = \min\{r \mid \nu_r > t_0 + s\}$.

The fact that all solutions $\phi(t, s, z, \xi)$ with $z \in V_M \setminus X$ and $\xi \in \Xi_0$ leave V_M follows directly from (3^0) arguing by contradiction: suppose $\phi(t, s, z, \xi) \in V_M$ for all $t \ge s$. If $s \ge s_0$ then

$$|\phi^{(2)}(\nu_i, s, z, \xi)| = |P(\nu_i, s, z, \xi)y| > c^i|y|, \quad \forall i \ge 0. \tag{3.18}$$

If $s < s_0$, let $\bar z = (\bar x, \bar y) = \phi(s_0, s, z, \xi)$ (note that $\bar y \ne 0$). Then $\phi(t + s_0, s_0, \bar z, \xi) \in V_M$ for all $t \ge 0$, hence there exists a sequence $s_0 = \tilde\nu_0, \tilde\nu_1, \dots, \tilde\nu_i, \dots$ such that

$$\begin{aligned}|\phi^{(2)}(\tilde\nu_i, s, z, \xi)| &= |\phi^{(2)}(\tilde\nu_i, s_0, \bar z, \xi)| \\ &= |P(\tilde\nu_i, s_0, \bar z, \xi)\bar y| > c^i|\bar y|, \quad \forall i \ge 0.\end{aligned} \tag{3.19}$$

Thus, either (3.18) or (3.19) gives a contradiction to V_M being bounded. □

Our main result, which we state next, provides sufficient conditions for robust persistence of the species that make up the vector y.

Theorem 3.5. *Assume that* (A1)–(A5) *hold. Then* (1.4) *is robustly uniformly persistent: there exist $\varepsilon > 0$ and Ξ_0 a bounded neighborhood of ξ_0 such that*

$$\liminf_{t\to\infty} |\phi^{(2)}(t + s, s, z, \xi)| > \varepsilon, \quad \forall s \in \mathbb{T},\ z \in Z \setminus X,\ \xi \in \Xi_0. \tag{3.20}$$

Same holds for (1.3) *if for any $\delta > 0$ there exists V a neighborhood of B and Ξ a neighborhood of ξ_0 such that*

$$\inf\{d(F(z, \xi), X) \mid z \in V, |y| \ge \delta, \xi \in \Xi\} > 0. \tag{3.21}$$

Proof. Since (A1)–(A4) hold, let V_M be a neighborhood of M, $\widetilde\Xi_0$ a neighborhood of ξ_0, and $c > 1$, $T > 0$, $s_0 \in \mathbb{T}$ given by Theorem 3.4. We claim first that there exists V_B^1 a neighborhood of B such that

$$\delta := \inf\{|y| \mid z = (x, y) \in \overline{V_B^1 \setminus V_M}\} > 0. \tag{3.22}$$

If (3.22) does not hold then we can find a sequence $(z^n)_n \subset Z \setminus V_M$ $(*)$ satisfying $|y^n| \to 0$ and $d(z^n, B) \to 0$. But then from (A5) we have that $(z^n)_n$ is bounded, thus it has a convergent subsequence $z^{n_k} \to z$. Hence $z \in X$. On the other hand, there exists a sequence $(b_n)_n \subset B$ such that $d(z^n, b_n) \to 0$. Then, since

$$d(z, b_{n_k}) \le d(z, z^{n_k}) + d(z^{n_k}, b_{n_k}),$$

we have that $d(z, b_{n_k}) \to 0$ as $k \to \infty$. This implies that $z \in B$ (because B is closed), hence $z \in M$. But, on the other hand, $z \notin V_M$ (see $(*)$), and so we have a contradiction to $M = B \cap X$. Thus the claim holds.

Now, for the δ given by (3.22), there exist $V_B^2 \subseteq V_B^1$ a neighborhood of B and $\Xi_0 \subseteq \widetilde\Xi_0$ a neighborhood of ξ_0 for which (3.21) holds (that is, (3.21) holds with V and Ξ replaced by V_B^2 and Ξ_0, respectively).

Let $\hat z \in Z \setminus X$, $\hat s \in \mathbb{T}$ and $\hat\xi \in \Xi_0$. From (3.4) and Theorem 3.4 we can assume that $\hat z \in V_B^2 \setminus V_M$. If $\phi(t + \hat s, \hat s, \hat z, \hat\xi)$ never enters V_M (for any $t \ge 0$) then obviously

$$\liminf_{t\to\infty} |\phi^{(2)}(t + \hat s, \hat s, \hat z, \hat\xi)| > \delta.$$

Otherwise there exist $\hat t \ge 0$ and $\tilde\varepsilon \in (0, \delta)$, $\tilde\varepsilon$ independent of $\hat z$ and $\hat\xi$, such that $\phi(\hat t + \hat s, \hat s, \hat z, \hat\xi) \in V_M$ and $|\phi(t + \hat s, \hat s, \hat z, \hat\xi)^{(2)}| \ge \tilde\varepsilon$ for all $t \in [0, \hat t]$. This is obvious in the continuous case, while in the discrete case it follows from (3.21). Thus, it suffices to prove (3.20) only for $z \in V_{\tilde\varepsilon} = \{z = (x, y) \in V_B^2 \cap V_M \mid |y| \ge \tilde\varepsilon\}$. So let $\tilde z \in V_{\tilde\varepsilon}$, $\tilde s \in \mathbb{T}$ and $\tilde\xi \in \Xi_0$. Without loss of generality, we can assume that $\tilde s \ge s_0$. Let $t_0 \in \mathbb{T}$ and assume that $\phi(t + \tilde s, \tilde s, \tilde z, \tilde\xi) \in V_M$ for all $t \in [0, t_0]$. Then let $\tilde s = \nu_0, \nu_1, \nu_2, \dots, \nu_n$ be as in Theorem 3.4. Thus, we have

$$|P(\nu_i, \tilde s, \tilde z, \tilde\xi)\tilde y| \ge c^i|\tilde y| \ge c^i\tilde\varepsilon \ge \tilde\varepsilon, \quad \forall i \in \{1, \dots, n\}. \tag{3.23}$$

Fix a $t \in [0, t_0]$. Then $\nu_{i-1} \le t + \tilde s \le \nu_i$, for some $i \in \{1, \dots, n\}$ (see (2^0) in Theorem 3.4). Using (2.4) we have

$$|P(\nu_i, \tilde s, \tilde z, \tilde\xi)\tilde y| \le \|P(\nu_i, t + \tilde s, \phi(t + \tilde s, \tilde s, \tilde z, \tilde\xi), \tilde\xi)\| |P(t + \tilde s, \tilde s, \tilde z, \tilde\xi)\tilde y|. \tag{3.24}$$

Using the assumptions (A3), (3.1)–(3.3) and that $0 \le \nu_i - \nu_{i-1} \le T$, similar to the way we obtained (3.12), we can find that there is a $C > 0$, independent of $\tilde z$, $\tilde s$ and $\tilde\xi$ (and depending only on V_M, Ξ_0 and T) such that $\|P(\nu_i, t + \tilde s, \phi(t + \tilde s, \tilde s, \tilde z, \tilde\xi), \tilde\xi)\| \le C$. Then, from (3.23) and (3.24) we have that

$$|P(t + \tilde s, \tilde s, \tilde z, \tilde\xi)\tilde y| \ge C^{-1}\tilde\varepsilon. \tag{3.25}$$

Hence, since $\phi^{(2)}(t + \tilde s, \tilde s, \tilde z, \tilde\xi) = P(t + \tilde s, \tilde s, \tilde z, \tilde\xi)\tilde y$, we have that (3.20) holds, where we can define $\varepsilon = \min\{\tilde\varepsilon, C^{-1}\tilde\varepsilon\}$. □

Remark 3.6. We need to mention that, in the continuous case, one does not need (A2) in order to prove (3.25). In fact, if V_B and Ξ are such that (A1), (A3) and (A4) hold, and $V_M \subset V_B$ is any bounded set containing M, it can be shown in this case (*i.e.*, $t \in \mathbb{R}$) that for every $t_0 \geq 0$ there exists a positive constant $C > 0$ (depending only on V_M, Ξ and t_0) such that

$$\phi(t+s,s,z,\xi) \in V_M, \quad \forall t \in [0,t_0] \Rightarrow |P(t+s,s,z,\xi)\eta| \geq C,\ \forall \eta \in U.$$

Thus we make the following assumption, only needed in the discrete case.

(D) For every $z \in M$, $\eta \in U$ and $t_0 \in \mathbb{Z}_+$, there exists $C > 0$ such that $|P(t_0+s,s,z,\xi_0)\eta| \geq C$, for all $s \in \mathbb{Z}_+$.

We give next a result that can be useful in applications where there is a finite number of orbits in X, that attract all the other orbits in X (we have such an application in Section 4.2). Let $O(s,z) := \{\phi(t,s,z) \mid t \geq s\}$ denote the (positive) orbit through (s,z).

Proposition 3.7. *Assume that* (A1), (A3), (A4) *and* (D) *hold and that there are finitely many orbits* $\mathcal{O}_1,\ldots,\mathcal{O}_r$ *in* X*, where each* $\mathcal{O}_i = O(0,a_i)$*, for some* $a_i \in M$*, with the following properties:*

(i) $\forall z \in M, \varepsilon > 0, \exists\, s_0, t_0 \in \mathbb{T}$ *such that,* $\forall s \geq s_0, \exists i \in \{1,\ldots,r\}$ *and* $|\phi(t+s,s,z) - \phi(t+s,0,a_i)| < \varepsilon, \forall t \geq t_0$*;*
(ii) $\forall \mathcal{O}_i, \exists\, c_i > 1, T_i > 0, s_i \in \mathbb{T}$ *such that* $\forall \hat{z} \in \mathcal{O}_i, \eta \in U, \exists\, T(\hat{z},\eta) \in (0,T_i]$ *satisfying* $|P(T(\hat{z},\eta)+\hat{s},\hat{s},\hat{z})\eta| > c_i$*, where* $O(\hat{s},\hat{z}) \subseteq \mathcal{O}_i, \hat{s} \geq s_i$*.*

Then (A2) *holds.*

Proof. Let $\tilde{z} \in M, \tilde{c} > 1$. From (A1) we have that

$$\exists s_0 \in \mathbb{T} \quad \text{such that } |\phi(t+s,0,a_i)| \in M,\ \forall s \geq s_0,\ t \geq 0,\ i \in \{1,\ldots,r\}. \tag{3.26}$$

Let $s_{\max} = \max\{s_0,s_1,\ldots,s_r\}$, $c = \min\{c_1,\ldots,c_r\}$, $T = \max\{T_1,\ldots,T_r\}$. Let $a \in (1,c)$. Let $\widetilde{V}_B$ as in (A4).

We claim that for each $i \in \{1,\ldots,r\}$ there is an $\varepsilon_i > 0$ such that, for every pair $(\hat{s},\hat{z})$ such that $O(\hat{s},\hat{z}) \subseteq \mathcal{O}_i, \hat{s} \geq s_{\max}$, and for every $\eta \in U$ and $z \in B_{\varepsilon_i}(\hat{z}) \cap X$

$$\phi(t+\hat{s},\hat{s},z) \in \widetilde{V}_B, \quad \forall t \in [0,T] \Rightarrow |P(T(\hat{z},\eta)+\hat{s},\hat{s},z)\eta| > a, \tag{3.27}$$

where $T(\hat{z},\eta)$ and $\hat{s}$ are as in (ii). Otherwise, there would be sequences $\hat{z}^k \in \mathcal{O}_i, z^k \in X, \eta_k \in U$, with $|z^k - \hat{z}^k| \to 0$, such that $|P(T(\hat{z}^k,\eta_k)+\hat{s}_k,\hat{s}_k,z^k)\eta_k| \leq a$, where $O(\hat{s}_k,\hat{z}^k) \subseteq \mathcal{O}_i, \hat{s}_k \geq s_{\max}$, and $\phi(t+\hat{s}_k,\hat{s}_k,z^k) \in \widetilde{V}_B$ for all $t \in [0,T]$. On the other hand, from (ii) we have $|P(T(\hat{z}^k,\eta_k)+\hat{s}_k,\hat{s}_k,\hat{z}^k)\eta_k| > c$. Hence

$$|P(T(\hat{z}^k,\eta_k)+\hat{s}_k,\hat{s}_k,\hat{z}^k)\eta_k - P(T(\hat{z}^k,\eta_k)+\hat{s}_k,\hat{s}_k,z^k)\eta_k| \geq c-a > 0, \quad \forall k. \tag{3.28}$$

But now one can show, the same way as in the proof of Lemma 3.3, that (3.28) cannot hold. Hence the claim holds. Thus let $0 < \varepsilon < \min\{\varepsilon_1,\ldots,\varepsilon_r\}$ such that $B_\varepsilon(z) \cap X \subseteq \widetilde{V}_B$ for all $z \in \mathcal{O}_i \cap M$ and all $i \in \{1,\ldots,r\}$. Then, from (i), there exists $t_0 > 0, \tilde{s}_0 \geq s_{\max}$ such that $\phi(t+s,s,\tilde{z}) \in \widetilde{V}_B$ for all $t \geq t_0$ and all $s \geq \tilde{s}_0$. Let $\tilde{\eta} \in U, s \geq \tilde{s}_0$ and assume that $\phi(t+s,s,\tilde{z})$ is attracted to some $\mathcal{O}_i$ (*i.e.*, (i) holds) and i is fixed with this property.

For the rest of the proof we will use the same idea as in the proof of Theorem 3.4. Thus, let $z^1 = \phi(t_0+s,s,\tilde{z})$, $\hat{z}^1 = \phi(t_0+s,0,a_i)$, $\hat{s}_1 = t_0+s$ and $\eta_1 = P(t_0+s,s,\tilde{z})\tilde{\eta}/|P(t_0+s,s,\tilde{z})\tilde{\eta}|$. Then, since $|z^1 - \hat{z}^1| < \varepsilon$ and $\phi(t+\hat{s}_1,\hat{s}_1,z^1) \in \widetilde{V}_B, \forall t \geq 0$, from (3.27) we have that there exists $T(\hat{z}^1,\eta_1) \in (0,T]$ such that $|P(T(\hat{z}^1,\eta_1)+\hat{s}_1,\hat{s}_1,z^1)\eta_1| > a$. This implies, using (2.4), that

$$|P(T(\hat{z}^1,\eta_1)+\hat{s}_1,s,\tilde{z})\tilde{\eta}| > a|P(t_0+s,s,\tilde{z})\tilde{\eta}|. \tag{3.29}$$

Now let $t_1 = T(\hat{z}^1,\eta_1)+t_0$, $z^2 = \phi(t_1+s,s,\tilde{z})$, $\hat{z}^2 = \phi(t_1+s,0,a_i)$, $\hat{s}_2 = t_1+s$ and $\eta_2 = P(t_1+s,s,\tilde{z})\tilde{\eta}/|P(t_1+s,s,\tilde{z})\tilde{\eta}|$. By repeating the previous step we obtain

$$|P(T(\hat{z}^2,\eta_2)+\hat{s}_2,s,\tilde{z})\tilde{\eta}| > a^2|P(t_0+s,s,\tilde{z})\tilde{\eta}|, \tag{3.30}$$

for some $T(\hat{z}^2,\eta_2) \in (0,T]$. Clearly, by continuing this algorithm, we obtain that

$$|P(T(\hat{z}^n,\eta_n)+\hat{s}_n,s,\tilde{z})\tilde{\eta}| > a^n|P(t_0+s,s,\tilde{z})\tilde{\eta}|, \quad \forall n \geq 1, \tag{3.31}$$

where $\hat{s}_n = T(\hat{z}^{n-1},\eta_{n-1}) + \cdots + T(\hat{z}^1,\eta_1) + t_0 + s$ and each $T(\hat{z}^i,\eta_i) \in (0,T]$, for $i = 1,\ldots,n$. But $|P(t_0+s,s,\tilde{z})\tilde{\eta}|$ is bounded below by a positive constant independent of s (see Remark 3.6). Thus, there exists an $\tilde{n}$, independent of s, such that $a^{\tilde{n}}|P(t_0+s,s,\tilde{z})\tilde{\eta}| > c$. Define $T(s) = T(\hat{z}^{\tilde{n}},\eta_{\tilde{n}}) + \cdots + T(\hat{z}^1,\eta_1) + t_0$. Hence $T(s) \in (0,\tilde{n}T+t_0]$. Note that t_0 is also independent of s, as it depends only on the attracting orbit $\mathcal{O}_i$. Since there are only finitely many such orbits, we conclude that (A2) holds. $\square$

3.1. Periodic systems

In this section we consider the case when $F(t,z)$ is periodic in t (*i.e.*, there exists $\tau \in \mathbb{T} \setminus \{0\}$ such that $F(t,z) = F(t+\tau,z)$, $\forall t \in \mathbb{T}, z \in Z$) and provide sufficient conditions for assumption (A2) to hold. Note that assumption (3.1) holds in this case (as A is assumed to be a continuous matrix function). Let $\mathcal{P}$ denote the collection of all periodic orbits in M.

Proposition 3.8. *Assume that $\mathcal{P}$ is not empty and every orbit in M is attracted to a member of $\mathcal{P}$. If for each periodic orbit $O(\hat{s},\hat{z})$ in $\mathcal{P}$ and each $\eta \in U$, $P(\tau+\hat{s},\hat{s},\hat{z})$ has a left eigenvector v corresponding to an eigenvalue r with $|r| > 1$, and $v\eta = v_1\eta_1 + \cdots + v_q\eta_q \neq 0$, then* (A2) *holds.*

Proof. Note that, from the definition of $P(t,s,z)$, it follows that

$$P(t+s,s,z)\eta = P(t+s+\tau,s+\tau,z)\eta, \quad \forall s,t \in \mathbb{T},\ z \in Z,\ \eta \in U. \tag{3.32}$$

Thus, it suffices to show that

$$\begin{aligned} &\forall z \in M, \quad \forall \eta \in U, \exists T_{z,\eta} > 0, \exists c(z,\eta) > 1, \text{ such that} \\ &\forall s \in [0,\tau], \quad \exists\, 0 < T(s) \le T_{z,\eta} \text{ and } |P(T(s)+s,s,z)\eta| > c(z,\eta). \end{aligned} \tag{3.33}$$

We claim that it is even enough to have only the following:

$$\begin{aligned} &\forall z \in M, \quad \forall \eta \in U, \forall s \in [0,\tau], \exists\, T > 0 \text{ such that} \\ &|P(T+s,s,z)\eta| > 1. \end{aligned} \tag{3.34}$$

In the discrete case, this is based on the fact that there are only finitely many s's in $[0,\tau]$. We now prove (3.33) for the continuous case, based on (3.34). Thus, let $z \in M$, $\eta \in U$ be fixed. Then, for all $\hat{s} \in [0,\tau]$, there exist $T(\hat{s}) > 0$ such that $|P(T(\hat{s})+\hat{s},\hat{s},z)\eta| > 1$. $s \mapsto |P(T(\hat{s})+s,s,z)\eta|$ being continuous, there exist $c(\hat{s}) > 1$ and $I_{\hat{s}}$ a neighborhood of $\hat{s}$, such that $|P(T(\hat{s})+s,s,z)\eta| > c(\hat{s})$, for all $s \in I_{\hat{s}}$. Since $[0,\tau]$ is compact and contained in $\cup_{s\in[0,\tau]} I_s$, there exists a finite set $\{s_1,\dots,s_k\} \subset [0,\tau]$ such that $[0,\tau] \subseteq I := \cup_{i=1}^{k} I_{s_i}$. Let $c(z,\eta) = \min\{c(s_1),\dots,c(s_k)\}$. Hence, for all $s \in [0,\tau]$ there is a $T(s) \in \{T(s_1),\dots,T(s_k)\}$ such that $|P(T(s)+s,s,z)\eta| > c(z,\eta)$. By defining $T_{z,\eta}$ to be $\max\{T(s_1),\dots,T(s_k)\}$ we have that (3.33) holds.

Now we prove (3.34). Let $z \in M$ and $\eta \in U$. Let $s \in [0,\tau]$ and let $\mathcal{O} = O(\hat{s},\hat{z})$ be the periodic orbit in $\mathcal{P}$ that attracts $\phi(t+s,s,z)$. Then

$$\begin{aligned} P(k\tau+\hat{s},\hat{s},\hat{z}) &= P(\tau+(k-1)\tau+\hat{s},\tau+\hat{s},\phi(\tau+\hat{s},\hat{s},\hat{z}))P(\tau+\hat{s},\hat{s},\hat{z}) \\ &= P((k-1)\tau+\hat{s},\hat{s},\hat{z})P(\tau+\hat{s},\hat{s},\hat{z}). \end{aligned}$$

From this it follows, by an induction argument, that

$$P(k\tau+\hat{s},\hat{s},\hat{z}) = [P(\tau+\hat{s},\hat{s},\hat{z})]^k. \tag{3.35}$$

Let v be an eigenvector of $P(\tau+\hat{s},\hat{s},\hat{z})$ (without loss of generality we assume it is a unit vector) such that $v\eta \neq 0$, and let r be an eigenvalue corresponding to v, $|r| > 1$. Then, using (3.35) we obtain

$$|P(k\tau+\hat{s},\hat{s},\hat{z})\eta| \ge |vP(k\tau+\hat{s},\hat{s},\hat{z})\eta| = |r|^k|v\eta|. \tag{3.36}$$

Then there exists $k \in \mathbb{Z}_+$ such that $|r|^k|v\eta| > 1$. Consequently we obtain

$$|P(k\tau+\hat{s},\hat{s},\hat{z})\eta| > 1. \tag{3.37}$$

Using (3.37) and that P is continuous in (s,z,η), in the continuous case (respectively continuous in (z,η) in the discrete case), and $[0,\tau] \times \mathcal{O} \times U$ is compact, it follows that there exist a neighborhood V of $\mathcal{O}$, $c > 1$ and $k_1,\dots,k_l \in \mathbb{Z}_+$ such that

$$\forall s \in [0,\tau], \quad z \in V, \eta \in U, \exists\, i \in \{1,\dots,l\}, |P(k_i\tau+s,s,z)\eta| \ge c. \tag{3.38}$$

Let $t(s,z)$ be such that $\phi(t+s,s,z) \in V$, for all $t \ge t(s,z)$. Let $\tilde{s} = (t(s,z)+s) \mod \tau$, $\tilde{z} = \phi(\tilde{s},s,z)$ and $\tilde{\eta} = P(\tilde{s},s,z)\eta/|P(\tilde{s},s,z)\eta|$. From Lemma 2.1 part (c) and assumption (D), $\tilde{\eta}$ is well defined and belongs to U. Note that (3.38) is similar to (3.15). Thus, since $\phi(t+\tilde{s},\tilde{s},\tilde{z}) \in V$, for all $t \ge 0$, as in the proof of Theorem 3.4, we can obtain that for every $i \in \mathbb{Z}_+$ there exists $\nu_i > 0$ such that $|P(\nu_i,\tilde{s},\tilde{z})\tilde{\eta}| \ge c^i$. This implies that

$$|P(\nu_i,s,z)\eta| \ge c^i|P(\tilde{s},s,z)\eta|. \tag{3.39}$$

We can choose i so large that the right hand side in (3.39) is greater than one. Hence (3.34) holds and with this the proof is complete. $\square$

4. Applications

4.1. An epidemic model of amphibians in periodic environment

We now apply our results from the previous section to the following discrete time amphibian (adult) population model with periodic coefficients (of period two), which belongs to Emmert and Allen [16].

$$\begin{cases} A_S(n+1) = [b_S(n)\varphi(N(n)) + p_S(n)e^{-\beta w\cdot I(n)}]A_S(n) \\ A_I(n+1) = p_S(n)(1-e^{-\beta w\cdot I(n)})A_S(n) + [b_I(n)\varphi(N(n)) + p_I(n)]A_I(n) \\ F(n+1) = b_F A_I(n) + p_F F(n) \end{cases} \tag{4.1}$$

A_S and A_I stand for *susceptible* and *infective* adults, respectively, while F represents the fungus (which is the source of the disease in the model). $p_K(n), K = S, I$, are probabilities that a susceptible ($K = S$), or an infective ($K = I$), adult survives over the time period from n to $n+1$, while $b_K(n) = 0$ for n odd and $b_K(n) = b_K$ for n even. $N(n) = c_S A_S(n) + c_I A_I(n)$ and $w \cdot I(n) = w_A A_I(n) + w_F F(n)$. Function φ could be, for example, of Beverton–Holt type:

$$\varphi(x) = 1/(1+\alpha x), \quad \alpha > 0, \tag{4.2}$$

or of Ricker type:

$$\varphi(x) = e^{-\alpha x}, \quad \alpha > 0. \tag{4.3}$$

All parameters are assumed to be positive (see [16] for a detailed explanation of the model). In addition, we denote the vector of parameters that appear in the model by ξ (ξ belongs to $\mathbb{R}^l$, for some $l \in \mathbb{Z}_+$). Let ξ_0 be a fixed vector of parameters.

Consider the equation

$$x(n+1) = [b_i(n)\varphi(c_i x(n)) + p_i(n)]x(n), \tag{4.4}$$

where i could be either S or I. For $i = S$, (4.4) gives the disease-free dynamics for (4.1). It is shown in [16] that (4.4) has a period-two orbit $\{A_i^0, A_i^1\}$ given by

$$A_i^0 = \frac{1}{c_i}\varphi^{-1}\left(\frac{\Lambda_S}{b_S}\right), \qquad A_i^1 = \frac{1}{c_i p_i^1}\varphi^{-1}\left(\frac{\Lambda_i}{b_i}\right), \tag{4.5}$$

where $p_i^0 := p_i(0)$, $p_i^1 := p_i(1)$ and

$$\Lambda_i = \frac{1 - p_i^0 p_i^1}{p_i^1}. \tag{4.6}$$

That is, any solution $x(n)$ of (4.4) with $x(0) = A_i^0$ satisfies $x(2n) = A_i^0$ and $x(2n+1) = A_i^1$, for all $n \geq 0$. In fact, sufficient conditions are provided for A_i^0 to be a globally asymptotically stable fixed point of

$$f_i(x) := [b_i\varphi(c_i x) + p_i^0]x \tag{4.7}$$

(in the sense that it is asymptotically stable and attracts all points in $\mathbb{R}_+ \setminus \{0\}$), hence $\{A_i^0, A_i^1\}$ is a globally attracting periodic orbit. Thus, for simplicity, we will occasionally make use of the following assumption:

(H_i) $\{A_i^0, A_i^1\}$ ($i = S, I$) is a period-two globally attracting orbit of (4.4).

Note that the period-two orbit (4.5) may exist only if $p_i^1(b_i + p_i^0) > 1$. Note also that (4.1) supports a "susceptible-free" (or "disease-only") environment, whose dynamics are given by

$$\begin{cases} A_I(n+1) = [b_I(n)\varphi(c_I A_I(n)) + p_I(n)]A_I(n) \\ F(n+1) = b_F A_I(n) + p_F F(n). \end{cases} \tag{4.8}$$

The equation for A_I in (4.8) is independent of F and is given by (4.4), with $i = I$.

Lemma 4.1. *If* (H_I) *holds then* $\{(A_I^0, F^0), (A_I^1, F^1)\}$ *is a period-two globally attracting orbit of* (4.8), *where*

$$F^0 = \frac{b_F(p_F A_I^0 + A_I^1)}{1 - p_F^2}; \qquad F^1 = \frac{b_F(p_F A_I^1 + A_I^0)}{1 - p_F^2}. \tag{4.9}$$

Proof. Let $(A_I(n), F(n))$, $n \geq s$, where s is even and $(A_I(s), F(s)) = (A_I^0, F^0)$. Then it is a simple calculation to show that $(A_I(s+2), F(s+2)) = (A_I^0, F^0)$, hence $(A_I(s+2k), F(s+2k)) = (A_I^0, F^0)$, $\forall k \geq 0$. Since $F(s+1) = F^1$, it also follows that $(A_I(s+2k+1), F(s+2k+1)) = (A_I^1, F^1)$, $\forall k \geq 0$.

Now we show that $\{(A_I^0, F^0), (A_I^1, F^1)\}$ is globally attracting. For this, it suffices to show that $(A_I(2n), F(2n)) \to (A_I^0, F^0)$ as $n \to \infty$, where again, $(A_I(n), F(n))$, $n \geq s$, $(A_I(s), F(s)) = (x_0, y_0)$, is an arbitrary non-zero solution of (4.8). We have that

$$F(2n+2) = b_F A_I(2n+1) + p_F b_F A_I(2n) + p_F^2 F(2n). \tag{4.10}$$

From (H_I), $A_I(2n) \to A_I^0$ and $A_I(2n+1) \to A_I^1$. Let $c = b_F A_I^1 + p_F b_F A_I^0$ and consider the equation

$$y(n+1) = c + p_F^2 y(n), \quad n \geq s, \tag{4.11}$$

which has the solution $y(n) = c/(1-p_F^2) + p_F^{2n} y(s)$. Thus $y(n) \to c/(1-p_F^2) = F^0$. From this it follows that $F(2n) \to F^0$, which completes the proof. $\square$

Our goal is to see when the disease can persist in the population. For this, we first prove that (4.1) is dissipative.

Proposition 4.2. *There exists a compact set B that attracts each trajectory of (4.1).*

Proof. For both choices of ϕ (Ricker or Beverton–Holt), there exists an $R \geq 0$ such that $R \geq \sup_{x \geq 0} x\phi(x)$. Let $a = (Rb_S/c_S, Rb_I/c_I, 0)$, $p_S = \max\{p_S^0, p_S^1\}$, $p_I = \max\{p_I^0, p_I^1\}$ and

$$A = \begin{pmatrix} p_S & 0 & 0 \\ p_S & p_I & 0 \\ 0 & b_F & p_F \end{pmatrix}.$$

Denote by $z(n)$, $n \geq s$ a solution of (4.1). Then

$$z(n+1) \leq Az(n) + a, \quad \forall\, n \geq s. \tag{4.12}$$

Iterating (4.12) we obtain

$$z(n) \leq A^{n-s} z(s) + (I-A)^{-1}(I - A^{n-s})a, \quad \forall n \geq s+1. \tag{4.13}$$

But $A^{n-s} \to 0$ as $n \to \infty$, so $z(n)$ is attracted to the set

$$B =: [0, b] = \{z \in \mathbb{R}_+^3 \mid 0 \leq z \leq b\}, \quad \text{where } b = (I-A)^{-1}a. \quad \square \tag{4.14}$$

It can be seen from (4.14) that B depends continuously on the parameters. This guarantees (3.4).

Next we provide sufficient conditions for persistence of the disease in the model. Let $E_S^0 = (A_S^0, 0, 0)$, $E_S^1 = (A_S^1, 0, 0)$, $E_I^0 = (0, A_I^0, F^0)$, $E_I^1 = (0, A_I^1, F^1)$ and define the following matrices:

$$J_S^I(n, z) = \left(b_S(n)\phi(c_I z_2) + p_S(n) e^{-\beta(w_A z_2 + w_F z_3)}\right) \tag{4.15}$$

and

$$J_{IF}^S(n, z) = \begin{pmatrix} \beta w_A p_S(n) z_1 + b_I(n)\varphi(c_S z_1) + p_I(n) & \beta w_F p_S(n) z_1 \\ b_F & p_F \end{pmatrix} \tag{4.16}$$

where $z = (z_1, z_2, z_3)$. Let $\mathcal{R}_S$ and $\mathcal{R}_I$ denote the spectral radii of matrices $J_S^I(1, E_I^1) J_S^I(0, E_I^0)$ and $J_{IF}^S(1, E_S^1) J_{IF}^S(0, E_S^0)$, respectively. Solutions corresponding to ξ_0 are simply denoted by $(A_S(n), A_I(n), F(n))$, while solutions corresponding to any other parameter ξ are denoted by $(A_S^\xi(n), A_I^\xi(n), F^\xi(n))$.

Proposition 4.3. *The following statements hold true:*

(a) *In any of the following two cases:*
 (i) *$p_I^1(b_I + p_I^0) < 1$ and $p_S^1(b_S + p_S^0) > 1$, or*
 (ii) *$p_S^1(b_S + p_S^0) > 1$, (H_I) and $\mathcal{R}_S > 1$,*
 there exists $\varepsilon > 0$ and a bounded neighborhood Ξ_0 of ξ_0 such that

$$\liminf_{n\to\infty} A_S^\xi(n) > \varepsilon, \tag{4.17}$$

 for all solutions $(A_S^\xi(n), A_I^\xi(n), F^\xi(n))$ of (4.1), $n \geq s$, with $\xi \in \Xi_0$ and $A_S^\xi(s) > 0$.

(b) *If, in addition to the hypotheses in* (a) (i) *or* (a) (ii), *(H_S) holds and $\mathcal{R}_I > 1$, then there exists $\varepsilon > 0$ and a bounded neighborhood Ξ_0 of ξ_0 such that*

$$\liminf_{n\to\infty} \min\{A_S(n)^\xi, A_I(n)^\xi, F(n)^\xi\} > \varepsilon, \tag{4.18}$$

 for all solutions $(A_S^\xi(n), A_I^\xi(n), F^\xi(n))$ of (4.1), $n \geq s$, with $\xi \in \Xi_0$, $A_S^\xi(s) > 0$ and $A_I^\xi(s) + F^\xi(s) > 0$.

Proof. (a) Let $X = \{z = (z_1, z_2, z_3) \in \mathbb{R}^3_+ \mid z_1 = 0\}$ and B given by (4.14). Let $M = B \cap X$.

Since the system (4.1) is periodic, using (3.32) it is straightforward to check assumption (D). Next we verify the assumptions (A1) and (A3)–(A5). (A1) and (A5) hold from Proposition 4.2.

Since the dynamics on X are given by (4.8), using that $\phi(x) \le 1$ for all $x \ge 0$ and that $p_I(n) \le 1$ for all n, we have $A_I(n+1) \le (b_I + 1)A_I(n)$. This implies that, whenever $A_I(s) \in M$, $A_I(n+s) \le K_1 := (b_I+1)^{n-1}\max\{|z| \mid z \in M\}$. Then $F(n+1) \le b_F K_1 + p_F F(n)$, which implies that $F(n+s) \le K_2$, for some K_2 independent of s. Hence (A3) holds.

Also, as previously mentioned, (3.4) follows from the proof of Proposition 4.2 (namely from (4.14)). The map that gives the right hand side in (4.1) is C^1 in z and ξ, and so is every entry in the matrix

$$\tilde{A}(n,z) = \begin{pmatrix} b_S(n)\varphi(c_S A_S + c_I A_I) + p_S(n)e^{-\beta w\cdot I} & 0 & 0 \\ p_S(n)(1-e^{-\beta w\cdot I}) & b_I\varphi(c_S A_S + c_I A_I) + p_I(n) & 0 \\ 0 & b_F & p_F \end{pmatrix},$$

where note that (4.1) can be written as $z(n+1) = \tilde{A}(z(n))z(n)$. Thus, because we can choose $\widetilde{V}_B$ (as in (A4)) bounded, assumptions (3.2) and (3.3) are satisfied. So assumption (A2) is the only one that remains to be checked, and then (4.17) and (4.18) follow from Theorem 3.5.

First, we consider the case (i). The dynamics on the set X satisfy

$$A_I(2n+2) = p_I^1(b_I\varphi(c_I A_I(2n)) + p_I^0)A_I(2n) \le p_I^1(b_I + p_I^0)A_I(2n). \tag{4.19}$$

Since $p_I^1(b_I + p_I^0) < 1$, $A_I(2n) \to 0$ as $n \to \infty$. This implies $A_I(n) \to 0$, from which it follows that $F(n) \to 0$. For any z in X, let $P(n+s, s, z)$ be the solution of $u(n+1) = J_S^I(n, z(n))u(n)$, $P(s,s,z) = (1)$, where $z(n)$ is the solution of (4.1), $z(s) = z$. Hence, for any $s \ge 0$, $P(2+s, s, 0)$ is the 1×1 matrix $J_S^I(1,0)J_S^I(0,0) = (p_S^1(b_S + p_S^0))$. Thus, the spectral radius of this matrix is greater than one (by hypothesis). Hence, from Proposition 3.8, we have that (A2) holds.

Now we consider the case (ii). From Lemma 4.1 we have that all orbits in X are attracted either to $\{0\}$ or $\{E_I^0, E_I^1\}$. We know from case (i) that the spectral radius of $P(2+s, s, 0)$ is greater than one. Now $P(2, 0, E_I^0) = P(3, 1, E_I^1) = J_S^I(1, E_I^1)J_S^I(0, E_I^0)$, which has spectral radius $\mathcal{R}_S > 1$, from hypothesis. So again, from Proposition 3.8 we have that (A2) holds.

Then (4.17) follows from Theorem 3.5.

(b) Let $X = \{z = (z_1, z_2, z_3) \in \mathbb{R}^3_+ \mid z_2 = z_3 = 0\}$ and B given by (4.14).

First we show that there exist $\varepsilon > 0$ and Ξ_0 a neighborhood of ξ_0 such that

$$\liminf_{n\to\infty} A_I(n)^\xi + F(n)^\xi > \varepsilon, \tag{4.20}$$

for all solutions $(A_S(n)^\xi, A_I(n)^\xi, F(n)^\xi)$ of (4.1), $n \ge s$ and $\xi \in \Xi_0$, with $A_I(s)^\xi + F(s)^\xi > 0$. From part (a), there exists an $\tilde{\varepsilon} > 0$ such that the set $\tilde{B} := \{z \in B \mid z_1 \ge \tilde{\varepsilon}\}$ absorbs all solutions of (4.1) with $A_S(s) > 0$, corresponding to all ξ close to ξ_0. A simple calculation shows that b in (4.14) has all components positive, so we can choose $\tilde{\varepsilon}$ sufficiently small, so that $\tilde{B} \ne \emptyset$. Let $M = \tilde{B} \cap X$. Assumptions (A1), (A3)–(A5), and (D) can be verified as above. Then, because (H_S) holds, all orbits originating in M are attracted to $\{E_S^0, E_S^1\}$. $P(2, 0, E_S^0) = J_{IF}^S(1, E_S^1)J_{IF}^S(0, E_S^0)$ and $P(3, 1, E_S^1) = J_{IF}^S(0, E_S^0)J_{IF}^S(1, E_S^1)$, hence both have spectral radius $\mathcal{R}_I > 1$ which corresponds to eigenvectors with both components positive (Perron–Frobenius). Thus, Proposition 3.8 says that (A2) holds, and then (4.20) comes from Theorem 3.5.

Then, using the equations for A_I and F in (4.1), one readily obtains that

$$\liminf_{n\to\infty} \min\{A_I(n)^\xi, F(n)^\xi\} > \varepsilon, \tag{4.21}$$

for all solutions $(A_S(n)^\xi, A_I(n)^\xi, F(n)^\xi)$ of (4.1), $n \ge s$ and $\xi \in \Xi_0$, with $A_I(s)^\xi + F(s)^\xi > 0$.

Now (4.18) follows from (4.21) and (4.17). □

4.2. A nonautonomous SIRS model with n infection strains

Here we consider a generalized version of the model of Teng et al. [17], with n infection strains I_j, $j = 1, \ldots, n$ and the corresponding transmission rates α_j, disease-induced mortality rates β_j and recovery (from the infection) rates γ_j (all time dependent). The other (time dependent) coefficients that appear in the model are population growth rate, G, instantaneous per capita natural death rate, μ, and instantaneous per capita rate of leaving the removed stage, ω. In order to address robust uniform persistence questions in regard to this model, we further let the time dependent coefficients mentioned above, depend also on a parameter $\xi \in \mathbb{R}^l$. Thus, the model is

$$\begin{aligned} S' &= G(t,\xi) - \mu(t,\xi)S - \sum_{j=1}^n \alpha_j(t,\xi)SI_j + \omega(t,\xi)R \\ I_j' &= (\alpha_j(t,\xi)SI_j - \mu(t,\xi)I_j - \gamma_j(t,\xi)I_j - \beta_j(t,\xi))I_j, \quad j = 1,\ldots,n \\ R' &= \sum_{j=1}^n \gamma_j(t,\xi)I_j - \mu(t,\xi)R - \omega(t,\xi)R. \end{aligned} \tag{4.22}$$

In order to avoid confusion with our previous notation, the notation in (4.22) has been slightly changed, compared to [17].

As before, we fix a parameter ξ_0. The total population $N = S + \sum_{j=1}^{n} I_j + R$ satisfies the differential inequality

$$N'(t, \xi) \leq G(t, \xi) - \mu(t, \xi)N(t, \xi). \tag{4.23}$$

For every fixed ξ, $G(t, \xi)$ is assumed to be bounded. G is also assumed to be locally Lipschitz continuous, namely

$$|G(t, \xi) - G(t, \xi_0)| \leq K_G|\xi - \xi_0|, \tag{4.24}$$

for all $t \geq 0$ and all ξ in some neighborhood of ξ_0, where K_G is a nonnegative constant. Analogous assumptions are made for all the other coefficients in the model.

Based on (4.23), certain additional restrictions can be put on the function μ to guarantee that (A1) holds with B being compact set (hence (A5) also holds). Such an assumption, that is analogous to the one made in [17], is

$$\exists\, a > 0 \quad \text{such that } \liminf_{s\to\infty} \int_s^{a+s} \mu(r, \xi_0)dr > 0. \tag{4.25}$$

Thus, let $\delta(\xi_0) = \liminf_{s\to\infty} \int_s^{a+s} \mu(r, \xi_0)dr > 0$. From (4.25) it follows that there exists an $s_0 \geq 0$ such that $\int_s^{na+s} \mu(r, \xi_0)dr \geq n\delta(\xi_0)$, for all $n \in \mathbb{Z}_+$ and all $s \geq s_0$. Hence

$$\int_s^{t+s} \mu(r, \xi_0)dr \to \infty \quad \text{as } t \to \infty, \text{ uniformly in } s. \tag{4.26}$$

Also, for any s and t, $s \leq t$, we have

$$\int_s^t \mu(r, \xi_0)dr \geq \int_s^{s+[\frac{t-s}{a}]a} \mu(r, \xi_0)dr \geq \left[\frac{t-s}{a}\right]\delta(\xi_0) \geq \left(\frac{t-s}{a} - 1\right)\delta(\xi_0), \tag{4.27}$$

where $[\cdot]$ above represents the least integer function. But from (4.23) we obtain that

$$N(t, \xi_0) \leq N(s)e^{-\int_s^t \mu(r,\xi_0)dr} + \int_s^t G(\alpha, \xi_0)e^{-\int_\alpha^t \mu(r,\xi_0)dr}d\alpha. \tag{4.28}$$

Let K_G^0 be a constant such that $G(t, \xi_0) \leq K_G^0$ for all $t \geq 0$. Then using (4.26)–(4.28) it follows, by straightforward calculation, that

$$\limsup_{t\to\infty} N(t, \xi_0) \leq aK_G^0 \frac{e^{\delta(\xi_0)}}{\delta(\xi_0)}. \tag{4.29}$$

Thus, we can take

$$B = \left\{x \in \mathbb{R}_+^{n+2} \mid |x| \leq aK_G^0 \frac{e^{\delta(\xi_0)}}{\delta(\xi_0)}\right\}.$$

In order to investigate the persistence of the disease in the population, we define $X = \{z \in \mathbb{R}^{n+2} \mid I_j = 0, j = 1, \ldots, n\}$, where $z = (S, I_1, \ldots, I_n, R)$ is the population vector. Let $M = B \cap X$ (hence M is compact). Since the disease-free dynamics satisfy (4.23) with $N(t) = S(t) + R(t)$ and "$\leq$" replaced by "$=$", assumption (A3) can be readily verified. Then, as in the previous application, V_B in (A4) can be taken to be bounded, and then using that all coefficients in (4.22) are locally Lipschitz continuous and that F and A corresponding to (4.22) are C^1 in z, we also obtain that (3.1)–(3.3) hold.

Finally, we give below a brief argument to show that the last part of assumption (A4), namely (3.4), is satisfied. Since μ satisfies an inequality analogous to (4.24), namely $|\mu(t, \xi) - \mu(t, \xi_0)| \leq K_\mu|\xi - \xi_0|$, for all $t \geq 0$ and all ξ in some neighborhood of ξ_0, it follows that

$$\delta(\xi_0) - aK_\mu|\xi - \xi_0| \leq \delta(\xi) = \liminf_{s\to\infty} \int_s^{a+s} \mu(r, \xi)dr \leq \delta(\xi_0) + aK_\mu|\xi - \xi_0|,$$

which shows that δ is continuous at ξ_0. Now, from (4.24) we have that $G(t, \xi) \leq K_G^0 + K_G|\xi - \xi_0|$. Then $N(t, \xi)$ satisfies an inequality analogous to (4.29), with $\delta(\xi_0)$ replaced by $\delta(\xi)$ and K_G^0 replaced by $K_G^0 + K_G|\xi - \xi_0|$. This shows that (3.4) holds.

Let $\tilde{S}(t)$, $\tilde{S}(0) = \tilde{S}_0$, be a fixed solution of

$$S'(t) = G(t, \xi_0) - \mu(t, \xi_0)S(t) \tag{4.30}$$

such that $(\tilde{S}_0, 0, \ldots, 0, R_0) \in M$, for some R_0 (where $M = B \cap X$).

Proposition 4.4. *Assume that there exists $T > 0$ such that, for all $j = 1, \ldots, n$,*

$$\liminf_{s\to\infty} \int_s^{s+T} \alpha_j(t,\xi_0)\tilde{S}(t) - \mu(t,\xi_0) - \gamma_j(t,\xi_0) - \beta_j(t,\xi_0)dt > 0. \tag{4.31}$$

Then there exists an $\varepsilon > 0$ and Ξ_0 a neighborhood of ξ_0, such that

$$\liminf_{t\to\infty} I_1^\xi(t) + \cdots + I_n^\xi(t) > \varepsilon, \quad \forall \xi \in \Xi_0, \tag{4.32}$$

where $(S(t)^\xi, I_1^\xi(t), \ldots, I_n^\xi(t), R^\xi(t))$, $t \geq s$, is any solution of (4.22) with $I_1^\xi(s) + \cdots + I_1^\xi(s) > 0$.

Proof. Let $J_{j_1,\ldots,j_k} = \{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$, $J_{j_1,\ldots,j_k} \neq \{1, \ldots, n\}$, and define

$$Z_{J_{j_1,\ldots,j_k}} = \{x = (S, I_1, \ldots, I_n, R) \in \mathbb{R}_+^{n+2} \mid I_{j_1} = \cdots = I_{j_k} = 0\}. \tag{4.33}$$

Note that $Z_{J_{j_1,\ldots,j_k}}$ is positively invariant for (4.22) and we take it to be our state space. Let $B_{J_{j_1,\ldots,j_k}} \subset Z_{J_{j_1,\ldots,j_k}}$ be the corresponding absorbing set (as in (A1)). Let $J^c_{j_1,\ldots,j_k}$ be the complement of $J_{j_1,\ldots,j_k}$ and $X_{J_{j_1,\ldots,j_k}} = \{x = (S, I_1, \ldots, I_n, R) \in Z_{J_{j_1,\ldots,j_k}} \mid I_j = 0, \forall j \in J^c_{j_1,\ldots,j_k}\}$, which is also a (non-empty) positively invariant set. Note that $B_{J_{j_1,\ldots,j_k}} \cap X_{J_{j_1,\ldots,j_k}} = M$.

As previously discussed, assumptions (A1) and (A3)–(A5) hold. So we only need to show that assumption (A2) holds, and for this we apply Proposition 3.7. Thus, let $z_0 = (S_0, 0, \ldots, 0, R_0) \in M$ and $P(t, s, z_0)$ be the fundamental matrix for

$$u'(t) = \operatorname{diag}(\alpha_j(t,\xi_0)S(t) - \delta_j(t,\xi_0))u(t), \tag{4.34}$$

where $\delta_j(t,\xi_0) = \mu(t,\xi_0) + \gamma_j(t,\xi_0) + \beta_j(t,\xi_0)$ and $(S(t), 0, \ldots, 0, R(t))$ is the solution of (4.22), corresponding to $\xi = \xi_0$, with $(S(s), I_1(s), \ldots, I_n(s), R(s)) = z_0$. Hence

$$P(t,s,z_0) = \operatorname{diag}\left(\exp\left(\int_s^t \alpha_j(r,\xi_0)S(r) - \delta_j(r,\xi_0)dr\right)\right). \tag{4.35}$$

If $\psi(t) = \psi(t, s, S_0)$ denotes the solution of (4.30) with $\psi(s) = S_0$, then $S(t) \geq \psi(t)$, for all $t \geq s$. Hence $P(t, s, z_0) \geq P^*(t, s, z_0) := \operatorname{diag}(\exp(\int_s^t \alpha_j(r,\xi_0)\psi(r) - \delta_j(r,\xi_0)dr))$. Let $M^* = \{z_1 \mid z = (z_1, \ldots, z_{n+2}) \in M\}$. Hence M^* is compact. Then it suffices to verify the hypotheses of Proposition 3.7 for M^*, ψ and P^*, in order to conclude that (A2) holds.

Let $\tilde{S}(t)$, $\tilde{S}(0) = \tilde{S}_0$, be a fixed solution of (4.30) such that $\tilde{S}_0 \in M^*$. Then

$$|\psi(t+s,s,S_0) - \tilde{S}(t+s)| = e^{-\int_s^{s+t}\mu(r,\xi_0)dr}|S_0 - \tilde{S}(s)|. \tag{4.36}$$

Thus, from (4.36), we have that (i) in Proposition 3.7 is satisfied. Also, (ii) in Proposition 3.7 follows from the hypothesis. Hence (A2) holds. Then, from Theorem 3.5 we have that there exist $\varepsilon > 0$ and Ξ_0 a neighborhood of ξ_0 in Ξ, both depending on $J_{j_1,\ldots,j_k}$, such that

$$\liminf_{t\to\infty} \sum_{i\in J^c_{j_1,\ldots,j_k}} I_i^\xi(t) > \varepsilon, \quad \forall \xi \in \Xi_0, \tag{4.37}$$

where $\sum_{i\in J^c_{j_1,\ldots,j_k}} I_i^\xi(s) > 0$. Since there are finitely many sets $J_{j_1,\ldots,j_k}$, we conclude that (4.32) holds. □

Restricted to a single infection strain, condition (4.31) is the same as the condition in Theorem 1 in [17], that the authors impose to obtain uniform persistence of the disease. However, we do not impose a similar restriction on α_j's, as the one in (H2) in [17].

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Appendix

Proof of Lemma 2.1. (a) Let $\xi \in R^l$. We have that $A(n,z,\xi) = \frac{\partial F}{\partial y}(n,z,\xi)$, $\forall z \in X$, $n \in \mathbb{Z}_+$, where $\frac{\partial F}{\partial y}(n,z,\xi) = (\frac{\partial F_{p+i}}{\partial y_j}(n,z,\xi))_{1\leq i,j\leq q}$. But, since X is positively invariant, $F_{p+i}(n,z,\xi) = 0$, $\forall z \in X$, $1 \leq i \leq q$. So,

$$\begin{aligned}\frac{\partial F_{p+i}}{\partial y_j}(n,z,\xi) &= \lim_{h\to 0^+} \frac{F_{p+i}(n,x_1,\ldots,x_p,0,\ldots,y_j = h,\ldots,0,\xi) - F_{p+i}(n,z,\xi)}{h} \\ &= \lim_{h\to 0^+} \frac{F_{p+i}(n,x_1,\ldots,x_p,0,\ldots,y_j = h,\ldots,0,\xi)}{h} \geq 0.\end{aligned}$$

(b) Let $\xi \in R^l$. Suppose there exist $\bar{t} \in \mathbb{R}, \bar{z} \in X$ such that $a_{ij}(\bar{t}, \bar{z}, \xi) < 0$ for some $i \neq j$. Then, by continuity, there exists a neighborhood $V(\bar{z})$ of $\bar{z}$ such that $a_{ij}(\bar{t}, z, \xi) < 0, \forall z \in V(\bar{z})$. Let $z = (x_1, \dots, x_p, 0, \dots, 0, y_j, 0, \dots, 0) \in V(\bar{z}), y_j > 0$, and consider the solution $z(t)$ of (1.4) that starts at z at $t = \bar{t}$. Then $y_i'(\bar{t}) = a_{ij}(\bar{t}, z, \xi)y_j < 0$. So there exists $\hat{t} > \bar{t}$ such that $y_i(\hat{t}) < y_i(\bar{t}) = 0$, which contradicts $\mathbb{R}_+^{p+q}$ being positively invariant. Hence, $a_{ij}(t, z, \xi) \geq 0, t \in \mathbb{R}, z \in X$, whenever $i \neq j$.

(c) The discrete case follows directly from the fact that X is positively invariant and from (a). For the continuous case, we can use Proposition B.7. in [18] to conclude that any solution $u(t)$ of (2.3) (where $z \in X$), with $u(s) \geq 0$, satisfies $u(t) \geq 0, \forall t \geq s$. Hence $P(t + s, s, z, \xi) \geq 0, \forall z \in X, \forall t, s \geq 0, \forall \xi \in \mathbb{R}^l$. □

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