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Uniqueness of boundary blowup solutions to k-curvature equation

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ABSTRACT

We consider the boundary blowup problem for the k-curvature equation, i.e., $H_k[u] = f(u)g(|Du|)$ in an n-dimensional domain Ω, with the boundary condition $u(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$. We prove the uniqueness result under some hypotheses.

1. Introduction

This paper deals with the so-called curvature equations of the form

$$H_k[u] = S_k(\kappa_1, \dots, \kappa_n) = f(u)g(|Du|) \quad \text{in } \Omega, \tag{1.1}$$

with the following boundary condition

$$u(x) \to \infty \quad \text{as } \operatorname{dist}(x, \partial\Omega) \to 0. \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ and for a function $u \in C^2(\Omega)$, $\kappa = (\kappa_1, \dots, \kappa_n)$ denotes the principal curvatures of the graph of the function u, namely, the eigenvalues of the matrix

$$\mathcal{C} = D\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}}\left(I - \frac{Du \otimes Du}{1+|Du|^2}\right) D^2u, \tag{1.3}$$

and S_k, $k = 1, \dots, n$, denotes the k-th elementary symmetric function, i.e.,

$$S_k(\kappa) = \sum \kappa_{i_1} \cdots \kappa_{i_k}, \tag{1.4}$$

where the sum is taken over increasing k-tuples, $1 \le i_1 < i_2 < \cdots < i_k \le n$. The mean, scalar and Gauss curvatures correspond respectively to the special cases $k = 1, 2, n$ in (1.4). In this paper we call Eq. (1.1) the "k-curvature equation".

In [1] we have studied the existence and non-existence result of a solution to (1.1)–(1.2). In addition, we have obtained a result for the asymptotic behavior near $\partial\Omega$ of such a solution. In this paper, we deal with the uniqueness of solutions to (1.1)–(1.2).

We remark that (1.1) is a quasilinear equation for $k = 1$ while it is a *fully nonlinear* equation for $k \ge 2$. In the particular case that $k = n$, it is an equation of the Monge–Ampère type. It is much harder to analyze fully nonlinear equations, but the study of the classical Dirichlet problem for k-curvature equations in the case that $2 \le k \le n-1$ has been developed in the last two decades, see for instance [2–4].

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The condition (1.2) is called the "boundary blowup condition", and a solution which satisfies (1.2) is called a "boundary blowup solution", a "large solution", or an "explosive solution". The boundary blowup problems arise from physics, geometry and many branches of mathematics, see for instance [5–7]. The existence and the asymptotic behavior of solutions for such problems starts from the pioneering works of Bieberbach [8] and Rademacher [7] who considered $\Delta u = e^u$ in the two- and three-dimensional domain respectively. For the case of semilinear equations, they have extensively been studied (see, for example, [6,9–18]). The case of quasilinear equations of divergence type to which the mean curvature equation ($k = 1$ in (1.1)) belongs has been treated in [19–21]. However, there are only a few results concerning such problems for fully nonlinear PDEs, such as [22–24] for the Monge–Ampère equation, [25] for k-Hessian equations, and [1] by the author for k-curvature equations.

In some works among them, the uniqueness of boundary blowup solutions has been also discussed, see [6,16,18,19,25] for example. But there were no results for the uniqueness of boundary blowup solutions for k-curvature equations, even for the mean curvature equation which corresponds to the case of $k = 1$ for (1.1). In this paper, we shall obtain the uniqueness result for (1.1)–(1.2), which is stated in Sections 3 and 4.

Throughout the paper, we assume the following conditions on f and g:

- Let $t_0 \in [-\infty, \infty)$. $f \in C^\infty(t_0, \infty)$ is a positive function and satisfies $f'(t) > 0$ for all $t \in (t_0, \infty)$.
- If $t_0 > -\infty$, then $f(t) \to 0$ as $t \to t_0 + 0$; otherwise (i.e., if $t_0 = -\infty$),

$$\int_{-\infty}^{t} f(s)\, ds < \infty \quad \text{for all } t \in \mathbb{R}. \tag{1.5}$$

- $g \in C^\infty[0, \infty)$ is a positive function.

The first condition assures us that the comparison principle for solutions to (1.1) holds. The typical examples of f are $f(t) = t^p$ $(p > 0)$, $t_0 = 0$ and $f(t) = e^t$, $t_0 = -\infty$.

This paper is divided as follows. In the next section, we state our results for the existence and the estimate of the asymptotic behavior of a solution near the boundary to the boundary blowup problem (1.1)–(1.2), for the sake of completeness. This includes the improved results for the asymptotic behavior of boundary blowup solutions. In Section 3, we state our uniqueness result and prove it. However, the case $k = n$ is excluded from these theorems. We consider the particular case in Section 4.

2. Results for existence and asymptotic behavior of a solution

In this section we review the results for the existence and the asymptotic behavior of a solution to (1.1)–(1.2). The following existence result has been proved in [1].

Theorem 2.1. *Let $2 \le k \le n - 1$. We assume that Ω, f and g satisfy the following conditions.*

(A1) *Ω is a bounded and uniformly k-convex domain with boundary $\partial\Omega \in C^\infty$.*
(A2) *There exists a constant $T > 0$ such that g is non-increasing in $[T, \infty)$, and $\lim_{t\to\infty} g(t) = 0$.*
(A3) *Set $\tilde{g}(t) = g(t)/t$ and $F(t) = \int_{t_0}^{t} f(s)\, ds$. Then*

$$\int^{\infty} \frac{dt}{\tilde{g}^{-1}\left(\frac{1}{F(t)}\right)} < \infty. \tag{2.1}$$

(A4) *Set*

$$H(t) = \int_0^t \frac{s^k}{g(s)\left(1+s^2\right)^{(k+2)/2}}\, ds. \tag{2.2}$$

Then $\lim_{t\to\infty} H(t) = \infty$.
(A5) *Set $\varphi(t) = g(t)(1+t^2)^{k/2}$. Then $\varphi(t)$ is a convex function in $[0, \infty)$.*
(A6) *$\limsup_{t\to\infty} |g'(t)|t^2 < \infty$.*

Then there exists a viscosity solution to (1.1)–(1.2).

We note that for $k = 1$ the existence has been already studied in [21], so that we focus here on the case $k \ge 2$. For the definition and the general theory of viscosity solutions to PDEs, we refer to, for example, [26–29]. For the viscosity theory for curvature equations in particular, see [4].

Example 2.1. Let $2 \le k \le n - 1$ and p, q be positive constants. Suppose Ω is a bounded and uniformly k-convex domain with boundary $\partial\Omega \in C^\infty$. We consider these two equations:

(i) $H_k[u] = u^p / \left(1 + |Du|^2\right)^{q/2}$ in Ω.
From Theorem 2.1, it follows that there exists a boundary blowup solution provided $p > q$ and $1 \le q \le k - 1$.

(ii) $H_k[u] = e^{pu} / \left(1 + |Du|^2\right)^{q/2}$ in Ω.
There exists a boundary blowup solution provided $1 \le q \le k - 1$.

Remark 2.1. In the preceding paper [1], we have also obtained a necessary condition for boundary blowup solutions to exist, so that we have given an example of f and g for which there does not exist any boundary blowup solution.

Next we establish the asymptotic behavior of a solution to (1.1)–(1.2) near $\partial\Omega$. We shall prove the following, which is slightly improved over the corresponding one in [1], so that we give its proof here.

Theorem 2.2. *Let $1 \le k \le n-1$. We assume that* (A1)–(A3) *in Theorem 2.1 and the conditions given below are satisfied.*

(B1) *$t_0 = -\infty$, or $t_0 > -\infty$ and $f^{1/k}$ is Lipschitz continuous at t_0.*
(B2) *There exists a constant $T' > t_0$ such that f is a convex function in $[T', \infty)$.*
(B3) *Set $h(t) = \frac{t}{g(t)^{1/k}\sqrt{1+t^2}}$. Then there exists a constant $\alpha > 0$ such that $h(t)/t^{\alpha}$ is non-decreasing in $(0, \infty)$.*
(B4) $\lim_{t\to\infty} \frac{g(t)}{(1+t^2)g'(t)} = 0.$

Then there exist positive constants C_1, C_2 such that every solution u to (1.1)–(1.2) satisfies

$$\psi^{-1}(C_1 \mathrm{dist}(x, \partial\Omega)) - O(1) \le u(x) \le \psi^{-1}(C_2 \mathrm{dist}(x, \partial\Omega)) + O(1) \tag{2.3}$$

near $\partial\Omega$, where ψ is defined by

$$\psi(t) = \int_t^{\infty} \frac{ds}{h^{-1}\left(f(s)^{1/k}\right)}. \tag{2.4}$$

Proof. Let u be a solution to (1.1)–(1.2). From now on, we use the following notation: $d(x) = \mathrm{dist}(x, \partial\Omega)$ and $\Omega_r = \{x \in \Omega \mid d(x) < r\}$ for $r > 0$.

It follows from (A1) that there exists a positive constant R such that the following conditions are satisfied:

(a) $d = d(x)$ is a C^{∞} function in Ω_R;
(b) For each point $x \in \Omega_R$, there exists a unique point $z(x) \in \partial\Omega$ such that $d(x) = |x - z(x)|$;
(c) There exist positive constants m, M such that for every point $x \in \Omega_R$, it holds that

$$\tilde{\kappa} = (\tilde{\kappa}_1, \ldots, \tilde{\kappa}_{n-1}) := \left(\frac{\kappa'_1}{1 - d(x)\kappa'_1}, \ldots, \frac{\kappa'_{n-1}}{1 - d(x)\kappa'_{n-1}}\right) \in \Gamma_k(\mathbb{R}^{n-1}) \tag{2.5}$$

and that

$$m \le S_k(\tilde{\kappa}) \le M, \tag{2.6}$$

where $\kappa'_1, \ldots, \kappa'_{n-1}$ denote the principal curvatures of $\partial\Omega$ at $z(x)$.

First, we prove the first inequality in (2.3). Let $\tilde{v}_1 = \tilde{v}_1(r)$ be a solution to the following problem

$$\begin{cases} \binom{n-1}{k-1} \dfrac{u''}{(1+u'^2)^{3/2}} \left(\dfrac{u'}{r\sqrt{1+u'^2}}\right)^{k-1} + \binom{n-1}{k} \left(\dfrac{u'}{r\sqrt{1+u'^2}}\right)^{k} = f(u)g(|u'|), & \text{in } (0, \mathrm{diam}\,\Omega), \\ u(0) = u_0 > t_0, \quad u'(0) = 0, & \\ u(r) \to \infty & \text{as } r \to \mathrm{diam}\,\Omega - 0. \end{cases} \tag{2.7}$$

The existence of the solution $\tilde{v}_1$ is guaranteed by the hypotheses (A2), (A3), (B1) and (B3); see [1, Theorem 3.6] for the proof. We set $v_1(x) = \tilde{v}_1(|x|)$, so that v_1 is a classical radially symmetric solution to (1.1) with the boundary blowup condition

$$v_1(x) \to \infty \quad \text{as } \mathrm{dist}(x, B_{\mathrm{diam}\Omega}(0)) \to 0. \tag{2.8}$$

For $y \in \Omega$ which satisfies $d(y) = 3R/4$, it follows from the comparison principle that

$$u(x) \ge v_1(x - y) \quad \text{in } \left\{x \in \Omega \,\middle|\, |x - y| < \frac{R}{2}\right\}. \tag{2.9}$$

Therefore, setting $C := v_1(0)$, we obtain that for any solution u to (1.1) and any point $y \in \Omega$ which satisfies $d(y) = 3R/4$,

$$u(y) \ge C. \tag{2.10}$$

Next, we see that there exists a constant $w_1 > t_0$ such that a non-increasing, convex solution w on $(0, R]$ to the following problem

$$\begin{cases} f(w)g(|w'|) = \left(\dfrac{|w'|}{\sqrt{1+w'^2}}\right)^k \cdot m & \text{in } (0, R), \\ w(r) \to \infty & \text{as } r \to +0, \\ w(R) = w_1 & \end{cases} \tag{2.11}$$

exists. Indeed, by the same argument as in [1, Section 3], one can prove the existence of such a solution. We omit its proof. For $\varepsilon \in (0, R/4)$, we define

$$v_{1\varepsilon}(x) = w(d(x)+\varepsilon) + L, \quad x \in \overline{\Omega_{3R/4}}, \tag{2.12}$$

where $L = \min\{C - w(3R/4), 0\}$. Then it follows that $u(x) \to \infty$ as $d(x) \to 0$ while $v_{1\varepsilon}(x)$ takes a finite value on the set $\{d(x) = 0\} = \partial\Omega$. Moreover, for any x which satisfies $d(x) = 3R/4$ we have

$$v_{1\varepsilon}(x) = w\left(\frac{3}{4}R + \varepsilon\right) + L \le w\left(\frac{3}{4}R\right) + L \le C \le u(x) \tag{2.13}$$

due to (2.10). Finally, it holds that for $x \in \Omega_{3R/4}$

$$\begin{aligned} H_k[v_{1\varepsilon}](x) &= \left(\frac{|w'(d(x)+\varepsilon)|}{\sqrt{1+w'(d(x)+\varepsilon)^2}}\right)^k S_k(\tilde\kappa) + \frac{w''(d(x)+\varepsilon)}{\left(1+w'(d(x)+\varepsilon)^2\right)^{3/2}}\left(\frac{|w'(d(x)+\varepsilon)|}{\sqrt{1+w'(d(x)+\varepsilon)^2}}\right)^{k-1} S_{k-1}(\tilde\kappa) \\ &\ge \left(\frac{|w'(d(x)+\varepsilon)|}{\sqrt{1+w'(d(x)+\varepsilon)^2}}\right)^k \cdot m \\ &= f(w(d(x)+\varepsilon))\, g(|w'(d(x)+\varepsilon)|) \\ &= f(v_{1\varepsilon}(x) - L)\, g(|Dv_{1\varepsilon}(x)|) \ge f(v_{1\varepsilon}(x))\, g(|Dv_{1\varepsilon}(x)|). \end{aligned} \tag{2.14}$$

Here we note that $L \le 0$. Therefore we can deduce by the comparison principle that

$$v_{1\varepsilon}(x) = w(d(x)+\varepsilon) + L \le u(x) \tag{2.15}$$

for $x \in \Omega_{3R/4}$. Taking the limit $\varepsilon \to +0$, we get that

$$w(d(x)) + L \le u(x) \tag{2.16}$$

for $x \in \Omega_{3R/4}$.

Using (2.11) and the condition (B3), we obtain that

$$\begin{aligned} |w'| = -w' &= h^{-1}\left(m^{-1/k} f(w)^{1/k}\right) \\ &\le \max\left\{1, m^{-1/\alpha k}\right\} h^{-1}(f(w)^{1/k}). \end{aligned} \tag{2.17}$$

Integrating from 0 to r yields that

$$\psi(w(r)) = \int_{w(r)}^{\infty} \frac{ds}{h^{-1}(f(s)^{1/k})} \le \max\left\{1, m^{-1/\alpha k}\right\} r \tag{2.18}$$

for $r \in (0, R)$. Combining (2.16) and (2.18), we conclude that the first inequality in (2.3) holds.

Next we prove the second inequality in (2.3). As we have argued before, we see that there exists a constant $w_2 > t_0$ such that a non-increasing, convex solution $\tilde w$ on $(0, R]$ to the following problem

$$\begin{cases} f(\tilde w) g(|\tilde w'|) = \left(\dfrac{|\tilde w'|}{\sqrt{1+(\tilde w')^2}}\right)^k \cdot M & \text{in } (0, R) \\ \tilde w(r) \to \infty & \text{as } r \to +0, \\ \tilde w(R) = w_2 \end{cases} \tag{2.19}$$

exists. We choose a constant $R' \in (0, R)$ such that $\tilde w(R') \ge T'$, where T' is a constant which appears in the condition (B2). For $\varepsilon \in (0, R'/4)$, we define

$$v_{2\varepsilon}(x) = \tilde w(d(x) - \varepsilon) + L', \quad x \in \overline{\Omega_{R'} \setminus \Omega_\varepsilon}, \tag{2.20}$$

where L' is a positive constant to be determined later.

Hereafter, we use the abbreviation: $v_{2\varepsilon} = v_{2\varepsilon}(x)$ and $\tilde w = \tilde w(d(x) - \varepsilon)$. Then it follows from (B2) that

$$f(\tilde w) = f(v_{2\varepsilon} - L') \le f(v_{2\varepsilon}) - L' f'(\tilde w) \quad \text{in } \Omega_{R'}. \tag{2.21}$$

By differentiating the ODE in (2.19), we have

$$\tilde w'' = \frac{f'(\tilde w)\tilde w' g(|\tilde w'|)^2 \left(1+(\tilde w')^2\right)^{3/2}}{M|\tilde w'|\left(1+(\tilde w')^2\right) g'(|\tilde w'|) - Mk g(|\tilde w'|)} \left(\frac{|\tilde w'|}{\sqrt{1+(\tilde w')^2}}\right)^{-(k-1)}, \tag{2.22}$$

which implies that

$$\begin{aligned} H_k[v_{2\varepsilon}] &= \left(\frac{|\tilde{w}'|}{\sqrt{1+(\tilde{w}')^2}}\right)^k S_k(\tilde{\kappa}) + \frac{\tilde{w}''}{\left(1+(\tilde{w}')^2\right)^{3/2}}\left(\frac{|\tilde{w}'|}{\sqrt{1+(\tilde{w}')^2}}\right)^{k-1} S_{k-1}(\tilde{\kappa}) \\ &\le f(\tilde{w})g(|\tilde{w}'|) + \frac{f'(\tilde{w})\tilde{w}'g(|\tilde{w}'|)^2}{M|\tilde{w}'|\left(1+(\tilde{w}')^2\right)g'(|\tilde{w}'|) - Mkg(|\tilde{w}'|)} S_{k-1}(\tilde{\kappa}) \\ &\le g(|Dv_{2\varepsilon}|)\left(f(v_{2\varepsilon}) - f'(\tilde{w})\left(L' + \frac{S_{k-1}(\tilde{\kappa})}{M\frac{(1+(\tilde{w}')^2)g'(|\tilde{w}'|)}{g(|\tilde{w}'|)} + \frac{Mk}{\tilde{w}'}}\right)\right). \end{aligned} \tag{2.23}$$

Here we used (2.21). By the boundedness of $S_{k-1}(\tilde{\kappa})$ in $\Omega_{R'}$ and the condition (B4), one sees that there exists $R'' \in (0, R')$ (which depends on L', but does not depend on ε) such that

$$H_k[v_{2\varepsilon}] \le f(v_{2\varepsilon})\, g(|Dv_{2\varepsilon}|) \quad \text{in } \Omega_{R''} \setminus \Omega_\varepsilon. \tag{2.24}$$

Now we choose L' sufficiently large so that $L' > \tilde{v}_2(0) - w_2$ where $\tilde{v}_2 = \tilde{v}_2(r)$ is a solution to

$$\begin{cases} \binom{n-1}{k-1}\dfrac{u''}{\left(1+u'^2\right)^{3/2}}\left(\dfrac{u'}{r\sqrt{1+u'^2}}\right)^{k-1} + \binom{n-1}{k}\left(\dfrac{u'}{r\sqrt{1+u'^2}}\right)^k = f(u)g(|u'|), & r > 0 \\ u(0) = \tilde{v}_2(0) > t_0, \quad u'(0) = 0, & \\ u(r) \to \infty & \text{as } r \to R''/2 - 0. \end{cases} \tag{2.25}$$

It is possible because as L' is larger and larger, we can choose R'' larger and larger so that $\tilde{v}_2(0)$ becomes smaller and smaller. We set $v_2(x) = \tilde{v}_2(|x|)$.

Then, it follows from the comparison principle that $u(y) \le v_2(0) = \tilde{v}_2(0)$ for any $y \in \Omega$ which satisfies $d(y) = R''$. Thus we have that

$$v_{2\varepsilon}(y) = \tilde{w}(R'' - \varepsilon) + L' \ge w_2 + L' > \tilde{v}_2(0) \ge u(y) \tag{2.26}$$

for any $y \in \Omega$ which satisfies $d(y) = R''$. Moreover, it holds that $v_{2\varepsilon}(x) \to \infty$ as $d(x) \to \varepsilon + 0$ while $u(x)$ takes finite value if $d(x) = \varepsilon$. Therefore, we can deduce by the comparison principle that

$$v_{2\varepsilon}(x) = \tilde{w}(d(x) - \varepsilon) + L' \ge u(x) \tag{2.27}$$

for $x \in \Omega_{R''} \setminus \Omega_\varepsilon$. Taking the limit $\varepsilon \to +0$, we get that

$$\tilde{w}(d(x)) + L' \ge u(x) \tag{2.28}$$

for $x \in \Omega_{R''}$.

Using (2.19) and the condition (B3), we obtain that

$$\begin{aligned} -\tilde{w}' &= h^{-1}\left(M^{-1/k} f(\tilde{w})^{1/k}\right) \\ &\ge \min\left\{1, M^{-1/\alpha k}\right\} h^{-1}(f(\tilde{w})^{1/k}). \end{aligned} \tag{2.29}$$

Integrating from 0 to r yields that

$$\psi(w(r)) = \int_{w(r)}^{\infty} \frac{ds}{h^{-1}(f(s)^{1/k})} \ge \min\left\{1, M^{-1/\alpha k}\right\} r \tag{2.30}$$

for $r \in (0, R'')$. Combining (2.28) and (2.30), we conclude that the second inequality in (2.3) holds. □

Example 2.2. Let $1 \le k \le n-1$ and $p, q > 0$. Suppose Ω is a bounded and uniformly k-convex domain with boundary $\partial\Omega \in C^\infty$. We consider the same equations as in Example 2.1:

(i) $H_k[u] = u^p / \left(1 + |Du|^2\right)^{q/2}$ in Ω.

Theorem 2.2 implies that a boundary blowup solution u (if it exists) satisfies

$$C_1 \mathrm{dist}(x, \partial\Omega)^{-\frac{q}{p-q}} \le u(x) \le C_2 \mathrm{dist}(x, \partial\Omega)^{-\frac{q}{p-q}} \quad \text{near } \partial\Omega \tag{2.31}$$

for some constants $C_1, C_2 > 0$, provided $p \ge k$ and $p > q$.

(ii) $H_k[u] = e^{pu} / \left(1 + |Du|^2\right)^{q/2}$ in Ω.

We can also see that a boundary blowup solution u (if it exists) satisfies

$$u(x) = -\frac{q}{p}\log \operatorname{dist}(x, \partial\Omega) + O(1) \quad \text{near } \partial\Omega, \tag{2.32}$$

provided $q > 0$.

Remark 2.2. The case $k = n$, which corresponds to the Gauss curvature equation, is excluded from Theorems 2.1 and 2.2. Alternative results for the case $k = n$ are given in Section 4.

3. Uniqueness results for the boundary blowup problem

In this section, we give the uniqueness result for the boundary blowup problem (1.1)–(1.2) for $1 \le k \le n-1$.

Theorem 3.1. *Let $1 \le k \le n-1$. We assume that the conditions in Theorem 2.2 are satisfied. Also, we assume the following.*

(C1) *Ω is star-shaped (with respect to some point $x_0 \in \Omega$).*
(C2) *There exists constants $\beta > 0$ and $T'' > 0$ such that $f(t)/t^\beta$ is non-decreasing in $[T'', \infty)$.*
(C3) *$\lim_{s\to+0} s\psi^{-1}(s) = 0$, where ψ is defined in Theorem 2.2.*

Then the problem (1.1)–(1.2) has at most one viscosity solution.

Proof. In this proof, we denote the notation $d(x) = \operatorname{dist}(x, \partial\Omega)$ again. Without loss of generality, we may assume that $x_0 = 0$. Suppose that u_1 and u_2 be solutions to (1.1)–(1.2). In the following proof, we argue in the classical sense, but one can justify it in the viscosity sense.

For $\lambda \in (1, 2)$, we define a function $\tilde{u}_{2,\lambda}$ in Ω by $\tilde{u}_{2,\lambda}(x) = \lambda u_2(x/\lambda) - \phi(\lambda)$, where $\phi(\lambda)$ is a positive constant to be determined later. It can be defined due to the condition (C1). Then it holds that

$$H_k[\tilde{u}_{2,\lambda}] = \frac{1}{\lambda^k} H_k[u_2]\left(\frac{x}{\lambda}\right). \tag{3.1}$$

Later, we will determine $\phi(\lambda)$ appropriately, in such a way as to satisfy that $\tilde{u}_{2,\lambda}$ is a subsolution to (1.1) and that $\phi(\lambda) \to 0$ as $\lambda \to 1+0$.

Meanwhile, we suppose that one can choose $\phi(\lambda)$ as above. Now $u_1(x) \to \infty$ as $d(x) \to 0$, while $\tilde{u}_{2,\lambda}$ has finite value on $\partial\Omega$. It follows from the comparison principle that

$$u_1(x) \ge \tilde{u}_{2,\lambda}\left(\frac{x}{\lambda}\right) - \phi(\lambda). \tag{3.2}$$

Letting $\lambda \to 1+0$, we get $u_1 \ge u_2$ in Ω.

Now we prove that $\phi(\lambda)$ can be chosen as desired. Noticing $D\tilde{u}_{2,\lambda}(x) = Du_2(x/\lambda)$, we have by (3.1) that

$$H_k[\tilde{u}_{2,\lambda}] = \frac{1}{\lambda^k} f\left(u_2\left(\frac{x}{\lambda}\right)\right) g(|D\tilde{u}_{2,\lambda}(x)|). \tag{3.3}$$

Therefore, $\tilde{u}_{2,\lambda}$ is a subsolution to (1.1) if and only if it holds that for any $x \in \Omega$,

$$\frac{1}{\lambda^k} f\left(u_2\left(\frac{x}{\lambda}\right)\right) \ge f(\tilde{u}_{2,\lambda}(x)) = f\left(\lambda u_2\left(\frac{x}{\lambda}\right) - \phi(\lambda)\right). \tag{3.4}$$

By Theorem 2.2, we obtain that there exist constants $c > t_0$ and $c_1, c_2 > 0$ such that for any $x \in \Omega$,

$$c \le u_2\left(\frac{x}{\lambda}\right) \le \psi^{-1}\left(\frac{\lambda-1}{\lambda}c_1\right) + c_2, \tag{3.5}$$

because $d(x/\lambda) \ge (1 - 1/\lambda)c_1$ for some $c_1 > 0$ which is independent of $x \in \Omega$. Therefore it is enough to prove that there exists $\phi(\lambda) > 0$ such that

$$f^{-1}\left(\frac{1}{\lambda^k} f(s)\right) \ge \lambda s - \phi(\lambda) \quad \text{for any } c \le s \le \psi^{-1}\left(\frac{\lambda-1}{\lambda}c_1\right) + c_2, \tag{3.6}$$

and that $\phi(\lambda) \to 0$ as $\lambda \to 1+0$.

We set

$$\psi(\lambda, s) = \lambda s - f^{-1}\left(\frac{1}{\lambda^k} f(s)\right), \quad \lambda \in [1, 2],\ s \in (s_0, \infty), \tag{3.7}$$

and

$$\eta(\lambda) = \sup_{c \le s \le 2^{k/\beta} T''} |\psi(\lambda, s)|, \quad \lambda \in [1, 2], \tag{3.8}$$

where β and T'' are constants which appear in the condition (C2). We note that $\psi(1, s) \equiv 0$ which implies $\eta(1) = 0$. Then it is easily seen that $\lim_{\lambda \to 1+0} \eta(\lambda) = 0$.

Now we define

$$\phi(\lambda) = \left(\lambda - \lambda^{-k/\beta}\right) \left(\psi^{-1}\left(\frac{\lambda - 1}{\lambda} c_1\right) + c_2\right) + \eta(\lambda). \tag{3.9}$$

We fix arbitrary s which satisfies $c \le s \le \psi^{-1}((\lambda - 1)c_1/\lambda) + c_2$. First, if $c \le s \le 2^{k/\beta} T''$, then we get that

$$f^{-1}\left(\frac{1}{\lambda^k} f(s)\right) = \lambda s - \psi(\lambda, s) \ge \lambda s - \eta(\lambda) \ge \lambda s - \phi(\lambda). \tag{3.10}$$

Next, if $2^{k/\beta} T'' \le s \le \psi^{-1}((\lambda - 1)c_1/\lambda) + c_2$, then it holds that

$$f^{-1}\left(\frac{1}{\lambda^k} f(s)\right) \ge \frac{s}{\lambda^{k/\beta}} = \lambda s - \left(\lambda - \lambda^{-k/\beta}\right) s \ge \lambda s - \phi(\lambda). \tag{3.11}$$

Here we used the condition (C2). Furthermore, it holds that

$$\phi(\lambda) = \frac{\lambda\left(\lambda - \lambda^{-k/\beta}\right)}{(\lambda - 1)c_1} \left[\frac{(\lambda - 1)c_1}{\lambda} \left(\psi^{-1}\left(\frac{\lambda - 1}{\lambda} c_1\right) + c_2\right)\right] + \eta(\lambda) \to 0 \tag{3.12}$$

as $\lambda \to 1 + 0$, due to the condition (C3). This completes the proof.

By a similar argument, we see that $u_1 \le u_2$ in Ω and hence $u_1 = u_2$ in Ω. □

Example 3.1. Let $1 \le k \le n - 1$ and $p, q > 0$. Suppose Ω is a bounded, star-shaped and uniformly k-convex domain with boundary $\partial\Omega \in C^\infty$. We consider again the same equations as in the last section:

(i) $H_k[u] = u^p / \left(1 + |Du|^2\right)^{q/2}$ in Ω.
Theorem 3.1 implies that there exists at most one boundary blowup solution, provided $p \ge k$ and $p > 2q$.

(ii) $H_k[u] = e^{pu} / \left(1 + |Du|^2\right)^{q/2}$ in Ω.
It follows that there exists at most one boundary blowup solution for any $p, q > 0$.

4. The case of Gauss curvature equation

The case $k = n$, which corresponds to the Gauss curvature equation

$$\frac{\det D^2 u}{(1 + |Du|^2)^{(n+2)/2}} = f(u) g(|Du|) \tag{4.1}$$

is excluded from all theorems in Sections 2 and 3. In this section, we shall obtain the alternative results for the case $k = n$.

First, we state results for the existence and for the asymptotic behavior of a boundary blowup solution, which we have already proved in [1].

Theorem 4.1. *Let $k = n$. We assume that Ω is a bounded and strictly convex domain with boundary $\partial\Omega \in C^\infty$. Furthermore, we also assume that the condition* (A3) *is satisfied and that* $\limsup_{t \to \infty} g(t)t < \infty$. *Then there exists a viscosity solution to* (1.1)–(1.2).

Theorem 4.2. *Let $k = n$. We assume that Ω is a bounded and strictly convex domain with boundary $\partial\Omega \in C^\infty$. Furthermore, we also assume that the conditions* (A3), (B1), (B2) *and*

(B5) *There exists a constant $\alpha > 0$ such that $H(t)/t^\alpha$ is non-decreasing for $t > 0$, where*

$$H(t) = \int_0^t \frac{s^n}{g(s)\left(1 + s^2\right)^{(n+2)/2}} \, ds, \tag{4.2}$$

are satisfied. Then there exist positive constants C_1, C_2 such that every solution u to (1.1)–(1.2) *satisfies*

$$C_1 \mathrm{dist}(x, \partial\Omega) \le \Psi(u(x)) \le C_2 \mathrm{dist}(x, \partial\Omega), \tag{4.3}$$

where Ψ is defined by

$$\Psi(t) = \int_t^\infty \frac{ds}{H^{-1}(F(s))}. \tag{4.4}$$

Next, we establish the uniqueness result for the case $k = n$.

Theorem 4.3. *Let $k = n$. We assume that the conditions in Theorem 4.2 are satisfied. Also, we assume that the conditions* (C2) *and*

(C3)′ $\lim_{s\to+0} s\Psi^{-1}(s) = 0$, *where Ψ is defined in Theorem 4.2,*

are satisfied. Then the problem (1.1)–(1.2) has at most one viscosity solution.

The proof of this theorem is mostly the same as that of Theorem 3.1, so we omit it. Finally, we give some examples.

Example 4.1. Let $k = n$ and $p, q > 0$. Suppose Ω is a bounded and strictly convex domain with boundary $\partial\Omega \in C^\infty$. We consider again the same equations given before:

(i) $H_n[u] = u^p / \left(1 + |Du|^2\right)^{q/2}$ in Ω.

Theorem 4.1 implies that if $p > q \geq 1$, then there exists a boundary blowup solution, and it follows from Theorem 4.2 that any boundary blowup solution satisfies

$$C_1 \operatorname{dist}(x, \partial\Omega)^{-\frac{q-1}{p-q+2}} \leq u(x) \leq C_2 \operatorname{dist}(x, \partial\Omega)^{-\frac{q-1}{p-q+2}} \quad \text{near } \partial\Omega \tag{4.5}$$

for some constants $C_1, C_2 > 0$, provided $p \geq n$ and $p > q > 1$. Moreover, Theorem 4.3 implies that there exists at most one boundary blowup solution, provided $p \geq n$, $p > 2q - 3$ and $q > 1$.

(ii) $H_n[u] = e^{pu} / \left(1 + |Du|^2\right)^{q/2}$ in Ω.

One can see that there exists a unique boundary blowup solution which satisfies

$$u(x) = -\frac{q-1}{p} \log \operatorname{dist}(x, \partial\Omega) + O(1) \quad \text{near } \partial\Omega, \tag{4.6}$$

provided $q > 1$.

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