J. Math. Anal. Appl. 407 (2013) 457–463

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Existence of BPS vortices in the theory of branes

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ARTICLE INFO

Article history:
Received 7 February 2013
Available online 24 May 2013
Submitted by V. Radulescu

Keywords:
Vortices
D-brane
BPS equations
Minimization

ABSTRACT

An existence theorem is established for the BPS equations arising in the theory of branes. For the doubly periodic domain case, we obtain an explicitly necessary and sufficient condition for the existence of a unique solution.

1. Introduction

Vortices are important objects in planar physics [1,8,5,7,9,3,6,10,17]. In this paper we aim to establish the existence of a multiple vortex solution of the BPS equations derived in [15] arising from the theory of D-branes.

Following [8], the BPS equations derived in [15] can be reduced as

$$\Delta U = 2e^U + e^V - 3 + 4\pi \sum_{s=1}^{M} \delta_{p_s}(x), \tag{1.1}$$

$$\Delta V = e^U + 2e^V - 3 + 4\pi \sum_{s=1}^{N} \delta_{q_s}(x), \tag{1.2}$$

where U, V are the unknown functions, and δ_p denotes the Dirac measure centered at p. We will consider the equations in two cases. In the first case the equations will be studied over a doubly periodic domain Ω, governing multiple vortices hosted in Ω such that the field configurations are subject to the 't Hooft boundary condition [16,18,19] under which periodicity is achieved modulo gauge transformations. In the second case the equations will be studied over the full plane $\mathbb{R}^2$ and the solutions satisfy the boundary condition

$$U, V \to 0 \quad \text{as } |x| \to \infty. \tag{1.3}$$

Our main result reads as follows.

Theorem 1.1. (i) *Consider Eqs. (1.1)–(1.2) over a doubly periodic domain Ω. For any given points $p_1, \ldots, p_M \in \Omega$, a solution exists if and only if*

$$\max\{M, N\} < \frac{3|\Omega|}{4\pi}. \tag{1.4}$$

Furthermore, if there exists a solution, it must be unique.

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http://dx.doi.org/10.1016/j.jmaa.2013.05.053

(ii) *Consider Eqs. (1.1)–(1.2) over the full plane $\mathbb{R}^2$ subjected to the boundary condition (1.3). For any given points $p_1, \ldots, p_M \in \mathbb{R}^2$, there exists a unique solution such that the boundary condition (1.3) is realized exponentially fast.*

(iii) *In both cases, there holds the quantized integrals*

$$\int \left(2e^U + e^V - 3\right) \mathrm{d}x = -4\pi M, \tag{1.5}$$

$$\int \left(e^U + 2e^V - 3\right) \mathrm{d}x = -4\pi N. \tag{1.6}$$

The rest of our paper is organized as follows. In Section 2 we prove the explicitly necessary and sufficient condition for the existence result in the doubly periodic domain case. In Section 3 we prove the existence result for the planar case.

2. Proof of existence for doubly periodic case

We consider the problem (1.1)–(1.2) over a doubly periodic domain Ω.

Let u_0 and v_0 be the solution to (see [2])

$$\Delta u_0 = 4\pi \sum_{s=1}^{M} \delta_{p_s} - \frac{4\pi M}{|\Omega|}, \tag{2.1}$$

$$\Delta v_0 = 4\pi \sum_{s=1}^{N} \delta_{q_s} - \frac{4\pi N}{|\Omega|}. \tag{2.2}$$

Set $U = u + u_0$ and $V = v + v_0$, and the equations are transformed into

$$\Delta u = 2e^{u_0+u} + e^{v_0+v} - 3 + \frac{4\pi M}{|\Omega|}, \tag{2.3}$$

$$\Delta v = e^{u_0+u} + 2e^{v_0+v} - 3 + \frac{4\pi N}{|\Omega|}. \tag{2.4}$$

If (u, v) is a solution to the above equations, then, integrating the equations over Ω, we obtain the necessary condition for existence

$$M < \frac{3|\Omega|}{4\pi}, \qquad N < \frac{3|\Omega|}{4\pi} \tag{2.5}$$

which is equivalent to (1.4).

Next we show that the above condition (1.4) is also sufficient.

Using the following transformation

$$f = u + v, \qquad g = u - v, \tag{2.6}$$

or equivalently

$$u = \frac{f+g}{2}, \qquad v = \frac{f-g}{2}, \tag{2.7}$$

we change Eqs. (2.3)–(2.4) into

$$\Delta f = 3\left(e^{u_0+\frac{f+g}{2}} + e^{v_0+\frac{f-g}{2}} - 2\right) + \frac{4\pi}{|\Omega|}(M+N), \tag{2.8}$$

$$\Delta g = e^{u_0+\frac{f+g}{2}} - e^{v_0+\frac{f-g}{2}} + \frac{4\pi}{|\Omega|}(M-N), \tag{2.9}$$

which are the Euler–Lagrange equations of the functional

$$I(f,g) = \int_\Omega \left\{\frac{1}{6}|\nabla f|^2 + \frac{1}{2}|\nabla g|^2 + 2e^{u_0+\frac{f+g}{2}} + 2e^{v_0+\frac{f-g}{2}} + 2\left[\frac{2\pi}{3|\Omega|}(M+N) - 1\right] f + \frac{4\pi}{|\Omega|}(M-N)g\right\} \mathrm{d}x. \tag{2.10}$$

Integrating Eqs. (2.8)–(2.9) over Ω, we obtain the conditions

$$\int_\Omega e^{u_0+\frac{f+g}{2}} \mathrm{d}x = |\Omega| - \frac{4\pi}{3}(2M-N) \equiv \eta_1 > 0, \tag{2.11}$$

$$\int_\Omega e^{v_0+\frac{f-g}{2}} \mathrm{d}x = |\Omega| - \frac{4\pi}{3}(2N-M) \equiv \eta_2 > 0, \tag{2.12}$$

which is equivalent to (2.5). Then (1.5)–(1.6) follows from (2.11)–(2.12).

We will deal with the doubly periodic domain case by two methods, namely, the first is the direct minimization method developed in [11,19], and the second method is a constrained minimization method, which was recently used in [14,13,12] to tackle the existence of non-Abelian vortices and dyons.

2.1. Direct minimization

Let $W^{1,2}(\Omega)$ be the usual Sobolev space of Ω-periodic L^2 functions with their derivatives also in $L^2(\Omega)$. For $W^{1,2}(\Omega)$ in the scalar case, we have the decomposition

$$W^{1,2}(\Omega) = \mathbb{R} + \dot{W}^{1,2}(\Omega)$$

such that any $w \in W^{1,2}(\Omega)$ can be expressed as

$$w = \underline{w} + \dot{w}, \quad \underline{w} \in \mathbb{R}, \qquad \dot{w} \in \dot{W}^{1,2}(\Omega), \quad \int_\Omega \dot{w}\mathrm{d}x = 0. \tag{2.13}$$

For the function $w \in \dot{W}^{1,2}(\Omega)$, there holds the Trudinger–Moser inequality [2,4]

$$\int_\Omega e^w \mathrm{d}x \le C \exp\left(\frac{1}{16\pi}\int_\Omega |\nabla w|^2 \mathrm{d}x\right), \tag{2.14}$$

which is important for our estimate.

As $(f, g) \in W^{1,2}(\Omega)$, using the above inequality (2.14) we see that the functional defined by (2.10) is a C^1 functional which is strictly convex and lower semi-continuous with respect to the weak topology of $W^{1,2}(\Omega)$.

For $(f, g) \in W^{1,2}(\Omega)$, applying the decomposition formula (2.13) and (2.6)–(2.7), we have

$$I(f, g) - \int_\Omega \left\{\frac{1}{6}|\nabla \dot{f}|^2 + \frac{1}{2}|\nabla \dot{g}|^2\right\} \mathrm{d}x = 2\int_\Omega (e^{u_0+\dot{u}+\underline{u}} + e^{v_0+\dot{v}+\underline{v}})\mathrm{d}x - 2(\eta_1\underline{u} + \eta_2\underline{v}). \tag{2.15}$$

It follows from Jensen's inequality that

$$\int_\Omega e^{u_0+\dot{u}+\underline{u}} \ge |\Omega| \exp\left(\frac{1}{|\Omega|}\int_\Omega (u_0 + \dot{u} + \underline{u})\mathrm{d}x\right) = |\Omega| \exp\left(\frac{1}{|\Omega|}\int_\Omega u_0 \mathrm{d}x\right) e^{\underline{u}} \equiv \sigma_1 e^{\underline{u}}. \tag{2.16}$$

Analogously,

$$\int_\Omega e^{v_0+\dot{v}+\underline{v}} \ge |\Omega| \exp\left(\frac{1}{|\Omega|}\int_\Omega v_0 \mathrm{d}x\right) e^{\underline{v}} \equiv \sigma_2 e^{\underline{v}}. \tag{2.17}$$

Combining (2.15)–(2.17), we have

$$\begin{aligned} I(f, g) - \int_\Omega \left\{\frac{1}{6}|\nabla \dot{f}|^2 + \frac{1}{2}|\nabla \dot{g}|^2\right\} \mathrm{d}x &\ge 2\big[(\sigma_1 e^{\underline{u}} - \eta_1\underline{u}) + (\sigma_2 e^{\underline{v}} - \eta_2\underline{v})\big] \\ &\ge 2\left(\eta_1 \ln\frac{\sigma_1}{\eta_1} + \eta_2 \ln\frac{\sigma_2}{\eta_2}\right). \end{aligned} \tag{2.18}$$

From (2.18) we can see that the functional $I(f, g)$ is bounded from below and the following minimization problem

$$\eta_0 \equiv \inf\big\{I(f, g) | f, g \in W^{1,2}(\Omega)\big\} \tag{2.19}$$

is well-defined.

Let (f_k, g_k) be a minimizing sequence of (2.19). It is easy to see that the function $F(t) = \sigma e^t - \eta t$, where σ, η are positive constants, satisfies the property that $F(t) \to +\infty$ as $t \to \pm\infty$. Then, we infer from (2.18) that $\{\underline{u}_k\}$ and $\{\underline{v}_k\}$ are bounded, as a result, $\{\underline{f}_k\}$ and $\{\underline{g}_k\}$ are bounded. Then, the sequences $\{\underline{f}_k\}$ and $\{\underline{g}_k\}$ admit convergent subsequences, still denoted by $\{\underline{f}_k\}$ and $\{\underline{g}_k\}$ for convenience. That is to say, there exist two real numbers $\underline{f}_\infty, \underline{g}_\infty \in \mathbb{R}$ such that $\underline{f}_k \to \underline{f}_\infty$ and $\underline{g}_k \to \underline{g}_\infty$ as $k \to \infty$.

In addition, using Poincaré inequality and (2.18), we conclude that the sequences $\{\dot{f}_k\}$ and $\{\dot{g}_k\}$ are bounded in $W^{1,2}(\Omega)$. Therefore, the sequences $\{\dot{f}_k\}$ and $\{\dot{g}_k\}$ admit weakly convergent subsequences, still denoted by $\{\dot{f}_k\}$ and $\{\dot{g}_k\}$ for convenience. In other words, there exist two functions $\dot{f}_\infty, \dot{g}_\infty \in W^{1,2}(\Omega)$ such that $\dot{f}_k \to \dot{f}_\infty$ and $\dot{g}_k \to \dot{g}_\infty$ weakly in $W^{1,2}(\Omega)$ as $k \to \infty$. Of course, $\dot{f}_\infty, \dot{g}_\infty \in \dot{W}^{1,2}(\Omega)$.

Set $f_\infty = \underline{f}_\infty + \dot{f}_\infty, g_\infty = \underline{g}_\infty + \dot{g}_\infty$, which are all in $W^{1,2}(\Omega)$ naturally. Then, the above convergence implies $f_k \to f_\infty, g_k \to g_\infty$ weakly in $W^{1,2}(\Omega)$ as $k \to \infty$. Since the functional $I(f, g)$ is weakly lower semi-continuous in $W^{1,2}(\Omega)$, we conclude that (f_∞, g_∞) is a solution of the minimization problem (2.19) and is a critical point of $I(f, g)$. As a critical point of $I(f, g)$, it satisfies Eqs. (2.8)–(2.9). Noting that $I(f, g)$ is strictly convex, we know that it has at most one critical point, which implies the uniqueness of the solutions to Eqs. (2.8)–(2.9).

2.2. Constrained minimization

The constraints (2.11)–(2.12) can be written as

$$J_1(f,g) \equiv \int_\Omega e^{u_0+u}\mathrm{d}x = \eta_1, \qquad J_2(f,g) = \int_\Omega e^{v_0+v}\mathrm{d}x = \eta_2. \tag{2.20}$$

We consider the constrained minimization problem

$$\eta_0 \equiv \inf\big\{I(f,g)|(f,g) \in W^{1,2}(\Omega) \text{ and satisfies (2.20)}\big\}. \tag{2.21}$$

Assume that $(\tilde f, \tilde g)$ is a solution of (2.21). Then there exist two real numbers, say λ_1, λ_2, such that

$$D(I+\lambda_1 J_1 + \lambda_2 J_2)(\tilde f, \tilde g)(f,g) = 0, \quad \forall (f,g) \in W^{1,2}(\Omega). \tag{2.22}$$

Noting $(\tilde f, \tilde g)$ and taking $(f,g) = (1,0), (0,1)$ in (2.22), we obtain

$$\lambda_1\eta_1 + \lambda_2\eta_2 = 0, \qquad \lambda_1\eta_1 - \lambda_2\eta_2 = 0. \tag{2.23}$$

Therefore, $\lambda_1 = \lambda_2 = 0$. That is to say, the constraints do not lead to a Lagrange problem so that a solution of the constrained minimization problem is a critical point of the functional $I(f,g)$ itself.

From (2.20) we obtain

$$\underline{u} = \ln\eta_1 - \ln\left(\int_\Omega e^{u_0+u}\mathrm{d}x\right), \qquad \underline{v} = \ln\eta_2 - \ln\left(\int_\Omega e^{v_0+v}\mathrm{d}x\right). \tag{2.24}$$

Then, we have

$$\begin{aligned} I(f,g) - \int_\Omega\left(\frac{1}{6}|\nabla\dot f| + \frac{1}{2}|\nabla\dot g|^2\right)\mathrm{d}x &= 2\int_\Omega e^{u_0+\dot u}\mathrm{d}x + 2\int_\Omega e^{v_0+\dot v}\mathrm{d}x - 2\eta_1\underline{u} - 2\eta_2\underline{v} \\ &\geq 2\left(\eta_1\ln\frac{\sigma_1}{\eta_1} + \eta_2\ln\frac{\sigma_2}{\eta_2}\right), \end{aligned} \tag{2.25}$$

where we have used (2.18) and Jensen's inequality. Hence, the functional $I(f,g)$ is bounded from below and the minimization problem (2.19) is well-defined.

Let $\{(f_k, g_k)\}$ be a minimizing sequence of (2.19). Then it follows from (2.25) that $\{(\dot f_k, \dot g_k)\}$ is bounded in $W^{1,2}(\Omega)$. Without loss of generality, we may assume that $\{(\dot f_k, \dot g_k)\}$ weakly converges to $(\dot f, \dot g)$ in $W^{1,2}(\Omega)$. Then from the Trudinger–Moser inequality (2.14) and (2.24) we obtain that $\{(\underline{f}_k, \underline{g}_k)\}$ converges to $(\underline{f}, \underline{g})$, where $\underline{f}, \underline{g} \in \mathbb{R}$. Hence, $\{(f_k, g_k)\}$ weakly converges to $(\underline{f} + \dot f, \underline{g} + \dot g)$ in $W^{1,2}(\Omega)$. Then by the Trudinger–Moser inequality (2.14) and the embedding theorem, we see that the weak limit $(f,g) \equiv (\underline{f} + \dot f, \underline{g} + \dot g)$ satisfies the constraints in (2.20). Since J_1 and J_2 are weakly continuous and the functional I is weakly lower semi-continuous, we can conclude that the weak limit of $\{(f_k, g_k)\}$ is a solution of (2.19). The uniqueness of the solutions follows from the strict convexity of the functional.

3. Proof of existence for the planar case

To deal with the existence of the planar case, we follow the approach of [8] and introduce the background functions

$$u_0 = -\sum_{s=1}^M \ln(1+\mu|x-p_s|^{-2}), \qquad v_0 = -\sum_{s=1}^N \ln(1+\mu|x-q_s|^{-2}). \tag{3.1}$$

Straight calculation leads to

$$\Delta u_0 = -\sum_{s=1}^M \frac{4\mu}{(1+\mu|x-p_s|^2)^2} + 4\pi\sum_{s=1}^M \delta_{p_s} \equiv -h_1(x) + 4\pi\sum_{s=1}^M\delta_{p_s}, \tag{3.2}$$

$$\Delta v_0 = -\sum_{s=1}^N \frac{4\mu}{(1+\mu|x-q_s|^2)^2} + 4\pi\sum_{s=1}^N \delta_{q_s} \equiv -h_2(x) + 4\pi\sum_{s=1}^N\delta_{q_s}. \tag{3.3}$$

Writing $U = u_0 + u$, $V = v_0 + v$, we have

$$\Delta u = 2e^{u_0+u} + e^{v_0+v} - 3 + h_1, \tag{3.4}$$

$$\Delta v = e^{u_0+u} + 2e^{v_0+v} - 3 + h_2. \tag{3.5}$$

Let $f = u + v$, $g = u - v$, which is $u = \frac{f+g}{2}$, $v = \frac{f-g}{2}$. Obviously, we have the relation

$$f^2 + g^2 = 2(u^2 + v^2), \qquad |\nabla f|^2 + |\nabla g|^2 = 2(|\nabla u|^2 + |\nabla v|^2). \tag{3.6}$$

Then we have

$$\Delta f = 3\left(e^{u_0+\frac{f+g}{2}} + e^{v_0+\frac{f-g}{2}} - 2\right) + h_1 + h_2, \tag{3.7}$$

$$\Delta g = e^{u_0+\frac{f+g}{2}} - e^{v_0+\frac{f-g}{2}} + h_1 - h_2 \tag{3.8}$$

over the full plane $\mathbb{R}^2$. The boundary condition for f, g reads

$$f \to 0, \qquad g \to 0 \quad \text{as } |x| \to \infty. \tag{3.9}$$

To solve the problem (3.7)–(3.9), we look for the critical point of the functional

$$I(f,g) = \int_{\mathbb{R}^2} \left\{ \frac{1}{6}|\nabla f|^2 + \frac{1}{2}|\nabla g|^2 + 2e^{u_0+\frac{f+g}{2}} + 2e^{v_0+\frac{f-g}{2}} + \left(\frac{h_1+h_2}{3} - 2\right) f + (h_1 - h_2) g \right\} \mathrm{d}x. \tag{3.10}$$

It is easy to see that the functional is C^1 and strictly convex over $W^{1,2}(\mathbb{R}^2)$.

By direct computation, we see that the Fréchet derivative of I satisfies

$$\begin{aligned}
&(DI(f,g))(f,g) - \int_{\mathbb{R}^2} \left(\frac{1}{6}|\nabla f|^2 + \frac{1}{2}|\nabla g|^2 \right) \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^2} \left\{ \frac{1}{6}|\nabla f|^2 + \frac{1}{2}|\nabla g|^2 + 2(e^{u_0+u} - 1 + l_1)u + 2(e^{v_0+v} - 1 + l_2)v \right\} \mathrm{d}x \\
&\quad \geq \int_{\mathbb{R}^2} \left\{ \frac{1}{3}(|\nabla u|^2 + |\nabla v|^2) + 2(e^{u_0+u} - 1 + l_1)u + 2(e^{v_0+v} - 1 + l_2)v \right\} \mathrm{d}x,
\end{aligned} \tag{3.11}$$

where we have used the notation

$$l_1 \equiv \frac{1}{3}(2h_1 - h_2), \qquad l_2 \equiv \frac{1}{3}(2h_2 - h_1) \tag{3.12}$$

and the relation (3.6).

Now we estimate the last two terms in (3.11). Let

$$M(u) = (e^{u_0+u} - 1 + l_1)u, \qquad M(v) = (e^{v_0+v} - 1 + l_2)v,$$

$u_+ = \max\{u, 0\}$, and $u_- = \max\{-u, 0\}$. Then it follows from the elementary inequality $e^t - 1 \geq t$ that

$$e^{u_0+u_+} - 1 + l_1 \geq u_0 + u_+ + l_1,$$

which implies

$$\begin{aligned}
\int_{\mathbb{R}^2} M(u_+)\mathrm{d}x &\geq \int_{\mathbb{R}^2} (u_0 + u_+ + l_1)u_+ \mathrm{d}x \\
&= \int_{\mathbb{R}^2} u_+^2 \mathrm{d}x + \int_{\mathbb{R}^2} (u_0 + l_1)u_+ \mathrm{d}x \\
&\geq \frac{1}{2}\int_{\mathbb{R}^2} u_+^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^2} (u_0 + l_1)^2 \mathrm{d}x.
\end{aligned} \tag{3.13}$$

Next, using the inequality $1 - e^{-t} \geq \frac{t}{1+t}$ $\forall t \geq 0$, we have

$$\begin{aligned}
M(-u_-) &= \big(1 - l_1 - e^{u_0-u_-}\big)u_- \\
&= \big(1 - l_1 + [1 - e^{-u_-}]e^{u_0} - e^{u_0}\big)u_- \\
&\geq \left(1 - l_1 + \frac{u_-}{1+u_-}e^{u_0} - e^{u_0}\right)u_- \\
&\geq [1 - l_1]\frac{u_-^2}{(1+u_-)^2} + \frac{u_-}{1+u_-}(1 - e^{u_0} - l_1).
\end{aligned} \tag{3.14}$$

We may take μ sufficiently large such that $\max\{l_1, l_2\} < \frac{1}{2}$ (say). Noting that $1 - e^{u_0}$ and l_1, l_2 are in $L^2(\Omega)$, we see that

$$\int_{\mathbb{R}^2} \frac{u_-}{1+u_-}(1 - e^{u_0} - l_1)\mathrm{d}x \geq -\frac{1}{4}\int_{\mathbb{R}^2} \frac{u_-^2}{(1+u_-)^2}\mathrm{d}x - \int_{\mathbb{R}^2} (1 - e^{u_0} - l_1)^2 \mathrm{d}x. \tag{3.15}$$

Then it follows from (3.14)–(3.15) that

$$\int_{\mathbb{R}^2} M(-u_-)\mathrm{d}x \geq \frac{1}{4}\int_{\mathbb{R}^2} \frac{u_-^2}{(1+u_-)^2}\mathrm{d}x - C. \tag{3.16}$$

Here and below, we use C and C with subscripts to denote positive constants. By using (3.13) and (3.16), we obtain

$$\int_{\mathbb{R}^2} M(u)\mathrm{d}x \geq \frac{1}{4}\int_{\mathbb{R}^2} \frac{u^2}{(1+|u|)^2}\mathrm{d}x - C. \tag{3.17}$$

Similarly, we can get

$$\int_{\mathbb{R}^2} M(v)\mathrm{d}x \geq \frac{1}{4}\int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2}\mathrm{d}x - C. \tag{3.18}$$

Therefore, it follows from (3.11), (3.17) and (3.18) that

$$\begin{aligned} &DI(f,g)(f,g) - \int_{\mathbb{R}^2}\left\{\frac{1}{6}|\nabla f|^2 + \frac{1}{2}|\nabla g|^2\right\}\mathrm{d}x \\ &\quad \geq \frac{1}{3}\int_{\mathbb{R}^2}(|\nabla u|^2 + |\nabla v|^2)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2}\left\{\frac{u^2}{(1+|u|)^2} + \frac{v^2}{(1+|v|)^2}\right\}\mathrm{d}x - C. \end{aligned} \tag{3.19}$$

Now we carry out an estimate of L^2 norms. Using the following Sobolev inequality

$$\int_{\mathbb{R}^2} w^4\mathrm{d}x \leq 2\int_{\mathbb{R}^2} w^2\mathrm{d}x \int_{\mathbb{R}^2}|\nabla w|^2\mathrm{d}x, \quad w \in W^{1,2}(\mathbb{R}^2),$$

we obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^2} w^2\mathrm{d}x\right)^2 &= \left(\int_{\mathbb{R}^2}\frac{|w|}{1+|w|}(1+|w|)|w|\mathrm{d}x\right)^2 \\ &\leq 2\int_{\mathbb{R}^2}\frac{|w|^2}{(1+|w|)^2}\mathrm{d}x\int_{\mathbb{R}^2}(|w|^2+|w|^4)\mathrm{d}x \\ &\leq 4\int_{\mathbb{R}^2}\frac{|w|^2}{(1+|w|)^2}\mathrm{d}x\int_{\mathbb{R}^2} w^2\mathrm{d}x\left(1+\int_{\mathbb{R}^2}|\nabla w|^2\mathrm{d}x\right) \\ &\leq \frac{1}{2}\left(\int_{\mathbb{R}^2} w^2\mathrm{d}x\right)^2 + C\left[1+\left(\int_{\mathbb{R}^2}\frac{|w|^2}{(1+|w|)^2}\mathrm{d}x\right)^4 + \left(\int_{\mathbb{R}^2}|\nabla w|^2\mathrm{d}x\right)^4\right]. \end{aligned} \tag{3.20}$$

Then, it follows from (3.20) that

$$\left(\int_{\mathbb{R}^2} w^2\mathrm{d}x\right)^{\frac{1}{2}} \leq C\left[1+\int_{\mathbb{R}^2}\frac{|w|^2}{(1+|w|)^2}\mathrm{d}x + \int_{\mathbb{R}^2}|\nabla w|^2\mathrm{d}x\right]. \tag{3.21}$$

Hence, from (3.19) and (3.21) we have

$$DI(f,g)(f,g) - \int_{\mathbb{R}^2}\left\{\frac{1}{6}|\nabla f|^2 + \frac{1}{2}|\nabla g|^2\right\}\mathrm{d}x \geq C_0(\|u\|_{L^2(\mathbb{R}^2)} + \|v\|_{L^2(\mathbb{R}^2)}) - C_1. \tag{3.22}$$

Then, by the relation (3.6) and (3.22) we get the following lower bound

$$DI(f,g)(f,g) \geq C_2(\|f\|_{W^{1,2}(\mathbb{R}^2)} + \|g\|_{W^{1,2}(\mathbb{R}^2)}) - C_3. \tag{3.23}$$

By using the estimate (3.23) the critical point of the functional I can be obtained. In fact, from (3.23) we may choose $R > 0$ sufficiently large such that

$$\inf\left\{DI(f,g)(f,g) \mid \|f\|_{W^{1,2}(\mathbb{R}^2)} + \|g\|_{W^{1,2}(\mathbb{R}^2)} = R\right\} \geq 1 \tag{3.24}$$

(say).

Now we consider the minimization problem

$$\eta_0 \equiv \inf\left\{I(f,g)(f,g) \mid \|f\|_{W^{1,2}(\mathbb{R}^2)} + \|g\|_{W^{1,2}(\mathbb{R}^2)} \leq R\right\}. \tag{3.25}$$

This problem obviously has a solution since the functional I is weakly lower semi-continuous. Let $\tilde{\mathbf{w}} = (\tilde{f}, \tilde{g})$ be a solution of (3.25). We prove that it must be a interior point. Otherwise, if

$$\|\tilde{f}\|_{W^{1,2}(\mathbb{R}^2)} + \|\tilde{g}\|_{W^{1,2}(\mathbb{R}^2)} = R,$$

then,

$$\lim_{t\to 0}\frac{I([1-t]\tilde{\mathbf{w}})-I(\tilde{\mathbf{w}})}{t}=\left.\frac{\mathrm{d}I([1-t]\tilde{\mathbf{w}})}{\mathrm{d}t}\right|_{t=0}=-DI(\tilde{\mathbf{w}})(\tilde{\mathbf{w}})\leq -1. \tag{3.26}$$

Hence, we can choose $t>0$ sufficiently small such that, with $\mathbf{w}^t=(f^t,g^t)=(1-t)\tilde{\mathbf{w}}$,

$$I(f^t,g^t)=I(\mathbf{w}^t)<I(\tilde{\mathbf{w}})=I(\tilde{f},\tilde{g})=\eta_0, \tag{3.27}$$

$$\|f^t\|_{\mathbb{R}^2}+\|g^t\|_{\mathbb{R}^2}=(1-t)(\|\tilde{f}\|_{W^{1,2}(\mathbb{R}^2)}+\|\tilde{g}\|_{W^{1,2}(\mathbb{R}^2)})=(1-t)R<R, \tag{3.28}$$

which contradicts the definition of η_0. Therefore, $(\tilde{f},\tilde{g})$ must be an interior point for the problem (3.25). As a result, it is a critical point of the functional I. The uniqueness of the critical point follows from the fact that the functional I is strictly convex.

Using the Sobolev embedding inequality

$$\|w\|_{L^p(\mathbb{R}^2)}\leq\left[\pi\left(\frac{p}{2}-1\right)\right]^{\frac{p-2}{2p}}\|w\|_{W^{1,2}(\mathbb{R}^2)},\quad \forall p>2,\ w\in W^{1,2}(\mathbb{R}^2), \tag{3.29}$$

we see that $e^w-1\in L^2(\mathbb{R}^2)$ as $w\in W^{1,2}(\mathbb{R}^2)$. Applying this fact in the equation and using an elliptic estimate, we obtain $f,g\in W^{2,2}(\mathbb{R}^2)$, which implies $f,g\to 0$ as $|x|\to\infty$. By the inequality (3.29) and the equations of f,g, we see that the right hand side of the equations are in $L^p(\mathbb{R}^2)$ for any $p>2$. Then the elliptic L^p estimate implies $f,g\in W^{2,p}(\mathbb{R}^2)$. Hence, $|\nabla f|,|\nabla g|\to 0$ as $|x|\to\infty$. Linearizing the equations of f,g, we see that f,g vanish exponentially fast and $|\nabla f|,|\nabla g|$ vanish like $O(|x|^{-3})$ as $|x|\to\infty$. Hence, we have

$$\int_{\mathbb{R}^2}\Delta f\,\mathrm{d}x=0,\qquad \int_{\mathbb{R}^2}\Delta g\,\mathrm{d}x=0.$$

Integrating Eqs. (3.7)–(3.8) over $\mathbb{R}^2$, we have

$$3\int_{\mathbb{R}^2}\left(e^{u_0+\frac{f+g}{2}}+e^{v_0+\frac{f-g}{2}}-2\right)\mathrm{d}x=-\int_{\mathbb{R}^2}(h_1+h_2)\mathrm{d}x=-4\pi(M+N), \tag{3.30}$$

$$\int_{\mathbb{R}^2}\left(e^{u_0+\frac{f+g}{2}}-e^{v_0+\frac{f-g}{2}}\right)\mathrm{d}x=-\int_{\mathbb{R}^2}(h_1-h_2)\mathrm{d}x=-4\pi(M-N). \tag{3.31}$$

Then, from (3.30)–(3.31), we obtain (1.5)–(1.6).

Acknowledgments

The author would like to thank the anonymous referees for their valuable comments and helpful suggestions. This work was supported by the scientific research project of colleges of Hebei Province under grant CZ2010208.

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