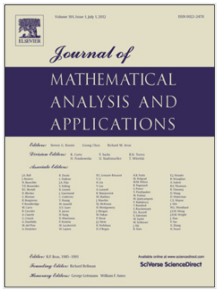

Lipschitz constants and logarithmic Sobolev inequality

Yasuhiro Fujita

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LIPSCHITZ CONSTANTS AND LOGARITHMIC SOBOLEV INEQUALITY

YASUHIRO FUJITA

Abstract. We revisit two results of [8]; they are a logarithmic Sobolev inequality on $\mathbb{R}^n$ with Lipschitz constants and an expression of Lipschitz constants as the limit of a functional by the entropy. We have two goals in this paper. The first goal is to clarify when the strict inequality holds in this inequality. The second goal is to investigate the asymptotic behavior of this functional by the Abelian and Tauberian theorems of Laplace transforms.

1. Introduction

Let $n \in \mathbb{N}$. For a smooth enough function $f \geq 0$ on $\mathbb{R}^n$, we define the entropy of f by

$$\mathrm{Ent}(f) = \int_{\mathbb{R}^n} f(x) \log f(x) dx - \int_{\mathbb{R}^n} f(x) dx \ \log \int_{\mathbb{R}^n} f(x) dx.$$

We interpret that $0 \log 0 = 0$. We denote by $\| \cdot \|_\infty$ the $L^\infty(\mathbb{R}^n)$-norm with respect to the Lebesgue measure on $\mathbb{R}^n$. Hence, if f is Lipschitz continuous on $\mathbb{R}^n$, $\|Df\|_\infty$ (the $L^\infty(\mathbb{R}^n)$-norm of the Euclidean length $|Df|$ of the gradient Df) is the Lipschitz constant of f.

In [8], the author showed the following logarithmic Sobolev inequality with Lipschitz constants: if f is Lipschitz continuous on $\mathbb{R}^n$ and $e^f \in L^\alpha(\mathbb{R}^n)$ for some $\alpha > 0$, then $e^f \in L^\beta(\mathbb{R}^n)$ for any $\beta > \alpha$, and

$$\mathrm{Ent}(e^{\beta f}) \leq n \int_{\mathbb{R}^n} e^{\beta f(x)} dx \ \log \left(\frac{k_n \beta \|Df\|_\infty}{e} \right). \tag{1.1}$$

Here, the constant k_n is given by

$$k_n = \left(\frac{1}{\sigma_{n-1}(n-1)!} \right)^{1/n} \tag{1.2}$$

and $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit ball of $\mathbb{R}^n$. Furthermore, when $f(x) = C - \theta|x - a|$ for some $C \in \mathbb{R}$, $\theta > 0$ and

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$a \in \mathbb{R}^n$, equality holds in (1.1) for all $\beta > 0$. Note that inequality (1.1) is equivalent to

$$\frac{e}{k_n \beta} \exp\left(\frac{1}{n} \frac{\mathrm{Ent}(e^{\beta f})}{\int_{\mathbb{R}^n} e^{\beta f(x)} dx}\right) \leq \|Df\|_\infty. \tag{1.3}$$

As an application of inequality (1.1), the author also showed that the Lipschitz constant $\|Df\|_\infty$ can be expressed by using the entropy of f. Indeed, if f is Lipschitz continuous on $\mathbb{R}^n$ with the Lipschitz constant $\theta := \|Df\|_\infty > 0$ and fulfills the condition

$$-\theta \log(1 + |x - a|) \geq f(x) - f(a) \geq -\theta |x - a|, \quad x \in \mathbb{R}^n \tag{1.4}$$

for some $a \in \mathbb{R}^n$, then

$$\|Df\|_\infty = \lim_{\beta \to \infty} \frac{e}{k_n \beta} \exp\left(\frac{1}{n} \frac{\mathrm{Ent}(e^{\beta f})}{\int_{\mathbb{R}^n} e^{\beta f(x)} dx}\right). \tag{1.5}$$

This is a link between the Lipschitz constant and the limit of the functional given by the entropy.

In this paper, we revisit inequality (1.1) and representation formula (1.5) by using the distribution function of a given function f. Let f be a continuous function on $\mathbb{R}^n$ such that f attains its maximum over $\mathbb{R}^n$ at some point $a \in \mathbb{R}^n$. Without a loss of generality in (1.1) and (1.5), we may assume that $f(a) = 0$. Indeed, if necessary, we consider $f(\cdot) - f(a)$ instead of f, since $\mathrm{Ent}(e^{\beta(f(\cdot) - f(a))}) = e^{-\beta f(a)} \mathrm{Ent}(e^{\beta f})$. We do not require that f is Lipschitz continuous on $\mathbb{R}^n$ in general. We define the distribution function F of f by

$$F(t) = |\{x \in \mathbb{R}^n \mid -f(x) \leq t\}|, \quad t \geq 0, \tag{1.6}$$

where $|A|$ denotes the Lebesgue measure of a set A of $\mathbb{R}^n$. We assume that

(1.7) there exist constants $\beta_0 > 0$ and $C_0 > 0$ such that $F(t) \leq C_0 e^{\beta_0 t}$ for all $t \geq 0$.

Let us define $I(\beta)$ by

$$I(\beta) = \frac{e}{k_n \beta} \exp\left(\frac{1}{n} \frac{\mathrm{Ent}(e^{\beta f})}{\int_{\mathbb{R}^n} e^{\beta f(x)} dx}\right), \quad \beta > \beta_0. \tag{1.8}$$

As shown in Lemma 2.1 below, it is well-defined under (1.7).

We have two goals in this paper. The first goal is to clarify when the strict inequality holds in (1.1). It is stated as follows: Let ω be the Laplace transform of F defined by

$$\omega(\beta) = \int_0^\infty e^{-\beta t} F(t) dt, \quad \beta > \beta_0. \tag{1.9}$$

We define the function ϕ by

$$\phi(\beta) = \log \omega(\beta), \quad \beta > \beta_0. \tag{1.10}$$

Note that ϕ is well-defined, since $\omega > 0$ on (β_0, ∞). As shown in Theorem 3.1 below, we have

$$\frac{d}{d\beta}\, I(\beta) = \frac{\beta}{n}\, I(\beta) \left(\phi''(\beta) - \frac{n+1}{\beta^2} \right), \quad \beta > \beta_0. \tag{1.11}$$

This equality provides an answer to the comment of [8, Remark 3.2] about monotone property of $I(\beta)$. That is, the monotone property of $I(\cdot)$ is determined by the sign of the function $\phi''(\beta) - (n+1)/\beta^2$. We provide examples to illustrate this result.

As an application of this result, we clarify when the strict inequality holds in (1.1). Our main result says the following: let f be Lipschitz continuous on $\mathbb{R}^n$. Then, the strict inequality holds in (1.1) for all $\beta \in D$, where D is the set defined by

$$D = \left\{ \beta \in (\beta_0, \infty) \;\middle|\; \phi''(\beta) \neq \frac{n+1}{\beta^2} \right\}. \tag{1.12}$$

Note that $\phi''(\beta)$ is expressed as $(\omega'(\beta)/\omega(\beta))'$. The problem is when $D \neq \emptyset$. When $D \neq \emptyset$, the strict inequality holds for some $\beta \in (\beta_0, \infty)$. We show that $D = \emptyset$ if and only if $F(t) = v_n (t/\theta)^n,\ t \geq 0$, for some constant $\theta > 0$. Here, v_n is the volume of the unit ball of $\mathbb{R}^n$ and $v_n = \sigma_{n-1}/n$. Note that the distribution function $F(t) = v_n (t/\theta)^n,\ t \geq 0$, for some constant $\theta > 0$ is that of the function $-\theta|x-a|$ for some constant $a \in \mathbb{R}^n$. Hence, $D = \emptyset$ if and only if the distribution function of f is same as that of the function $-\theta|x-a|$ for some constants $\theta > 0$ and $a \in \mathbb{R}^n$. Therefore, when this is not the case, $D \neq \emptyset$. Furthermore, we show that if $D \neq \emptyset$, then the set D is very large in the sense that $(\beta_0, \infty) \setminus D$ is at most countable.

The second goal is to investigate the asymptotic behavior of $I(\beta)$ as $\beta \to \infty$. The key is the Abelian and Tauberian theorems about the Laplace transform of the distribution function F. We show that if there exist constants $\alpha > 0$ and $C > 0$ such that

$$F(t) \sim \frac{C t^\alpha}{\Gamma(\alpha+1)} \quad \text{as } t \to 0^+, \tag{1.13}$$

then

$$I(\beta) \sim \frac{(\beta/e)^{\frac{\alpha-n}{n}}}{k_n C^{\frac{1}{n}}} \quad \text{as } \beta \to \infty. \tag{1.14}$$

Here, we used the symbol $A(s) \sim B(s)$ as $s \to s_0$ to indicate that B is positive in a neighborhood of s_0 and $A(s)/B(s) \to 1$ as $s \to s_0$. Note that the asymptotic behavior of $I(\beta)$ as $\beta \to \infty$ is determined by a local property of $F(t)$ as $t \to 0^+$. As the special case for $\alpha = n$, we conclude that if there exists a constant $\theta > 0$ such that

$$F(t) \sim v_n \left(\frac{t}{\theta}\right)^n \quad \text{as } t \to 0^+, \tag{1.15}$$

then

$$I(\beta) \sim \theta \quad \text{as } \beta \to \infty. \tag{1.16}$$

If $f(x)/|x-a| \to -\theta$ as $|x-a| \to 0$, then (1.15) is fulfilled, so that we have (1.16). Note that this corresponds to (1.5) under (1.4), since $\log(1+|x-a|)/|x-a| \to 1$ as $|x-a| \to 0$. The implication from (1.13) to (1.14) is an Abelian theorem.

As a Tauberian theorem, we show that if the condition

$$\lim_{\beta\to\infty} \frac{\phi(\beta)}{\beta} = 0 \tag{1.17}$$

is fulfilled, then (1.14) with constants $\alpha > 0$ and $C > 0$ implies (1.13). Condition (1.17) is considered as a Tauberian condition. We provide a sufficient condition on F in order that (1.17) is fulfilled. As a particular case, if (1.17) is fulfilled, then (1.16) with a constant $\theta > 0$ implies (1.15).

The contents of this paper are as follows: in Section 2, we provide preliminaries. In Section 3, we clarify when the strict inequality holds in (1.1). In Section 4, we investigate the asymptotic behavior of $I(\beta)$ as $\beta \to \infty$.

2. Preliminaries

Throughout this paper, we assume (1.7) for the function F defined by (1.6). Recall that f is a continuous function on $\mathbb{R}^n$ such that f attains its maximum 0 over $\mathbb{R}^n$ at some point $a \in \mathbb{R}^n$. We do not require that it is Lipschitz continuous on $\mathbb{R}^n$.

Lemma 2.1. *$\omega \in C^\infty((\beta_0, \infty))$ and we have, for $\beta > \beta_0$,*

$$I(\beta) = \frac{e}{k_n \beta} \, (\beta\omega(\beta))^{-\frac{1}{n}} \, \exp\left(\frac{1}{n} + \frac{1}{n}\,\frac{\beta\omega'(\beta)}{\omega(\beta)}\right). \tag{2.1}$$

Proof. By (1.7) and (1.9), we see that $\omega \in C^\infty((\beta_0, \infty))$ and

$$\omega^{(n)}(\beta) = (-1)^n \int_0^\infty e^{-\beta t}\, t^n F(t) dt, \quad \beta > \beta_0. \tag{2.2}$$

Next, let χ_A be the indicator function of a set A. When $\beta > \beta_0$, we have

$$\beta\omega(\beta) = \beta \int_0^\infty e^{-\beta t} \left[\int_{\mathbb{R}^n} \chi_{\{-f(x)\le t\}}\ dx\right] dt$$
$$= \beta \int_{\mathbb{R}^n} \left[\int_0^\infty e^{-\beta t}\chi_{\{-f(x)\le t\}}\ dt\right] dx$$
$$= \beta \int_{\mathbb{R}^n} \left[\int_{-f(x)}^\infty e^{-\beta t}\ dt\right] dx = \int_{\mathbb{R}^n} e^{\beta f(x)}\ dx.$$

Similarly, we have

$$\omega(\beta) + \beta\omega'(\beta) = \int_0^\infty (1-\beta t)e^{-\beta t}F(t)\ dt$$
$$= \int_{\mathbb{R}^n} \left[\int_{-f(x)}^\infty (1-\beta t)e^{-\beta t}\ dt\right] dx = \int_{\mathbb{R}^n} f(x)e^{\beta f(x)}\ dx.$$

By the definition of $\mathrm{Ent}(e^{\beta f})$, we have

$$I(\beta) = \frac{e}{k_n\beta}\left(\int_{\mathbb{R}^n} e^{\beta f(x)}dx\right)^{-1/n} \exp\left(\frac{\beta}{n}\ \frac{\int_{\mathbb{R}^n} f(x)e^{\beta f(x)}dx}{\int_{\mathbb{R}^n} e^{\beta f(x)}dx}\right). \tag{2.3}$$

Putting these results together, we conclude that the lemma holds. $\square$

Next, we provide a sufficient condition on F in order that (1.17) is fulfilled.

Lemma 2.2. *Assume that there are constants $\ell > 0$, $\rho > 0$ and $\delta > 0$ such that*

$$F(t) \ge \ell t^\rho \quad \textit{in } (0,\delta).$$

Then, (1.17) is fulfilled.

Proof. By (1.7), we have

$$\omega(\beta) \le \frac{C_0}{\beta-\beta_0}, \quad \beta > \beta_0. \tag{2.4}$$

On the other hand, by our assumption, we have, for $\beta > 1$,

$$\omega(\beta) \ge \int_0^\delta e^{-\beta t}\ell t^\rho\ dt = \frac{\ell}{\beta^{\rho+1}} \int_0^{\beta\delta} e^{-s}s^\rho\ ds$$
$$> \frac{\ell}{\beta^{\rho+1}} \int_0^\delta e^{-s}s^\rho\ ds =: C_1\beta^{-(\rho+1)}.$$

Thus, for $\beta > \max\{\beta_0, 1\}$, we obtain

$$C_1 \beta^{-(\rho+1)} \le \omega(\beta) \le \frac{C_0}{\beta - \beta_0}.$$

From this result, we conclude that the lemma holds. $\square$

3. Strict inequality in (1.1)

In this section, we clarify when the strict inequality holds in (1.1).

Theorem 3.1. *Equality (1.11) holds.*

Proof. Let $\beta > \beta_0$. By (1.10) and (2.1), we have

$$I(\beta) = \frac{e}{k_n} \exp\left\{ \frac{1}{n}\Big(1 + \beta\phi'(\beta) - \phi(\beta) - (n+1)\log\beta\Big) \right\}. \tag{3.1}$$

Then, we obtain

$$\begin{aligned} \frac{d}{d\beta} I(\beta) &= \frac{1}{n} I(\beta) \frac{d}{d\beta}\Big(1 + \beta\phi'(\beta) - \phi(\beta) - (n+1)\log\beta\Big) \\ &= \frac{\beta}{n} I(\beta) \left(\phi''(\beta) - \frac{n+1}{\beta^2} \right). \end{aligned}$$

$\square$

We provide two examples.

Example 3.2. For $\theta > 0$, $\gamma > 0$ and $a \in \mathbb{R}^n$, let $f(x) = -\theta|x-a|^\gamma$ in $\mathbb{R}^n$. Then we have

$$F(t) = \left|\left\{ x \in \mathbb{R}^n \mid |x-a| \le (t/\theta)^{1/\gamma} \right\}\right| = v_n \left(\frac{t}{\theta}\right)^{\frac{n}{\gamma}}, \quad t \ge 0.$$

Hence, we have

$$\omega(\beta) = \frac{v_n}{\theta^{\frac{n}{\gamma}}} \beta^{-\frac{n}{\gamma}-1} \Gamma\left(\frac{n}{\gamma} + 1\right), \quad \beta > 0.$$

Therefore, (1.17) is fulfilled, and

$$\phi''(\beta) = \frac{\frac{n}{\gamma} + 1}{\beta^2}, \quad \beta > 0. \tag{3.2}$$

This implies that $I(\cdot)$ is strictly increasing (resp. strictly decreasing) on $(0, \infty)$ if and only if $0 < \gamma < 1$ (resp. $1 < \gamma$). When $\gamma = 1$, it is easy to check that $I(\beta) = \theta$ for any $\beta > 0$.

Example 3.3. Let $n = 1$. For $\theta > 0$ and $a \in \mathbb{R}^n$, let $f(x) = -\theta \log(1 + |x - a|)$ in $\mathbb{R}$. Then, we have

$$F(t) = \left| \left\{ x \in \mathbb{R} \mid |x - a| \le e^{\frac{t}{\theta}} - 1 \right\} \right| = 2\left(e^{\frac{t}{\theta}} - 1\right), \quad t \ge 0.$$

Hence, we have

$$\omega(\beta) = \frac{2}{\beta(\beta\theta - 1)}, \quad \beta > \frac{1}{\theta}.$$

Therefore, (1.17) is fulfilled, and

$$\phi''(\beta) = \frac{1}{\beta^2} + \frac{\theta^2}{(\beta\theta - 1)^2} > \frac{2}{\beta^2}, \quad \beta > \frac{1}{\theta}. \tag{3.3}$$

This implies that $I(\cdot)$ is strictly increasing on $(1/\theta, \infty)$.

Recall that D is the set defined by (1.12).

Theorem 3.4. *The following conditions are equivalent:*

(3.4) *there exists an open interval E of (β_0, ∞) such that $I(\cdot)$ is a constant function on E.*

(3.5) $D = \emptyset$.

(3.6) $F(t) = v_n (t/\theta)^n,\ t \ge 0$, *for some constant $\theta > 0$.*

Proof. (3.4)$\Longrightarrow$(3.5). By (2.1), $I(\cdot)^n$ is holomorphic in the domain $\{z \in \mathbb{C} |\, \mathrm{Re} z > \beta_0\}$. Hence, by (3.4) and the theorem of identity of holomorphic functions, $I(\cdot)^n \equiv C$ in this domain for some constant $C > 0$. Therefore, by (3.1) and (1.10), we obtain

$$n \log\left(\frac{e}{k_n}\right) + 1 + \beta\phi'(\beta) - \phi(\beta) - (n+1)\log\beta = \log C, \quad \beta > \beta_0.$$

Differentiating this equality, we have

$$\phi''(\beta) = \frac{n+1}{\beta^2}, \quad \beta > \beta_0.$$

This implies (3.5).

(3.5)$\Longrightarrow$(3.6). By (3.5), we have

$$\phi(\beta) = K - L\beta - (n+1)\log\beta, \quad \beta > \beta_0$$

for some constants $K, L \in \mathbb{R}$, and

$$\omega(\beta) = e^{K - L\beta}\beta^{-(n+1)}, \quad \beta > \beta_0.$$

Since (2.4) is fulfilled under (1.7), we see that $L \ge 0$. Let

$$G(t) = \frac{e^K}{n!}\chi_{[L,\infty)}(t)(t - L)^n, \quad t \ge 0.$$

Since

$$\int_0^\infty e^{-\beta t} G(t) dt = \omega(\beta), \quad \beta > \beta_0,$$

we have $F(t) = G(t)$ a.e. in $[0, \infty)$ by the uniqueness of Laplace transforms. Since F and G are right-continuous on $[0, \infty)$, we have $F(t) = G(t), t \geq 0$.

Now, we derive a contradiction to suppose that $L > 0$. If $L > 0$, then $F(t) = 0$ on $[0, L)$. Let $a \in \mathbb{R}^n$ be a point where f attains its maximum over $\mathbb{R}^n$. Since f is continuous on $\mathbb{R}^n$, we can choose $\delta > 0$ so that

$$|f(x) - f(a)| < \frac{L}{2}, \quad |x - a| < \delta.$$

Since $f(a) = 0$, we have

$$\{x \mid |x - a| \leq \delta\} \subset \left\{ x \mid -f(x) < \frac{L}{2} \right\},$$

so that $v_n \delta^n \leq F(L/2)$. This is a contradiction to the hypothesis that $F(t) = 0$ on $[0, L)$. Therefore, we have proved that $L = 0$, and $F(t) = e^K t^n / n!$, $t \geq 0$. By letting $\theta = (n! v_n e^{-K})^{1/n}$, we conclude (3.6).

(3.6)$\Longrightarrow$(3.4). If (3.6) is fulfilled, we have $I(\beta) = \theta$ on (β_0, ∞) by Example 3.2. Thus, (3.4) holds for $E = (\beta_0, \infty)$. $\square$

Now, we state the main results of this section. By the proof of Lemma 2.1, we note that, under (1.7), $e^f \in L^\beta(\mathbb{R}^n)$ and $\mathrm{Ent}(e^{\beta f}) < \infty$ for any $\beta > \beta_0$.

Theorem 3.5. *Let f be Lipschitz continuous on $\mathbb{R}^n$. Then, the strict inequality holds in (1.1) for any $\beta \in D$.*

Proof. We derive a contradiction to suppose that equality holds in (1.1) for some $\hat{\beta} \in D$, i.e., $I(\hat{\beta}) = \|Df\|_\infty$.

First, we consider the case such that $\phi''(\hat{\beta}) > (n+1)/\hat{\beta}^2$. Then, there exists an open interval (β_1, β_2) such that $\beta_1 < \hat{\beta} < \beta_2$ and $\phi''(\beta) > (n+1)/\beta^2$ for all $\beta \in (\beta_1, \beta_2)$. By Theorem 3.1 and (1.3), we see that $I(\beta) = \|Df\|_\infty$ on $[\hat{\beta}, \beta_2)$. By Theorem 3.4, $D = \emptyset$. This is a contradiction to the hypothesis that $\hat{\beta} \in D$.

Next, we consider the case such that $\phi''(\hat{\beta}) < (n+1)/\hat{\beta}^2$. Then, there exists an open interval (β_1, β_2) such that $\beta_1 < \hat{\beta} < \beta_2$ and $\phi''(\beta) < (n+1)/\beta^2$ for all $\beta \in (\beta_1, \beta_2)$. By Theorem 3.1 and (1.3), we see that $I(\beta) = \|Df\|_\infty$ on $(\beta_1, \hat{\beta}]$. By Theorem 3.4, $D = \emptyset$. This is a contradiction to the hypothesis that $\hat{\beta} \in D$.

We conclude that, for any $\beta \in D$, equality never holds in (1.1). $\square$

Remark 3.6. When f is not Lipschitz continuous on $\mathbb{R}^n$, we interpret that $\|Df\|_\infty = \infty$ and the right-hand side of (1.1) is equal to ∞. Then, we see that the strict inequality holds in (1.1) for all $\beta \in (\beta_0, \infty)$.

Next, we show that when $D \neq \emptyset$, it is very large in the sense that the set $(\beta_0, \infty) \setminus D$ is at most countable.

Theorem 3.7. *If $D \neq \emptyset$, then the set $(\beta_0, \infty) \setminus D$ is at most countable. In particular, for a function f which does not satisfy (3.6), the strict inequality holds in (1.1) for uncountably many $\beta \in (\beta_0, \infty)$.*

Proof. Assume that $D \neq \emptyset$. Since ϕ is holomorphic in the domain $\{z \in \mathbb{C} \,|\, \mathrm{Re} z > \beta_0\}$, we see that every point of $(\beta_0, \infty) \setminus D$ is an isolated point of this set; otherwise $D = \emptyset$ by the theorem of identity. This implies that the set $(\beta_0, \infty) \setminus D$ is at most countable.

Next, let f be a function which does not satisfy (3.6). When $\|Df\|_\infty = \infty$, the strict inequality holds in (1.1) for all $\beta \in (\beta_0, \infty)$ by Remark 3.6. When $\|Df\|_\infty < \infty$, the strict inequality holds in (1.1) for uncountably many $\beta \in (\beta_0, \infty)$ by Theorem 3.5. $\square$

Example 3.8. We consider the function f from Example 3.2. When $0 < \gamma < 1$ and $1 < \gamma$, the strict inequality holds in (1.1) for all $\beta \in (0, \infty)$ by (3.2), Theorem 3.5 and Remark 3.6. When $\gamma = 1$, equality holds in (1.1) for all $\beta \in (0, \infty)$.

Example 3.9. We consider the function f from Example 3.3. Then, the strict inequality holds in (1.1) for all $\beta \in (1/\theta, \infty)$ by (3.3) and Theorem 3.5.

4. Asymptotic behavior of $I(\beta)$ as $\beta \to \infty$

In this section, we study asymptotic behavior of $I(\beta)$ as $\beta \to \infty$. The key proposition is the following Abelian and Tauberian theorem (cf. [1, Theorem 1.7.1′], [7, Theorem 3 of Chapter XIII-5]).

Proposition 4.1. *Let U be a non-decreasing function on $[0,\infty)$ satisfying $U(0)=0$ and $U(t)\le K_1 e^{\beta_1 t}$, $t\ge 0$, for some constants $\beta_1>0$ and $K_1>0$. Let $c,\rho>0$. Then, the following are equivalent:*

$$U(t)\sim \frac{ct^{\rho}}{\Gamma(\rho+1)} \quad \text{as } t\to 0^+.$$

$$\int_0^{\infty} e^{-\beta t}U(t)dt \sim c\beta^{-(\rho+1)} \quad \text{as } \beta\to\infty.$$

Remark 4.2. Proposition 4.1 is stated more generally by using regularly varying functions (see the books above).

Now, we provide our first result of this section. Recall that, throughout this paper, we assume (1.7) for the function F defined by (1.6). Recall also that we use the symbol $A(s)\sim B(s)$ as $s\to s_0$ to indicate that B is positive in a neighborhood of s_0 and $A(s)/B(s)\to 1$ as $s\to s_0$.

Theorem 4.3. *If there exist constants $\alpha>0$ and $C>0$ such that (1.13) is fulfilled, then we have (1.14).*

Proof. By (1.9) and (2.2), we have

$$\begin{aligned} &\omega(\beta)\sim C\beta^{-(\alpha+1)} \quad \text{as } \beta\to\infty, \\ &\omega'(\beta)\sim -C(\alpha+1)\beta^{-(\alpha+2)} \quad \text{as } \beta\to\infty. \end{aligned}$$

Hence, by (2.1), it is not difficult to see that

$$\begin{aligned} I(\beta) \;&\sim\; \frac{e}{k_n\beta}\left(C\beta^{-\alpha}\right)^{-\frac{1}{n}} \exp\left(\frac{1}{n} - \frac{\alpha+1}{n}\right) \\ &=\; \frac{(\beta/e)^{\frac{\alpha-n}{n}}}{k_n C^{\frac{1}{n}}} \quad \text{as } \beta\to\infty. \end{aligned}$$

□

Corollary 4.4. *If there exists a constant $\theta>0$ such that (1.15) is fulfilled, then we have (1.16). In particular, if there exist a constant $\theta>0$ and a point $a\in\mathbb{R}^n$ such that $f(x)/|x-a|\to-\theta$ as $|x-a|\to 0$, then we have (1.16).*

Proof. The first part is a consequence of Theorem 4.3. Next, note that if $f(x)/|x-a|\to-\theta$ as $|x-a|\to 0$, then (1.15) holds. Thus, the second part follows. □

Example 4.5. Consider the function f satisfying (1.7) and

$$f(x) = -\theta|x-a|^\gamma, \quad x \in D,$$

where $\theta > 0$, $\gamma > 0$ and $a \in \mathbb{R}^n$ are constant, and D is a bounded domain of $\mathbb{R}^n$ such that $a \in D$. Then, we have

$$F(t) = \left|\left\{x \in \mathbb{R}^n \mid |x-a| \le (t/\theta)^{1/\gamma}\right\}\right| = v_n \left(\frac{t}{\theta}\right)^{\frac{n}{\gamma}} \quad \text{as } t \to 0^+.$$

Since $k_n v_n^{1/n} = (1/n!)^{1/n}$, we have

$$I(\beta) \sim \frac{\theta^{\frac{1}{\gamma}}(\beta/e)^{\frac{1-\gamma}{\gamma}}}{[\Gamma(\frac{n}{\gamma}+1)/n!]^{\frac{1}{n}}} \quad \text{as } \beta \to \infty.$$

Therefore,

$$\lim_{\beta\to\infty} I(\beta) = \begin{cases} \infty, & 0 < \gamma < 1, \\ 0, & 1 < \gamma, \\ \theta, & \gamma = 1. \end{cases} \tag{4.1}$$

Next, let

$$f(x) = -\theta|x-a|^\gamma, \quad x \in \mathbb{R}^n.$$

By Examples 3.2, 3.8 and Theorem 4.3, we see that when $0 < \gamma < 1$, $I(\beta)$ increases strictly and diverges to ∞ as $\beta \to \infty$. When $1 < \gamma$, $I(\beta)$ decreases strictly and converges to 0 as $\beta \to \infty$. When $\gamma = 1$, $I(\beta) = \theta$ on $(0, \infty)$.

Example 4.6. Consider the function f satisfying (1.7) and

$$f(x) = -\theta \log(1 + \log(1 + |x-a|)), \quad x \in D,$$

where $\theta > 0$ and $a \in \mathbb{R}^n$ are constant, and D is a bounded domain of $\mathbb{R}^n$ such that $a \in D$. In this case, $f(x)$ does not satisfy (1.4), but $f(x)/|x-a| \to -\theta$ as $|x-a| \to 0$. Hence, we have $\lim_{\beta\to\infty} I(\beta) = \theta$ by Corollary 4.4.

Example 4.7. We consider the function f of Example 3.3. By Examples 3.3, 3.9 and Theorem 4.3, $I(\beta)$ strictly increases and converges to θ as $\beta \to \infty$

Next, we show that the converse of Theorem 4.3 holds provided that condition (1.17) is fulfilled. This condition is considered as a Tauberian condition.

Theorem 4.8. *Assume that (1.17) is fulfilled. If (1.14) is satisfied for some constants $\alpha > 0$ and $C > 0$, then we have (1.13).*

Proof. Let $0 < \epsilon < 1$ be arbitrary. By (1.14), we can find a constant $R > 0$ so that

$$(1-\epsilon)\frac{(\beta/e)^{\frac{\alpha-n}{n}}}{k_n C^{\frac{1}{n}}} < I(\beta) < (1+\epsilon)\frac{(\beta/e)^{\frac{\alpha-n}{n}}}{k_n C^{\frac{1}{n}}}, \quad \beta > R.$$

By (2.1), we have, for $\beta > \beta_0$,

$$I(\beta) = \frac{1}{k_n} \, \exp\left(\frac{n+1}{n} + \frac{1}{n}\left[\beta^2\left(\frac{\phi(\beta)}{\beta}\right)' - (n+1)\log\beta\right]\right).$$

These implies that if $\beta > \max\{\beta_0, R\}$,

$$n\log(1-\epsilon) < \alpha + 1 + \log C + \beta^2\left(\frac{\phi(\beta)}{\beta}\right)' - (\alpha+1)\log\beta < n\log(1+\epsilon).$$

Let $\gamma > \beta > \max\{\beta_0, R\}$. Then, we have

$$\begin{aligned} & n\log(1-\epsilon)\int_\beta^\gamma t^{-2}\,dt \\ < \; & (\alpha+1+\log C)\int_\beta^\gamma t^{-2}\,dt + \frac{\phi(\gamma)}{\gamma} - \frac{\phi(\beta)}{\beta} - (\alpha+1)\int_\beta^\gamma t^{-2}\log t\,dt \\ < \; & n\log(1+\epsilon)\int_\beta^\gamma t^{-2}\,dt, \end{aligned}$$

so that

$$\begin{aligned} & n\log(1-\epsilon)\left(-\frac{1}{\gamma}+\frac{1}{\beta}\right) \\ < \; & (\alpha+1+\log C)\left(-\frac{1}{\gamma}+\frac{1}{\beta}\right) + \frac{\phi(\gamma)}{\gamma} - \frac{\phi(\beta)}{\beta} \\ & -(\alpha+1)\left(-\frac{1}{\gamma}\log\gamma - \frac{1}{\gamma} + \frac{1}{\beta}\log\beta + \frac{1}{\beta}\right) \\ < \; & n\log(1+\epsilon)\left(-\frac{1}{\gamma}+\frac{1}{\beta}\right). \end{aligned}$$

Letting $\gamma \to \infty$ and using (1.17), we obtain, for $\beta > \max\{\beta_0, R\}$,

$$-n\log(1-\epsilon) > -\log C + \phi(\beta) + (\alpha+1)\log\beta > -n\log(1+\epsilon),$$

which implies that

$$(1+\epsilon)^{-n} < \frac{\omega(\beta)}{C\beta^{-(\alpha+1)}} < (1-\epsilon)^{-n}, \quad \beta > \max\{\beta_0, R\}.$$

Hence

$$\omega(\beta) \sim C\beta^{-(\alpha+1)} \quad \text{as } \beta \to \infty.$$

By Proposition 3.1, we conclude (1.13). $\square$

Corollary 4.9. *Assume that (1.17) is fulfilled. If (1.16) is fulfilled for some constant $\theta > 0$, then we have (1.15).*

References

[1] N. H. Bingham, C. M. Goldie and J. L. Teugels, Regular variation, Encyclopedia of Mathematics and its Applications, 27. Cambridge University Press, Cambridge, 1987.

[2] A. Blanchet, M. Bonforte J. Dolbeault, G. Grillo and J. L. Vázquez, *Asymptotics of the fast diffusion equation via entropy estimates.* Arch. Ration. Mech. Anal. 191 (2009), pp.347–385.

[3] S. G. Bobkov, I. Gentil and M. Ledoux, *Hypercontractivity of Hamilton-Jacobi equations*, J. Math. Pures Appl. 80 (2001), pp. 669–696.

[4] M. Del Pino and J. Dolbeault, *The optimal Euclidean L^p-Sobolev logarithmic inequality*, J. Funct. Anal. 197 (2003) pp.151–161.

[5] J. Dolbeault, B. Nazaret and G. Savaré, *From Poincare to Logarithmic Sobolev Inequalities: A Gradient Flow Approach*, SIAM J. Math. Anal. 44 (2012), pp.3186–3216.

[6] L. Evans, Partial differential equations, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.

[7] W. Feller, An introduction to probability theory and its applications Vol. II, John Wiley & Sons, Inc., New York-London-Sydney, 1966.

[8] Y. Fujita, *An optimal logarithmic Sobolev inequality with Lipschitz constants*, Journal of Functional Analysis 261 (2011) pp.1133-1144

[9] I. Gentil, *Ultracontractive bounds on Hamilton—Jacobi equations*, Bull. Sci. Math. 126 (2002), pp. 507–524.

[10] I. Gentil, *The general optimal L^p-Euclidean logarithmic Sobolev inequality by Hamilton-Jacobi equations*, J. Funct. Anal. 202 (2003) pp. 591–599.

[11] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. 97 (1975) 1061–1083.

[12] D. V. Widder, The Laplace transforms, Princeton Mathematical Series 6, Princeton University Press, Princeton, 1972.

Department of Mathematics, University of Toyama, Toyama 930-8555, Japan
E-mail address: yfujita@sci.u-toyama.ac.jp