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Iterative roots of piecewise monotonic functions with finite nonmonotonicity height ☆

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ABSTRACT

It is known that any continuous piecewise monotonic function with nonmonotonicity height not less than 2 has no continuous iterative roots of order n greater than the number of forts of the function. In this paper, we consider the problem of iterative roots in the case that the order n is less than or equal to the number of forts. By investigating the trajectory of possible continuous roots, we give a general method to find all iterative roots of those functions with finite nonmonotonicity height.

1. Introduction

Given a nonempty set X and a positive integer $n \in \mathbb{N}$, a function $f : X \to X$ is said to be an iterative root of $F : X \to X$ of order n if

$$f^n(x) = F(x), \quad \forall x \in X, \tag{1.1}$$

where f^n denotes the nth iterate of f, i.e., $f^n(x) = f \circ f^{n-1}(x)$ and $f^0(x) \equiv x$ for any $x \in X$. The existence of iterative roots has been intensively studied for almost 200 years, starting from Ch. Babbage [1], and great advance were made to find the solutions of Eq. (1.1) (see [2–7,11–13]). Among these works, there are plentiful results on iterative roots for monotonic self-mapping on compact interval, in which the roots are defined piece by piece from a small neighborhood without fixed points to the whole domain [5–7]. However, the method is invalid without the assumption of monotonicity and thus finding iterative roots for non-monotonic mapping is treated as a difficult problem. In 1983, Jingzhong Zhang and Lu Yang [15] studied a class of non-monotonic continuous functions, called strictly piecewise monotonic functions (abbreviated as PM functions). By introducing the idea of "characteristic interval", the problem of iterative roots for PM function can be reduced to that on its characteristic interval, which becomes the monotone case. In this paper, being different from these works on searching existence conditions for iterative roots, we try to give a general method to find all roots.

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2. Preliminaries

Let $I := [a, b]$ for $a, b \in \mathbb{R}$, $a < b$, and let $F: I \to I$ be a continuous function. A point $c \in (a, b)$ is called a *fort of* F if F is strictly monotonic in no neighborhood of c. The set of forts of F is denoted by $S(F)$. A function F is said to be *piecewise monotonic* if the number of forts of F, denoted by $N(F)$ is finite. By $\mathcal{PM}(I, I)$ we denote the set of all piecewise monotonic self-mappings of I.

Lemma 2.1. *(See Lemma 2.3 in [9].) Let $p, q \in \mathbb{R}$ be such that $p < q$ and $F([a, b]) \subset [p, q]$ and let $F : [a, b] \to \mathbb{R}$ and $G : [p, q] \to \mathbb{R}$ be continuous functions. Then*

$$S(G \circ F) = S(F) \cup \{c \in (a, b):\ F(c) \in S(G)\}.$$

In particular, the function $G \circ F$ is piecewise monotonic if and only if so are the functions F and $G|_{F([a,b])}$.

By Lemma 2.1, we know that every continuous iterative root of a piecewise monotonic (strictly monotonic) self-mapping is also piecewise monotonic (strictly monotonic). Moreover, for each function $F \in \mathcal{PM}(I, I)$, $N(F)$ is nondecreasing under iteration, i.e.,

$$0 = N(F^0) \leqslant N(F) \leqslant N(F^2) \leqslant \cdots \leqslant N(F^n) \leqslant \cdots.$$

Then we define the *nonmonotonicity height* (or simple height) $H(F)$ of F as the least $k \in \mathbb{N} \cup \{0\}$ satisfying $N(F^k) = N(F^{k+1})$ if such a k exists and ∞ otherwise.

Example 2.1. (See Example 2.9 in [9].) For the classical hat function $F : [0, 1] \to [0, 1]$, given by

$$F(x) = \min\{2x, 2 - 2x\},$$

one can check that $S(F^k) = \{\frac{1}{2^k}, \ldots, \frac{2^k-1}{2^k}\}$, $k \in \mathbb{N}$, and thus $H(F) = \infty$.

Example 2.2. Let $F : [0, 1] \to [0, 1]$ be defined by

$$F(x) = \begin{cases} \frac{1}{2}x, & \forall x \in [0, \frac{1}{2}), \\ -\frac{1}{2}x + \frac{1}{2}, & \forall x \in [\frac{1}{2}, 1]. \end{cases}$$

Obviously, F maps $[0, 1]$ onto $[0, 1/4]$, which implies that $H(F) = 1$.

Example 2.3. Consider the function $F : [0, 1] \to [0, 1]$, given by

$$F(x) = \begin{cases} x, & \forall x \in [0, \frac{1}{3}), \\ -x + \frac{2}{3}, & \forall x \in [\frac{1}{3}, \frac{2}{3}), \\ 2x - \frac{4}{3}, & \forall x \in [\frac{2}{3}, 1]. \end{cases}$$

We have $S(F^3) = S(F^2) = \{\frac{1}{3}, \frac{2}{3}, \frac{5}{6}\}$ and then $H(F) = 2$.

The simplest case for the nonmonotonicity height is $H(F) = 0$, which means that F is strictly monotonic. When $H(F) = 1$, the problem of iterative roots was reduced to be discussed on its characteristic interval (see [8,10,15,14]). More concretely, for every $F \in \mathcal{PM}(I, I)$ with $H(F) = 1$, there exists a sub-interval of I denoted by $K(F)$, covering the range of F such that F is strictly monotonic on it. Such a maximal sub-interval bounded by either forts or end-points, is called the *characteristic interval* of F. For instance, the characteristic interval of F given in Example 2.2 is $K(F) = [0, \frac{1}{2}]$, and F in Example 2.3 has no characteristic interval since $H(F) > 1$. When F is strictly increasing on its characteristic interval, the following results are obtained:

Theorem 2.1. *(See Theorem 4 in [14].) Let $F \in \mathcal{PM}(I, I)$ and $H(F) \leqslant 1$. Suppose that* (i) *F is strictly increasing on its characteristic interval $[a', b']$ and* (ii) *$F(x)$ on I cannot reach a' and b' unless $F(a') = a'$ or $F(b') = b'$. Then for any integer $n > 1$, F has a continuous iterative root of order n. Moreover, these conditions are necessary for integers $n > N(F) + 1$.*

In addition, when $H(F) > 1$, we also have the following nonexistence result of iterative roots for order $n > N(F)$.

Theorem 2.2. *(See Theorem 1 in [14].) Let $F \in \mathcal{PM}(I, I)$ and $H(F) > 1$. Then F has no continuous iterative roots of order n for $n > N(F)$.*

Therefore, in [15,14] two open problems were raised naturally:

(P1) Does $F \in \mathcal{PM}(I,I)$ with $H(F) > 1$ have an iterative root of order n for $n \leqslant N(F)$?
(P2) Does $F \in \mathcal{PM}(I,I)$ with $H(F) \leqslant 1$ have an iterative root of order n for $n \leqslant N(F)+1$ if $F(x') = a'$ (or b') at $x' \in I \backslash K(F)$?

Problem (P2) was solved partly for the case that the continuous iterative root f of order n satisfying $H(f) = n$ [8]. When $H(F) \leqslant 1$, it was proved that every continuous iterative root is an extension from the characteristic interval [10], which generalized the results of Theorem 3 in [14]. Recently, Problem (P1) in the case of equal to was discussed in [9]. The authors showed that only types $\mathcal{T}_1$ ('almost increasing') and $\mathcal{T}_2$ ('almost decreasing') of continuous iterative roots possibly appear and they characterized all type $\mathcal{T}_1$ roots of order n. A full description of type $\mathcal{T}_2$ is still an open question.

In this paper, we continue to study Problem (P1). Our purpose is to find all iterative roots of order $n \leqslant N(F)$ under the assumption that the nonmonotonicity height is finite, i.e., $H(F) < \infty$ (see Examples 2.2–2.3). Moreover, an algorithm for computing those iterative roots of such functions is also given and we illustrate our results by two examples in the last section.

3. Nonmonotonicity height and iterative roots

For each $F \in \mathcal{PM}(I,I)$, let $H(F) = k$, $2 \leqslant k < \infty$ is an integer. From the definition of nonmonotonicity height $H(F)$, we know

$$N\left(F^k\right) = N\left(F^{k+1}\right) = \cdots = N\left(F^{2k}\right) = \cdots,$$

which implies that $H(F^k) = 1$. Hence, F^k has a characteristic interval denoted by I' such that F^k is strictly monotonic on it. Clearly, F is also strictly monotonic on I' by Lemma 2.1.

Lemma 3.1. *Let $F \in \mathcal{PM}(I,I)$ and $H(F) = k$. Then k is the smallest nonnegative integer such that $F^k(I) \subset I'$.*

Proof. It is easy to see that $F^k(I) \subset I'$ because of the fact $H(F^k) = 1$. Now, suppose that $F^{k'}(I) \subset I'$ for a nonnegative integer k'. Since F is strictly monotone on I', we have

$$S\left(F^{k'+1}\right) = S\left(F \circ F^{k'}\right) = S\left(F^{k'}\right),$$

which implies that $k' \geqslant k$. $\square$

On the other hand, the set $S(F)$ partitions the whole interval I into $N(F)+1$ sub-intervals. Let I_i be an open interval of the ith sub-interval. Then F is strictly monotone on each I_i. We say that F admits the partition $I(F) := \{I_i\colon\ i = 0,1,\dots,N(F)\}$. Similarly, let J_i be an open interval of the ith sub-interval between two consecutive forts of F^k and F^k admits the partition $I(F^k) := \{J_i\colon\ i = 0,1,\dots,N(F^k)\}$. Thus, $I = \bigcup_{i=0}^{N(F)} cl(I_i) = \bigcup_{i=0}^{N(F^k)} cl(J_i)$ and F is also strictly monotone on each J_i. Moreover, for every J_i, $i = 0,1,\dots,N(F^k)$, there exists a unique I_t, $t \in \{0,1,\dots,N(F)\}$, such that $J_i \subset I_t$. Obviously, $I' \in I(F^k)$.

In what follows, assume that F has a continuous iterative root f of order $n \geqslant 1$. By Lemma 2.1 we know that $f \in \mathcal{PM}(I,I)$. Furthermore, the following lemma shows some properties of f.

Lemma 3.2. *Let $F \in \mathcal{PM}(I,I)$ and $H(F) = k$. If f is a continuous iterative root of F of order $n \geqslant 1$, then*:

(i) *f is strictly monotone on each $J_i \in I(F^k)$, $i = \{0,1,2,\dots,N(F^k)\}$.*
(ii) *For every sub-interval $J_i \in I(F^k)$, there is an integer $j \in \{0,1,2,\dots,N(F^k)\}$ such that $f(J_i) \subset J_j$.*
(iii) *$f^{nk}(I) \subset I'$.*

Proof. Since F is strictly monotonic on each $J_i \in I(F^k)$, it follows from Lemma 2.1 that (i) holds.

In order to prove (ii), we assume that there exists $i \in \{0,1,\dots,N(F^k)\}$ such that $f(J_i) \not\subset J_j$ for all $j = 0,1,\dots,N(F^k)$. It follows that there are two interior points $x_1, x_2 \in J_i$ such that $f(x_1) \in J_m$ and $f(x_2) \in J_{m+1}$, where J_m and J_{m+1} are two consecutive sub-intervals in $I(F^k)$ because of the continuity of f. Let $\{c\} := cl(J_m) \cap cl(J_{m+1})$, i.e., the common end-point of the closure of two sub-intervals. Clearly, $c \in S(F^k)$. By the continuity of f on J_i, there exists x_3 between x_1 and x_2 such that

$$f(x_3) = c.$$

Notice

$$S\big(f\circ F^k\big)=S(f)\cup\big\{x\in(a,b)\colon\ f(x)\in S\big(F^k\big)\big\},$$

which implies that $x_3\in S(f\circ F^k)$. However, being an interior point of J_i, $x_3\notin S(F^k)$. It follows that $S(F^k)<S(f\circ F^k)\leqslant S(F^k\circ F^k)$, a contradiction to the fact that $H(F^k)=1$.

(iii) is obtained directly by the fact that $f^{nk}(I)=F^k(I)\subset I'$. □

According to Lemmas 3.1–3.2, it seems that the iteration orbit of F on each $J_i\subset I(F^k)$ proceeds in a regular manner. We first give the following definition.

Definition 3.1. A finite sequence

$$Ins_F:=\big\{\big\{J_i,J_{i1},J_{i2},\dots,J_{i(r-1)},I'\big\},\ i=0,1,\dots,N\big(F^k\big)\big\}$$

of each pairwise disjoint intervals is called an interval sequence with respect to $F:I\to I$ if

$$F(J_i)\subset J_{i1},\qquad F(J_{ij})\subset J_{i(j+1)},\quad j=1,2,\dots,r-2,\quad\text{and}\quad F(J_{i(r-1)})\subset I'.$$

Here $r+1:=\operatorname{card}Ins_F(J_i)$ is called the number of interval sequence beginning with $J_i\in I(F^k)$. In other words, $r\leqslant N(F^k)-1$ is the smallest nonzero integer such that $F^r(J_i)\subset I'$.

We further denote the iteration of F on $Ins_F(J_i)=\{J_i,J_{i1},J_{i2},\dots,I'\}$ by

$$J_i\xrightarrow{F}J_{i1}\xrightarrow{F}J_{i2}\xrightarrow{F}\cdots\xrightarrow{F}I'.$$

Example 3.1. Consider mapping $F:[0,1]\to[0,1]$, given by

$$F(x)=\begin{cases}x, & \forall x\in[0,\frac14),\\ -\frac12x+\frac38, & \forall x\in[\frac14,\frac12),\\ \frac94x-1, & \forall x\in[\frac12,\frac23),\\ -\frac32x+\frac32, & \forall x\in[\frac23,1].\end{cases}$$

By calculating, we obtain

$$F^3(x)=F^2(x)=\begin{cases}x, & \forall x\in[0,\frac14),\\ -\frac12x+\frac38, & \forall x\in[\frac14,\frac12),\\ \frac94x-1, & \forall x\in[\frac12,\frac59),\\ -\frac98x+\frac78, & \forall x\in[\frac59,\frac23),\\ \frac34x-\frac38, & \forall x\in[\frac23,\frac56),\\ -\frac32x+\frac32, & \forall x\in[\frac56,1],\end{cases}$$

which implies that $S(F^3)=S(F^2)=\{\frac14,\frac12,\frac59,\frac23,\frac56\}$ and thus $H(F)=2$.

Furthermore, F^2 admits the partition $I(F^2)=\sum_{i=0}^5 cl\,J_i$, where $I'=cl(J_0)=[0,\frac14]$, $J_1=(\frac14,\frac12)$, $J_2=(\frac12,\frac59)$, $J_3=(\frac59,\frac23)$, $J_4=(\frac23,\frac56)$ and $J_5=(\frac56,1)$. Then by Definition 3.1, it is easy to check that the interval sequence with respect to F is

$$Ins_F=\big\{\big\{I'\big\},\big\{J_1,I'\big\},\big\{J_2,I'\big\},\big\{J_3,J_1,I'\big\},\big\{J_4,J_1,I'\big\},\big\{J_5,I'\big\}\big\}.$$

This sequence describes the orbit of F among the partition $I(F^2)$.

Proposition 3.1. *Let $F\in\mathcal{PM}(I,I)$ with $H(F)=k$ and s denote the smallest nonzero integer such that $F^s(I')\subset I'$; then s is a divisor of k. In particular, $s=1$ means that F is a self-mapping on I'.*

The proof of Proposition 3.1 follows from Lemma 3.1 and (ii) in Lemma 3.2 directly. Similarly, let s' be the smallest nonzero integer such that $f^{s'}(I')\subset I'$, then s' is a divisor of nk by (ii)–(iii) in Lemma 3.2.

4. Construction of iterative roots

Let $\mathrm{Fix}(F)$ be the set of all fixed points of F and $S'(F) := S(F) \cup \{a\} \cup \{b\}$.

Lemma 4.1. *Let $F \in \mathcal{PM}(I, I)$ and $H(F) = k$. Then $\#\{S'(F) \cap \mathrm{Fix}(F)\} \leqslant 2$. Furthermore, two fixed forts (or end-points) are consecutive if $\#\{S'(F) \cap \mathrm{Fix}(F)\} = 2$.*

Proof. For an indirect proof, suppose that $\#\{S'(F) \cap \mathrm{Fix}(F)\} \geqslant 3$. Then for every point $c \in S'(F) \cap \mathrm{Fix}(F)$, we have $c \in \mathrm{Fix}(F^k)$, which is a contradiction to the fact that $F^k(I) \subset I' \in I(F^k)$ since F^k is strictly monotone on I'. Hence, $\#\{S'(F) \cap \mathrm{Fix}(F)\} \leqslant 2$. Furthermore, if $\#\{S'(F) \cap \mathrm{Fix}(F)\} = 2$, it follows from the same reason that there is no forts between the two fixed forts (or end-points). $\square$

By Lemma 4.1, for $\#\{S'(F) \cap \mathrm{Fix}(F)\} \leqslant 2$ we have three cases: (i) $\#\{S'(F) \cap \mathrm{Fix}(F)\} = 2$; (ii) $\#\{S'(F) \cap \mathrm{Fix}(F)\} = 1$; (iii) $S'(F) \cap \mathrm{Fix}(F) = \emptyset$. Before presenting Theorem 4.1, we first give a useful lemma.

Lemma 4.2. *Let $F \in \mathcal{PM}(I, I)$ and $H(F) = k$. If there is a sub-interval $J_i \in I(F^k)$ for $i \in \{0, 1, 2, \dots, N(F^k)\}$ such that $F(J_i) \subset J_i$, then $cl(J_i)$ is the characteristic interval of F^k and $s = 1$.*

Proof. From the assumption $F(J_i) \subset J_i$, we have $F^k(J_i) \subset J_i$. Notice $J_i \in I(F^k)$, which implies that $F^k(I) \subset J_i$ by the fact that $H(F^k) = 1$. Therefore, $cl(J_i)$ is the characteristic interval of F^k. The second assertion $s = 1$ follows from the condition $F(J_i) \subset J_i$ directly. $\square$

Theorem 4.1. *Let $F \in \mathcal{PM}(I, I)$ and $H(F) = k$. Then $s = 1$.*

Proof. In what follows, we consider cases (i)–(iii) respectively. In case (i), denote the two consecutive forts which are also fixed points by a' and b'. Since F^k is strictly monotonic on $[a', b']$ and $F([a', b']) \subset [a', b']$, it follows from Lemma 4.2 that $I' = [a', b']$ is the characteristic interval of F^k and thus $s = 1$.

In case (ii), let x_0 be the unique fixed fort (or end-point) of F, then $x_0 \in S'(F^k) \cap \mathrm{Fix}(F^k)$ is one of the end-points of I'. Assume that $I' = [x_0, x_1]$, $x_0 < x_1$ and $x_1 \in S'(F^k)$ (the discussion for the other case $I' = [x_1, x_0]$ is similar). We claim that

$$F\big([x_0, x_1]\big) \subset [x_0, x_1]. \tag{4.1}$$

Otherwise, by the continuity of F and the fact that $F(x_0) = x_0$, there exists a point $x_2 \in (x_0, x_1)$ such that $F(x_2) = x_1$. Notice

$$S\big(F^{k+1}\big) = S(F) \cup \big\{x \in (a, b)\colon\ F(x) \in S\big(F^k\big)\big\},$$

which implies that $x_2 \in S(F^{k+1})$ since $x_1 \in S'(F^k)$. However, x_2 is a monotonic point of F^k and thus $x_2 \in S(F^{k+1}) \backslash S(F^k)$ is a contradiction to the assumption $H(F) = k$. Therefore, (4.1) is proved and $s = 1$.

In case (iii), since F is a continuous self-mapping on I, it follows that F has at least one fixed point, denoted by y^*, which is surely a monotonic point of F. Then there exists a sub-interval $I^* := [y_0, y_1] \in I(F^k)$ such that $y^* \in (y_0, y_1)$. Obviously $y^* \in F(I^*) \cap I^* \neq \emptyset$. We further claim that

$$F\big(I^*\big) \subset I^*. \tag{4.2}$$

Otherwise, by the continuity of F we have

$$F(y_0) \cap I^* = \emptyset \quad \text{or} \quad F(y_1) \cap I^* = \emptyset.$$

Consequently, we can find a point $y_2 \in (y_0, y_1)$ such that $F(y_2) = y_0$ (or $F(y_2) = y_1$), which follows that $y_2 \in S(F^{k+1}) \backslash S(F^k)$, a contradiction to the fact that $H(F^k) = 1$. Therefore, (4.2) is proved and $I^* \in I(F^k)$ is the characteristic interval of F^k. The proof is completed. $\square$

Lemma 4.3. *Let $F \in \mathcal{PM}(I, I)$ and $H(F) = k$. If F has a continuous iterative root f, then $s' = 1$, i.e., $f(I') \subset I'$.*

Proof. It follows from Theorem 4.1 that $s = 1$. Suppose that f is a continuous iterative root of F of order n. For an indirect proof, assume $s' > 1$. Then there is a sub-interval $J' \in I(F^k)$, $J' \neq I'$ such that

$$f^{s'}\big(J'\big) \subset J', \tag{4.3}$$

by the definition of s' and Ins_f. On the other hand, since s' is a divisor of nk, let $nk = s'd$ for d is a positive integer. In view of (4.3), we have

$$F^k\big(J'\big) = f^{nk}\big(J'\big) = f^{s'd}\big(J'\big) \subset J',$$

which is a contradiction to the fact that $F^k(J') \subset I'$. Therefore, $s' = 1$ and f is a self-mapping on I'. $\square$

Theorem 4.2. *Let $F \in \mathcal{PM}(I, I)$ and $H(F) = k$. Then every continuous iterative root of F is an extension from an iterative root of F of the same order on sub-interval I'.*

Proof. If F has a continuous iterative root f of order n, it follows from Lemma 4.3 that $f|_{I'}$ is an iterative root of F on I', satisfies $f^n|_{I'} = F|_{I'}$.

By Lemma 3.2 and Theorem 4.1, for every sub-interval $J \in I(F^k)$, $J \neq I'$, there is a finite interval sequence with respect to F as

$$Ins_F(J) = \{J, F(J), F^2(J), \dots, F^{r-1}(J), I'\} \subset I(F^k). \tag{4.4}$$

It is clear that each J belongs to a certain $Ins_F(J)$, with a fixed positive integer r.

On the other hand, according to (ii) in Lemma 3.2, Lemma 4.3 and the fact that $f^n = F$, each $J \in I(F^k)$ also belongs to a finite interval sequence with respect to f as

$$Ins_f(J) := \{J, J_{1,1}, J_{1,2}, \dots, J_{1,n-1}, F(J), J_{2,1}, J_{2,2}, \dots, F^2(J), \dots, I'\} \subset I(F^k), \tag{4.5}$$

such that

$$J \xrightarrow{f} J_{1,1} \xrightarrow{f} J_{1,2} \xrightarrow{f} \cdots \xrightarrow{f} J_{1,n-1} \xrightarrow{f} F(J) \xrightarrow{f} \cdots \xrightarrow{f} I'. \tag{4.6}$$

Furthermore, (iii) in Lemma 3.2 and (4.5) imply that $\operatorname{card} Ins_f(J) \leqslant nk + 1$.

If $r = 1$ in (4.4), i.e., $F(J) \subset I'$, then the number of sub-intervals between J and I' in $Ins_f(J)$ does not exceed $n - 1$. More concretely, for these J we have

$$J \xrightarrow{f} J_1 \xrightarrow{f} J_2 \xrightarrow{f} \cdots \xrightarrow{f} J_{m-1} \xrightarrow{f} I', \quad 2 \leqslant m \leqslant n. \tag{4.7}$$

Thus, for every $x \in J$,

$$F(x) = f^n(x) = \underbrace{f|_{I'} \circ \cdots \circ f|_{I'}}_{n-m \text{ times}} \circ f|_{J_{m-1}} \circ \cdots \circ f|_{J_2} \circ f|_{J_1} \circ f|_J(x),$$

and then

$$f|_J(x) = f|_{J_1}^{-1} \circ f|_{J_2}^{-1} \circ \cdots \circ f|_{J_{m-1}}^{-1} \circ f|_{I'}^{-(n-m)} \circ F|_J(x). \tag{4.8}$$

Similarly, for those sub-intervals $J \in I(F^k)$ with $r > 1$ and $Ins_f(J)$ as (4.6), we obtain

$$F(x) = f^n(x) = f|_{J_{1,n-1}} \circ \cdots \circ f|_{J_{1,2}} \circ f|_{J_{1,1}} \circ f|_J(x),$$

for every $x \in J$. Hence,

$$f|_J(x) = f|_{J_{1,1}}^{-1} \circ f|_{J_{1,2}}^{-1} \circ \cdots \circ f|_{J_{1,n-1}}^{-1} \circ F|_J(x). \tag{4.9}$$

Therefore, the continuous iterative root f is an extension of $f|_{I'}$ from I', which is defined by

$$f(x) := \begin{cases} f|_{I'}(x), & x \in I', \\ f|_J(x), & x \in J \in I(F^k) \backslash \{I'\}. \end{cases}$$

The proof is completed. □

Remark that the equality (4.9) requires

$$F(J) \subset f(J_{1,n-1}), \tag{4.10}$$

which is a necessary condition for the existence of such continuous iterative roots.

According to the proof of Theorem 4.2, if F has continuous iterative roots, each of them can be found by the following algorithm:

Algorithm 4.1. Input: continuous piecewise monotonic function $F \in \mathcal{PM}(I,I)$; a given large enough integer $M > 0$; the set of forts $S(F)$; a given positive integer $n \geqslant 2$; Output: iterative roots of F.

Let $S := S(F)$.

Step1: Compute the nonmonotonicity height of F. For each fort $y \in S$ let

$$S := S \cup \bigl\{x \in (a,b)\colon\ F(x) = y\bigr\}.$$

Repeat this step until equation $F(x) = y$ has no new solutions. Record the times as $H(F)$. Go to the next step.

If equation $F(x) = y$ still provides new solutions in (a,b) after M times (it means $H(F) = \infty$), terminate this algorithm and quit.

Step2: Sort the points in S as $\{x_1, x_2, \dots, x_{p-1}\}$, where $a < x_1 < x_2 < \cdots < x_{p-1} < b$. Let $J_1 := (a, x_1)$, $J_i := (x_{i-1}, x_i)$, $i = 2, \dots, p-1$, $J_p := (x_{p-1}, b)$ and $I(F^k) := \{J_1, \dots, J_p\}$.

Step3: Compute $F(J_i)$ for $i = 1, \dots, p$. Find the interval sequence of each J_i with respect to F and record it as $Ins_F(J_i)$. If $F(J) \subset J$, record $cl(J)$ as the characteristic interval I'.

Step4: According to $Ins_F(J_i)$, choose possible interval sequences of J_i with respect to f. If $r_i = \#Ins_F(J_i) - 1 > 1$, insert possible sub-intervals which belong to $I(F^k) \backslash Ins_F(J_i)$ between J_i, $F(J_i)$, $F^2(J_i), \dots, F^{r_i - 1}(J_i)$ and I' as (4.5)–(4.6). If $r_i = 1$, insert possible sub-intervals from $I(F^k) \backslash Ins_F(J_i)$ between J_i and I' as (4.7). Then record all interval sequences of J_i with respect to f for $i = 1, \dots, p$ in the set Ins_f.

Step5: $Ins_f = \phi$? Yes, return "F has no continuous iterative roots" and quit.

No, go to the next step.

Step6: Let $Ins_f := \{Ins_1, \dots, Ins_t\}$ for $t > 0$ is a finite integer. Find an iterative root of F of order n on I' as the initial root (see [6,7]). The special case for F is strictly monotonic and piecewise linear was investigated in Algorithm 1-3 in [16].

Take arbitrary interval sequence from Ins_f and then calculate the iterative roots of F on each J_i by (4.8) if $r_i = 1$, or by (4.9) when $r_i > 1$. Note that condition (4.10) must be considered during the procedure of interval sequences finding. Repeat this step until each interval sequence in Ins_f was discussed.

Output iterative roots of F and quit.

Remark 4.1. For those functions with finite nonmonotonicity height, we see that the construction of their iterative roots can also be reduced to the characteristic intervals. Hence, these continuous iterative roots are determined by the initial roots on characteristic intervals.

Remark 4.2. According to Algorithm 4.1, the choosing of interval sequences between sub-intervals in Ins_F is not unique, which implies the non-uniqueness of iterative roots. Moreover, many interval sequences also provide us discontinuous iterative roots.

5. Examples

Example 5.1. Consider the mapping $F : [0,1] \to [0,1]$, given in Example 2.3 (see Fig. 1):

$$F(x) = \begin{cases} x, & \text{for } x \in [0, \frac{1}{3}), \\ -x + \frac{2}{3}, & \text{for } x \in [\frac{1}{3}, \frac{2}{3}), \\ 2x - \frac{4}{3}, & \text{for } x \in [\frac{2}{3}, 1]. \end{cases}$$

It is easy to see that $S(F) = \{\frac{1}{3}, \frac{2}{3}\}$ and $N(F) = 2$.

Take each point $y \in S(F)$; we obtain a new fort $\frac{5}{6}$ from equation $F(x) = y$. Since equation $F(x) = \frac{5}{6}$ provides no new solution in $(0,1)$, $S(F^3) = S(F^2) = \{\frac{1}{3}, \frac{2}{3}, \frac{5}{6}\}$ and $H(F) = 2$.

Let $I_0 := [0, \frac{1}{3}]$, $I_1 := (\frac{1}{3}, \frac{2}{3})$, $I_2 := (\frac{2}{3}, \frac{5}{6})$, $I_3 := (\frac{5}{6}, 1)$.

By computing $F(I_i)$ for $i = 0, \dots, 3$, we see that the characteristic interval of F^2 is I_0, and the interval sequence with respect to F is $Ins_F = \{\{I_3, I_1, I_0\}, \{I_2, I_0\}\}$.

Then, according to step4 in Algorithm 4.1, we have only one possible interval sequence with respect to f for $n = 2$:

$$Ins_f = \bigl\{\{I_3, I_2, I_1, I_0\}\bigr\}.$$

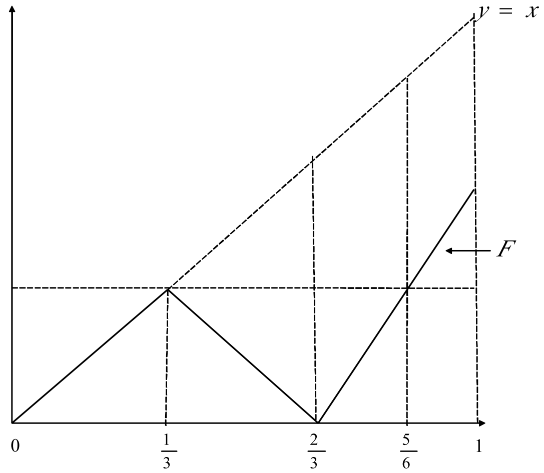

Fig. 1. F.

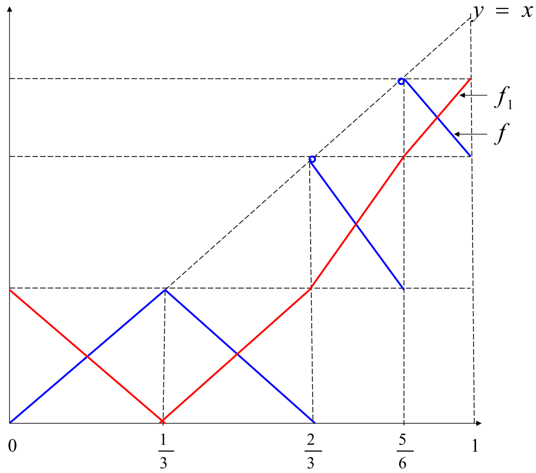

Fig. 2. f and f_1.

Hence, it follows from step6 that the iterative root extended from the initial root $f|_{I_0}(x) = x$ is

$$f(x) = \begin{cases} x, & \text{for } x \in [0, \frac{1}{3}], \\ -x + \frac{2}{3}, & \text{for } x \in (\frac{1}{3}, \frac{2}{3}], \\ 2 - 2x, & \text{for } x \in (\frac{2}{3}, \frac{5}{6}], \\ -x + \frac{5}{3}, & \text{for } x \in (\frac{5}{6}, 1]. \end{cases}$$

On can check that f (see Fig. 2) is a discontinuous square iterative root of F. Therefore, F has no continuous iterative roots of order 2, extended from the initial iterative root $f|_{I_0}(x) = x$.

Take another initial iterative root on I_0, say $f|_{I_0}(x) = \frac{1}{3} - x$. Similarly, we get

$$f_1(x) = \begin{cases} -x + \frac{1}{3}, & \text{for } x \in [0, \frac{1}{3}], \\ x - \frac{1}{3}, & \text{for } x \in (\frac{1}{3}, \frac{2}{3}], \\ 2x - 1, & \text{for } x \in (\frac{2}{3}, \frac{5}{6}], \\ x - \frac{1}{6}, & \text{for } x \in (\frac{5}{6}, 1], \end{cases}$$

which is a continuous square iterative root of F (see Fig. 2). Actually, according to Theorem 4.1 in [9], f_1 is a square iterative root of type $\mathcal{T}_1$. This fact shows that different initial iterative roots from characteristic interval may lead to different kinds of roots.

Example 5.2. Consider the mapping $F_1 : [0, 1] \to [0, 1]$, defined by (see Fig. 3)

$$F_1(x) = \begin{cases} x, & \text{for } x \in [0, \frac{1}{3}), \\ -x + \frac{2}{3}, & \text{for } x \in [\frac{1}{3}, \frac{2}{3}), \\ \frac{5}{2}x - \frac{5}{3}, & \text{for } x \in [\frac{2}{3}, \frac{4}{5}), \\ -\frac{5}{2}x + \frac{7}{3}, & \text{for } x \in [\frac{4}{5}, \frac{14}{15}), \\ \frac{25}{4}x - \frac{35}{6}, & \text{for } x \in [\frac{14}{15}, 1]. \end{cases}$$

One can check that $S(F_1) = \{\frac{1}{3}, \frac{2}{3}, \frac{4}{5}, \frac{14}{15}\}$ and $N(F_1) = 4$.

Take each point y from $S(F_1)$ and solve the equation $F_1(x) = y$; we obtain a new fort $\frac{74}{75}$. Furthermore, $S(F_1^3) = S(F_1^2) = \{\frac{1}{3}, \frac{2}{3}, \frac{4}{5}, \frac{14}{15}, \frac{74}{75}\}$ implies that $H(F_1) = 2$.

Let $I_0 := [0, \frac{1}{3}]$, $I_1 := (\frac{1}{3}, \frac{2}{3})$, $I_2 := (\frac{2}{3}, \frac{4}{5})$, $I_3 := (\frac{4}{5}, \frac{14}{15})$, $I_4 := (\frac{14}{15}, \frac{74}{75})$, $I_5 := (\frac{74}{75}, 1)$.

It is easy to see that the characteristic interval of F_1^2 is I_0 and the interval sequence with respect to F_1 is

$$Ins_{F_1} = \big\{\{I_5, I_1, I_0\}, \{I_2, I_0\}, \{I_3, I_0\}, \{I_4, I_0\}\big\}.$$

Moreover, by step4 in Algorithm 4.1, we have nine interval sequences with respect to f for $n = 2$:

$$Ins_1 = \big\{\{I_5, I_4, I_1, I_0\}, \{I_2, I_0\}, \{I_3, I_0\}\big\}, \qquad Ins_2 = \big\{\{I_5, I_4, I_1, I_0\}, \{I_3, I_2, I_0\}\big\},$$

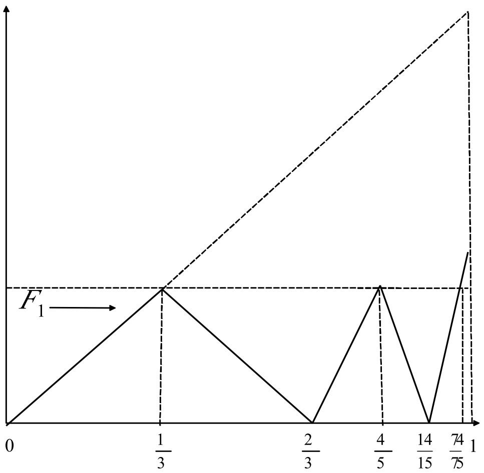

Fig. 3. F_1.

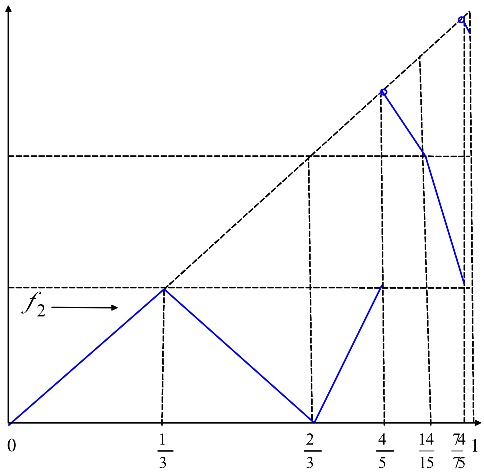

Fig. 4. f_2.

$$Ins_3 = \big\{\{I_5, I_4, I_1, I_0\}, \{I_2, I_3, I_0\}\big\}, \qquad Ins_4 = \big\{\{I_5, I_3, I_1, I_0\}, \{I_2, I_0\}, \{I_4, I_0\}\big\},$$
$$Ins_5 = \big\{\{I_5, I_3, I_1, I_0\}, \{I_4, I_2, I_0\}\big\}, \qquad Ins_6 = \big\{\{I_5, I_3, I_1, I_0\}, \{I_2, I_4, I_0\}\big\},$$
$$Ins_7 = \big\{\{I_5, I_2, I_1, I_0\}, \{I_3, I_0\}, \{I_4, I_0\}\big\}, \qquad Ins_8 = \big\{\{I_5, I_2, I_1, I_0\}, \{I_4, I_3, I_0\}\big\},$$
$$Ins_9 = \big\{\{I_5, I_2, I_1, I_0\}, \{I_3, I_4, I_0\}\big\}.$$

For instance, in view of sequence Ins_2 with the initial root $f|_{I_0}(x) = x$, we get $f|_{I_1} = f|_{I_0}^{-1} \circ F_1|_{I_1}$, $f|_{I_4} = f|_{I_1}^{-1} \circ F_1|_{I_4}$, $f|_{I_5} = f|_{I_4}^{-1} \circ F_1|_{I_5}$, $f|_{I_2} = f|_{I_0}^{-1} \circ F_1|_{I_2}$, $f|_{I_3} = f|_{I_2}^{-1} \circ F_1|_{I_3}$ and such square iterative root of F_1 is defined by (see Fig. 4)

$$f_2(x) = \begin{cases} x, & \text{for } x \in [0, \frac{1}{3}], \\ -x + \frac{2}{3}, & \text{for } x \in (\frac{1}{3}, \frac{2}{3}], \\ \frac{5}{2}x - \frac{5}{3}, & \text{for } x \in (\frac{2}{3}, \frac{4}{5}], \\ -x + \frac{8}{5}, & \text{for } x \in (\frac{4}{5}, \frac{14}{15}), \\ -\frac{25}{4}x + \frac{13}{2}, & \text{for } x \in [\frac{14}{15}, \frac{74}{75}], \\ -x + \frac{148}{75}, & \text{for } x \in (\frac{74}{75}, 1]. \end{cases}$$

One can check that f_2 is a discontinuous square iterative root of F_1. Fortunately, from sequence Ins_5 we can find one continuous square iterative root of F_1, defined by (see Fig. 5)

$$f_5(x) = \begin{cases} x, & \text{for } x \in [0, \frac{1}{3}), \\ -x + \frac{2}{3}, & \text{for } x \in [\frac{1}{3}, \frac{2}{3}), \\ \frac{5}{2}x - \frac{5}{3}, & \text{for } x \in [\frac{2}{3}, 1]. \end{cases}$$

On the other hand, it follows from Theorem 4.1 in [9] that F_1 has continuous iterative roots of order $n = N(F_1) = 4$ of type $\mathcal{T}_1$. In fact, according to our algorithm, we have six interval sequences with respect to f for $n = 4$:

$$Ins_1 = \{I_5, I_2, I_3, I_4, I_1, I_0\}, \qquad Ins_2 = \{I_5, I_2, I_4, I_3, I_1, I_0\},$$
$$Ins_3 = \{I_5, I_3, I_2, I_4, I_1, I_0\}, \qquad Ins_4 = \{I_5, I_3, I_4, I_2, I_1, I_0\},$$
$$Ins_5 = \{I_5, I_4, I_2, I_3, I_1, I_0\}, \qquad Ins_6 = \{I_5, I_4, I_3, I_2, I_1, I_0\}.$$

In view of sequence Ins_6 with the initial root $f|_{I_0}(x) = -x + \frac{1}{3}$, we get $f|_{I_1} = f|_{I_0}^{-3} \circ F_1|_{I_1}$, $f|_{I_2} = f|_{I_1}^{-1} \circ f|_{I_0}^{-2} \circ F_1|_{I_2}$, $f|_{I_3} = f|_{I_2}^{-1} \circ f|_{I_1}^{-1} \circ f|_{I_0}^{-1} \circ F_1|_{I_3}$, $f|_{I_4} = f|_{I_3}^{-1} \circ f|_{I_2}^{-1} \circ f|_{I_1}^{-1} \circ F_1|_{I_4}$, $f|_{I_5} = f|_{I_4}^{-1} \circ f|_{I_3}^{-1} \circ f|_{I_2}^{-1} \circ F_1|_{I_5}$. Hence, such iterative root of order 4 is defined by (see Fig. 6)

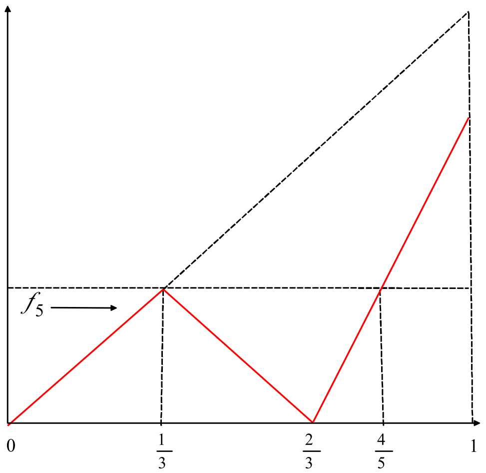

Fig. 5. f_5.

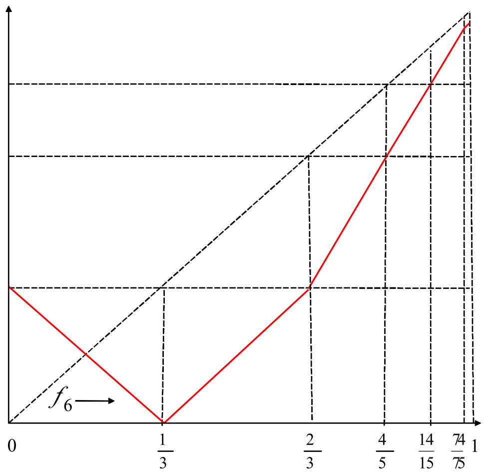

Fig. 6. f_6.

$$f_6(x) = \begin{cases} -x + \frac{1}{3}, & \text{for } x \in [0, \frac{1}{3}], \\ x - \frac{1}{3}, & \text{for } x \in (\frac{1}{3}, \frac{2}{3}], \\ \frac{5}{2}x - \frac{4}{3}, & \text{for } x \in (\frac{2}{3}, \frac{4}{5}], \\ x - \frac{2}{15}, & \text{for } x \in (\frac{4}{5}, \frac{14}{15}], \\ \frac{5}{2}x - \frac{23}{15}, & \text{for } x \in (\frac{14}{15}, \frac{74}{75}], \\ x - \frac{4}{75}, & \text{for } x \in (\frac{74}{75}, 1], \end{cases}$$

which is of type $\mathcal{T}_1$. Moreover, according to sequence Ins_4 we can also find one discontinuous iterative root of order 4, given by

$$f_4(x) = \begin{cases} -x + \frac{1}{3}, & \text{for } x \in [0, \frac{1}{3}], \\ x - \frac{1}{3}, & \text{for } x \in (\frac{1}{3}, \frac{2}{3}], \\ \frac{5}{2}x - \frac{4}{3}, & \text{for } x \in (\frac{2}{3}, \frac{4}{5}], \\ \frac{2}{5}x + \frac{46}{75}, & \text{for } x \in (\frac{4}{5}, \frac{14}{15}), \\ -\frac{5}{2}x + \frac{47}{15}, & \text{for } x \in [\frac{14}{15}, \frac{74}{75}], \\ -\frac{5}{2}x + \frac{17}{5}, & \text{for } x \in (\frac{74}{75}, 1]. \end{cases}$$

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