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Global existence and decay estimates of the Boltzmann equation with frictional force: The general results

Zaihong Jiang [a,*], Ying Wang [b]

[a] *College of Mathematics, Physics and Information Engineering, Zhejiang Normal University, Jinhua, PR China*
[b] *Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong*

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ABSTRACT

In this paper, we give the existence theory and the optimal time convergence rates of the solutions to the Boltzmann equation with frictional force near a global Maxwellian. We generalize our previous results on the same problem for hard sphere model into both hard potential and soft potential case. The main method used in this paper is the classic energy method combined with some new time–velocity weight functions to control the large velocity growth in the nonlinear term for the case of interactions with hard potentials and to deal with the singularity of the cross-section at zero relative velocity for the soft potential case.

1. Introduction

Consider the Boltzmann equation with frictional force

$$f_t + \xi \cdot \nabla_x f - \alpha u \cdot \nabla_\xi f = Q(f,f), \tag{1.1}$$

with the initial data

$$f(0,x,\xi) = f_0(x,\xi). \tag{1.2}$$

Here, $f = f(t,x,\xi) \in \mathbb{R}$ represents the probability (mass, number) density of gas particles around position $x \in \mathbb{R}^3$ with velocity $\xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3$ at time $t \in \mathbb{R}^+$. The frictional force $-\alpha u$ $(\alpha > 0)$ is proportional to the macroscopic velocity $u = u(x,t) = \frac{\int_{\mathbb{R}^3} \xi f\, d\xi}{\int_{\mathbb{R}^3} f\, d\xi}$. Without loss of generality, we take $\alpha = 1$ throughout this paper. Q is the nonlinear collision operator which is defined by

* Corresponding author.
E-mail address: jzhong@zjnu.cn (Z. Jiang).

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$$Q(f,g) = \frac{1}{2}\int\limits_{\mathbb{R}^3\times S^2} \big(f(\xi')g(\xi'_*) + f(\xi'_*)g(\xi') - f(\xi)g(\xi_*) - f(\xi_*)g(\xi)\big)B\big(|\xi-\xi_*|,\vartheta\big)\,d\xi_*\,d\Omega.$$

S^2 denotes the unit sphere and $\Omega \in S^2$. The conservation of momentum and energy gives the following relations:

$$\xi' = \xi - \big[(\xi-\xi_*)\cdot\Omega\big]\Omega,$$
$$\xi'_* = \xi_* + \big[(\xi-\xi_*)\cdot\Omega\big]\Omega,$$

in which ξ, ξ_* are the velocities before collision while ξ', ξ'_* are the velocities after collision. Under the angular cut-off assumption the cross-section $B(|\xi-\xi_*|,\vartheta)$ takes the form

$$B\big(|\xi-\xi_*|,\vartheta\big) = B(\vartheta)|\xi-\xi_*|^\gamma, \qquad \cos\vartheta = \frac{(\xi-\xi_*)\cdot\Omega}{|\xi-\xi_*|}, \quad -3<\gamma\leqslant 1,$$

where $0 < B(\vartheta) \leqslant const.|\cos\vartheta|$. The exponent γ is determined by the interaction potential between two colliding particles which is called the soft potential when $-3<\gamma<0$ and the hard potential when $0\leqslant\gamma\leqslant 1$, including the Maxwell model when $\gamma=0$ and the hard-sphere model when $\gamma=1$. In our previous work [15], we studied the hard-sphere model. Inspired by recent results in [5] and [6], in this paper we deal with the more general case when $-2\leqslant\gamma\leqslant 1$.

Clearly, the following global Maxwellian

$$M = \frac{1}{(2\pi)^{3/2}}\exp\big(-|\xi|^2/2\big) \tag{1.3}$$

is a stationary solution to (1.1). We set g as a perturbation of our solution f around M:

$$f = M + \sqrt{M}g. \tag{1.4}$$

Then the Boltzmann equation (1.1) can be reformulated into

$$\partial_t g + \xi\cdot\nabla_x g - u\cdot\nabla_\xi g + u\cdot\xi\sqrt{M} + \frac{1}{2}u\cdot\xi g = Lg + \Gamma(g,g), \tag{1.5}$$

where L is the linearized collision operator and Γ is the corresponding nonlinear collision operator, given by

$$Lg = \frac{1}{\sqrt{M}}\big(Q(M,\sqrt{M}g) + Q(\sqrt{M}g,M)\big),$$
$$\Gamma(g,g) = \frac{1}{\sqrt{M}}Q(\sqrt{M}g,\sqrt{M}g). \tag{1.6}$$

Now we consider the Cauchy problem of (1.5) with the corresponding initial data

$$g(0,x,\xi) = g_0(x,\xi) = \frac{1}{\sqrt{M}}\big(f_0(x,\xi) - M\big), \quad (x,\xi)\in\mathbb{R}^3\times\mathbb{R}^3. \tag{1.7}$$

Now we list some notation that will be used in this paper. We use $H^N_{x,\xi}$, H^N_x, H^N_ξ to denote the Hilbert spaces $H^N(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)$, $H^N(\mathbb{R}^3_x)$, $H^N(\mathbb{R}^3_\xi)$, respectively, and L^2, L^2_x, L^2_ξ are used for the case when $N=0$. When there is no confusion, we use H^N to denote H^N_x or $H^N_{x,\xi}$ and use L^2 to denote L^2_x, $L^2_{x,\xi}$. $\langle\cdot,\cdot\rangle$ denotes the inner product in the Hilbert space $L^2_{x,\xi}$. We use $\|\cdot\|$ to denote L^2 norm. When the norms need to be distinguished from each other, we write $\|\cdot\|_{L^2_\xi}$, $\|\cdot\|_{L^2_x}$ and $\|\cdot\|_{L^2_{x,\xi}}$ respectively. For the multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, we denote

$$\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3},\quad \text{and}\quad |\alpha|=\sum_{i=1}^{3}\alpha_i.$$

For simplicity, we use ∂_i to denote ∂_{x_i} for each $i=1,2,3$. For multi-indices $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $\beta=(\beta_1,\beta_2,\beta_3)$, we denote $\partial_\beta^\alpha=\partial_x^\alpha\partial_\xi^\beta$, that is $\partial_\beta^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\partial_{\xi_3}^{\beta_3}$. $A\sim B$ means $\iota A\leqslant B\leqslant\frac{B}{\iota}$, for generic constant $0<\iota<1$. For $q\geqslant 1$, we also define the mixed velocity–space Lebesgue space $Z_q=L^2_\xi(L^q_x)=L^2(\mathbb{R}^3_\xi;L^q(\mathbb{R}^3_x))$ with the norm

$$\|g\|_{Z_q}=\left(\int\limits_{\mathbb{R}^3}\left(\int\limits_{\mathbb{R}^3}\left|g(x,\xi)\right|^q dx\right)^{2/q}d\xi\right)^{1/2},\quad g=g(x,\xi)\in Z_q.$$

We use $\hat{g}$ to denote the Fourier transform of g. $(a|b)=a\cdot\bar{b}$ denotes the complex inner product of complex vectors a and b. Throughout this paper, C denotes a generic positive (generally large) constant and κ,η denote generic positive (generally small) constants. They may be different from line to line.

The Boltzmann equation with frictional force was first studied in [14], where the authors proved that there exists a unique global-in-time classical solution which approaches to the global Maxwellian time asymptotically when the initial perturbation is smooth and close to this given Maxwellian. The analysis in that paper was based on the macro–micro decomposition introduced in [11,12] through energy estimates. Recently in [15], we give the existence result of the same model by using a different kind of energy method which was first developed in [8], we also obtain the optimal time convergence rate of the solutions towards the equilibrium. For the time-decay part, we are inspired by [3,4], where the time-decay rate can be obtained by a combination of Fourier analysis and the energy estimates. However, the above results only hold for the hard-sphere model, the general case when $-3<\gamma\leqslant 1$ remains open. One of the main difficulties lies in the fact that the nonlinear velocity growth effect of the term $\frac{1}{2}u\cdot\xi g$ cannot be controlled by the dissipation of the linearized collision operator L when $\gamma<1$. For this, we introduce some mixed time–velocity weight functions as in [5,6] to capture the dissipation for controlling the nonlinear velocity growth. Besides, a time–frequency/time weighted method is also needed to overcome the large velocity degeneracy in the energy dissipation for the soft potential case when $\gamma<0$. Note that for the soft potentials, as in [6], our method in this paper only deals with the situation when $-2\leqslant\gamma<0$, the very soft potential case when $-3<\gamma<-2$ is left for future discussion.

The rest of the paper is organized as follows. In the next section, we review some basic properties of the linearized operator L, and give the macro–micro decomposition to build up the working system. In Section 3, we give a detailed proof of our results for the soft potential case, Theorem 3.1. The results for the hard potentials, Theorem 4.1, can be proved in a similar way yet much simpler, thus we only give a sketch of its proof in the last section.

2. Preliminaries

2.1. Basic properties of L

The properties of L are very crucial in energy estimation for the Boltzmann equation. We list them here for later use.

(I) $L = -\nu(\xi) + K$, where $\nu(\xi)$ is a nonnegative measurable function called the collision frequency, while K is a self-adjoint compact operator on $L^2(\mathbb{R}^3)$ with a real symmetric integral kernel $K(\xi, \xi_*)$. The explicit representations of ν and K are as follows:

$$\nu(\xi) = \int_{\mathbb{R}^3 \times S^2} B(\vartheta)|\xi - \xi_*|^\gamma M(\xi_*)\, d\Omega\, d\xi_* \sim \left(1 + |\xi|\right)^\gamma,$$

and

$$\begin{aligned} Ku(\xi) = &\int_{\mathbb{R}^3 \times S^2} B(\vartheta)|\xi - \xi_*|^\gamma M^{1/2}(\xi_*) M^{1/2}\left(\xi'_*\right) u\left(\xi'\right) d\Omega\, d\xi_* \\ &+ \int_{\mathbb{R}^3 \times S^2} B(\vartheta)|\xi - \xi_*|^\gamma M^{1/2}(\xi_*) M^{1/2}\left(\xi'\right) u\left(\xi'_*\right) d\Omega\, d\xi_* \\ &- \int_{\mathbb{R}^3 \times S^2} B(\vartheta)|\xi - \xi_*|^\gamma M^{1/2}(\xi_*) M^{1/2}(\xi) u(\xi_*)\, d\Omega\, d\xi_* \\ = &\int_{\mathbb{R}^3} K(\xi, \xi_*) u(\xi_*)\, d\xi_*. \end{aligned}$$

(II) The null space of the operator L is the 5-dimensional space of collision invariants:

$$\mathcal{N} = \mathrm{Ker}(L) = \mathrm{span}\left\{\sqrt{M};\ \xi_i\sqrt{M},\ i = 1, 2, 3;\ |\xi|^2\sqrt{M}\right\}.$$

(III) Following from the Boltzmann H-theorem, L is self-adjoint and non-positive in $L^2(\mathbb{R}^3)$. Furthermore, there exists a constant $\lambda_0 > 0$ such that:

$$-\int_{\mathbb{R}^3} gLg\, d\xi \geqslant \lambda_0 \int_{\mathbb{R}^3} \nu(\xi)\left(\{\mathbf{I} - \mathbf{P}\}g\right)^2 d\xi, \quad \forall g \in D(L), \tag{2.1}$$

where $\mathbf{P}$ denotes the projection operator from L^2_ξ to $\mathcal{N}$ and $D(L)$ is the domain of L given by $D(L) = \{g \in L^2_\xi \mid \nu(\xi)g \in L^2_\xi\}$, cf. [1,7].

2.2. Macro–micro decomposition

Now we give the macro–micro decomposition to prepare for the later energy estimates.

$$\begin{cases} g(t, x, \xi) = g_1 + g_2, \\ g_1 = \mathbf{P}g \in \mathcal{N}, \\ g_1 = \left\{a(t, x) + \displaystyle\sum_{i=1}^3 b_i(t, x)\xi_i + c(t, x)|\xi|^2\right\}\sqrt{M}, \\ g_2 = g - g_1 = (\mathbf{I} - \mathbf{P})g \in \mathcal{N}^\perp. \end{cases} \tag{2.2}$$

Then Eq. (1.5) can be rewritten as

$$\partial_t g_1 + \xi \cdot \nabla_x g_1 - u \cdot \nabla_\xi g_1 + u \cdot \xi\sqrt{M} + \frac{1}{2}u \cdot \xi g_1 = r + l + h, \tag{2.3}$$

with

$$
\begin{aligned}
&r = -\partial_t g_2, \\
&l = -\xi \cdot \nabla_x g_2 + u \cdot \nabla_\xi g_2 - \frac{1}{2} u \cdot \xi g_2 + L g_2, \\
&h = \Gamma(g, g).
\end{aligned}
$$

Next we derive the evolution equations for (a, b, c). In fact, by putting (2.2)$_3$ into (2.3) and collecting the coefficients w.r.t. the basis $\{e_k,\ k = 13\}$ consisting of

$$
\sqrt{M},\ (\xi_i\sqrt{M})_{1\leqslant i\leqslant 3},\ \big(|\xi_i|^2\sqrt{M}\big)_{1\leqslant i\leqslant 3},\ (\xi_i\xi_j\sqrt{M})_{1\leqslant i<j\leqslant 3},\ \big(|\xi|^2\xi_i\sqrt{M}\big)_{1\leqslant i\leqslant 3}, \tag{2.4}
$$

we have the following macroscopic equations on the coefficients (a, b, c) of g_1:

$$
\begin{aligned}
&\partial_t a - u \cdot b = \gamma^{(0)}, \\
&\partial_t b_i + \partial_i a + u_i a - 2u_i c + u_i = \gamma_i^{(1)}, \\
&\partial_t c + \partial_i b_i + u_i b_i = \gamma_i^{(2)}, \\
&\partial_i b_j + \partial_j b_i + u_i b_j + u_j b_i = \gamma_{ij}^{(2)}, \\
&\partial_i c + c u_i = \gamma_i^{(3)}, \\
&\quad \text{where } i \neq j,\ \text{with } u = \frac{b}{1 + a + 3c}.
\end{aligned} \tag{2.5}
$$

All terms on the right hand side are the coefficients of $r + l + h$ with respect to the corresponding elements in the basis $\{e_{13}\}$ and are given by:

$$
\begin{aligned}
&\gamma^{(0)} = -\partial_t \bar{r}^{(0)} + l^{(0)} + h^{(0)}, \\
&\gamma_i^{(1)} = -\partial_t \bar{r}_i^{(1)} + l_i^{(1)} + h_i^{(1)}, \\
&\gamma_i^{(2)} = -\partial_t \bar{r}_i^{(2)} + l_i^{(2)} + h_i^{(2)}, \\
&\gamma_{ij}^{(2)} = -\partial_t \bar{r}_{ij}^{(2)} + l_{ij}^{(2)} + h_{ij}^{(2)}, \\
&\gamma_i^{(3)} = -\partial_t \bar{r}_i^{(3)} + l_i^{(3)} + h_i^{(3)}, \\
&\quad \text{where } i \neq j,\ \text{with } r = -\partial_t \bar{r}.
\end{aligned} \tag{2.6}
$$

Based on (2.5)$_3$ and (2.5)$_4$, the macroscopic component $b = (b_1, b_2, b_3)$ satisfies an elliptic-type equation

$$
-\Delta_x b_j - \partial_j\partial_j b_j = -\sum_{i\neq j} \partial_i\big(\gamma_{ij}^{(2)} - (u_i b_j + u_j b_i)\big) + \sum_{i\neq j} \partial_j\big(\gamma_i^{(2)} - u_i b_i\big) - 2\partial_j\big(\gamma_j^{(2)} - u_j b_j\big). \tag{2.7}
$$

Also, (a, b, c) satisfies the local macroscopic balance laws. In fact, multiplying Eq. (1.1) by the collision invariants $1, \xi, \frac{1}{2}|\xi|^2$, integrating the products on ξ, then, using the perturbation (1.4) and the decomposition (2.2), direct calculation one has the macroscopic balance laws on the coefficients (a, b, c) of g_1:

$$
\begin{aligned}
&\partial_t a - \nabla_x \cdot \int_{\mathbb{R}^3} \frac{1}{2}|\xi|^2 \xi \sqrt{M} g_2 \, d\xi = u \cdot b, \\
&\partial_t b_i + \partial_i(a + 5c) + \nabla_x \cdot \int_{\mathbb{R}^3} \xi\xi_i \sqrt{M} g_2 \, d\xi = -b_i, \\
&\partial_t c + \frac{1}{3}\nabla_x \cdot b + \frac{1}{3}\nabla_x \cdot \int_{\mathbb{R}^3} \frac{1}{2}|\xi|^2 \xi \sqrt{M} g_2 \, d\xi = -\frac{1}{3} u \cdot b.
\end{aligned}
$$

3. The soft potential case

In this section we give a detailed proof of our results for the soft potential case, i.e. $-2 \leqslant \gamma < 0$. First, let us introduce the following mixed time–velocity weight function

$$w_\tau(t,\xi) = \langle\xi\rangle^{\gamma\tau} e^{\langle\xi\rangle^2[\frac{\lambda}{(1+t)^\theta}]}, \tag{3.1}$$

where $\tau \in \mathbb{R}$, $0 < \lambda \ll 1$, $\theta > 0$, and $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$. We need to define the following energy functionals:

$$\big[\big[g(t)\big]\big]_{N,l} := \sum_{|\alpha|+|\beta|\leqslant N} \big\|\omega_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g(t)\big\|, \tag{3.2}$$

where $N \geqslant 8$ is an integer, and $l \geqslant N$ is a constant.

$$\mathcal{E}_{N,l} \sim \big[\big[g(t)\big]\big]^2_{N,l}. \tag{3.3}$$

$$\begin{aligned}\mathcal{D}_{N,l} = &\sum_{|\alpha|+|\beta|\leqslant N} \big\|\nu^{\frac12}(\xi)\omega_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\big\|^2 + \frac{1}{(1+t)^\theta}\sum_{|\alpha|+|\beta|\leqslant N}\big\|\langle\xi\rangle\omega_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\big\|^2 \\ &+ \|b\|^2 + \sum_{|\alpha|\leqslant N-1}\big\|\nabla_x\partial^\alpha(a,b,c)\big\|^2,\end{aligned} \tag{3.4}$$

and the time-weighted temporal sup-energy

$$X_{N,l} = \sup_{0\leqslant s\leqslant t} \mathcal{E}_{N,l} + \sup_{0\leqslant s\leqslant t}(1+t)^{\frac32}\mathcal{E}_{N,l-1}. \tag{3.5}$$

Now the main result for the soft potential case when $-2 \leqslant \gamma < 0$ is stated as follows:

Theorem 3.1. *Let $-2 \leqslant \gamma < 0$, $N \geqslant 8$, $l_0 > \frac32$ and $l \geqslant 1 + \max\{N, \frac{l_0}{2} - \frac{1}{\gamma}\}$ with $0 \leqslant \lambda \ll 1$ and $0 < \theta \leqslant \frac14$. Assume that $f_0 = M + M^{\frac12}g_0$. There exist constants $\epsilon_0 > 0$, $C_0 > 0$ such that if*

$$\sum_{|\alpha|+|\beta|\leqslant N}\big\|\omega_{|\beta|-l}(0,\xi)\partial^\alpha_\beta g_0\big\| + \big\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}} g_0\big\|_{Z_1} \leqslant \epsilon_0, \tag{3.6}$$

then the Cauchy problem (1.5), (1.7) admits a unique global solution $g(t,x,\xi)$ satisfying $f(t,x,\xi) = M + M^{\frac12}g(t,x,\xi)$ and

$$\sup_{t\geqslant0}\big\{\big[\big[g(t)\big]\big]_{N,l} + (1+t)^{\frac34}\big[\big[g(t)\big]\big]_{N,l-1}\big\} \leqslant C\epsilon_0. \tag{3.7}$$

3.1. Some basic properties

Lemma 3.1. *(See [13].) Let $-3 < \gamma < 0$, $\tau \in \mathbb{R}$, $0 \leqslant \tilde q \ll 1$, and $\omega_{\tau,\tilde q}(\xi) = \langle\xi\rangle^{\gamma\tau}e^{\tilde q\langle\xi\rangle^2}$ be the velocity weight function. If $|\beta| > 0$, then for any $\eta > 0$, there is $C_\eta > 0$ such that*

$$\begin{aligned}\int_{\mathbb{R}^3}\omega^2_{\tau,\tilde q}(\xi)\partial_\beta(\nu g)\partial_\beta g\,d\xi \geqslant &\int_{\mathbb{R}^3}\nu(\xi)\omega^2_{\tau,\tilde q}(\xi)|\partial_\beta g|^2\,d\xi \\ &- \eta\sum_{|\beta_1|\leqslant|\beta|}\int_{\mathbb{R}^3}\nu(\xi)\omega^2_{\tau,\tilde q}(\xi)|\partial_{\beta_1}g|^2\,d\xi - C_\eta\int_{\mathbb{R}^3}\chi_{|\xi|\leqslant 2C_\eta}\langle\xi\rangle^{2\gamma\tau}|g|^2\,d\xi.\end{aligned} \tag{3.8}$$

If $|\beta| \geqslant 0$, then for any $\eta > 0$, there is $C_\eta > 0$ such that

$$\left|\int_{\mathbb{R}^3} \omega^2_{\tau,\tilde{q}}(\xi)\partial_\beta(Kg)h\,d\xi\right| \leqslant \left\{\eta \sum_{|\beta_1|\leqslant|\beta|}\left(\int_{\mathbb{R}^3} \nu(\xi)\omega^2_{\tau,\tilde{q}}(\xi)|\partial_{\beta_1}g|^2\,d\xi\right)^{\frac{1}{2}} + C_\eta\left(\int_{\mathbb{R}^3} \chi_{|\xi|\leqslant 2C_\eta}\langle\xi\rangle^{2\gamma\tau}|g|^2\,d\xi\right)^{\frac{1}{2}}\right\} \times \left(\int_{\mathbb{R}^3} \nu(\xi)\omega^2_{\tau,\tilde{q}}(\xi)|h|^2\,d\xi\right)^{\frac{1}{2}}. \tag{3.9}$$

Lemma 3.2. *(See [6].) Let $l \geqslant 0$, $0 < \lambda \ll 1$. It holds that*

$$\begin{aligned}\langle \Gamma(g,g), g\rangle &\leqslant C\|(a,b,c)\|_{H^1}\|\nabla_x(a,b,c)\|\|\nu^{\frac{1}{2}}g_2\| \\ &\quad + C\left\{\|\nabla_x(a,b,c)\|_{H^1} + \sum_{|\alpha|+|\beta|\leqslant N}\|\partial^\alpha_\beta g_2\|\right\}\|\nu^{\frac{1}{2}}g_2\|^2,\end{aligned} \tag{3.10}$$

and

$$\langle \Gamma(g,g), \omega^2_{-l}(t,\xi)g_2\rangle \leqslant C\|(a,b,c)\|_{H^1}\|\nabla_x(a,b,c)\|\|\nu^{\frac{1}{2}}g_2\| + C\mathcal{E}^{\frac{1}{2}}_{N,l}\|\nu^{\frac{1}{2}}\omega_{-l}(t,\xi)g_2\|^2. \tag{3.11}$$

Lemma 3.3. *(See [6].) Let $l \geqslant |\beta|$, $1 \leqslant |\alpha| + |\beta| \leqslant N$ and $0 < \lambda \ll 1$. For given $g = g(t,x,\xi)$, define $g_{\alpha\beta}$ as $g_{\alpha\beta} = \partial^\alpha g$ if $|\beta| = 0$ and $g_{\alpha\beta} = \partial^\alpha_\beta g_2$ if $|\beta| \geqslant 1$, then, it holds that*

$$\langle \partial^\alpha_\beta\Gamma(g,g), \omega^2_{-l}(t,\xi)g_{\alpha\beta}\rangle \leqslant C\|(a,b,c)\|_{H^N}\|\nabla_x(a,b,c)\|_{H^{N-1}}\|\nu^{\frac{1}{2}}g_{\alpha\beta}\| + C\mathcal{E}^{\frac{1}{2}}_{N,l}(t)\mathcal{D}_{N,l}(t). \tag{3.12}$$

Lemma 3.4. *(See [2].) For the nonlinear part h represented as $h^{(0)}, h^{(1)}_i, h^{(2)}_i, h^{(2)}_{ij}, h^{(3)}_i$ given in (2.6) we have the following estimate for the soft potential case:*

$$\sum_{|\alpha|\leqslant N}\sum_{ij}\|\partial^\alpha_x[h^{(0)}, h^{(1)}_i, h^{(2)}_i, h^{(2)}_{ij}, h^{(3)}_i]\|^2 \leqslant C\mathcal{E}_{N,l}\mathcal{D}^2_{N,l}. \tag{3.13}$$

For the coefficients of the separated part $\bar{r}$ and the linear part l, it holds that

$$\begin{aligned}\sum_{|\alpha|\leqslant N-1}\sum_{ij}\|\partial^\alpha_x[\bar{r}^{(0)}, \bar{r}^{(1)}_i, \bar{r}^{(2)}_i, \bar{r}^{(2)}_{ij}, \bar{r}^{(3)}_i]\|^2 &\leqslant C\sum_{|\alpha|\leqslant N-1}\|\nu^{\frac{1}{2}}\partial^\alpha_x g_2\|^2, \\ \sum_{|\alpha|\leqslant N-1}\sum_{ij}\|\partial^\alpha_x[l^{(0)}, l^{(1)}_i, l^{(2)}_i, l^{(2)}_{ij}, l^{(3)}_i]\|^2 &\leqslant C\sum_{|\alpha|\leqslant N}\|\nu^{\frac{1}{2}}\partial^\alpha_x g_2\|^2.\end{aligned}$$

Those terms containing the microscopic part g_2 can be bounded by the microscopic dissipation rate:

$$\sum_{|\alpha|\leqslant N-1}\|\partial^\alpha_x\nabla_x\cdot[\langle|\xi|^2\xi\sqrt{M}, g_2\rangle, \langle\xi\otimes\xi\sqrt{M}, g_2\rangle]\|^2 \leqslant C\sum_{0<|\alpha|\leqslant N}\|\nu^{\frac{1}{2}}\partial^\alpha_x g_2\|^2.$$

3.2. Time decay for the evolution equation

Consider the linearized homogeneous equation

$$\partial_t g + \xi\cdot\nabla_x g + b\cdot\xi\sqrt{M} = Lg, \tag{3.14}$$

with initial data $g_0(x,\xi)$. We define e^{tB} as the solution operator to (3.14).

Lemma 3.5. *Set* $\mu = \mu(\xi) := \langle\xi\rangle^{-\frac{\gamma}{2}}$. *Let* $-3 < \gamma < 0$, $l \geqslant 0$, $l_0 \geqslant \frac{3}{2}$, $\alpha \geqslant 0$, $m = |\alpha|$, *and suppose*

$$\left\|\mu^{l+l_0} g_0\right\|_{Z_1} + \left\|\mu^{l+l_0}\partial^\alpha g_0\right\| < \infty. \tag{3.15}$$

Then, the evolution operator e^{tB} *satisfies*

$$\left\|\mu^l \partial^\alpha e^{tB} g_0\right\| \leqslant C(1+t)^{-\sigma[1,m]}\left(\left\|\mu^{l+l_0} g_0\right\|_{Z_1} + \left\|\mu^{l+l_0}\partial^\alpha g_0\right\|\right), \tag{3.16}$$

with $\sigma[q,m] = \frac{3}{2}(\frac{1}{q} - \frac{1}{2}) + \frac{m}{2}$.

Proof. We prove this lemma in two steps.

Step 1. The Fourier transform of (3.14) gives

$$\partial_t \hat{g} + i\xi \cdot k\hat{g} + \hat{b} \cdot \xi\sqrt{M} = L\hat{g}. \tag{3.17}$$

Taking the complex inner product with $\hat{g}$, integrating it over $\mathbb{R}^3_\xi$ and taking the real part, we obtain

$$\frac{1}{2}\partial_t\|\hat{g}\|^2_{L^2_\xi} + Re\int_{\mathbb{R}^3}(\hat{b}\cdot\xi\sqrt{M}|\hat{g})\,d\xi = Re\int_{\mathbb{R}^3}(L\hat{g}|\hat{g})\,d\xi. \tag{3.18}$$

Then, we can deduce

$$\partial_t\|\hat{g}\|^2_{L^2_\xi} + \kappa\left\|\nu^{\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\hat{g}\}\right\|^2_{L^2_\xi} + |\hat{b}|^2 \leqslant 0. \tag{3.19}$$

Similar to the procedure in Section 2, one can derive a hydrodynamic system of linear equations

$$\partial_t(a+3c) + \nabla_x\cdot b = 0, \tag{3.20}$$

$$\partial_t b_j + \partial_j(a+3c) + 2\partial_j c + \sum_m \partial_m A_{jm}\big(\{\mathbf{I}-\mathbf{P}\}g\big) + b_j = 0, \tag{3.21}$$

$$\partial_t c + \frac{1}{3}\nabla_x\cdot b + \frac{1}{6}\sum_j \partial_j B_j\big(\{\mathbf{I}-\mathbf{P}\}g\big) = 0, \tag{3.22}$$

and

$$\partial_t\big[A_{jj}\big(\{\mathbf{I}-\mathbf{P}\}g\big) + 2c\big] + 2\partial_j b_j = A_{jj}(R), \tag{3.23}$$

$$\partial_t\big[A_{jm}\big(\{\mathbf{I}-\mathbf{P}\}g\big)\big] + \partial_j b_m + \partial_m b_j = A_{jm}(R), \quad j \neq m, \tag{3.24}$$

$$\partial_t\big[B_j\big(\{\mathbf{I}-\mathbf{P}\}g\big)\big] + 10\partial_j c = B_j(R), \quad 1 \leqslant j,m \leqslant 3, \tag{3.25}$$

$$A_{jm}(g) := \big\langle(\xi_j\xi_m - 1)\sqrt{M}, g\big\rangle,$$
$$B_j(g) := \big\langle\big(|\xi|^2 - 5\big)\xi_j\sqrt{M}, g\big\rangle,$$
$$R = -\xi\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}g + L\{\mathbf{I}-\mathbf{P}\}g,$$

and with direct calculation we can obtain

$$\begin{aligned} &-\partial_t\left[\sum_j \partial_j A_{jm}\big(\{\mathbf{I}-\mathbf{P}\}g\big) + \frac{1}{2}\partial_m A_{mm}\big(\{\mathbf{I}-\mathbf{P}\}g\big)\right] - \Delta_x b_m - \partial_m\partial_m b_m \\ &= \frac{1}{2}\sum_{j\neq m}\partial_m A_{jj}(R) - \sum_j \partial_j A_{jm}(R), \end{aligned} \tag{3.26}$$

for some fixed m. Similar as in Lemma 4.1 in [15], using (3.20) to (3.26) we have for any $t \geqslant 0$, $k \in \mathbb{R}^3$ that

$$\partial_t \, Re\, \mathcal{E}^{int}(\hat g) + \kappa \frac{|k|^2}{1+|k|^2}\big(|\hat b|^2 + |\hat c|^2 + |\hat a + 3\hat c|^2\big) \leqslant C\big\|\nu^{\frac12}\{\mathbf{I}-\mathbf{P}\}\hat g\big\|^2_{L^2_\xi} + C|\hat b|^2 \tag{3.27}$$

for some $\kappa_1 > 0$. Where $\mathcal{E}^{int}(\hat g)$ takes the form

$$\begin{aligned}\mathcal{E}^{int}(\hat g) = {}& \kappa_1 \sum_m \bigg(\sum_j \frac{ik_j}{1+|k|^2} A_{jm}\big(\{\mathbf{I}-\mathbf{P}\}\hat g\big) + \frac12 \frac{ik_m}{1+|k|^2} A_{mm}\big(\{\mathbf{I}-\mathbf{P}\}\hat g\big) \bigg| - \hat b_m \bigg) \\ & + \kappa_1 \sum_j \bigg(B_j\big(\{\mathbf{I}-\mathbf{P}\}\hat g\big) \bigg| \frac{ik_j}{1+|k|^2}\hat c \bigg) + \sum_m \bigg(\hat b_m \bigg| \frac{ik_m}{1+|k|^2}(\hat a + 3\hat c) \bigg).\end{aligned} \tag{3.28}$$

Therefore, for $0 < \kappa_2 \ll 1$, a suitable linear combination of (3.19) and (3.27) gives

$$\partial_t\big[\|\hat g\|^2_{L^2_\xi} + \kappa_2\, Re\, \mathcal{E}^{int}(\hat g)\big] + \kappa \bigg\{ \big\|\nu^{\frac12}\{\mathbf{I}-\mathbf{P}\}\hat g\big\|^2_{L^2_\xi} + \frac{|k|^2}{1+|k|^2}\big(|\hat b|^2 + |\hat c|^2 + |\hat a + 3\hat c|^2\big) + |\hat b|^2 \bigg\} \leqslant 0. \tag{3.29}$$

Step 2. Taking the complex inner product of (3.17) with $\mu^{2l}\hat g$ and integrating it over $\mathbb{R}^3_\xi$, one has

$$\begin{aligned}\frac12 \partial_t \big\|\mu^l \hat g\big\|^2_{L^2_\xi} + \big\|\nu^{\frac12}\mu^l\{\mathbf{I}-\mathbf{P}\}\hat g\big\|^2_{L^2_\xi} = {}& - Re \int_{\mathbb{R}^3} \big(\nu(\xi)\{\mathbf{I}-\mathbf{P}\}\hat g \big| \mu^{2l}\mathbf{P}\hat g\big) + Re \int_{\mathbb{R}^3} \big(K\{\mathbf{I}-\mathbf{P}\}\hat g \big| \mu^{2l}\{\mathbf{I}-\mathbf{P}\}\hat g\big)\, d\xi \\ & - Re \int_{\mathbb{R}^3} \big(\hat b\cdot \xi\sqrt{M} \big| \mu^{2l}\hat g\big)\, d\xi.\end{aligned} \tag{3.30}$$

Here, the estimation for the right-hand terms comes from Cauchy–Schwarz's inequality and (3.9) cf. [6]. By using the fact $\frac{2|k|^2}{1+|k|^2}\chi_{|k|^2\geqslant 1} \geqslant 1$, we can deduce

$$\partial_t\big\|\mu^l \hat g\big\|^2_{L^2_\xi}\chi_{|k|^2\geqslant 1} + \kappa\big\|\nu^{\frac12}\mu^l\{\mathbf{I}-\mathbf{P}\}\hat g\big\|^2_{L^2_\xi}\chi_{|k|^2\geqslant 1} \leqslant C\bigg(\big\|\nu^{\frac12}\{\mathbf{I}-\mathbf{P}\}\hat g\big\|^2_{L^2_\xi} + \frac{|k|^2}{1+|k|^2}\big|\widehat{(a,b,c)}\big|^2 \bigg). \tag{3.31}$$

Applying $\{\mathbf{I}-\mathbf{P}\}$ to (3.17) we have

$$\partial_t\{\mathbf{I}-\mathbf{P}\}\hat g + i\xi\cdot k\{\mathbf{I}-\mathbf{P}\}\hat g = L\{\mathbf{I}-\mathbf{P}\}\hat g + \mathbf{P}i\xi\cdot k\hat g - i\xi\cdot k\mathbf{P}\hat g. \tag{3.32}$$

By further taking the complex inner product of the above equation with $\mu^{2l}\{\mathbf{I}-\mathbf{P}\}\hat g$ and integrating it over $\mathbb{R}^3_\xi$, one has

$$\begin{aligned}& \frac12 \partial_t\big\|\mu^l\{\mathbf{I}-\mathbf{P}\}\hat g\big\|^2_{L^2_\xi} + \kappa\big\|\mu^{l-1}\{\mathbf{I}-\mathbf{P}\}\hat g\big\|^2 \\ & \quad \leqslant Re \int_{\mathbb{R}^3} \big(K\{\mathbf{I}-\mathbf{P}\}\hat g \big| \mu^{2l}\{\mathbf{I}-\mathbf{P}\}\hat g\big)\, d\xi + Re \int_{\mathbb{R}^3} \big(\mathbf{P}i\xi\cdot k\hat g - i\xi\cdot k\mathbf{P}\hat g \big| \mu^{2l}\{\mathbf{I}-\mathbf{P}\}\hat g\big)\, d\xi.\end{aligned} \tag{3.33}$$

Similar as in [6] we have

$$\begin{aligned}& \partial_t\big\|\mu^l\{\mathbf{I}-\mathbf{P}\}\hat g\big\|^2_{L^2_\xi}\chi_{|k|^2\leqslant 1} + \kappa\big\|\mu^{l-1}\{\mathbf{I}-\mathbf{P}\}\hat g\big\|^2\chi_{|k|^2\leqslant 1} \\ & \quad \leqslant C\bigg(\big\|\nu^{\frac12}\{\mathbf{I}-\mathbf{P}\}\hat g\big\|^2_{L^2_\xi} + \frac{|k|^2}{1+|k|^2}\big|\widehat{(a,b,c)}\big|^2 \bigg).\end{aligned} \tag{3.34}$$

For properly chosen constants $0 < \kappa_3, \kappa_4 \leqslant 1$, set

$$E_l(\hat{g}) = \|\hat{g}\|^2_{L^2_\xi} + \kappa_2 \, Re\, \mathcal{E}^{int}(\hat{g}) + \kappa_3 \big\|\mu^l \hat{g}\big\|^2_{L^2_\xi} \chi_{|k|^2 \geqslant 1} + \kappa_4 \big\|\mu^l \{\mathbf{I} - \mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi} \chi_{|k|^2 \leqslant 1},$$

$$D_l(\hat{g}) = \big\|\mu^{l-1}\{\mathbf{I} - \mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi} + \frac{|k|^2}{1+|k|^2}\big(|\hat{b}|^2 + |\hat{c}|^2 + |\hat{a} + 3\hat{c}|^2\big) + |\hat{b}|^2.$$

We have

$$\partial_t E_l(\hat{g}) + \kappa D_l(\hat{g}) \leqslant 0. \tag{3.35}$$

As $\|\hat{g}\|^2_{L^2_\xi} + \kappa_2 \, Re\, \mathcal{E}^{int}(\hat{g}) \sim \|\hat{g}\|^2_{L^2_\xi}$, it is straightforward to check that

$$E_l(\hat{g}) \sim \big\|\mu^l \hat{g}\big\|^2_{L^2_\xi}, \qquad D_l(\hat{g}) \geqslant \kappa \rho(k) E_{l-1}(\hat{g}), \quad \rho(k) = \frac{|k|^2}{1+|k|^2}. \tag{3.36}$$

Therefore, we arrive at

$$\partial_t E_l(\hat{g}) + \kappa \rho(k) E_{l-1}(\hat{g}) \leqslant 0, \tag{3.37}$$

for any $t \geqslant 0$, $k \in \mathbb{R}^3$ and for any $l \geqslant 0$. The same as in Theorem 3.1 in [6], we can deduce for given $l_0 > 3/2$, we fixing $J > 3/2$, $p > 1$ such that $l_0 = J + p - 1$, and hence

$$E_l(\hat{g}) \leqslant C\big[1 + \rho(k)t\big]^{-J} E_{l+l_0}(\hat{g}_0).$$

Since $J > 3/2$, in the completely same way as in [5] and [3] by considering the frequency integration over $\mathbb{R}^3_k = \{|k| \leqslant 1\} \cup \{|k| \geqslant 1\}$ with a little modification of the proof in [9,10], one can derive the desired time-decay property. This completes the proof of our lemma. □

3.3. Energy estimates

Proposition 3.1. *Assume $-2 \leqslant \gamma < 0$, let $N \geqslant 8$, $l \geqslant N$, $0 \leqslant \lambda \ll 1$ and $0 < \theta \leqslant \frac{1}{4}$. Suppose that the a priori assumption*

$$\sup_{0 \leqslant t \leqslant T} X_{N,l}(t) \leqslant \delta \tag{3.38}$$

holds for $\delta > 0$ small enough. Then, there is $\mathcal{E}_{N,l}(t)$ such that

$$\frac{d}{dt}\mathcal{E}_{N,l}(t) + \kappa \mathcal{D}_{N,l}(t) \leqslant 0 \tag{3.39}$$

holds for any $0 \leqslant t \leqslant T$.

Proof. *Step 1.* Energy estimates without any weight:

In the proof we use the a priori assumption (3.38) and the equivalent relationship between u and b:

$$u = \frac{b}{1 + a + 3c}. \tag{3.40}$$

Also, the following Sobolev inequality is frequently used in performing our energy estimates:

$$\|g\|_{L^\infty} \leqslant C\|\nabla g\|^{\frac{1}{2}} \big\|\nabla^2 g\big\|^{\frac{1}{2}} \leqslant C\|\nabla g\|_{H^1}.$$

1). First, we multiply (1.5) by g and take the integration over $\mathbb{R}^3 \times \mathbb{R}^3$:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} g^2\,d\xi\,dx + \frac{1}{2}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \xi\cdot\nabla_x g^2\,d\xi\,dx - \frac{1}{2}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} u\cdot\nabla_\xi g^2\,d\xi\,dx \\
&\quad + \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} u\cdot\xi\sqrt{M}g\,d\xi\,dx + \frac{1}{2}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} u\cdot\xi g^2\,d\xi\,dx \\
&= \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} gLg\,d\xi\,dx + \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} g\Gamma(g,g)\,d\xi\,dx. \qquad (3.41)
\end{aligned}$$

First we notice $\int_{\mathbb{R}^3\times\mathbb{R}^3} \xi\cdot\nabla_x g^2\,d\xi\,dx = \int_{\mathbb{R}^3\times\mathbb{R}^3} u\cdot\nabla_\xi g^2\,d\xi\,dx = 0$. Then we use (2.1) and Lemma 3.2 to obtain the estimates involving Lg and $\Gamma(g,g)$. Terms involving the frictional force u are estimated as follows:

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} u\cdot\xi\sqrt{M}g\,d\xi\,dx = \int\limits_{\mathbb{R}^3} |b|^2\,dx - \int\limits_{\mathbb{R}^3} \frac{(a+3c)|b|^2}{1+a+3c}\,dx \geqslant (1-C\delta)\|b\|^2,$$

$$\frac{1}{2}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} u\cdot\xi g^2\,d\xi\,dx = \frac{1}{2}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} u\cdot\xi g_1^2\,d\xi\,dx + \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} u\cdot\xi g_1 g_2\,d\xi\,dx + \frac{1}{2}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} u\cdot\xi g_2^2\,d\xi\,dx.$$

Using (3.38) and (3.40), direct calculation gives

$$\frac{1}{2}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} u\cdot\xi g_1^2\,d\xi\,dx = \frac{1}{2}\int\limits_{\mathbb{R}^3}(a+5c)u\cdot b\,dx = \frac{1}{2}\int\limits_{\mathbb{R}^3}\frac{a+5c}{1+a+3c}|b|^2\,dx \leqslant C\delta\|b\|^2.$$

Using the Cauchy inequality and again the a priori assumption (3.38) together with (3.40) we have

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} u\cdot\xi g_1 g_2\,d\xi\,dx \leqslant C\delta\left\|\nu^{\frac{1}{2}}g_2\right\|^2 + C\delta\|b\|^2.$$

At last use (3.40) to obtain

$$\frac{1}{2}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} u\cdot\xi g_2^2\,d\xi\,dx \leqslant C\sup_x|u|\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|g_2^2\,dx\,d\xi \leqslant \frac{C\delta}{(1+t)^\theta}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|g_2^2\,dx\,d\xi.$$

Combining estimates of all terms we obtain the estimates on the zero-th order:

$$\frac{1}{2}\frac{d}{dt}\left\|g(t)\right\|^2 + \kappa\|b\|^2 + \kappa\left\|\nu^{\frac{1}{2}}g_2\right\|^2 \leqslant \frac{C\delta}{(1+t)^\theta}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|g_2^2\,dx\,d\xi + C\delta\left\|\nabla_x(a,b,c)\right\|^2.$$

2). We rewrite (1.5) as

$$\partial_t g + b\cdot\xi\sqrt{M} = -\xi\cdot\nabla_x g - \frac{1}{2}u\cdot\xi g + u\cdot\nabla_\xi g + \frac{a+3c}{1+a+3c}b\cdot\xi\sqrt{M} + Lg + \Gamma(g,g). \qquad (3.42)$$

We take derivatives ∂^α $(1 \leqslant |\alpha| \leqslant N)$ of (3.42) and multiply the resulting equation by $\partial^\alpha g$, then integrating over $\mathbb{R}^3 \times \mathbb{R}^3$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\partial^\alpha g\right\|^2+\kappa_0\left\|\nu^{\frac{1}{2}}\partial^\alpha g_2\right\|^2+\left\|\partial^\alpha b\right\|^2\\
&\quad=-\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\partial^\alpha g\xi\cdot\nabla_x\left(\partial^\alpha g\right)dx\,d\xi+\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\partial^\alpha g\partial^\alpha(u\cdot\nabla_\xi g)\,dx\,d\xi\\
&\qquad+\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\partial^\alpha g\partial^\alpha\left(\frac{a+3c}{1+a+3c}b\right)\cdot\xi\sqrt{M}\,dx\,d\xi-\frac{1}{2}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\partial^\alpha g\partial^\alpha(u\cdot\xi g)\,dx\,d\xi\\
&\qquad+\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\partial^\alpha g\partial^\alpha\Gamma(g,g)\,dx\,d\xi.
\end{aligned}\tag{3.43}
$$

The first term on the right hand side equals to 0. Using Lemma 3.3 we can obtain the estimate of the last term on the right hand. For the fourth term on the right hand of (3.43) we have

Lemma 3.6. *Let $N\geqslant 4$, $1\leqslant|\alpha|\leqslant N$ and $l\geqslant 0$, then it holds that*

$$
\begin{aligned}
&-\frac{1}{2}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\omega_{-l}^2(t,\xi)\partial^\alpha g\partial^\alpha(u\cdot\xi g)\,dx\,d\xi\\
&\quad\leqslant C\left\|\nabla_x(a,b,c)\right\|_{H^{N-1}}\sum_{1\leqslant|\alpha|\leqslant N}\left\{\left\|\langle\xi\rangle^{\frac{1}{2}}\omega_{-l}(t,\xi)\partial^\alpha g_2\right\|^2+\left\|\partial^\alpha(a,b,c)\right\|^2\right\}.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&-\frac{1}{2}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\omega_{-l}^2(t,\xi)\partial^\alpha g\partial^\alpha(u\cdot\xi g)\,dx\,d\xi\\
&\quad=-\left\langle\partial^\alpha\left(\frac{1}{2}u\cdot\xi g_1\right),\omega_{-l}^2(t,\xi)\partial^\alpha g\right\rangle-\left\langle\partial^\alpha\left(\frac{1}{2}u\cdot\xi g_2\right),\omega_{-l}^2(t,\xi)\partial^\alpha g_1\right\rangle-\left\langle\partial^\alpha\left(\frac{1}{2}u\cdot\xi g_2\right),\omega_{-l}^2(t,\xi)\partial^\alpha g_2\right\rangle\\
&\quad:=\sum_{i=1}^{3}J_{1i}.
\end{aligned}
$$

Now, we do calculation as follows:

$$
\begin{aligned}
J_{11}&=-\left\langle\partial^\alpha\left(\frac{1}{2}u\cdot\xi g_1\right),\omega_{-l}^2(t,\xi)\partial^\alpha g\right\rangle\\
&=-\sum_{|\alpha_1|\leqslant|\alpha|}C_\alpha^{\alpha_1}\left\langle\frac{1}{2}\xi\cdot\partial^{\alpha-\alpha_1}u\partial^{\alpha_1}g_1,\omega_{-l}^2(t,\xi)\partial^\alpha g\right\rangle\\
&\leqslant C\sum_{|\alpha_1|\leqslant|\alpha|}\int\limits_{\mathbb{R}^3}\left|\partial^{\alpha-\alpha_1}u\right|\left|\partial^{\alpha_1}(a,b,c)\right|\left\|\partial^\alpha g\right\|_{L^2_\xi}dx\\
&\leqslant\begin{cases}C\sup_x\left|\partial^{\alpha_1}(a,b,c)\right|\left\|\partial^{\alpha-\alpha_1}u\right\|\left\|\partial^\alpha g\right\|, & \text{if } |\alpha_1|\leqslant\frac{|\alpha|}{2},\\ C\sup_x\left|\partial^{\alpha-\alpha_1}u\right|\left\|\partial^{\alpha_1}(a,b,c)\right\|\left\|\partial^\alpha g\right\|, & \text{if } |\alpha_1|>\frac{|\alpha|}{2},\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
J_{12}&=-\left\langle\partial^\alpha\left(\frac{1}{2}u\cdot\xi g_2\right),\omega_{-l}^2(t,\xi)\partial^\alpha g_1\right\rangle\\
&=-\sum_{|\alpha_1|\leqslant|\alpha|}C_\alpha^{\alpha_1}\left\langle\frac{1}{2}\xi\cdot\partial^{\alpha-\alpha_1}u\partial^{\alpha_1}g_2,\omega_{-l}^2(t,\xi)\partial^\alpha g_1\right\rangle
\end{aligned}
$$

$$\leqslant C\sum_{|\alpha_1|\leqslant|\alpha|}\int_{\mathbb{R}^3}\left|\partial^{\alpha-\alpha_1}u\right|\left|\partial^\alpha(a,b,c)\right|\left\|\partial^{\alpha_1}g_2\right\|_{L^2_\xi}dx$$

$$\leqslant\begin{cases}C\sup_x\|\partial^{\alpha_1}g_2\|_{L^2_\xi}\|\partial^\alpha(a,b,c)\|\|\partial^{\alpha-\alpha_1}u\|, & \text{if } |\alpha_1|\leqslant\frac{|\alpha|}{2},\\ C\sup_x|\partial^{\alpha-\alpha_1}u|\|\partial^\alpha(a,b,c)\|\|\partial^{\alpha_1}g\|, & \text{if } |\alpha_1|>\frac{|\alpha|}{2},\end{cases}$$

$$\begin{aligned}J_{13}&=-\left\langle\partial^\alpha\left(\frac12 u\cdot\xi g_2\right),-\omega^2_{-l}(t,\xi)\partial^\alpha g_2\right\rangle\\&=\sum_{|\alpha_1|\leqslant|\alpha|}C^{\alpha_1}_\alpha\left\langle\frac12\xi\cdot\partial^{\alpha-\alpha_1}u\partial^{\alpha_1}g_2,\omega^2_{-l}(t,\xi)\partial^\alpha g_2\right\rangle\\&\leqslant C\sum_{|\alpha_1|\leqslant|\alpha|}\int_{\mathbb{R}^3}\left|\partial^{\alpha-\alpha_1}u\right|\left\||\xi|^{\frac12}\omega_{-l}(t,\xi)\partial^{\alpha_1}g_2\right\|_{L^2_\xi}\left\||\xi|^{\frac12}\omega_{-l}(t,\xi)\partial^\alpha g_2\right\|_{L^2_\xi}dx\\&\leqslant\begin{cases}C\sup_x\||\xi|^{\frac12}\omega_{-l}(t,\xi)\partial^{\alpha_1}g_2\|_{L^2_\xi}\||\xi|^{\frac12}\omega_{-l}(t,\xi)\partial^\alpha g_2\|\|\partial^{\alpha-\alpha_1}u\| & \text{if } |\alpha_1|\leqslant\frac{|\alpha|}{2},\\ C\sup_x|\partial^{\alpha-\alpha_1}u|\||\xi|^{\frac12}\omega_{-l}(t,\xi)\partial^{\alpha_1}g_2\|\||\xi|^{\frac12}\omega_{-l}(t,\xi)\partial^\alpha g_2\|, & \text{if } |\alpha_1|>\frac{|\alpha|}{2}.\end{cases}\end{aligned}$$

Therefore, Lemma 3.6 follows by collecting all above estimates. □

For the second term on the right hand of (3.43) we have

Lemma 3.7. *Assume $-2\leqslant\gamma<0$. Let $N\geqslant4$, $1\leqslant|\alpha|\leqslant N$ and $l\geqslant0$, then it holds that*

$$\begin{aligned}&\left\langle\partial^\alpha(u\cdot\nabla_\xi g),\omega^2_{-l}(t,\xi)\partial^\alpha g\right\rangle\\&\quad\leqslant C\left\|\nabla_x(a,b,c)\right\|_{H^{N-1}}\sum_{1\leqslant|\alpha|+|\beta|\leqslant N,|\beta|\leqslant1}\left\{\left\|\langle\xi\rangle\omega_{|\beta|-l}(t,\xi)\partial^\alpha g_2\right\|^2+\left\|\partial^\alpha(a,b,c)\right\|^2\right\}.\end{aligned}$$

Proof. We write it as

$$\begin{aligned}&\left\langle\partial^\alpha(u\cdot\nabla_\xi g),\omega^2_{-l}(t,\xi)\partial^\alpha g\right\rangle\\&\quad=\left\langle u\cdot\nabla_\xi\partial^\alpha g,\omega^2_{-l}(t,\xi)\partial^\alpha g\right\rangle+\sum_{1\leqslant|\alpha_1|\leqslant|\alpha|}C^{\alpha_1}_\alpha\left[\left\langle\partial^{\alpha_1}u\cdot\nabla_\xi\partial^{\alpha-\alpha_1}g_1,\omega^2_{-l}(t,\xi)\partial^\alpha g\right\rangle\right.\\&\qquad\left.+\left\langle\partial^{\alpha_1}u\cdot\nabla_\xi\partial^{\alpha-\alpha_1}g_2,\omega^2_{-l}(t,\xi)\partial^\alpha g\right\rangle\right]\\&\quad:=\sum_{i=1}^3J_{2i}.\end{aligned}\tag{3.44}$$

Here

$$J_{21}=\left\langle u\cdot\nabla_\xi\partial^\alpha g,\omega^2_{-l}(t,\xi)\partial^\alpha g\right\rangle=-\frac12\int_{\mathbb{R}^3\times\mathbb{R}^3}u\cdot\nabla_\xi\omega^2_{-l}(t,\xi)\left|\partial^\alpha g\right|^2dx\,d\xi.$$

To estimate J_{21}, notice

$$\begin{aligned}\nabla_\xi\omega^2_{-l}(t,\xi)&=(-2\gamma l)\langle\xi\rangle^{-2\gamma l-1}\nabla_\xi\langle\xi\rangle e^{\frac{2\lambda\langle\xi\rangle^2}{(1+t)^\theta}}+4\lambda/(1+t)^\theta\langle\xi\rangle^{-2\gamma l+1}\nabla_\xi\langle\xi\rangle e^{\frac{2\lambda\langle\xi\rangle^2}{(1+t)^\theta}}\\&\leqslant C\langle\xi\rangle^{-2\gamma l+1}e^{\frac{2\lambda\langle\xi\rangle^2}{(1+t)^\theta}}\leqslant C\langle\xi\rangle\omega^2_{-l}(t,\xi),\end{aligned}\tag{3.45}$$

where $\langle\xi\rangle \geqslant 1$ and the fact that both $\lambda/(1+t)^\theta$ and $\nabla_\xi\langle\xi\rangle$ are bounded by a constant independent of t and ξ was used. Then we have

$$\begin{aligned} J_{21} &\leqslant C\sup_x |u|\big(\big\|\langle\xi\rangle^{\frac{1}{2}}\omega_{-l}(t,\xi)\partial^\alpha g_2\big\|^2 + \big\|\nabla_x(a,b,c)\big\|_{H^{N-1}}\big) \\ &\leqslant C\sup_x |b|\big(\big\|\langle\xi\rangle^{\frac{1}{2}}\omega_{-l}(t,\xi)\partial^\alpha g_2\big\|^2 + \big\|\nabla_x(a,b,c)\big\|_{H^{N-1}}\big). \end{aligned}$$

For $J_{22\alpha_1}$, it is straightforward to estimate it by

$$\begin{aligned} J_{22\alpha_1} &= C_\alpha^{\alpha_1}\big\langle\partial^{\alpha_1}u\cdot\nabla_\xi\partial^{\alpha-\alpha_1}g_1, \omega_{-l}^2(t,\xi)\partial^\alpha g\big\rangle \\ &\leqslant C\int_{\mathbb{R}^3}\big|\partial^{\alpha_1}u\big|\big|\partial^{\alpha-\alpha_1}(a,b,c)\big|\big(\big\|\omega_{-l}(t,\xi)\partial^\alpha g_2\big\|_{L^2_\xi} + \big|\partial^\alpha(a,b,c)\big|\big)\,dx \\ &\leqslant \begin{cases} C\sup_x|\partial^{\alpha_1}u|\|\partial^{\alpha-\alpha_1}(a,b,c)\|(\|\omega_{-l}(t,\xi)\partial^\alpha g_2\| + \|\partial^\alpha(a,b,c)\|), & \text{if } |\alpha_1| \leqslant \frac{|\alpha|}{2}, \\ C\sup_x|\partial^{\alpha-\alpha_1}(a,b,c)|\|\partial^{\alpha_1}u\|(\|\omega_{-l}(t,\xi)\partial^\alpha g_2\| + \|\partial^\alpha(a,b,c)\|), & \text{if } |\alpha_1| > \frac{|\alpha|}{2}. \end{cases} \end{aligned}$$

Now we have

$$J_{22\alpha_1} \leqslant C\big\|\nabla_x(a,b,c)\big\|_{H^{N-1}}\big(\big\|\nabla_x(a,b,c)\big\|_{H^{N-1}}^2 + \big\|\omega_{-l}(t,\xi)\partial^\alpha g_2\big\|^2\big). \tag{3.46}$$

Note that $\langle\xi\rangle^{\gamma+2} \geqslant 1$ due to $-2 \leqslant \gamma < 0$ so that

$$\omega_{-l}^2(t,\xi) \leqslant \langle\xi\rangle\omega_{1-l}\langle\xi\rangle\omega_{-l}. \tag{3.47}$$

Thus

$$\begin{aligned} J_{23\alpha_1} &= \big\langle\partial^{\alpha_1}u\cdot\nabla_\xi\partial^{\alpha-\alpha_1}g_2, \omega_{-l}^2(t,\xi)\partial^\alpha g\big\rangle \\ &\leqslant C\int_{\mathbb{R}^3}\big|\partial^{\alpha_1}u\big|\big|\langle\xi\rangle\omega_{1-l}\nabla_\xi\partial^{\alpha-\alpha_1}g_2\big|_{L^2_\xi}\big\|\langle\xi\rangle\omega_{-l}\partial^\alpha g\big\|_{L^2_\xi}\,dx \\ &\leqslant C\big\|\nabla_x(a,b,c)\big\|_{H^{N-1}}\bigg(\sum_{|\alpha|+|\beta|\leqslant N, |\beta|\leqslant 1}\big\|\langle\xi\rangle\omega_{|\beta|-l}\partial_\beta^\alpha g_2\big\|^2 + \big\|\langle\xi\rangle\omega_{-l}\partial^\alpha g\big\|^2\bigg). \end{aligned}$$

Here and in the following proof we have used the facts

$$\sup_x\big|\partial^{\alpha_1}u\big| \leqslant C\big\|\nabla_x(a,b,c)\big\|_{H^{N-1}}, \qquad \big\|\partial^{\alpha_2}u\big\| \leqslant C\big\|\nabla_x(a,b,c)\big\|_{H^{N-1}},$$

for $|\alpha_1| \leqslant \frac{N}{2}$, $|\alpha_2| > \frac{N}{2}$.

By collecting all the above estimates, it then completes the proof of the lemma. □

For the third term on the right hand of (3.43), we have

$$\begin{aligned} &\int_{\mathbb{R}^3\times\mathbb{R}^3}\partial^\alpha g\partial^\alpha\bigg(\frac{a+3c}{1+a+3c}b\bigg)\cdot\xi\sqrt{M}\,dx\,d\xi \\ &\quad\leqslant \int_{\mathbb{R}^3}\partial^\alpha b\cdot\partial^\alpha\bigg(\frac{a+3c}{1+a+3c}b\bigg) \leqslant C\big\|\nabla_x(a,b,c)\big\|_{H^{N-1}}\big\|\nabla_x(a,b,c)\big\|_{H^{N-1}}^2. \end{aligned} \tag{3.48}$$

Now we conclude

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha g\|^2+\kappa\big[\|\nu^{\frac{1}{2}}\partial^\alpha g_2\|^2+\|\partial^\alpha b\|^2\big]\leqslant C\delta\mathcal{D}_{N,l}(t).$$

Step 2. Energy estimates with the weight function $\omega_{|\beta|-l}(t,\xi)$:

First we rewrite (1.5) as

$$\partial_t g_2+\xi\cdot\nabla_x g_2-u\cdot\nabla_\xi g_2+\nu(\xi)g_2=Kg_2+\Gamma(g,g)-\frac{1}{2}u\cdot\xi g_2+[\mathbf{P},\Gamma_u]g, \tag{3.49}$$

where $\Gamma_u=\xi\cdot\nabla_x-u\cdot\nabla_\xi+\frac{1}{2}u\cdot\xi$. By multiplying the above equation by $\omega_{-l}^2(t,\xi)g_2$ and integrating over $\mathbb{R}^3\times\mathbb{R}^3$ we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\omega_{-l}(t,\xi)g_2\|^2+\|\nu^{\frac{1}{2}}(\xi)\omega_{-l}(t,\xi)g_2\|^2-\frac{1}{2}\left\langle\frac{d}{dt}\omega_{-l}^2(t,\xi),|g_2|^2\right\rangle\\
&\quad=\langle Kg_2,\omega_{-l}^2(t,\xi)g_2\rangle+\langle\Gamma(g,g),\omega_{-l}^2(t,\xi)g_2\rangle-\left\langle\frac{1}{2}u\cdot\xi g_2,\omega_{-l}^2(t,\xi)g_2\right\rangle\\
&\qquad+\langle u\cdot\nabla_\xi g_2,\omega_{-l}^2(t,\xi)g_2\rangle+\langle[\mathbf{P},\Gamma_u]g,\omega_{-l}^2(t,\xi)g_2\rangle\\
&\quad:=\sum_{i=1}^5 J_{3i}.
\end{aligned}\tag{3.50}$$

For the left-hand third term, notice

$$-\frac{d}{dt}\omega_{-l}^2(t,\xi)=\frac{\lambda\theta}{(1+t)^{1+\theta}}\langle\xi\rangle^2\omega_{-l}^2(t,\xi), \tag{3.51}$$

then we have $-\frac{1}{2}\langle\frac{d}{dt}\omega_{-l}^2(t,\xi),|g_2|^2\rangle=\frac{\lambda\theta}{2(1+t)^{1+\theta}}\|\langle\xi\rangle\omega_{-l}(t,\xi)g_2\|^2$. For the right-hand side terms, we have the following estimates:

Using Lemma 3.1 we have

$$\begin{aligned}
J_{31}&=\langle Kg_2,\omega_{-l}^2(t,\xi)g_2\rangle\\
&\leqslant\eta\|\nu^{\frac{1}{2}}\omega_{-l}(t,\xi)g_2\|^2+C_\eta\|\chi_{|\xi|\leqslant C_\eta}\langle\xi\rangle^{-\gamma l}g_2\|\|\nu^{\frac{1}{2}}\omega_{-l}(t,\xi)g_2\|\\
&\leqslant 2\eta\|\nu^{\frac{1}{2}}\omega_{-l}(t,\xi)g_2\|^2+C_\eta\|\nu^{\frac{1}{2}}g_2\|^2.
\end{aligned}$$

Lemma 3.2 implies

$$\begin{aligned}
J_{32}&=\langle\Gamma(g,g),\omega_{-l}^2(t,\xi)g_2\rangle\\
&\leqslant C\big[\mathcal{E}_{N,l}(t)\big]^{\frac{1}{2}}\|\nu^{\frac{1}{2}}\omega_{-l}(t,\xi)g_2\|^2+C\|(a,b,c)\|_{H^1}\|\nabla_x(a,b,c)\|\|\nu^{\frac{1}{2}}g_2\|.
\end{aligned}$$

By the Cauchy–Schwarz inequality we have

$$\begin{aligned}
J_{33}&=-\left\langle\frac{1}{2}u\cdot\xi g_2,\omega_{-l}^2(t,\xi)g_2\right\rangle\leqslant C\sup_x|u|\|\langle\xi\rangle^{\frac{1}{2}}\omega_{-l}(t,\xi)g_2\|^2,\\
J_{34}&=\langle u\cdot\nabla_\xi g_2,\omega_{-l,q}^2(t,\xi)g_2\rangle=-\frac{1}{2}\langle u\cdot\nabla_\xi\omega_{-l}^2(t,\xi),|g_2|^2\rangle\leqslant C\sup_x|u|\|\langle\xi\rangle^{\frac{1}{2}}\omega_{-l}(t,\xi)g_2\|^2,\\
J_{35}&=\langle[\mathbf{P},\Gamma_u]g,\omega_{-l}^2(t,\xi)g_2\rangle\\
&\leqslant\eta\|\nu^{\frac{1}{2}}g_2\|^2+C_\eta\{\|\nu^{\frac{1}{2}}\nabla_x g_2\|^2+\|\nabla_x(a,b,c)\|^2\}+C_\eta\|u\|_{H^2}^2\{\|\nu^{\frac{1}{2}}\omega_{-l}(t,\xi)g_2\|^2+\|\nabla_x(a,b,c)\|_{H^1}^2\}.
\end{aligned}$$

By collecting all above estimates we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\|\omega_{-l}(t,\xi)g_2\big\|^2+\kappa\big\|\nu^{\frac{1}{2}}(\xi)\omega_{-l}(t,\xi)g_2\big\|^2+\frac{\lambda\theta}{(1+t)^{1+\theta}}\big\|\langle\xi\rangle\omega_{-l}(t,\xi)g_2\big\|^2\\&\quad\leqslant C\delta\mathcal{D}_{N,l}(t)+C\big\{\big\|\nabla_x(a,b,c)\big\|^2+\big\|\nu^{\frac{1}{2}}\nabla_x g_2\big\|^2+\big\|\nu^{\frac{1}{2}}g_2\big\|^2\big\}.\end{aligned}$$

For the weighted estimate on the terms containing only x derivatives, we directly use (1.5). In fact, take $1\leqslant|\alpha|\leqslant N$, and by applying ∂_x^α to (1.5) and multiplying it by $\omega_{-l}^2(t,\xi)\partial^\alpha$ and integrating over $\mathbb{R}^3\times\mathbb{R}^3$, one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\|\omega_{-l}(t,\xi)\partial^\alpha g\big\|^2+\big\langle\nu(\xi)\partial^\alpha g_2,\omega_{-l}^2(t,\xi)\partial^\alpha g\big\rangle+\frac{\lambda\theta}{(1+t)^{1+\theta}}\big\|\langle\xi\rangle\omega_{-l}(t,\xi)\partial^\alpha g\big\|^2\\&\quad=\big\langle K\partial^\alpha g_2,\omega_{-l}^2(t,\xi)\partial^\alpha g\big\rangle-\big\langle\partial^\alpha u\cdot\xi\sqrt{M},\omega_{-l}^2(t,\xi)\partial^\alpha g\big\rangle\\&\qquad+\left\langle\partial^\alpha\Gamma(g,g)-\partial^\alpha\left(\frac{1}{2}u\cdot\xi g\right)+\partial^\alpha(u\cdot\nabla_\xi g),\omega_{-l}^2(t,\xi)\partial^\alpha g\right\rangle.\end{aligned}\tag{3.52}$$

For the left-hand terms, one has

$$\begin{aligned}\big\langle\nu(\xi)\partial^\alpha g_2,\omega_{-l}^2(t,\xi)\partial^\alpha g\big\rangle&=\big\|\nu(\xi)\omega_{-l}(t,\xi)\partial^\alpha g_2\big\|^2+\big\langle\nu(\xi)\partial^\alpha g_2,\omega_{-l}^2(t,\xi)\partial^\alpha g_1\big\rangle\\&\geqslant\frac{1}{2}\big\|\nu(\xi)\omega_{-l}(t,\xi)\partial^\alpha g_2\big\|^2-C\big\|\partial^\alpha(a,b,c)\big\|^2,\end{aligned}\tag{3.53}$$

and

$$\frac{\lambda\theta}{(1+t)^{1+\theta}}\big\|\langle\xi\rangle\omega_{-l}(t,\xi)\partial^\alpha g\big\|^2\geqslant\frac{\lambda\theta}{2(1+t)^{1+\theta}}\big\|\langle\xi\rangle\omega_{-l}(t,\xi)\partial^\alpha g_2\big\|^2-C\big\|\partial^\alpha(a,b,c)\big\|^2.\tag{3.54}$$

For the right-hand side we only give the estimates of

$$\begin{aligned}&-\big\langle\partial^\alpha u\cdot\xi\sqrt{M},\omega_{-l}^2(t,\xi)\partial^\alpha g\big\rangle\\&\quad\leqslant\eta\big\|\nu^{\frac{1}{2}}\omega_{-l}(t,\xi)\partial^\alpha g\big\|^2+C_\eta\big\|\partial^\alpha u\big\|^2\leqslant\eta\big\|\nu^{\frac{1}{2}}\omega_{-l}(t,\xi)\partial^\alpha g_2\big\|^2+C_\eta\big\|\nabla_x(a,b,c)\big\|_{H^{N-1}}^2.\end{aligned}\tag{3.55}$$

The rest terms on the right hand side follow from Lemma 3.1, Lemma 3.3, Lemma 3.7 and Lemma 3.6.

Thus,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{1\leqslant|\alpha|\leqslant N}\big\|\omega_{-l}(t,\xi)\partial^\alpha g\big\|^2+\kappa\sum_{1\leqslant|\alpha|\leqslant N}\big\|\nu^{\frac{1}{2}}(\xi)\omega_{-l}(t,\xi)\partial^\alpha g_2\big\|^2+\frac{\kappa}{(1+t)^{1+\theta}}\sum_{1\leqslant|\alpha|\leqslant N}\big\|\langle\xi\rangle\omega_{-l}(t,\xi)g_2\big\|^2\\&\quad\leqslant C\delta\mathcal{D}_{N,l}+C\left\{\sum_{1\leqslant|\alpha|\leqslant N}\big\|\nu^{\frac{1}{2}}(\xi)\partial^\alpha g_2\big\|^2+\big\|\nabla_x(a,b,c)\big\|_{H^{N-1}}^2\right\}.\end{aligned}$$

For the weighted estimate on the mixed $x-\xi$ derivatives, we use Eq. (3.49) of g_2. Let $1\leqslant m\leqslant N$. By applying ∂_β^α with $|\beta|=m$ and $|\alpha|+|\beta|\leqslant N$ to (3.49), multiplying it by $\omega_{|\beta|-l}^2(t,\xi)\partial_\beta^\alpha g_2$ and integrating over $\mathbb{R}^3\times\mathbb{R}^3$, one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\|\omega_{|\beta|-l}(t,\xi)\partial_\beta^\alpha g_2\big\|^2+\big\langle\partial_\beta\{\nu\partial^\alpha g_2\},\omega_{|\beta|-l}^2(t,\xi)\partial_\beta^\alpha g_2\big\rangle+\left\langle-\frac{1}{2}\frac{d}{dt}\omega_{|\beta|-l}^2(t,\xi),\big|\partial_\beta^\alpha g_2\big|^2\right\rangle\\&\quad=\left\langle\partial_\beta^\alpha\left(\Gamma(g,g)+u\cdot\nabla_\xi g_2-\frac{1}{2}u\cdot\xi g_2\right),\omega_{|\beta|-l}^2(t,\xi)\partial_\beta^\alpha g_2\right\rangle\end{aligned}$$

$$+\left\langle \partial_\beta K\partial^\alpha g_2, \omega^2_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\right\rangle + \left\langle \partial^\alpha_\beta[\mathbf{P}, \Gamma_u]g, \omega^2_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\right\rangle$$
$$+\left\langle -\partial^\alpha_\beta(\xi\cdot\nabla_x g_2), \omega^2_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\right\rangle. \tag{3.56}$$

For the left-hand second term of (3.56), from Lemma 3.1 one has

$$\begin{aligned}
&\left\langle \partial_\beta\{\nu\partial^\alpha g_2\}, \omega^2_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\right\rangle\\
&\geqslant \left\|\nu^{\frac12}\omega_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\right\|^2 - \eta\sum_{|\beta_1|\leqslant|\beta|}\left\|\nu^{\frac12}\omega_{|\beta|-l}(t,\xi)\partial^\alpha_{\beta_1} g_2\right\|^2 - C_\eta\left\|\chi_{|\xi|\leqslant 2C_\eta}\langle\xi\rangle^{\gamma(|\beta|-l)}\partial^\alpha g_2\right\|\\
&\geqslant \left\|\nu^{\frac12}\omega_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\right\|^2 - \eta\sum_{|\beta_1|\leqslant|\beta|}\left\|\nu^{\frac12}\omega_{|\beta|-l}(t,\xi)\partial^\alpha_{\beta_1} g_2\right\|^2 - C\left\|\nu^{\frac12}\partial^\alpha g_2\right\|.
\end{aligned}\tag{3.57}$$

The right-hand first term is bounded by $C\delta\mathcal{D}_{N,l}(t)$, it follows from Lemma 3.3 and similar method to the proof of Lemma 3.7 and Lemma 3.6. The right-hand first term follows from Lemma 3.1. For the right-hand third term, we have

$$\begin{aligned}
&\left\langle \partial^\alpha_\beta[\mathbf{P}, \Gamma_u]g, \omega^2_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\right\rangle\\
&\leqslant \eta\left\|\nu^{\frac12}\partial^\alpha_\beta g_2\right\|^2 + C\left(1+\left\|(a,b,c)\right\|_{H^N}\right)\left(\left\|\nabla_x(a,b,c)\right\|^2_{H^{N-1}} + \sum_{|\alpha|\leqslant N}\left\|\nu^{\frac12}\partial^\alpha g_2\right\|^2\right).
\end{aligned}\tag{3.58}$$

The right-hand third term is estimated as

$$\begin{aligned}
&\left\langle -\partial^\alpha_\beta(\xi\cdot\nabla_x g_2), \omega^2_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\right\rangle\\
&= -\sum_{\beta_1=1} C^{\beta_1}_\beta\left\langle \partial_{\beta_1}\xi\cdot\nabla_x\partial^\alpha_{\beta-\beta_1} g_2, \omega^2_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\right\rangle\\
&\leqslant \eta\left\|\nu^{\frac12}\omega_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\right\|^2 + C_\eta\Sigma_{|\alpha|+|\beta|\leqslant N, |\beta_2|=|\beta|-1}\left\|\nu^{\frac12}\omega_{|\beta_2|-l}(t,\xi)\partial^\alpha_{\beta_2} g_2\right\|^2,
\end{aligned}$$

where we have used the fact that for $\beta_2 = \beta-\beta_1, |\beta_1| = 1$

$$\omega^2_{|\beta|-l}(t,\xi) = \omega_{|\beta|+\frac12-l}(t,\xi)\omega_{|\beta_2|+\frac12-l}(t,\xi) \leqslant C\nu^{\frac12}\omega_{|\beta|-l}(t,\xi)\nu^{\frac12}\omega_{|\beta_2|-l}(t,\xi).$$

Therefore, summing over $\{|\beta| = m, |\alpha|+|\beta|\leqslant N\}$ for each given $1\leqslant m\leqslant N$ and then taking the proper linear combination of those $N-1$ estimates with properly chosen constants $C_m > 0$ $(1\leqslant m\leqslant N)$ and $\eta > 0$ small enough, one has

$$\begin{aligned}
&\frac12\frac{d}{dt}C_m\sum_{|\alpha|+|\beta|\leqslant N, |\beta|=m}\left\|\omega_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\right\|^2 + \kappa\sum_{|\alpha|+|\beta|\leqslant N, |\beta|\geqslant 1}\left\|\nu^{\frac12}\omega_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\right\|^2\\
&\quad + \frac{\kappa}{(1+t)^{1+\theta}}\sum_{|\alpha|+|\beta|\leqslant N, |\beta|\geqslant 1}\left\|\langle\xi\rangle\omega_{|\beta|-l}(t,\xi)\partial^\alpha_\beta g_2\right\|^2\\
&\leqslant C\delta\mathcal{D}_{N,l} + C\left\{\sum_{|\alpha|\leqslant N}\left\|\nu^{\frac12}\omega_{-l}(t,\xi)\partial^\alpha g_2\right\|^2 + \left\|\nabla_x(a,b,c)\right\|^2_{H^{N-1}}\right\}.
\end{aligned}$$

Step 3. Now we turn to the estimation on the macroscopic dissipation rate. The main result of this part is the following lemma:

Lemma 3.8.

$$\frac{d}{dt}\mathcal{G}\big(g(t)\big)+\sum_{|\alpha|\leqslant N-1}\big\|\partial^\alpha\nabla_x(a,b,c)\big\|^2\leqslant C\bigg\{\sum_{|\alpha|\leqslant N}\big\|\nu^{\frac12}\partial^\alpha g_2\big\|^2+\mathcal{E}_{N,l}\mathcal{D}_{N,l}+\sum_{|\alpha|\leqslant N-1}\big\|\partial_x^\alpha b\big\|^2\bigg\}. \tag{3.59}$$

Here,

$$\mathcal{G}\big(g(t)\big):=\sum_{|\alpha|\leqslant N-1}\sum_{i=1}^{3}\big[\mathcal{G}_{\alpha,i}^a\big(g(t)\big)+\mathcal{G}_{\alpha,i}^b\big(g(t)\big)+\mathcal{G}_{\alpha,i}^c\big(g(t)\big)+\mathcal{G}_{\alpha,i}^{ab}\big(g(t)\big)\big],$$

with

$$\begin{aligned}
\mathcal{G}_{\alpha,i}^a\big(g(t)\big)&=\big\langle\partial_x^\alpha\bar{r}_i^{(1)},\partial_i\partial_x^\alpha a\big\rangle,\\
\mathcal{G}_{\alpha,i}^b\big(g(t)\big)&=-\sum_{j\neq i}\big\langle\partial_x^\alpha\bar{r}_j^{(2)},\partial_i\partial_x^\alpha b_i\big\rangle+\sum_{j\neq i}\big\langle\partial_x^\alpha\bar{r}_{ji}^{(2)},\partial_j\partial_x^\alpha b_i\big\rangle+2\big\langle\partial_x^\alpha\bar{r}_i^{(2)},\partial_i\partial_x^\alpha b_i\big\rangle,\\
\mathcal{G}_{\alpha,i}^c\big(g(t)\big)&=\big\langle\partial_x^\alpha\bar{r}_i^{(3)},\partial_i\partial_x^\alpha c\big\rangle,\\
\mathcal{G}_{\alpha,i}^{ab}\big(g(t)\big)&=\big\langle\partial_x^\alpha b_i,\partial_i\partial_x^\alpha a\big\rangle.
\end{aligned}$$

$\mathcal{G}_{\alpha,i}^a(g(t))$, $\mathcal{G}_{\alpha,i}^b(g(t))$, and $\mathcal{G}_{\alpha,i}^c(g(t))$ stand for the interactive energy functionals between the microscopic part g_2 with the macroscopic part a, b and c separately, while $\mathcal{G}_{\alpha,i}^{ab}(g(t))$ stands for the interactive energy functional between a and b.

The proof of Lemma 3.8 is similar to the proof of Lemma 3.2 in [15], the only difference lies in Lemma 3.4.

Step 4. Now we define

$$\begin{aligned}
\mathcal{E}_{N,l}={}&M_3\bigg[M_2\bigg\{\frac{M_1}{2}\bigg[\sum_{|\alpha|\leqslant N}\big\|\partial^\alpha g\big\|^2\bigg]+\mathcal{G}\big(g(t)\big)\bigg\}+\|\omega_{-l}g_2\|^2+\sum_{1\leqslant|\alpha|\leqslant N}\big\|\omega_{-l}\partial^\alpha g_2\big\|^2\bigg]\\
&+C_m\sum_{|\alpha|+|\beta|\leqslant N,|\beta|=m}\big\|\omega_{|\beta|-l}(t,\xi)\partial_\beta^\alpha g_2\big\|^2.
\end{aligned}\tag{3.60}$$

Here $M_i>0$, $i=1,2,3$ are suitably large. Then, taking a proper linear combination of those estimates in the previous three steps we can prove

$$\frac{d}{dt}\mathcal{E}_{N,l}+\kappa\mathcal{D}_{N,l}\leqslant 0. \tag{3.61}$$

This completes the proof of Proposition 3.1. □

3.4. Global existence and optimal convergence rate

To close the energy estimates under the priori assumption (3.38), one has to obtain the time-decay of $\mathcal{E}_{N,l-1}$. The following lemma is crucial in this direction.

Lemma 3.9. *Assume $-2\leqslant\gamma<0$. Fix parameters N,l_0,l, θ and λ as stated in Theorem 3.1. Suppose that the priori assumption holds true for $\delta>0$ small enough. Then, one has*

$$X_{N,l}(t)\leqslant C\big\{\epsilon_{N,l}^2+X_{N,l}^2(t)\big\}, \tag{3.62}$$

for any $t \geqslant 0$, *where* $\epsilon_{N,l}$ *is defined by*

$$\epsilon_{N,l} = \sum_{|\alpha|+|\beta|\leqslant N} \left\|\omega_{|\beta|-l}(0,\xi)\partial_\beta^\alpha g_0\right\| + \left\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}} g_0\right\|_{Z_1}. \tag{3.63}$$

As preparation, we need to prove the following lemma firstly.

Lemma 3.10. *Under the assumptions of Lemma 3.9, one has*

$$\left\|g(t)\right\| \leqslant C(1+t)^{-\frac{3}{4}}\left\{\epsilon_{N,l} + X_{N,l}(t)\right\}, \tag{3.64}$$

and

$$\left\|\nabla_x g(t)\right\|_{L^2_\xi(H^{N-2}_x)} \leqslant C(1+t)^{-\frac{5}{4}}\left\{\epsilon_{N,l} + X_{N,l}(t)\right\}, \tag{3.65}$$

for any $0 \leqslant t \leqslant T$.

Proof. By the Duhamel principle, the solution g to the Cauchy problem can be written as the mild form

$$g(t) = e^{tB} g_0 + \int_0^t e^{(t-s)B} G(s)\, ds, \tag{3.66}$$

where $G(s) = \Gamma(g,g) + u\cdot\nabla_\xi g - \frac{1}{2}u\cdot\xi g + (b-u)\cdot\xi\sqrt{M}$. Using Lemma 3.5, one has

$$\begin{aligned}
\left\|g(t)\right\| &\leqslant \left\|e^{tB}g_0\right\| + \left\|\int_0^t e^{(t-s)B}G(s)\,ds\right\| \\
&\leqslant C(1+t)^{-\frac{3}{4}}\left\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}g_0\right\|_{Z_1} + \int_0^t (1+t-s)^{-\frac{3}{4}}\left\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}G(s)\right\|_{Z_1}ds, \\
\left\|\nabla_x g(t)\right\|_{L^2_\xi(H^{N-2}_x)} &\leqslant C(1+t)^{-\frac{5}{4}}\left(\left\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}g_0\right\|_{Z_1} + \left\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}\nabla_x g_0\right\|_{L^2_\xi(H^{N-2}_x)}\right) \\
&\quad + \int_0^t (1+t-s)^{-\frac{5}{4}}\left(\left\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}G(s)\right\|_{Z_1} + \left\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}\nabla_x G(s)\right\|_{L^2_\xi(H^{N-2}_x)}\right)ds.
\end{aligned}$$

We claim that

$$\left\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}G(t)\right\|_{Z_1} + \sum_{|\alpha|\leqslant N-1}\left\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}\partial^\alpha G(t)\right\| \leqslant C\mathcal{E}_{N,l-1}(t), \tag{3.67}$$

for any $0 \leqslant t < T$, $l \geqslant 1 + \max\{N, \frac{l_0}{2} - \frac{1}{\gamma}\}$.

The term containing $\Gamma(g,g)$ can be bounded by $C\mathcal{E}_{N,l-1}(t)$, this can be found in [6]. We only give the estimate on terms containing u. Direct calculations yield

$$\begin{aligned}
&\left\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}\left(u\cdot\nabla_\xi g - \frac{1}{2}u\cdot\xi g + (b-u)\cdot\xi\sqrt{M}\right)\right\|_{Z_1} \\
&\quad\leqslant C\left(\left\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}\|u\|_{L^2_x}\left\{\|\nabla_\xi g\|_{L^2_x} + \langle\xi\rangle\|g\|_{L^2_x}\right\}\right\|_{L^2_\xi} + \|b\|\|a+3c\|\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\big(\|u\|\big\{\big\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}\nabla_\xi g\big\|+\big\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}+1}g\big\|\big\}+\|b\|\|a+3c\|\big)\\
&\leqslant C\big(\|b\|\big\{\|\omega_{-\frac{l_0}{2}}\nabla_\xi g\|+\|\omega_{-\frac{l_0}{2}+\frac{1}{\gamma}}g\|\big\}+\|b\|\|a+3c\|\big)\\
&\leqslant C\mathcal{E}_{N,l-1}(t),
\end{aligned}
$$

where we have used the assumption $l-1>\frac{l_0}{2}-\frac{1}{\gamma}$. By applying L^∞_x-norm to the lower-order derivative term and using Sobolev's inequality, one has

$$
\begin{aligned}
&\sum_{|\alpha|\leqslant N-1}\left\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}\partial^\alpha\left(u\cdot\nabla_\xi g-\frac{1}{2}u\cdot\xi g+(b-u)\cdot\xi\sqrt{M}\right)\right\|\\
&\quad\leqslant C\big\|(a,b,c)\big\|_{H^N}\sum_{|\alpha|+|\beta|\leqslant N}\big\|\omega_{|\beta|-(l-1)}(t,\xi)\partial^\alpha_\beta g\big\|+C\|b-u\|_{H^{N-1}}\leqslant C\mathcal{E}_{N,l-1}(t).
\end{aligned}
\tag{3.68}
$$

Now we can conclude

$$
\begin{aligned}
&\big\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}G(s)\big\|_{Z_1}+\sum_{|\alpha|\leqslant N-1}\big\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}\partial^\alpha G(s)\big\|\\
&\quad\leqslant C\mathcal{E}_{N,l-1}(s)\leqslant C(1+s)^{-\frac{3}{2}}\sup_{0\leqslant s\leqslant t}(1+s)^{\frac{3}{2}}\mathcal{E}_{N,l-1}(s)\leqslant C(1+s)^{-\frac{3}{2}}X_{N,l}(t).
\end{aligned}
$$

Hence we obtain

$$
\big\|g(t)\big\|\leqslant C(1+t)^{-\frac{3}{4}}\big\{\big\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}g_0\big\|_{Z_1}+X_{N,l}(t)\big\},
\tag{3.69}
$$

and

$$
\big\|\nabla_x g(t)\big\|_{L^2_\xi(H^{N-2}_x)}\leqslant C(1+t)^{-\frac{5}{4}}\big\{\big\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}g_0\big\|_{Z_1}+\big\|\langle\xi\rangle^{-\frac{\gamma l_0}{2}}\nabla_x g_0\big\|_{L^2_\xi(H^{N-2}_x)}+X_{N,l}(t)\big\}.
\tag{3.70}
$$

This completes the proof of Lemma 3.10. □

Proof of Lemma 3.9. From Proposition 3.1, we know

$$
\sup_{0\leqslant s\leqslant t}\mathcal{E}_{N,l}(s)\leqslant\mathcal{E}_{N,l}(0)\leqslant C\epsilon^2_{N,l}.
\tag{3.71}
$$

Take $0<\epsilon<\frac{1}{2}$ small enough. Notice that (3.39) also holds true when l is replaced by $l-1$ since all the conditions of Proposition 3.1 are still satisfied under the assumption that $l\geqslant N+1$ and $\sup_{0\leqslant s<T}X_{N,l}(s)\leqslant\delta$ with $\delta>0$ small enough. Thus, it holds that

$$
\frac{d}{dt}\mathcal{E}_{N,l-1}+\kappa\mathcal{D}_{N,l-1}\leqslant 0.
\tag{3.72}
$$

Multiplying the above inequality by $(1+t)^{\frac{3}{2}+\epsilon}$ gives

$$
\frac{d}{dt}\big[(1+t)^{\frac{3}{2}+\epsilon}\mathcal{E}_{N,l-1}\big]+\kappa(1+t)^{\frac{3}{2}+\epsilon}\mathcal{D}_{N,l-1}\leqslant\left(\frac{3}{2}+\epsilon\right)(1+t)^{\frac{1}{2}+\epsilon}\mathcal{E}_{N,l-1}.
\tag{3.73}
$$

Similarly, we have

$$
\begin{aligned}
\frac{d}{dt}\big[(1+t)^{\frac{1}{2}+\epsilon}\mathcal{E}_{N,l-\frac{1}{2}}\big]+\kappa(1+t)^{\frac{1}{2}+\epsilon}\mathcal{D}_{N,l-\frac{1}{2}}&\leqslant\left(\frac{1}{2}+\epsilon\right)(1+t)^{-\frac{1}{2}+\epsilon}\mathcal{E}_{N,l-\frac{1}{2}}\\
&\leqslant C\mathcal{E}_{N,l-\frac{1}{2}}.
\end{aligned}
\tag{3.74}
$$

For any given $\tilde{l}$

$$\mathcal{D}_{N,\tilde{l}}(t) + \left\|(a,c)\right\|^2 \geqslant \kappa \mathcal{E}_{N,\tilde{l}-\frac{1}{2}}(t). \tag{3.75}$$

Then, from taking the time integration over $[0,t]$ of (3.74), (3.73) and (3.39) and further taking the appropriate linear combination one has

$$\begin{aligned}(1+t)^{\frac{3}{2}+\epsilon}\mathcal{E}_{N,l-1}(t) &\leqslant C\mathcal{E}_{N,l}(0) + C\int_0^t (1+s)^{\frac{1}{2}+\epsilon}\left\|(a,c)\right\|^2 ds \\ &\leqslant C\mathcal{E}_{N,l}(0) + C(1+t)^{\epsilon}\left(\epsilon_{N,l}^2 + X_{N,l}^2(t)\right),\end{aligned} \tag{3.76}$$

which implies

$$\sup_{1\leqslant s\leqslant t}(1+s)^{\frac{3}{2}}\mathcal{E}_{N,l-1}(s) \leqslant C\left(\epsilon_{N,l}^2 + X_{N,l}^2(t)\right). \tag{3.77}$$

This completes the proof of this lemma. □

Proof of Theorem 3.1. In fact, by the continuity argument, Lemma 3.9 implies that under the priori assumption (3.38) for $\delta > 0$ small enough, one has

$$X_{N,\ell}(t) \leqslant C\epsilon_{N,\ell}^2, \quad 0 \leqslant t < T, \tag{3.78}$$

provided that $\epsilon_{N,\ell}$ defined by (3.9) is sufficiently small. Recalling the condition (3.6) for initial data g_0 which coincides with (3.9), the priori assumption (3.38) can be closed. Then, the global existence follows, and $X_{N,l} \leqslant C\epsilon_0$ holds true from (3.2), (3.3), (3.5) and (3.78). This completes the proof of Theorem 3.1. □

4. The hard potential case

In this section we give the main results for the hard potential case. Inspired by [5], we introduce a mixed time–velocity weight function

$$\tilde{w}_\ell(t,\xi) = \langle\xi\rangle^{\frac{\ell}{2}} e^{\frac{\lambda|\xi|}{(1+t)^\theta}}, \tag{4.1}$$

where $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ and $\ell \in \mathbb{R}$, $\lambda > 0, \theta > 0$ are suitably chosen constants.

Define the energy functionals as

$$\mathscr{E}_{N,\ell}(t) \sim |||g|||_{N,\ell}(t) = \sum_{|\alpha|+|\beta|\leqslant N}\left\|\tilde{\omega}_\ell(t,\xi)\partial_\beta^\alpha g(t)\right\|,$$

$$\mathscr{D}_{N,\ell}(t) = \sum_{|\alpha|+|\beta|\leqslant N}\left\|\nu^{\frac{1}{2}}(\xi)\tilde{\omega}_\ell(t,\xi)\partial_\beta^\alpha g_2\right\|^2 + \|b\|^2 + \sum_{|\alpha|\leqslant N-1}\left\|\nabla_x\partial^\alpha(a,b,c)\right\|^2,$$

and

$$\mathscr{X}_{N,\ell}(t) = \sup_{0\leqslant s\leqslant t}(1+t)^{\frac{3}{2}}\mathscr{E}_{N,\ell}. \tag{4.2}$$

We make the a priori assumption:

$$\sup_{0\leqslant t\leqslant T} \mathscr{X}_{N,\ell}(t) \leqslant \delta, \tag{4.3}$$

where $\delta > 0$ is a sufficiently small positive constant. Then, we have

Theorem 4.1. *Let $0 \leqslant \gamma < 1$, $N \geqslant 4$, $\ell \geqslant 2$ and $0 < \theta \leqslant \frac{1}{4}$. Assume that $f_0 = M + M^{\frac{1}{2}} g_0$. There are constants $\epsilon_0 > 0$, $C_0 > 0$ such that if*

$$\sum_{|\alpha|+|\beta|\leqslant N} \left\| \tilde{\omega}_\ell(0,\xi)\partial_\beta^\alpha g_0 \right\| + \|g_0\|_{Z_1} \leqslant \epsilon_0, \tag{4.4}$$

then the Cauchy problem (1.5), (1.7) admits a unique global solution $g(t,x,\xi)$ satisfying $f(t,x,\xi) = M + M^{\frac{1}{2}} g(t,x,\xi)$ and

$$\sup_{t\geqslant 0}\left\{(1+t)^{\frac{3}{4}} |||g|||_{N,\ell}(t)\right\} \leqslant C\epsilon_0. \tag{4.5}$$

In order to prove Theorem 4.1, we only need to prove the following lemma which is similar to Lemma 3.9 in the last section:

Lemma 4.1. *Assume $0 \leqslant \gamma < 1$. Fix parameters N, ℓ as stated in Theorem 4.1. Suppose that the priori assumption (4.3) holds true for $\delta > 0$ small enough. Then, one has*

$$\mathscr{X}_{N,\ell}(t) \leqslant C\left\{\epsilon_{N,\ell}^2 + \mathscr{X}_{N,\ell}^2(t)\right\}, \tag{4.6}$$

for any $t \geqslant 0$, where $\epsilon_{N,\ell}$ is defined by

$$\epsilon_{N,\ell} = \sum_{|\alpha|+|\beta|\leqslant N} \left\| \tilde{\omega}_\ell(0,\xi)\partial_\beta^\alpha g_0 \right\| + \|g_0\|_{Z_1}. \tag{4.7}$$

Remark 4.1. The proof of Lemma 4.1 and Theorem 4.1 is very similar to the soft potential case, only much simpler. We need the time decay result, the non-weight energy estimates, the weighted energy estimates and the macroscopic dissipation estimates to close the a priori estimates. For the hard potential case, we can directly adopt some of the estimates in our previous work [15] on the hard sphere model, such as the time decay result, the macroscopic estimation. While for the weighted part, the estimation relies on the weighted estimates on the integral operator K and the nonlinear collision term $\Gamma(f,g)$ with respect to the weight function $\tilde{w}_\ell(t,\xi)$ defined in (4.1). In order to prevent the duplication, we do not give the details here.

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References

[1] C. Cercignani, R. Illner, M. Pulvirenti, The Mathematical Theory of Dilute Gases, Appl. Math. Sci., vol. 106, Springer-Verlag, New York, 1994.
[2] R.-J. Duan, Some mathematical theories on the gas motion under the influence of external forcing, PhD thesis, CityU of Hong Kong, 2008.
[3] R.-J. Duan, R.-M. Strain, Optimal time decay of the Vlasov–Poisson–Boltzmann system in $\mathbb{R}^3$, Arch. Ration. Mech. Anal. 199 (2010) 291–328.
[4] R.-J. Duan, S. Ukai, T. Yang, A combination of energy method and spectral analysis for studies on systems for gas motions, Front. Math. China 4 (2) (2009) 253–282.
[5] R.-J. Duan, T. Yang, H.-J. Zhao, The Vlasov–Poisson–Boltzmann system in the whole space: the hard potential case, preprint.

[6] R.-J. Duan, T. Yang, H.-J. Zhao, The Vlasov–Poisson–Boltzmann system for soft potentials, preprint.
[7] Y. Guo, Classical solutions to the Boltzmann equation for molecules with an angular cutoff, Arch. Ration. Mech. Anal. 169 (2003) 305–353.
[8] Y. Guo, The Boltzmann equation in the whole space, Indiana Univ. Math. J. 53 (2004) 1081–1094.
[9] S. Kawashima, Systems of a hyperbolic–parabolic composite type, with applications to the equations of magnetohydrodynamics, thesis, Kyoto University, 1983.
[10] S. Kawashima, The Boltzmann equation and thirteen moments, Japan J. Appl. Math. 7 (1990) 301–320.
[11] T.-P. Liu, T. Yang, S.-H. Yu, Energy method for the Boltzmann equation, Phys. D 188 (3–4) (2004) 178–192.
[12] T.-P. Liu, S.-H. Yu, Boltzmann equation: micro–macro decompositions and positivity of shock profiles, Comm. Math. Phys. 246 (1) (2004) 133–179.
[13] R.-M. Strain, Y. Guo, Exponential decay for soft potentials near Maxwellian, Arch. Ration. Mech. Anal. 187 (2008) 287–339.
[14] S.-W. Vong, The Boltzmann equation with frictional force, J. Differential Equations 222 (2006) 95–136.
[15] Y. Wang, Z.-H. Jiang, Optimal time decay of the Boltzmann equation with frictional force, J. Math. Anal. Appl. 374 (2) (2011) 499–515.