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Rearranging series of vectors on a small set

Paweł Klinga

Institute of Mathematics, University of Gdańsk, Wita Stwosza 57, 80-952 Gdańsk, Poland

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ABSTRACT

We consider a variation of the Lévy–Steinitz theorem where one rearranges only a small number of terms. In particular, we study the set of obtainable rearrangements of a multidimensional series by a permutation $\sigma : \omega \to \omega$ such that $\{n \in \omega : \sigma(n) \neq n\} \in \mathcal{I}$ for some proper ideal $\mathcal{I} \subset \mathcal{P}(\omega)$.

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1. Introduction

The classic theorem of Riemann [5] states that for any conditionally convergent series of reals $\sum_{n\in\omega} a_n$ and any real number a there exists a permutation of terms σ such that $\sum_{n\in\omega} a_{\sigma(n)} = a$.

In [8] Wilczyński strengthened the result, adding that we can take the permutation such that $\{n \in \omega : \sigma(n) \neq n\} \in \mathcal{I}_d$, where $\mathcal{I}_d$ denotes the ideal of sets of asymptotic density zero. One might say that this means that the permutation changes only a small number of terms. In the same paper Wilczyński asked what other ideals can be put in the place of $\mathcal{I}_d$. This question was answered by Filipów and Szuca in [1], where they proved that those ideals are exactly the ones which cannot be extended to a summable ideal.

Riemann's theorem has been generalized to a multidimensional case by Lévy [4] and Steinitz [7].

In this paper we consider an approach similar to the one-dimensional case, i.e. we study the sets of those points in a multidimensional Euclidean space which are obtained by rearranging a small amount of the series' terms. While the original Lévy–Steinitz theorem works for $\mathbb{R}^m$, where m is any positive natural, we decided to focus solely on the case $m = 2$ (i.e. the plane). We divided this into two parts – in Section 3 we work on a situation where the set of obtainable sums is a line on the plane, while in Section 4 it is the entire plane (which turned out to be a much more delicate counterpart).

E-mail address: pklinga@mat.ug.edu.pl.

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2. Preliminaries

By ω we denote the set of natural numbers, and by $\mathcal{P}(\omega)$ its powerset. An ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ is a family which is closed under taking subsets and finite unions. If $\omega \notin \mathcal{I}$ then an ideal is proper. In this paper we will assume that all ideals are proper.

We will use the following notation: $a^+ = \max\{a, 0\}$, $a^- = \min\{a, 0\}$.

Definition 2.1. Let $(v_n)_{n\in\omega}$ be a sequence in $\mathbb{R}^m$, $m \in \{1, 2, 3, ...\}$.

$$S\left(\sum_{n\in\omega} v_n\right) = \left\{v \in \mathbb{R}^m : \exists_{\sigma:\omega\to\omega}\ \sigma - \text{permutation},\ \sum_{n\in\omega} v_{\sigma(n)} = v\right\}.$$

Using this notation we can formulate Riemann's derangement theorem in the following way.

Theorem 2.2. *(See [5].) Let $\sum_{n\in\omega} a_n$ be a conditionally convergent series of reals. Then*

$$S\left(\sum_{n\in\omega} a_n\right) = \mathbb{R}.$$

By *supp*(σ) we'll denote the support of the permutation σ, that is the set $\{n \in \omega : \sigma(n) \neq n\}$.

Definition 2.3. Let $(v_n)_{n\in\omega}$ be a sequence in $\mathbb{R}^m$, $m \in \{1, 2, 3, ...\}$. Also let $\mathcal{I} \subset \mathcal{P}(\omega)$ be an ideal.

$$S_{\mathcal{I}}\left(\sum_{n\in\omega} v_n\right) = \left\{v \in \mathbb{R}^m : \exists_{\sigma:\omega\to\omega}\ \sigma - \text{permutation},\ \sum_{n\in\omega} v_{\sigma(n)} = v, \mathit{supp}(\sigma) \in \mathcal{I}\right\}.$$

Recall the definition of the ideal of sets of asymptotic density zero.

$$\mathcal{I}_d = \left\{A \subset \omega : \limsup_{n\to\infty} \frac{|\{0, 1, \ldots, n-1\} \cap A|}{n} = 0\right\}.$$

Theorem 2.4. *(See [8].) Let $\sum_{n\in\omega} a_n$ be a conditionally convergent series of reals. Then*

$$S_{\mathcal{I}_d}\left(\sum_{n\in\omega} a_n\right) = \mathbb{R}.$$

We will say that an ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ has the (W) property if for any conditionally convergent series of reals $\sum_{n\in\omega} a_n$ there exists $W \in \mathcal{I}$ such that $\sum_{n\in W} a_n$ is also conditionally convergent. Additionally, we will say that an ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ has the (R) property if for any conditionally convergent series of reals $\sum_{n\in\omega} a_n$ and any $a \in \mathbb{R}$ there exists a permutation $\sigma : \omega \to \omega$ such that $\sum_{n\in\omega} a_{\sigma(n)} = a$ and $\mathit{supp}(\sigma) \in \mathcal{I}$.

Hence we can express Theorem 2.4 by saying that $\mathcal{I}_d$ has the (R) property. In [8] Wilczyński asked to find a full characterization of ideals fulfilling this property. This question was answered by Filipów and Szuca [1].

First, recall the definition of the summable ideals. Let $(a_n)_{n\in\omega}$ be such sequence of reals that $a_n \geqslant 0$, $n \in \omega$, $a_n \to 0$ and $\sum_{n\in\omega} a_n = +\infty$. The family

$$\mathcal{I}_{a_n} = \left\{A \subset \omega : \sum_{n\in A} a_n < \infty\right\}$$

is an ideal which we call the summable ideal (determined by the sequence a_n).

Theorem 2.5. *(See [1].) Let $\mathcal{I} \subset \mathcal{P}(\omega)$ be an ideal. The following are equivalent.*

(1) *$\mathcal{I}$ has the (R) property.*
(2) *$\mathcal{I}$ cannot be extended to a summable ideal.*
(3) *$\mathcal{I}$ has the (W) property.*

This theorem closed the case of series of reals. However, the following theorem has been proved by Lévy and Steinitz for series of vectors.

Theorem 2.6. *(See [6].) Let $(v_n)_{n\in\omega}$ be a sequence in $\mathbb{R}^m$, $m \in \{1, 2, 3, ...\}$. The set $S(\sum_{n\in\omega} v_n)$ is either the empty set or a translate of a subspace (i.e., a set of the form $v + L$, where v is a given vector and L is a linear subspace).*

For instance, in the case of $\mathbb{R}^2$, depending on the series, the set of obtainable sums can be an empty set, a point on the plane, a line on the plane or the entire plane. The form of such set can be expressed explicitly using the following theorems.

Lemma 2.7. *(See [3].) Let $(v_n)_{n\in\omega} \subset \mathbb{R}^m$ and let $F = \{w \in \mathbb{R}^m : \sum_{n\in\omega} \langle w, v_n\rangle^+ < \infty\}$, where $\langle , \rangle$ denotes the real inner product. If $S(\sum_{n\in\omega} v_n)$ is nonempty then:*

(P_1) $\lim_{n\to\infty} v_n = 0$,
(P_2) $\forall_{w\in F} \ -w \in F$.

Theorem 2.8. *(See [3].) If conditions (P_1) and (P_2) hold, then $S(\sum_{n\in\omega} v_n)$ is nonempty and*

$$\forall_{s_0\in S(\sum_{n\in\omega} v_n)} \quad S\left(\sum_{n\in\omega} v_n\right) = s_0 + F^{\perp},$$

where $F^{\perp}$ is the orthogonal complement of F, that is

$$F^{\perp} = \left\{v \in \mathbb{R}^m : \ \forall_{w\in F} \ \langle v, w\rangle = 0\right\}.$$

Additionally, we'll be using the following facts later on.

Claim 2.9. *(See [3].) Let $(v_n)_{n\in\omega} \subset \mathbb{R}^m$ be such that $\lim_{n\to\infty} v_n = 0$. Let $v_n = v'_n + v''_n$ be an orthogonal decomposition such that $v'_n \in F$, $v''_n \in F^{\perp}$. Then the series $\sum_{n\in\omega} v'_n$ is absolutely convergent.*

Theorem 2.10. *(See [3].) Suppose that:*

(1) $\lim_{n\to\infty} v_n = 0$,
(2) $\forall_{w\neq 0} \ \sum_{n\in\omega} \langle w, v_n\rangle^+ = \infty$.

Then

$$S\left(\sum_{n\in\omega} v_n\right) = R^m.$$

The converse theorem appears in [6], i.e. if the set of obtainable sums is the entire space, then conditions (1)–(2) hold.

3. One-dimensional case

Theorem 3.1. *Let $\mathcal{I} \subset \mathcal{P}(\omega)$ be an ideal which has the (R) property. Suppose that $l \subset \mathbb{R}^2$ is such line on the plane and $\sum_{n\in\omega} v_n$ (where $(v_n)_{n\in\omega} \subset \mathbb{R}^2$) is such series that*

$$S\Big(\sum_{n\in\omega} v_n\Big) = l.$$

Then

$$S_{\mathcal{I}}\Big(\sum_{n\in\omega} v_n\Big) = l.$$

Proof. Let $v_n = (x_n, y_n)$. Without the loss of generality we may assume that the series $\sum_{n\in\omega} x_n$ and $\sum_{n\in\omega} y_n$ are conditionally convergent (in the case of absolute convergence on one of the coordinates, the line l is either horizontal or vertical and the conclusion is immediate).

According to Lemma 2.7 and Theorem 2.8 there exists such vector $w \neq 0$ that

$$\sum_{n\in\omega} \langle v_n, w\rangle^+ < \infty,$$
$$\sum_{n\in\omega} \langle v_n, w\rangle^- < \infty$$

and $l = v + F^\perp$, where $F^\perp$ is like in Lemma 2.7 and $v \in l$.

Fix $c \in l$ and let $v = \alpha w + \beta w^\perp$ be the orthogonal decomposition of v such that $w \in F$, $w^\perp \in F^\perp$, $\alpha, \beta \in \mathbb{R}$. Analogously, $v_n = \alpha_n w + \beta_n w^\perp$, $n \in \omega$ and $c = \alpha_c w + \beta_c w^\perp$ (it's easy to see that since v and c are both on l then $\alpha_c = \alpha$).

Due to the definition of F the series $\sum_{n\in\omega} \alpha_n w$ is absolutely convergent. Therefore we have

$$\sum_{n\in\omega} v_n = \sum_{n\in\omega} \big(\alpha_n w + \beta_n w^\perp\big) = \sum_{n\in\omega} \alpha_n w + \sum_{n\in\omega} \beta_n w^\perp,$$

where only the second element is conditionally convergent. Using the fact that $\mathcal{I}$ has the (R) property, we see that

$$S_{\mathcal{I}}\Big(\sum_{n\in\omega} \beta_n\Big) = \mathbb{R},$$

so there exists a permutation $\sigma : \omega \to \omega$ such that

$$\sum_{n\in\omega} \beta_{\sigma(n)} = \beta_c$$

and $supp(\sigma) \in \mathcal{I}$. Finally, we get

$$\sum_{n\in\omega} v_{\sigma(n)} = w \sum_{n\in\omega} \alpha_{\sigma(n)} + w^\perp \sum_{n\in\omega} \beta_{\sigma(n)} = w \sum_{n\in\omega} \alpha_n + w^\perp \beta_c = \alpha w + \beta_c w^\perp = c,$$

and hence

$$S_{\mathcal{I}}\Big(\sum_{n\in\omega} v_n\Big) = l. \qquad \square$$

4. Two-dimensional case

Theorem 4.1. *Let $(v_n)_{n\in\omega} \subset \mathbb{R}^2$ be a sequence such that $\lim_{n\to\infty} v_n = 0$. Let P be the set of partial sums of $\sum_{n\in\omega} v_n$, i.e.*

$$P = \left\{ \sum_{n\in F} v_n : F \subset \omega, |F| < \omega \right\}.$$

The following are equivalent.

(1) $S(\sum_{n\in\omega} v_n) = \mathbb{R}^2$,
(2) P *is dense in* $\mathbb{R}^2$.

Proof. (1) $\Rightarrow$ (2) Fix $a \in \mathbb{R}^2$ and $\varepsilon > 0$. There exists a permutation $\sigma : \omega \to \omega$ such that $\sum_{n\in\omega} v_{\sigma(n)} = a$. There exists $n_0 \in \omega$ such that $\sum_{n<n_0} v_{\sigma(n)} \in B(a,\varepsilon)$, where $B(a,\varepsilon) = \{x \in \mathbb{R}^2 : \|x - r\| < \varepsilon\}$. Then for $F = \{\sigma(0), \ldots, \sigma(n_0 - 1)\}$ we have $\sum_{n\in F} v_n \in B(a,\varepsilon)$.

(2) $\Rightarrow$ (1) Fix $M \in \omega$ and $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 > 0$. There exists $B(a,\varepsilon)$ such that if $(x,y) \in B(a,\varepsilon)$ then $\alpha x + \beta y > M$. Due to the density of P there exists $F \subset \omega$, $|F| < \omega$ such that $\sum_{n\in F} v_n \in B(a,\varepsilon)$, and so $\sum_{n\in F} \alpha x_n + \beta y_n > M$, where $(x_n, y_n) = v_n$. Hence $\sum_{n\in F} (\alpha x_n + \beta y_n)^+ > M$, where $x^+ = \max\{0, x\}$. Since M was arbitrary, we obtain $\sum_{n\in\omega} (\alpha x_n + \beta y_n)^+ = \infty$ which, by Theorem 2.10, implies that $S(\sum_{n\in\omega} v_n) = \mathbb{R}^2$. □

Theorem 4.2. *For any conditionally convergent series of reals $\sum_{n\in\omega} a_n$ there exists a partition A, B of ω such that both series $\sum_{n\in A} a_n$, $\sum_{n\in B} a_n$ are conditionally convergent.*

Proof. Let $\sum_{n\in\omega} a_n$ be a conditionally convergent series of reals. By induction, we can pick natural numbers $N_0 < N_1 < \ldots$ so that the following conditions are met.

(1) $\sum_{n=N_i}^{N_{i+1}-1} |a_n| > 1$;
(2) $\forall_{k,l\in\omega}\ l > k > N_i \Rightarrow |\sum_{n=k}^{l} a_n| < 10^{-i}$;

Indeed, (1) can be obtained using the conditional convergence and (2) using the fact that the series is Cauchy.

Put $A = \bigcup_{i=2k,\, k\in\omega} [N_i, N_{i+1}) \cap \omega$. We will show that the series $\sum_{n\in A} a_n$ is conditionally convergent. It is not absolutely convergent, since (1) holds. We will prove that it is Cauchy. Fix $\varepsilon > 0$. There is i such that $\sum_{n=i}^{\infty} 10^{-n} < \varepsilon$. For $l > k \geqslant N_{i+1}$ we have

$$\left| \sum_{n=k, n\in A}^{l} a_n \right| \leqslant 10^{-i} + 10^{-(i+1)} + \ldots + 10^{-j} < \varepsilon$$

for some $j \in \omega$.

A similar argument works for $B = \omega \setminus A$. □

Notice that one can easily obtain (by induction) an infinite partition $\{A_k\}_{k\in\omega}$ such that each series $\sum_{n\in A_k} a_n$ is conditionally convergent. This strengthens the result of Grahl and Nevo (see [2, Lemma 4]).

Theorem 4.3. *Let $(v_n)_{n\in\omega} \subset \mathbb{R}^2$ be such that $S(\sum_{n\in\omega} v_n) = \mathbb{R}^2$ and $\sum v_n$ is conditionally convergent. There exists a partition A, B of ω such that both series $\sum_{n\in A} v_n$, $\sum_{n\in B} v_n$ are convergent and $S(\sum_{n\in A} v_n) = S(\sum_{n\in B} v_n) = \mathbb{R}^2$.*

Proof. Let $v_n = (x_n, y_n)$. We will show that there exists $A \subset \omega$ such that for any reals a, b such that $a^2 + b^2 > 0$, the series $\sum_{n\in A} ax_n + by_n$ is conditionally convergent. According to Theorem 2.10, this will be sufficient. For B we will again put $\omega \setminus A$.

Similarly to the previous proof, one can choose inductively natural numbers $N_0, N_1, \dots$ so that putting $A = \bigcup_{i=2k,\, k\in\omega}[N_i, N_{i+1}) \cap \omega$ will finish the proof. We will show how to do this.

(1) For any $(a,b) \in [-i,i]^2 \setminus (-\frac{1}{i+1}, \frac{1}{i+1})^2$ there exists $N_{i+1}^{a,b}$ such that $\sum_{n=N_i}^{N_{i+1}^{a,b}} |ax_n + by_n| > 1$ (it follows from the fact that $\sum_{n\in\omega} ax_n + by_n$ is conditionally convergent). There exists an open neighbourhood $W_{a,b}$ of (a,b) such that for any $(a',b') \in W_{a,b}$ we have $\sum_{n=N_i}^{N_{i+1}^{a,b}} |a'x_n + b'y_n| > 1$. Since $[-i,i]^2 \setminus (-\frac{1}{i+1}, \frac{1}{i+1})^2$ is compact, it is possible to choose finitely many such neighbourhoods which suffice to cover it. Then put $N_{i+1} = \max N_{i+1}^{a,b}$. Therefore $\sum_{n=N_i}^{N_{i+1}} |ax_n + by_n| > 1$ for $(a,b) \in [-i,i]^2 \setminus (-\frac{1}{i+1}, \frac{1}{i+1})^2$.

(2) Increase N_i sufficiently enough, so we can have

$$\forall_{k,l\in\omega} \quad l > k > N_i \quad \Rightarrow \quad \left|\sum_{n=k}^{l} x_n\right| < 10^{-i}, \quad \left|\sum_{n=k}^{l} y_n\right| < 10^{-i}.$$

Then

$$\left|\sum_{n=k}^{l} ax_n + by_n\right| \leqslant 10^{-i}\big(|a| + |b|\big).$$

Like before, condition (1) guarantees that the series is not absolutely convergent and (2) assures it is Cauchy. □

Recall that $\mathcal{I}$ is a maximal ideal, if it's dual to some ultrafilter. Moreover, for any $A \subset \omega$ either A belongs to such ideal, or its complement does.

Corollary 4.4. *If $\mathcal{I} \subset \mathcal{P}(\omega)$ is a maximal ideal, then for every conditionally convergent $\sum_{n\in\omega} v_n$, $(v_n)_{n\in\omega} \subset \mathbb{R}^2$, such that $S(\sum_{n\in\omega} v_n) = \mathbb{R}^2$, there exists $W \in \mathcal{I}$ such that $\sum_{n\in W} v_n$ is conditionally convergent and $S(\sum_{n\in W} v_n) = \mathbb{R}^2$.*

While the two previous theorems indicate a strong correspondence between the nature of series of reals and those of vectors, below we will show that their behaviour is not necessarily symmetrical.

Notice that if a series $\sum_{n\in\omega} a_n$ of reals is such that $a_n \to 0$ and $\sum_{n\in\omega} a_n^+ = \infty$ and $\sum_{n\in\omega} a_n^- = -\infty$, it doesn't have to be conditionally convergent (as an example, put $\frac{(-1)^{n+1}}{n+1}$ for even n and $\frac{(-1)^{n+1}}{\sqrt{n+1}}$ for odd). However, in such situation, you can always find a subsequence to obtain the desired conditional convergence. An analogous hypothesis for vectors does not hold, i.e. it is not true that if x_n, y_n are sequences of reals such that for any $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 > 0$, we have $\sum_{n\in\omega}(\alpha x_n + \beta y_n)^+ = \infty$, there exists a subsequence $(n_k)_{k\in\omega}$ such that for any $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 > 0$, we still have $\sum_{k\in\omega}(\alpha x_{n_k} + \beta y_{n_k})^+ = \infty$ and $\sum_{k\in\omega}(x_{n_k}, y_{n_k})$ is convergent.

To construct a counterexample, we will take $x_n = \frac{(-1)^{n+1}}{n+1}$, $y_n = \frac{(-1)^{n+1}}{\sqrt{n+1}}$ and rearrange it in a certain way. For $n \in \omega$ define sets

$$A_n = \big\{k \in \omega : 2^n \leqslant k < 2^{n+1}, k - \text{even}\big\},$$
$$B_n = \big\{k \in \omega : 2^n \leqslant k < 2^{n+1}, k - \text{odd}\big\}.$$

Denote their elements in the following way: $A_0 = \{a_1^{(0)}, \dots, a_{r_0}^{(0)}\}$, $B_0 = \{b_1^{(0)}, \dots, b_{q_0}^{(0)}\}$, and so on. Define the following permutation $\sigma : \omega \to \omega$.

$$\begin{pmatrix} 1 & 2 & \dots & r_0 & r_0+1 & \dots & r_0+q_0 & r_0+q_0+1 & \dots \\ a_1^{(0)} & a_2^{(0)} & \dots & a_{r_0}^{(0)} & b_1^{(0)} & \dots & b_{q_0}^{(0)} & a_1^{(1)} & \dots \end{pmatrix}$$

Take $v_n = (x_n, y_n)$. The series $\sum_{n\in\omega} v_{\sigma(n)}$ satisfies the assumption of the hypothesis (i.e. for any $\alpha, \beta \in \mathbb{R}$, $\alpha^2+\beta^2 > 0$, we have $\sum_{n\in\omega}(\alpha x_{\sigma(n)} + \beta y_{\sigma(n)})^+ = \infty$), as $\sum_{n\in\omega} v_n$ did. Assume there exists a subsequence $(n_k)_{k\in\omega}$ such that for any $\alpha, \beta \in \mathbb{R}$, $\alpha^2+\beta^2 > 0$, we have $\sum_{k\in\omega}(\alpha x_{\sigma(n_k)} + \beta y_{\sigma(n_k)})^+ = \infty$ and $\sum_{k\in\omega} v_{\sigma(n_k)}$ is convergent. Take $K_n = \{k \in \omega : \sigma(n_k) \in A_n\}$ and $K = \bigcup_{n\in\omega} K_n$. Then $\sum_{k\in K} x_{\sigma(n_k)} = \infty$.

Suppose that there is $n_0 \in \omega$ such that for every $n > n_0$ we have $|K_n| \leqslant (\sqrt{2})^n$. Then for $n > n_0$ we would have

$$\sum_{k\in K_n} x_{\sigma(n_k)} \leqslant \frac{1}{\min A_n}|K_n| \leqslant \frac{1}{2^n}(\sqrt{2})^n = \left(\frac{1}{\sqrt{2}}\right)^n,$$

and therefore

$$\sum_{k\in K} x_{\sigma(n_k)} = \sum_{n\in\omega}\sum_{k\in K_n} x_{\sigma(n_k)} \leqslant \sum_{n=0}^{n_0}\sum_{k\in K_n} x_{\sigma(n_k)} + \sum_{n=n_0+1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n < \infty,$$

which is a contradiction. So there must be infinitely many $n \in \omega$ such that $|K_n| > (\sqrt{2})^n$. For every such n we have

$$\sum_{k\in K_n} y_{\sigma(n_k)} \geqslant \frac{1}{\sqrt{\max A_n}}(\sqrt{2})^n \geqslant \frac{1}{\sqrt{2^{n+1}}}(\sqrt{2})^n = \frac{1}{\sqrt{2}}.$$

As every $\{n_k : k \in K_n\}$ is an interval in $\{n_k : k \in \omega\}$, this shows that $\sum_{k\in\omega} y_{\sigma(n_k)}$ is not Cauchy. This completes the construction of the counterexample.

Lemma 4.5. *Let $x_n \to 0$ be such sequence of reals that $\sum_{n\in\omega} x_n^+ = \infty$ and $\sum_{n\in\omega} x_n^- = -\infty$. Then there exists $y_n \to 0$ such that for all $\alpha^2+\beta^2 > 0$ we have $\sum_{n\in\omega}(\alpha x_n + \beta y_n)^+ = \infty$.*

Proof. Without the loss of generality we may assume that $|x_n| \leqslant 1$ for all $n \in \omega$. Let $(N_k)_{k\in\omega}$ be such that

1. $N_{k+1} > N_k \geqslant k$,
2. $\forall_{n\geqslant N_k}\ |x_n| \leqslant \frac{1}{k}$,
3. $N_1 = 1$.

Put $y_n = \frac{(-1)^n}{\sqrt{k(n)}}$, where $k(n)$ is the unique k such that $n \in [N_k, N_{k+1})$. Obviously $y_n \to 0$. Fix $\alpha^2+\beta^2 > 0$.

Consider the following cases.

1) $\beta > 0$. Then for n even we have $\alpha x_n + \beta y_n = \alpha x_n + \beta\frac{1}{\sqrt{k(n)}}$. But $|\alpha x_n| \leqslant \frac{|\alpha|}{k(n)}$, so for n even and such great that $\frac{2|\alpha|}{\beta} \leqslant \sqrt{k(n)}$ we have $|\alpha x_n| \leqslant \frac{\beta}{2\sqrt{k(n)}}$. Therefore for such n even it is true that $\alpha x_n + \beta y_n > 0$ and also $(\alpha x_n + \beta y_n)^+ = \alpha x_n + \beta y_n \geqslant -\frac{1}{2}\beta y_n + \beta y_n = \frac{1}{2}\beta y_n$. It suffices to show that $\sum_{n\in\omega} y_n^+ = \infty$ and $\sum_{n\in\omega} y_n^- = -\infty$, which we will do later.

2) $\beta < 0$ – analogous for n odd.

3) $\beta = 0$. Then $\alpha \neq 0$ and $\sum_{n\in\omega}(\alpha x_n + \beta y_n)^+ = \sum_{n\in\omega}(\alpha x_n)^+ = \infty$.

We will show that $\sum_{n\in\omega} y_n^+ = \infty$ and $\sum_{n\in\omega} y_n^- = -\infty$. Take n even. According to condition 1 we have $N_n \geqslant n$. Hence $k(n) \leqslant n$. Therefore $y_n = \frac{1}{\sqrt{k(n)}} \geqslant \frac{1}{\sqrt{n}}$ and so $\sum_{n\in\omega} y_n^+ = \infty$.

The same argument works for n odd and $\sum_{n\in\omega} y_n^- = -\infty$. □

The following definition is due to [3].

Definition 4.6. Let $\sum_{n\in\omega} v_n$ be a series of vectors. Vector u is Lévy vector (determined by the series) if $\|u\| = 1$ and for all $\varepsilon > 0$ we have $\sum'_{n\in\omega} \|v_n\| = \infty$, where $\sum'$ denotes the sum of those nonzero v_n for which the angle between u and v_n is less than ε.

Lemma 4.7. *(See [3].) Let $\sum_{n\in\omega} v_n$ be a series of vectors and let u be any of its Lévy vectors. Then we can choose a subsequence of $(v_n)_{n\in\omega}$ of the form $(\alpha_n u + w_n)_{n\in\omega}$ with $\alpha_n > 0$, $n \in \omega$, $\sum_{n\in\omega} \alpha_n = \infty$ and $\sum_{n\in\omega} \|w_n\| < \infty$. Without the loss of generality we may even suppose that the subsequence has the form $(\alpha_n u)_{n\in\omega}$.*

Theorem 4.8. *Let $\mathcal{I} \subset \mathcal{P}(\omega)$ be an ideal. The following are equivalent.*

(1) *Let $(v_n)_{n\in\omega} \subset \mathbb{R}^2$, $v_n = (x_n, y_n) \to 0$ be such sequence that for all $\alpha^2+\beta^2 > 0$ $\sum_{n\in\omega}(\alpha x_n+\beta y_n)^+ = \infty$. Then there exists $W \in \mathcal{I}$ such that for all $\alpha^2 + \beta^2 > 0$ $\sum_{n\in W}(\alpha x_n + \beta y_n)^+ = \infty$.*
(2) *$\mathcal{I}$ cannot be extended to a summable ideal.*

Proof. (1) $\Rightarrow$ (2) Fix $(a_n)_{n\in\omega} \subset \mathbb{R}$ such that $a_n \to 0$, $a_n \geqslant 0$, $n \in \omega$ and $\sum_{n\in\omega} a_n = \infty$. We can construct a sequence $x_n = \pm a_n$ such that $\sum_{n\in\omega} x_n^+ = +\infty$ and $\sum_{n\in\omega} x_n^- = -\infty$. Using Lemma 4.5 we see that there exists $(y_n) \subset \mathbb{R}$, $y_n \to 0$ such that for any $\alpha^2 + \beta^2 > 0$ we have $\sum_{n\in\omega}(\alpha x_n + \beta y_n)^+ = \infty$. Since (1) holds, there exists $W \in \mathcal{I}$ such that for any $\alpha^2 + \beta^2 > 0$ $\sum_{n\in W}(\alpha x_n + \beta y_n)^+ = \infty$. Put $\alpha = 1$, $\beta = 0$. Then $\sum_{n\in W} x_n^+ = \infty$, and hence $\sum_{n\in W} a_n = \infty$.

(2) $\Rightarrow$ (1) Let $(v_n)_{n\in\omega}$ be like in (1). By $Lv(\sum_{n\in\omega} v_n)$ we shall denote its set of Lévy vectors. Since, by Theorem 2.10, $S(\sum_{n\in\omega} v_n) = \mathbb{R}^2$, then every closed semicircle of the unit circle must contain at least one Lévy vector (it follows from [3, 3.2]).

Consider the following cases.

1) For any v in the unit circle there exists $u \in Lv(\sum_{n\in\omega} v_n)$ such that the angle between v and u is less than $\frac{\pi}{2}$. Then there are at least three Lévy vectors and we can pick $u_1, u_2, u_3 \in Lv(\sum_{n\in\omega} v_n)$ such that for any v in the unit circle the angle between v and at least one of such u_i, $i = 1, 2, 3$, is less than $\frac{\pi}{2}$ (in the case of four vectors such as $(\pm 1, 0)$, $(0, \pm 1)$ it is not true, however, this special case would be analogous and the proof would be the same with picking an additional vector u_4). By Lemma 4.7, there is a subsequence $(\alpha_i u_1+w_i)_{i\in\omega} \subset (v_n)_{n\in\omega}$, where $\alpha_i > 0$, $i \in \omega$, $\sum_{i\in\omega} \alpha_i = \infty$, $\sum_{i\in\omega} \|w_i\| < \infty$ and we may assume that there is a subsequence of the form $(\alpha_i u_1)_{i\in\omega}$. Similarly, there are subsequences $(\beta_i u_2)_{i\in\omega}$, $(\gamma_i u_3)_{i\in\omega}$, $\beta_i, \gamma_i > 0$, $i \in \omega$, $\sum_{i\in\omega} \beta_i = \sum_{i\in\omega} \gamma_i = \infty$. Put $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$, $C = \{c_1, c_2, \ldots\}$, where $(a_i)_{i\in\omega}$, $(b_i)_{i\in\omega}$, $(c_i)_{i\in\omega}$ are such increasing sequences of naturals that $v_{a_i} = \alpha_i u_1$, $v_{b_i} = \beta_i u_2$, $v_{c_i} = \gamma_i u_3$ for some α_i, β_i, γ_i. Without the loss of generality, we can assume that A, B, C are pairwise disjoint (as v_n cannot be at the same time of the form αu_1 and βu_2, since u_1, u_2 are different). Put $\alpha_n^* = 0$ for $n \notin A$ and $\alpha_n^* = \alpha_i$ for $n \in A$ (then $n = a_i$ for some i). Then $\sum_{n\in\omega} \alpha_n^* = \infty$, so there is $W_1^* \in \mathcal{I}$ such that $\sum_{n\in W_1^*} \alpha_n^* = \infty$. Obviously we can assume that $W_1^* \subset A$. Analogously we find $W_2^* \subset B$ and $W_3^* \subset C$. Put $W = W_1^* \cup W_2^* \cup W_3^*$ and fix $\alpha^2 + \beta^2 > 0$. For some $i \in \{1, 2, 3\}$ the angle between (α, β) and u_i is less than $\frac{\pi}{2}$ – for simplicity's sake we can assume it's u_1. If $n \in A$, then $n = a_i$ for some $i \in \omega$. Then $v_n = \alpha_i u_1$, hence $\langle(\alpha, \beta), v_n\rangle = \alpha_i\langle(\alpha, \beta), u_1\rangle$. Because $\langle(\alpha, \beta), u_1\rangle > 0$, then $\sum_{n\in W}(\alpha x_n + \beta y_n)^+ = \sum_{n\in W}(\langle(\alpha, \beta), v_n\rangle)^+ = \infty$, which finishes this part of the proof.

2) There are only two Lévy vectors u_1, u_2 and then the angle between them is π – we can assume that $u_1 = (0, -1)$, $u_2 = (0, 1)$. Like in the previous case we have disjoint sets A, B such that $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$, $v_{a_i} = \alpha_i u_1$, $v_{b_i} = \beta_i u_2$ and $W_1^* \in \mathcal{I}$, $W_2^* \in \mathcal{I}$, $W_1^* \subset A$, $W_2^* \subset B$, such that $\sum_{a_i\in W_1^*} \alpha_i = \infty$, $\sum_{b_i\in W_2^*} \beta_i = \infty$. Using the assumption in (1), we have that $\sum_{n\in\omega}(1x_n+0y_n)^+ = \sum_{n\in\omega}(-1x_n+0y_n)^+ = \infty$. Since (2) holds, there exists $W^* \in \mathcal{I}$, $\sum_{n\in W^*} x_n^+ = \sum_{n\in W^*}(-x_n)^+ = \infty$. Put $W = W_1^* \cup W_2^* \cup W^*$ and

fix $\alpha^2 + \beta^2 > 0$. If $\beta \neq 0$ then $\langle (\alpha, \beta), u_i \rangle > 0$ for some $i \in \{1, 2\}$, since u_1, u_2 are vertical. Similarly to the previous case, $\sum_{n \in W} (\alpha x_n + \beta y_n)^+ = \infty$. On the other hand, if $\beta = 0$, then $\alpha \neq 0$. If $\alpha > 0$, then $\sum_{n \in W^*} (\alpha x_n + \beta y_n)^+ = \infty$, and hence $\sum_{n \in W} (\alpha x_n + \beta y_n)^+ = \infty$. Similarly for $\alpha < 0$. □

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