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Necessary condition for compactness of a difference of composition operators on the Dirichlet space

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A B S T R A C T

Let φ be a self-map of the unit disk and let C_φ denote the composition operator acting on the standard Dirichlet space $\mathcal{D}$. A necessary condition for compactness of a difference of two bounded composition operators acting on $\mathcal{D}$ is given. As an application, a characterization of disk automorphisms φ and ψ, for which the commutator $[C_\psi^*, C_\varphi]$ is compact, is given.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$ and let $\mathbb{T} = \{z : |z| = 1\}$ denote the unit circle in $\mathbb{C}$. The Dirichlet space $\mathcal{D}$ is the space of all analytic functions f in $\mathbb{D}$, such that

$$\|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty,$$

where $dA(z) = \pi^{-1} dxdy$ is the normalized two dimensional Lebesgue measure on $\mathbb{D}$. The Dirichlet space is a Hilbert space with inner product

$$\langle f, g\rangle_{\mathcal{D}} := f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} dA(z).$$

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The Dirichlet space has the reproducing kernel property and the kernel function is defined as

$$K_w(z) := 1 + \log \frac{1}{1 - \overline{w}z}, \tag{1.1}$$

where the branch of the logarithm is chosen such that

$$\log \frac{1}{1 - \overline{w}z} = \sum_{n=1}^{\infty} \frac{(\overline{w}z)^n}{n}.$$

By a self-map of $\mathbb{D}$ we mean an analytic function φ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. We will also assume that a self-map φ is not a constant function. For a self-map of the unit disk φ, the composition operator C_φ on the Dirichlet space $\mathcal{D}$ is defined by $C_\varphi f := f \circ \varphi$. The composition operator C_φ on Dirichlet space is not necessarily bounded for an arbitrary self-map of the unit disk. However, C_φ is bounded on $\mathcal{D}$ if, for example, φ is a finitely valent function (see, e.g., [9,13]). More is known about the composition operator C_φ when the symbol φ is a linear-fractional self-map of the unit disk of the form

$$\varphi(z) := \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$. In that case C_φ is compact on $\mathcal{D}$ if and only if $\|\varphi\|_\infty < 1$ (see, e.g., [3,11,13]).

For an arbitrary self-map of the unit disk φ, if the operator C_φ is bounded, then the adjoint operator C_φ^* satisfies

$$C_\varphi^* f(w) = \langle f, K_w \circ \varphi \rangle_{\mathcal{D}},$$

which yields a useful equality

$$C_\varphi^* K_w = K_{\varphi(w)}. \tag{1.2}$$

For φ a linear-fractional self-map of $\mathbb{D}$, Gallardo-Gutiérrez and Montes-Rodríguez in [4] (see also [8]) proved that the adjoint of the composition operator is given by formula

$$C_\varphi^* f = f(0) K_{\varphi(0)} - (C_{\varphi^*} f)(0) + C_{\varphi^*} f, \tag{1.3}$$

where

$$\varphi^*(z) := \frac{1}{\overline{\varphi^{-1}(\frac{1}{\bar{z}})}}, \quad z \in \mathbb{D},$$

is the Krein adjoint of φ. It is worth to note that φ^* is a linear-fractional self-map of the unit disk, in fact

$$\varphi^*(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}.$$

It is easy to check that w is a fixed point of φ if and only if $1/\overline{w}$ is a fixed point of φ^*. In particular, if φ has a fixed point on $\mathbb{T}$, then it is a fixed point of both φ and φ^*.

Let φ be a disk automorphism, which is of the form

$$\varphi(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D}, \tag{1.4}$$

where $a \in \mathbb{D}$ and $\theta \in (-\pi, \pi]$. We will say that

- φ is elliptic if and only if $|a| < \cos\frac{\theta}{2}$,
- φ is parabolic if and only if $|a| = \cos\frac{\theta}{2}$,
- φ is hyperbolic if and only if $|a| > \cos\frac{\theta}{2}$

(see, e.g., [11, Ex. 4, p. 7]). One can easily verify that if φ is elliptic, then φ^* is also elliptic.

For φ and ψ, two linear-fractional self-maps of $\mathbb{D}$, we consider the commutator

$$[C_\psi^*, C_\varphi] := C_\psi^* C_\varphi - C_\varphi C_\psi^*$$

on $\mathcal{D}$. The compactness of the commutator can be expressed by setting conditions on the maps ψ and φ. The commutator $[C_\psi^*, C_\varphi]$ is trivially compact on $\mathcal{D}$ if it is equal to zero, or when $C_\psi^* C_\varphi$ and $C_\varphi C_\psi^*$ are both compact. In particular, this happens when $\|\psi\|_\infty < 1$ or $\|\varphi\|_\infty < 1$. Thus, to avoid triviality, we will consider only composition operators, and their adjoints, whose symbols are the linear-fractional self-maps of $\mathbb{D}$ with $\|\psi\|_\infty = \|\varphi\|_\infty = 1$.

We should mention, that if φ and ψ are two linear-fractional self-maps of $\mathbb{D}$, then there are known conditions for compactness of the commutator $[C_\psi^*, C_\varphi]$ acting on the Hardy space H^2 obtained by Clifford et al. [2], and acting on the weighted Bergman spaces $A^2_\alpha(\mathbb{D})$ obtained by MacCluer et al. [6]. Their results were obtained for $\|\psi\|_\infty = \|\varphi\|_\infty = 1$ in the case when both φ and ψ are disk automorphisms, and in the case when at least one of the maps is not an automorphism. In particular, the authors proved that if φ and ψ are automorphisms of $\mathbb{D}$ and neither of them is equal to the identity map, then the commutator $[C_\psi^*, C_\varphi]$ is compact on $A^2_\alpha(\mathbb{D})$ if and only if both maps are rotations. We refer the reader to [6] for more background information.

In this paper we study properties of the difference of two composition operators defined on the Dirichlet space. In Section 2 we give a necessary condition for compactness of the difference of two bounded composition operators. In Section 3, as an application of our necessary condition for compactness, we determine when the commutator $[C_\psi^*, C_\varphi]$, with both symbols φ and ψ being disk automorphisms and not equal to the identity, is compact.

2. Difference of two composition operators

To study compactness of the commutator $[C_\psi^*, C_\varphi]$ we need to know when a difference of two composition operators is compact. There are known conditions for compactness of a difference of composition operators for weighted Dirichlet spaces obtained by Moorhouse in [10]. Unfortunately, these results do not apply to the classical Dirichlet space $\mathcal{D}$. In Theorem 2.2 we give a necessary condition for compactness of the difference of two bounded composition operators on $\mathcal{D}$. First, we give an elementary technical lemma.

Lemma 2.1. *Let the sequences $\mathbb{N} \ni n \mapsto a_n \in (0,1)$ and $\mathbb{N} \ni n \mapsto b_n \in (0,1)$ converge to 0 and let $\lim_{n\to\infty} b_n/a_n = 0$. Then there exists a positive integer N such that*

$$0 < \frac{\ln a_n}{\ln b_n} < 1,$$

for all $n > N$.

Now, we are ready to state our main result.

Theorem 2.2. *Let φ and ψ be self-maps of the unit disk $\mathbb{D}$ such that the composition operators C_φ, C_ψ induced by φ and ψ, respectively, are bounded. If $C_\varphi - C_\psi$ is compact on $\mathcal{D}$, then*

$$\lim_{|w|\to 1^-} \left\{ \frac{1-|w|^2}{1-|\varphi(w)|^2} + \frac{1-|w|^2}{1-|\psi(w)|^2} \right\} |\varphi(w) - \psi(w)| = 0. \tag{2.1}$$

Proof. Clearly, $C_\varphi - C_\psi$ is compact if and only if $C_\varphi^* - C_\psi^*$ is compact. Therefore, it is enough to prove that if (2.1) does not hold, then the operator $C_\varphi^* - C_\psi^*$ is not compact on $\mathcal{D}$. Assume that the limit in (2.1) does not exist or it exists, but it is not equal to 0. In both cases one can find a sequence $\mathbb{N} \ni n \mapsto w_n \in \mathbb{D} \setminus \{0\}$, with $|w_n| \to 1^-$, such that the following conditions are satisfied:

(i) $\displaystyle\lim_{n\to\infty} \left\{ \frac{1-|w_n|^2}{1-|\varphi(w_n)|^2} + \frac{1-|w_n|^2}{1-|\psi(w_n)|^2} \right\} |\varphi(w_n) - \psi(w_n)| \neq 0,$

(ii) the limits $\psi_0 := \lim_{n\to\infty} \psi(w_n)$ and $\varphi_0 := \lim_{n\to\infty} \varphi(w_n)$ exist,

(iii) the limits $\displaystyle\Phi_0 := \lim_{n\to\infty} \frac{1-|w_n|^2}{1-|\varphi(w_n)|^2}$ and $\displaystyle\Psi_0 := \lim_{n\to\infty} \frac{1-|w_n|^2}{1-|\psi(w_n)|^2}$ exist.

Indeed, such a sequence exists. Observe, that if φ is a self-map of the unit disk, then as a consequence of Schwarz–Pick lemma we have (see, e.g., [3, Corollary 2.40])

$$\frac{1-|\varphi(w_n)|}{1-|w_n|} \geq \frac{1-|\varphi(0)|}{1+|\varphi(0)|}, \quad w \in \mathbb{D}, \tag{2.2}$$

and thus both factors in the limit in condition (i) are bounded. Consequently, by passing to a subsequence if necessary, we obtain a sequence satisfying (i)–(iii).

Now, we consider a sequence of normalized kernel functions $K_{w_n}/\|K_{w_n}\|$, where K_{w_n} is given by (1.1), and we show that $\|(C_\varphi^* - C_\psi^*)K_{w_n}\|/\|K_{w_n}\|$ does not tend to 0. Since $K_{w_n}/\|K_{w_n}\| \to 0$ weakly, this disproves that $C_\varphi^* - C_\psi^*$ is compact (see, e.g., [12, Theorem 1.3.4]).

Note, that $\|K_{w_n}\|^2 = 1 + \log(1/(1-|w_n|^2))$, which, together with (1.2), yields

$$\begin{aligned} \frac{\left\|(C_\varphi^* - C_\psi^*)K_{w_n}\right\|^2}{\|K_{w_n}\|^2} &= \frac{\|K_{\varphi(w_n)}\|^2 + \|K_{\psi(w_n)}\|^2 - 2\operatorname{Re}\langle K_{\varphi(w_n)}, K_{\psi(w_n)}\rangle}{\|K_{w_n}\|^2} \\ &= \frac{\ln\frac{1}{1-|\varphi(w_n)|^2} + \ln\frac{1}{1-|\psi(w_n)|^2} - 2\ln\frac{1}{|1-\overline{\varphi(w_n)}\psi(w_n)|}}{1+\ln\frac{1}{1-|w_n|^2}}. \end{aligned} \tag{2.3}$$

Observe, that in view of (2.2) and its analogue for ψ, we must have $\varphi_0 \neq \psi_0$ and

$$0 < \frac{|1-\overline{\varphi_0}\psi_0|}{2} < |1-\overline{\varphi(w_n)}\psi(w_n)| \leq 2, \tag{2.4}$$

for sufficiently large n.

It is enough to consider three cases:

Case I:	$|\varphi_0| = 1$	and	$|\psi_0| < 1$, or
Case II:	$|\varphi_0| < 1$	and	$|\psi_0| = 1$, or
Case III:	$|\psi_0| = |\varphi_0| = 1$	and	$\psi_0 \neq \varphi_0$.

Case I. Let $|\varphi_0| = 1$ and $|\psi_0| < 1$. Then $\Psi_0 = 0$ and $\Phi_0 > 0$, by (i). So, for a sufficiently large n, say $n > N$, we have

$$0 < \frac{1-|\psi_0|^2}{2} < 1-|\psi(w_n)|^2 < 1,$$

and

$$0 < \frac{\Phi_0}{2} < \frac{1-|w_n|^2}{1-|\varphi(w_n)|^2} < 2\frac{1+|\varphi(0)|}{1-|\varphi(0)|}, \tag{2.5}$$

where the last inequality follows from (2.2). Hence,

$$\kappa(w_n) := \ln \frac{1}{1-|\psi(w_n)|^2} - 2\ln \frac{1}{|1-\overline{\varphi(w_n)}\psi(w_n)|} + \ln \frac{1-|w_n|^2}{1-|\varphi(w_n)|^2}$$

is bounded and, by (2.3), we get

$$\lim_{n\to\infty} \frac{\left\|(C_\varphi^* - C_\psi^*)K_{w_n}\right\|^2}{\|K_{w_n}\|^2} = \lim_{n\to\infty} \frac{\ln \frac{1}{1-|w_n|^2} + \kappa(w_n)}{1+\ln \frac{1}{1-|w_n|^2}} = 1.$$

Case II. If $|\varphi_0| < 1$ and $|\psi_0| = 1$, then the proof of the following equality

$$\lim_{n\to\infty} \frac{\left\|(C_\varphi^* - C_\psi^*)K_{w_n}\right\|^2}{\|K_{w_n}\|^2} = 1$$

proceeds analogously to the proof in Case I.

Case III. Let $|\varphi_0| = |\psi_0| = 1$ and $\varphi_0 \neq \psi_0$. Note, that Φ_0 and Ψ_0 cannot both be equal to 0.

If $\Phi_0 = 0$ and $\Psi_0 \neq 0$, then there exists a positive integer N, such that for $n > N$

$$0 < \frac{\Psi_0}{2} < \frac{1-|w_n|^2}{1-|\psi(w_n)|^2} < 2\frac{1+|\psi(0)|}{1-|\psi(0)|} \tag{2.6}$$

and (2.4) hold. Moreover, since $|\varphi_0| = |\psi_0| = 1$, we may assume, by passing to a subsequence if necessary, that $\varphi(w_n) \neq 0$ and $w_n \neq 0$ for each n. Now, we can use Lemma 2.1 with $a_n = 1-|\varphi(w_n)|^2$ and $b_n = 1-|w_n|^2$ and get that there exists a positive integer $N_1 > N$ such that

$$0 < \frac{\ln(1-|\varphi(w_n)|^2)}{\ln(1-|w_n|^2)} = \frac{\ln \frac{1}{1-|\varphi(w_n)|^2}}{\ln \frac{1}{1-|w_n|^2}} < 1,$$

for all $n > N_1$. By passing to a subsequence, if necessary, we can assume that the limit $\lim_{n\to\infty} \ln(1 - |\varphi(w_n)|^2)/\ln(1-|w_n|^2)$ exists. Hence,

$$\lambda(w_n) := -2\ln \frac{1}{|1-\overline{\varphi(w_n)}\psi(w_n)|} + \ln \frac{1-|w_n|^2}{1-|\psi(w_n)|^2}$$

is bounded and, by (2.3), we get

$$\lim_{n\to\infty} \frac{\left\|(C_\varphi^* - C_\psi^*)K_{w_n}\right\|^2}{\|K_{w_n}\|^2} = \lim_{n\to\infty} \frac{\ln \frac{1}{1-|\varphi(w_n)|^2} + \ln \frac{1}{1-|w_n|^2} + \lambda(w_n)}{1+\ln \frac{1}{1-|w_n|^2}} \geq 1.$$

If $\Phi_0 \neq 0$ and $\Psi_0 = 0$, then (2.4) and (2.5) hold for n sufficiently large and

$$\widetilde{\lambda}(w_n) := -2\ln \frac{1}{|1-\overline{\varphi(w_n)}\psi(w_n)|} + \ln \frac{1-|w_n|^2}{1-|\varphi(w_n)|^2}$$

is bounded. Another application of Lemma 2.1 ensures that

$$\lim_{n\to\infty} \frac{\left\|(C_\varphi^* - C_\psi^*)K_{w_n}\right\|^2}{\|K_{w_n}\|^2} = \lim_{n\to\infty} \frac{\ln \frac{1}{1-|\psi(w_n)|^2} + \ln \frac{1}{1-|w_n|^2} + \widetilde{\lambda}(w_n)}{1+\ln \frac{1}{1-|w_n|^2}} \geq 1.$$

Finally, if $\Phi_0 \neq 0$ and $\Psi_0 \neq 0$, then (2.4), (2.5) and (2.6) hold for sufficiently large n. Hence,

$$\widehat{\lambda}(w_n) := -2 \ln \frac{1}{|1 - \overline{\varphi(w_n)}\psi(w_n)|} + \ln \frac{1 - |w_n|^2}{1 - |\varphi(w_n)|^2} + \ln \frac{1 - |w_n|^2}{1 - |\psi(w_n)|^2}$$

is bounded and, by (2.3), we get

$$\lim_{n \to \infty} \frac{\left\| (C_\varphi^* - C_\psi^*) K_{w_n} \right\|^2}{\|K_{w_n}\|^2} = \lim_{n \to \infty} \frac{2 \ln \frac{1}{1 - |w_n|^2} + \widehat{\lambda}(w_n)}{1 + \ln \frac{1}{1 - |w_n|^2}} = 2.$$

This completes the proof. $\square$

The above theorem is in particular true for all finitely valent self-maps of the unit disk. Moreover, in the case of disk automorphisms of the form (1.4) we can obtain a much simpler condition.

Corollary 2.3. *Let φ and ψ be disk automorphisms given by (1.4). If $C_\varphi - C_\psi$ is compact on $\mathcal{D}$, then $\varphi = \psi$.*

Proof. Let φ and ψ be disk automorphisms given by (1.4) and assume that $C_\varphi - C_\psi$ is compact on $\mathcal{D}$. We show that $\varphi(\zeta) = \psi(\zeta)$ for all $\zeta \in \mathbb{T}$.

Fix $\zeta \in \mathbb{T}$. By Theorem 2.2 we know that the compactness of the difference $C_\varphi - C_\psi$ implies

$$\lim_{z \to \zeta} \left\{ \frac{1 - |z|^2}{1 - |\varphi(z)|^2} + \frac{1 - |z|^2}{1 - |\psi(z)|^2} \right\} |\varphi(z) - \psi(z)| = 0. \tag{2.7}$$

We show that neither $(1 - |z|^2)(1 - |\varphi(z)|^2)^{-1}$ nor $(1 - |z|^2)(1 - |\psi(z)|^2)^{-1}$ can tend to 0 as z tends to ζ. Indeed, for φ given by (1.4), we have

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2} = \frac{|1 - \bar{a}z|^2}{1 - |a|^2} \geq \frac{1 - |a|}{1 + |a|} > 0,$$

for all $z \in \overline{\mathbb{D}}$. Hence, $(1 - |z|^2)(1 - |\varphi(z)|^2)^{-1}$ is bounded away from zero as $z \to \zeta$. The same argument can be used to show that $(1 - |z|^2)(1 - |\psi(z)|^2)^{-1}$ is bounded away from zero as $z \to \zeta$. Thus, (2.7) implies

$$\lim_{z \to \zeta} |\varphi(z) - \psi(z)| = 0,$$

and $\varphi(\zeta) = \psi(\zeta)$. Since ζ was chosen arbitrarily, our claim follows. $\square$

3. Commutator

In this section we study some properties of the commutator $[C_\psi^*, C_\varphi]$ with φ and ψ being disk automorphisms.

For $f, g \in \mathcal{D}$ one can define the following rank-one operator

$$f \otimes g(h) := \langle h, g \rangle_{\mathcal{D}} f, \quad h \in \mathcal{D}.$$

By (1.3), for an arbitrary linear-fractional self-map ψ, the adjoint of the composition operator C_ψ can be written as

$$C_\psi^* = C_{\psi^*} + K, \tag{3.1}$$

where $Kf := (K_{\psi(0)} \otimes K_0)(f) - (K_0 \otimes K_0)(C_{\psi^*} f)$, and K_w is a kernel function given by (1.1). Obviously, K is a compact operator on $\mathcal{D}$. Hence, we have

$$[C_\psi^*, C_\varphi]f = C_\psi^* C_\varphi f - C_\varphi C_\psi^* f = (C_{\psi^* \circ \varphi} - C_{\varphi \circ \psi^*})f + Lf, \tag{3.2}$$

where

$$L := KC_\varphi - C_\varphi K = [K, C_\varphi] \tag{3.3}$$

is again compact. It is easy to verify that $L = 0$ if and only if φ is a rotation or ψ is the identity.

Theorem 3.1. *Let φ, ψ be two disk automorphisms given by (1.4), none of which is the identity. Then the commutator $[C_\psi^*, C_\varphi]$ is compact if and only if both φ and ψ^* have the same set of fixed points.*

Proof. Let φ, ψ be two disk automorphisms, none of which is the identity. Assume first, that φ and ψ^* have the same set of fixed points. Then, by [5, Theorem 1, p. 72], we know that φ and ψ^* commute, that is $\psi^* \circ \varphi = \varphi \circ \psi^*$. Thus, the difference $C_{\psi^* \circ \varphi} - C_{\varphi \circ \psi^*}$ in (3.2) is equal to zero and the commutator $[C_\psi^*, C_\varphi]$ is compact.

Now, assume that the commutator $[C_\psi^*, C_\varphi]$ is compact. Then, by formula (3.2), the difference $C_{\psi^* \circ \varphi} - C_{\varphi \circ \psi^*}$ is also compact and Corollary 2.3 implies that $\psi^* \circ \varphi = \varphi \circ \psi^*$. Finally, by [5, Theorem 2, p. 72] and the assumption that φ, ψ are two disk automorphisms not equal to the identity, we obtain that both φ and ψ^* have the same set of fixed points. □

We say that the composition operator C_φ is essentially normal if the self-commutator $[C_\varphi^*, C_\varphi]$ is compact. This property was studied in [1,14] for composition operator defined on the Hardy space and in [7] for composition operators defined on the weighted Bergman spaces. As a consequence of Theorem 3.1 we get the following sufficient condition for C_φ to be essentially normal on the Dirichlet space $\mathcal{D}$.

Corollary 3.2. *If φ is a disk automorphism given by (1.4), then the composition operator C_φ is essentially normal.*

Proof. Let φ be given by (1.4). If φ is equal to the identity, then $[C_\varphi^*, C_\varphi] = 0$.

Now assume that φ is not the identity map. We show that φ and φ^* have the same set of fixed points, which follows from our observation that w is a fixed point of φ if and only if $1/\overline{w}$ is a fixed point of φ^*. Indeed, if φ is a rotation, then φ^* is also a rotation and they have the same set of fixed points. Assume that φ is not a rotation. If φ is an elliptic automorphism, then it has two fixed points $z_k = e^{i\theta/2}(\bar{a})^{-1}(\cos\theta/2 + (-1)^k\sqrt{\cos^2\theta/2 - |a|^2})$, $k = 1, 2$ satisfying $w_1 = 1/\overline{w_2}$. If φ is a parabolic automorphism, then it has only one fixed point $z = (1 + e^{i\theta})/(2\bar{a}) \in \mathbb{T}$, and if φ is a hyperbolic automorphism, then it has two fixed points $z_k = e^{i\theta/2}(\bar{a})^{-1}(\cos\theta/2 + (-1)^k i\sqrt{|a|^2 - \cos^2\theta/2}) \in \mathbb{T}$, $k = 1, 2$. Thus, by Theorem 3.1, the commutator $[C_\varphi^*, C_\varphi]$ is compact, which completes the proof. □

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