J. Math. Anal. Appl. 432 (2015) 367–373

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On algebraic reflexivity of sets of surjective isometries between spaces of weak* continuous functions ☆

Fernanda Botelho [a,*], T.S.S.R.K. Rao [b]

[a] Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, USA
[b] Stat-Math Unit, Indian Statistical Institute, R.V. College P.O., Bangalore 560059, India

ARTICLE INFO

Article history:
Received 18 June 2015
Available online 2 July 2015
Submitted by Richard M. Aron

Dedicated to the memory of Professor James Jamison

Keywords:
Algebraic reflexivity
Isometries
Weak-* continuous functions

ABSTRACT

We establish the algebraic reflexivity of sets of surjective isometries between Banach spaces of weak* or weakly continuous functions. We also derive some results on the algebraic structure of the isometry group of $WC^*(X, E^*)$.

1. Introduction

Given two complex Banach spaces E and F, $\mathcal{G}(E,F)$ represents the space of all linear isometries from E onto F. If $E = F$ we denote the isometry group of E by $\mathcal{G}(E)$. For a non-empty set $\mathcal{W} \subset \mathcal{G}(E,F)$ and $\mathbf{v} \in E$, we set $\mathcal{W}\mathbf{v} = \{A\,\mathbf{v} : A \in \mathcal{W}\}$. We recall that the algebraic closure of $\mathcal{W}$, is the set, $\overline{\mathcal{W}} = \{T \in \mathcal{B}(E,F) : T\mathbf{v} \in \mathcal{W}\mathbf{v},\ \forall\,\mathbf{v} \in E\}$. An operator $T \in \overline{\mathcal{W}}$ is said to be locally in $\mathcal{W}$. Further, fixing a $\mathcal{G}_1 \subset \mathcal{G}(E,F)$, the algebraic closure of $\mathcal{W}$ in $\mathcal{G}_1$, is defined to be $\overline{\mathcal{W}}^{\mathcal{G}_1} = \{T \in \mathcal{G}_1 : T\mathbf{v} \in \mathcal{W}\mathbf{v},\ \forall\,\mathbf{v} \in E\}$.

Definition 1.1. (Cf. [6].) The set $\mathcal{W}$ is said to be algebraically reflexive relative to $\mathcal{G}_1$ if and only if every element of $\mathcal{G}_1$ that is locally in $\mathcal{W}$ is in $\mathcal{W}$, i.e. $\overline{\mathcal{W}}^{\mathcal{G}_1} = \mathcal{W}$. If $\mathcal{G}_1 = \mathcal{G}(E,F)$, then $\mathcal{W}$ is algebraically reflexive if and only if $\overline{\mathcal{W}} = \mathcal{W}$.

In this paper we study the algebraic reflexivity of some natural subsets of the isometry space, when $F = C(X)$, the space of complex-valued continuous functions on a compact Hausdorff space X. We recall

☆ The first author acknowledges research support from a Pearson Award and Dunavant Professorship Funds. Research of the second author was partially supported by a travel grant from the Simons Foundation.

* Corresponding author.
E-mail addresses: mbotelho@memphis.edu (F. Botelho), tss@isibang.ac.in, srin@fulbrightmail.org (T.S.S.R.K. Rao).

http://dx.doi.org/10.1016/j.jmaa.2015.06.063
0022-247X/

that X can be canonically embedded as a subset of the unit ball of $C(X)^*$, equipped with the weak-* topology, see [7] for information on the weak-* topology. If one considers the space of all functions $f : X \to E^*$ which are continuous relative to the weak*-topology on E^*, endowed with the norm $\|f\|_\infty = \sup_{x\in X} \|f(x)\|$, this is a Banach space, and we denote it by $WC^*(X, E^*)$. It is well known that the mapping $T \to T^*|X$ is a surjective isometry between $\mathcal{B}(E, C(X))$ and $WC^*(X, E^*)$. In order to make use of the structure of the surjective isometries (see [2] and the monograph [4]) we work under some natural assumptions on the range spaces. Similar questions for spaces of compact operators were studied in [5] and [10]. There are no satisfactory answers for algebraic reflexivity problem for the group of isometries on the space of bounded operators. In this paper we focus on the algebraic reflexivity of some subsets of the isometry group and try to do away with conventional assumptions of metrizability or the availability of approximation properties for the range or domain space. The monograph [8] also deals with problems similar to the ones considered here.

In Section 2, we prove the algebraic reflexivity of the set of surjective isometries in $\mathcal{G}(WC^*(X, E^*), WC^*(Y, F^*))$ preserving the constant functions. This result follows from the representation for the surjective isometries given by Cambern and Jarosz in [2]. In Section 3, we consider the isometric isomorphism between $WC^*(X, E^*)$ and $\mathcal{B}(E, C(X))$, to study the algebraic reflexivity of subsets of the isometry group on a class of closed ideals of $\mathcal{B}(E, C(X))$ that contain all compact operators. For the ideal of weakly compact operators whose adjoint has separable range, under a reasonable continuity assumption on the class of isometries, we show that they are algebraically reflexive in the group of isometries.

In the last section we derive the algebraic structure of the isometry group $\mathcal{G}(WC^*(X, E^*))$ and of certain subgroups of the isometry group. We show that under certain assumptions, the isometry group is a direct product of the group of $C(X)$-modulo isomorphisms on $WC^*(X, E^*)$, with the group of homeomorphisms on X.

2. The algebraic reflexivity property

We first recall a result due to Cambern and Jarosz that characterizes the linear isometries from $WC^*(X, E^*)$ onto $WC^*(Y, F^*)$. The scalar field $\mathbb{R}$ or $\mathbb{C}$ is denoted by $\mathbf{K}$.

Theorem 2.1. *(Cf. [2].) Let X and Y be compact metric spaces and E^* and F^* be Banach dual spaces with trivial centralizer, $Z(E^*), Z(F^*) = \mathbf{K}$, and let $T : WC^*(X, E^*) \to WC^*(Y, F^*)$ be a surjective linear isometry. Then there exist a homeomorphism $\Phi : Y \to X$ and a function $U : Y \to \mathcal{G}(E^*, F^*)$ such that*

$$(Tf)(y) = U(y)f \circ \Phi(y), \tag{1}$$

for all $f \in WC^*(X, E^*)$ and $y \in Y$.

It is easy to see that the surjective isometry T^{-1} is given by $(T^{-1}G)(x) = (U(\Phi^{-1}(x)))^{-1}G \circ \Phi^{-1}(x)$, for all $G \in WC^*(Y, E^*)$ and $x \in X$. If $Const(X, E^*)$ denotes the set of all constant functions, i.e. $\mathbf{e}^*(x) = e^* \in E^*$, then $T \in \mathcal{G}(WC^*(X, E^*), WC^*(Y, F^*))$ is said to preserve (fix) the space of constant functions if $T(Const(X, E^*)) \subseteq Const(Y, F^*)$, $(T(Const(X, E^*) = Const(Y, F^*)$, respectively). This is equivalent to say that the function U in equation (1) is constant. The single element in the range of U is an isometry in $\mathcal{G}(E^*, F^*)$. For simplicity of notation, it will also be denoted by U. We set $\mathcal{A}$ ($\mathcal{A}_c$) to be the set of all isometries $T \in \mathcal{G}(WC^*(X, E^*), WC^*(Y, F^*))$ that preserve (fix, respectively) the constant functions. If $X = Y$ and $E = F$ then the isometry group will be denoted by $\mathcal{G}(WC^*(X, E^*))$. We consider the subspace of $WC^*(X, E^*)$ consisting of all functions with finite rank, this means all functions f such that for all x, $f(x) = \sum_{i=1}^n \lambda_i(x)\,\mathbf{v}_i^*$ with $\lambda_i \in C(X)$ and $\mathbf{v}_i^* \in E^*$. This subspace is denoted by $WC_F^*(X, E^*)$ and $\overline{WC_F^*}(X, E^*)$ represents its closure, the space of functions that are continuous when E^* has the norm topology, in $WC^*(X, E^*)$. The form for the isometries in Theorem 2.1 implies that, given $T \in \mathcal{G}(WC^*(X, E^*), WC^*(Y, F^*))$ that fixes the constant functions, then T preserves the subspace of all finite rank functions, thus $T\,\overline{WC_F^*}(X, E^*) = \overline{WC_F^*}(Y, F^*)$. The next proposition states the algebraic reflexivity of $\mathcal{A}_c$ when restricted to $\overline{WC_F^*}(X, E^*)$.

Proposition 2.2. *Let X and Y be compact metric spaces and E^* and F^* are Banach dual spaces with trivial centralizers $Z(E^*), Z(F^*) = \mathbf{K}$. Then $\mathcal{A}_c$ restricted to $\overline{WC_F^*}(X, E^*)$ is algebraically reflexive.*

Proof. Let $T \in \mathcal{G}(WC^*(X, E^*), WC^*(Y, F^*))$ be locally in $\mathcal{A}_c$. The representation for T given in Theorem 2.1 asserts the existence of U and Φ such that $(Tf)(y) = U(y)f \circ \Phi(y)$, for all $f \in WC^*(X, E^*)$ and $y \in Y$. Since T is locally in $\mathcal{A}_c$ then $(Tf)(y) = V_f f \circ \Phi_f(y)$, for some $V_f \in \mathcal{G}(E^*, F^*)$ and Φ_f a homeomorphism from Y onto X. Thus the equation $U(y)f \circ \Phi(y) = V_f f \circ \Phi_f(y)$ applied to $\mathbf{e}^*$, a constant function, yields $U(y)e^* = V_{\mathbf{e}^*}e^*$. We observe that the value $V_{\mathbf{e}^*}e^*$ is independent of y. We define $V : E^* \to F^*$, as follows: $V(e^*) = V_{\mathbf{e}^*}e^*$. The operator V is a surjective linear isometry. First we show that V is linear. Given e^* and $e_1^* \in E^*$ and $a, b \in \mathbf{K}$, then $V(a \cdot e^* + b \cdot e_1^*) = U(y)(a \cdot e^* + b \cdot e_1^*) = a \cdot U(y)(e^*) + b \cdot U(y)(e_1^*) = a \cdot V(e^*) + b \cdot V(e_1^*)$. The surjectivity of V follows from the surjectivity of T. We consider S defined on $WC^*(X, E^*)$, given by $(SF)(y) = VF \circ \Phi(y)$. The operator S is in $\mathcal{A}_c$. For a function of the form $\lambda \cdot e^*$ with $\lambda \in C(X)$ and positive, we have

$$T(\lambda \cdot e^*)(y) = (\lambda \circ \Phi(y)) \cdot U(y)e^* = (\lambda \circ \Phi(y)) \cdot Ve^* = V_{\lambda \cdot e^*}\lambda \circ \Phi_{\lambda \cdot e^*}(y)e^*, \tag{2}$$

for all $y \in Y$. Since V and $U(y)$ are isometries then $\lambda \circ \Phi(y) = \lambda \circ \Phi_{\lambda \cdot e^*}(y)$ and $Ve^* = V_{\lambda \cdot e^*}e^*$. This implies that $Tf = Sf$ for rank one functions, i.e. functions of the form $\lambda \cdot e^*$ with λ a positive and continuous function on X. For an arbitrary rank one function $f = \lambda \cdot e^*$, there exists a positive real number a such that $a + \lambda$ is positive then

$$T((a + \lambda) \cdot e^*) = aT(\mathbf{e}^*) + T(\lambda \cdot e^*) = aVe^* + T(\lambda \cdot e^*) = (a + \lambda) \circ \Phi \cdot Ve^*$$

implying that $T(\lambda \cdot e^*) = (\lambda \circ \Phi) \cdot Ve^* = S(\lambda \cdot e^*)$. Inductively we show that T and S are equal when applied to finite rank functions in $WC^*(X, E^*)$. This completes the proof. $\square$

The proof given for Proposition 2.2 allows us to derive the algebraic reflexivity of $\mathcal{A}$ provided the isometry group of the range space E^*, is algebraically reflexive. See [8] for several examples of such spaces. In particular, for $1 < p < \infty$, $p \neq 2$, the ℓ^p spaces have this property. It may be noted that they also have trivial centralizer. The following result is easy to deduce.

Proposition 2.3. *Let X and Y be compact metric spaces and E^* and F^* are Banach dual spaces with trivial centralizers. If $\mathcal{G}(E^*)$ is algebraically reflexive, then $\mathcal{A}$ restricted to $\overline{WC_F^*}(X, E^*)$ is algebraically reflexive.*

We now recall a theorem due to Cambern and Jarosz that we will employ to derive the algebraic reflexivity of the subset of the isometry space of $C(Y)$-module isomorphisms. A $C(Y)$-module isomorphism is an isometry $S : WC^*(Y, E^*) \to WC^*(Y, F^*)$ such that for all $f \in WC^*(Y, E^*)$ and $\lambda \in C(Y)$ we have $S(\lambda \cdot f) = \lambda \cdot Sf$.

Theorem 2.4. *(Cf. Theorem 1 in [2].) Let X and Y be compact topological spaces and let E^* and F^* be Banach dual spaces with trivial centralizer. If $T : WC^*(X, E^*) \to WC^*(Y, F^*)$ be a surjective linear isometry, then there exist a homeomorphism $\Phi : Y \to X$ and a $C(Y)$-module isomorphism $S : WC^*(Y, E^*) \to WC^*(Y, F^*)$ such that $T = S \circ R$ with $R : WC^*(X, E^*) \to WC^*(Y, E^*)$ given by $Rf = f \circ \Phi$.*

For the remainder of this paper we assume $X = Y$. We start with a general lemma about $C(X)$-modulo isomorphisms, these are surjective isometries $S : WC^*(X, E^*) \to WC^*(X, F^*)$ such that for all

$\lambda \in C(X)$ and $f \in WC^*(X, E^*)$ we have $S(\lambda \cdot f) = \lambda \cdot S(f)$. We denote this class of isometries by $\mathcal{S}(WC^*(X, E^*), WC^*(X, F^*))$ and $\mathcal{S}(WC^*(X, E^*))$ when $E = F$.

Lemma 2.5. *Let X be a compact Hausdorff topological space and E^* a Banach dual space. Let $e^* \in E^*$ be a unit functional and $\mathbf{e}^*$ the corresponding constant function. If S is a $C(X)$-modulo isomorphism on $WC^*(X, E^*)$ then the set $\{x \in X : S(\mathbf{e}^*)(x) \neq 0\}$ is dense in X.*

Proof. Suppose $\{x \in X : S(\mathbf{e}^*)(x) \neq 0\}$ is not dense then there exists an open subset of X, O contained in its complement. Hence $S(\mathbf{e}^*)(x) = 0$ for all $x \in O$. Since S is an isometry $\|S(\mathbf{e}^*)\|_\infty = 1$. Let $\lambda : X \to [0, 1]$ be a continuous map with support contained in $\mathcal{O}$ attaining the value 1 at some point in $\mathcal{O}$. Then the function $\lambda\mathbf{e}^* \in WC^*(X, E^*)$ has norm 1 but $S(\lambda\mathbf{e}^*)$ is the zero functional. This absurdity proves the statement of the lemma. □

We now prove the algebraic reflexivity of $\mathcal{S}$, the set of $C(X)$-module isomorphisms.

Corollary 2.6. *Let X be compact first countable topological space and let E^* be Banach dual space with trivial centralizer. Then $\mathcal{S}$ is algebraically reflexive.*

Proof. Let T be a surjective isometry in $\mathcal{G}(C^*(X, E^*))$ that is locally in $\mathcal{S}$. An application of Theorem 2.4 implies that $T = S \circ R$. Therefore, for all $f \in WC^*(X, E^*)$, we have $S \circ R(f) = S_f(f)$. In particular for a constant function $\mathbf{e}^*$, $S(\mathbf{e}^*) = S_{\mathbf{e}^*}(\mathbf{e}^*)$. There exists a positive and continuous function $\lambda \in C(X)$ with range in the interval $[0, 1]$, attaining its maximum value equal to 1 at x and such that $\lambda(y) < \lambda(x)$ for $y \neq x$. We define $f = \lambda \cdot \mathbf{e}^*$. Then $\lambda \circ \Phi \cdot S(\mathbf{e}^*) = \lambda \cdot S_f(\mathbf{e}^*)$. Since S and S_f are isometries and $S(\mathbf{e}^*)(x) = S_{\mathbf{e}^*}(\mathbf{e}^*)(x) \neq 0$ for a dense set subset of X, then $\lambda \circ \Phi(x) = \lambda(x)$, for all $x \in X$. This implies that Φ is the identity map and $T = S$. This completes the proof. □

Remark 2.7. It is easy to see that the group $\mathcal{R} = \{T \in \mathcal{G}(WC^*(X, E^*)) : Tf = f \circ \Phi,\ \Phi \text{ a homeomorphism of } X\}$ is algebraically reflexive.

3. Algebraic reflexivity of the isometry group of operator ideals in $\mathcal{B}(E, F)$

In this section we study the algebraic reflexivity of some subsets of the isometry group $\mathcal{G}(\mathcal{I})$ of operator ideals $\mathcal{I}$ in $\mathcal{B}(E, F)$. There are several results in the literature that, under some additional conditions, completely describe the isometry group of the ideal of compact operators $\mathcal{K}(E, F)$ and also several partial results are available on the algebraic reflexivity of this group, [5] and [10]. Here we consider other operator ideals $\mathcal{I}$ of $\mathcal{B}(E, F)$ that, as before, contain the compact operators and study the algebraic reflexivity problem for certain sets of isometries that preserve the compact operators.

Let $\mathcal{W}(E, C(X))$ denote the space of weakly compact operators. Let $\mathcal{I}$ be the ideal $\{\tau \in \mathcal{W}(E, C(X)) : \tau^* \text{ has separable range}\}$. Clearly $\mathcal{I}$ contains all compact operators. We let $\mathcal{S}$, be the set of all isometries $\Phi \in \mathcal{G}(\mathcal{I})$ that preserves the constant functions and are sequentially continuous with respect to the adjoint strong operator topology. We recall that a map $\Phi : \mathcal{I} \to \mathcal{I}$ is sequentially continuous with respect to the adjoint strong operator topology, if and only if, for any $x \in X$, τ and a sequence $\{\tau_n\} \subset \mathcal{I}$ such that

$$\|\tau_n^*(\delta(x)) - \tau^*(\delta(x))\| \to 0$$

then

$$\|\Phi(\tau_n)^*(\delta(x)) - \Phi(\tau)^*(\delta(x))\| \to 0,$$

for all $x \in X$. From the isometric isomorphism between $WC^*(X, E^*)$ and $\mathcal{B}(E, C(X))$ we have the following

$$\mathcal{I} = \{\tau : X \to E^* \text{ such that } \tau \text{ is weak-}^* \text{ continuous and } \tau(X) \text{ is norm separable}\}.$$

In the following theorem we again assume that E^* has trivial centralizer, but do not assume metrizability of X. We prove our result only using local properties of the elements of the operator ideal.

In what follows we use the notations used in the proof of Proposition 2.2.

Theorem 3.1. *Let X be a compact topological space, E be a Banach space such that E^* has trivial centralizer and is algebraically reflexive. Let $\mathcal{S} \subset \mathcal{G}(\mathcal{I})$ be as above. Then $\mathcal{S}$ is algebraically reflexive in $\mathcal{G}(\mathcal{I})$.*

Proof. Let Φ be an isometry in the algebraic closure of $\mathcal{S}$. Clearly Φ is an into isometry and preserves the constant functions. By continuity we thus have that $\Phi(\mathcal{K}(E, C(X)) = \mathcal{K}(E, C(X))$. Now as E^* has trivial centralizer, instead of using Theorem 2.1, by applying Behrends' version of vector-valued Banach–Stone theorem [1, Theorem 8.10] on norm continuous functions, and proceeding as in the proof of Proposition 2.2 (see also Proposition 2.3), we have, given a norm continuous function $\tau : X \to E^*$, for $x \in X$,

$$\Phi(\tau)(x) = V(\tau(\phi)(x))$$

for a fixed homeomorphism ϕ and surjective isometry V of E^*. We next claim that for any $\tau \in \mathcal{I}$, there is a sequence of norm continuous functions, $\tau_n : X \to E^*$, such that $\|\tau_n(x) - \tau(x)\| \to 0$, for all $x \in X$. From this claim and from the continuity assumption on Φ, we have that $\Phi(\tau)(x) = V(\tau(\phi)(x))$, for all $x \in X$. This implies that $\Phi \in \mathcal{S}$.

Now we prove the claim. Let $\tau \in \mathcal{I}$. Since $\tau(X)$ is a weakly compact set that is norm separable, it is easy to see that $\tau(X)$ is a metric space in the weak topology. Let $\delta : \tau(X) \to C(\tau(X))^*$ be the evaluation map. Since $C(\tau(X))$ has a Schauder basis, there exists a norm bounded sequence of finite rank projections $P_n : C(\tau(X)) \to C(\tau(X))$ such that, for all $x \in X$, $P_n^*(\delta(\tau(x))) \to \delta(\tau(x))$, in the weak*-topology. As the projections P_n's are finite rank operators, we now have by composition and restriction, a sequence $g_n : X \to E^*$ of norm continuous functions such that, for $x \in X$, $g_n(x) \to f(x)$, in the weak topology of E^*. A standard argument using Mazur's theorem and the norm separability of $\tau(X)$ (see the proof given for the corollary in [9]) will lead to a sequence $\tau_n : X \to E^*$, of norm continuous functions, such that $\|\tau(x) - \tau_n(x)\| \to 0$ as claimed. $\square$

Remark 3.2. If we assume the global hypothesis that there is a sequence τ_n of compact operators on E such that $\tau_n^* \to I$ in the strong operator topology, then the τ_n in the above proof is easy to produce. This hypothesis in particular implies that E^* is separable. If E^* has a Schauder basis, then clearly such τ_n can be chosen to be of finite rank.

Remark 3.3. When E is reflexive, the space of weakly compact operators, $\mathcal{W}(E, C(X)) = \mathcal{B}(E, C(X))$. Thus when X is also metrizable, the above operator ideal coincides with $\mathcal{B}(E, C(X))$ and our result is valid for the entire space of operators.

Open problems:

(1) If E is a Banach space such that E^* has a Schauder basis, when X is a metric space, again by Theorem 4 in [2], our result holds for the entire space, $\mathcal{B}(E, C(X))$. For a metric space X, we do not know if our result is valid for $\mathcal{B}(E, C(X))$ under the assumption E^* is separable or at least for the larger ideal, $\mathcal{I}' = \{\tau \in \mathcal{B}(E, C(X)) : \tau^*(X) \text{ is norm separable}\}$ in the non-metrizable case?

(2) Now consider a general situation when $\mathcal{K}(E, F)$ is the norm closure of the space of finite rank operators and also that bounded operators are point-wise limits of bounded nets of compact operators. Suppose we know that isometry group of $\mathcal{K}(E, F)$ is described by isometries of the form UTV, where $U : F \to F$ and $V : E \to E$ are surjective isometries. What are the analogues of the results above for operator ideals containing compact operators?

4. Algebraic structure of $\mathcal{G}(WC^*(X, E^*))$

In this section we investigate the algebraic structure of the group of surjective isometries on $WC^*(X, E^*)$. As before we consider X a compact Hausdorff space and E^* a Banach dual with trivial centralizer. We define a bijection between $\mathcal{G}(WC^*(X, E^*))$ and $\mathcal{S}(WC^*(X, E^*)) \times Hom(X)$, with $\mathcal{S}(WC^*(X, E^*))$ denoting all the $C(X)$-isomorphisms on $WC^*(X, E^*)$ and $Hom(X)$ denoting all homeomorphisms on X. We first observe that the representation given in Theorem 2.4 is unique. If

$$S \circ R = S_1 \circ R_1, \tag{3}$$

then (3) applied to the constant function $\mathbf{e}^*$ yields $S(\mathbf{e}^*) = S_1(\mathbf{e}^*)$. The equation (3) applied to a rank one function, $\lambda \cdot \mathbf{e}^*$ with $\lambda \in C(X)$, yields $\lambda \circ \Phi \cdot S(\mathbf{e}^*) = \lambda \circ \Phi_1 \cdot S_1(\mathbf{e}^*)$. An application of Lemma 2.5 implies that $\lambda \circ \Phi = \lambda \circ \Phi_1$. Then $\Phi = \Phi_1$ and $S = S_1$. This allows us to define the map $\Psi : \mathcal{G}(WC^*(X, E^*)) \to \mathcal{S}(WC^*(X, E^*)) \times Hom(X)$, given by $\Psi(T) = (S, \Phi^{-1})$, with $T = S \circ R$ and R the composition operator, $R(f) = f \circ \Phi$.

Theorem 4.1. *Let X be compact metric space and let E^* be Banach dual space with trivial centralizer, $Z(E^*) = \mathbf{K}$. Then $\mathcal{A}$ is a subgroup of $\mathcal{G}((WC^*(X, E^*))$ isomorphic to the direct product $\mathcal{G}(E^*) \times Hom(X)$.*

Proof. We observe that for $T = S \circ R$ with $S(F)(x) = U(F(x))$ and $R(F)(x) = F(\Phi(x))$ then we define $\Psi : \mathcal{A} \to \mathcal{G}(E^*) \times Hom(X)$, given by $\Psi(T) = (U, \Phi^{-1})$. We show that Ψ is a group isomorphism. Since $T(F)(x) = UF(\Phi(x))$ then $T^{-1}(F)(x) = U^{-1}F(\Phi^{-1}(x)$ and $T_1T_2(F)(x) = U_1[T_2(F)](\Phi_1(x)) = U_1U_2(F)(\Phi_2(\Phi_1(x)))$. Therefore $\Psi(T^{-1}) = (U^{-1}, \Phi)$ and $\Psi(T_1T_2) = (U_1U_2, \Phi_1^{-1} \circ \Phi_2^{-1}) = \Psi(T_1) \star \Psi(T_2)$, with $\star$ denoting the standard group operation on $\mathcal{G}(E^*) \times Hom(X)$. $\square$

Remark 4.2. In particular if X is a rigid space, or a space supporting only one homeomorphism, then $\mathcal{A}$ is isomorphic to $\mathcal{G}(E^*)$.

Lemma 4.3. *Let X be compact Hausdorff space and let E^* be Banach dual space with trivial centralizer, $Z(E^*) = \mathbf{K}$. Then $\mathcal{S}$ is a normal subgroup of $\mathcal{G}(WC^*(X, E^*))$.*

Proof. We show that $T^{-1}\mathcal{S}T = \mathcal{S}$, with T a surjective isometry on $WC^*(X, E^*)$. Theorem 2.4 implies that $T = S_1 \circ R_1$, with S_1 a $C(X)$-module and $R_1(f) = f \circ \Phi_1$. Then it is sufficient to show that given $S \in \mathcal{S}$ and R we have $R^{-1} \circ S \circ R \in \mathcal{S}$. We observe that $R^{-1} \circ S$ is a surjective isometry which can be represented as follows: $R^{-1} \circ S = S_0 \circ R_0$. This equation applied to a constant function $\mathbf{e}^*$ and rank one function $\lambda\mathbf{e}^*$ yields $R^{-1} \circ S\mathbf{e}^* = S_0\mathbf{e}^*$ and $(\lambda \circ \Phi^{-1})R^{-1} \circ S\mathbf{e}^* = (\lambda \circ \Phi_0)S_0\mathbf{e}^*$. An application of Lemma 2.5 implies that $\lambda \circ \Phi^{-1} = \lambda \circ \Phi_0$ and an argument as in the proof for Corollary 2.6 implies that $\Phi^{-1} = \Phi_0$. Therefore $R^{-1} \circ S \circ R = S_0$. $\square$

We use the notation $G = H \rtimes K$ to denote the semidirect product of the subgroups of G, H and K where H is a normal subgroup of G. We appeal to Theorem 12 (p. 180) in [3] to derive the algebraic structure of $\mathcal{G}(WC^*(X, E^*))$.

Theorem 4.4. *(Cf. [3].) Suppose G is a group with subgroups H and K such that H is a normal subgroup of G and $H \cap K = 1$ Then HK is isomorphic to $H \rtimes K$. In particular, if $G = HK$ then G is the semidirect product of H and K.*

We denote by $\mathcal{R}$ the subgroup of all surjective isometries $Rf = f \circ \Phi$ with Φ a homeomorphism of X. It is an easy observation that $\mathcal{R} \cap \mathcal{S}$ is the identity, then we have the following result.

Corollary 4.5. *Let X be compact Hausdorff space and let E^* be Banach dual space with trivial centralizer, $Z(E^*) = \mathbf{K}$. Then $\mathcal{G}(WC^*(X, E^*))$ is isomorphic to the semidirect product of $\mathcal{S}$ with $\mathcal{R}$, $\mathcal{S} \rtimes \mathcal{R}$.*

References

[1] E. Behrends, M-Structures and the Banach Stone Theorem, Lecture Notes in Mathematics, vol. 736, Springer-Verlag, New York, 1979.
[2] M. Cambern, K. Jarosz, Isometries of spaces of weak* continuous functions, Proc. Amer. Math. Soc. 106 (1989) 707–712.
[3] D.S. Dummit, R.M. Foote, Abstract Algebra, third edition, John Wiley & Sons, Inc., USA, 2004.
[4] R. Fleming, J.E. Jamison, Isometries on Banach Spaces. Vol. 2. Vector-Valued Function Spaces, Chapman and Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 138, Chapman and Hall/CRC, Boca Raton, FL, 2008.
[5] K. Jarosz, T.S.S.R.K. Rao, Local isometries of function spaces, Math. Z. 243 (2003) 449–469.
[6] J. Li, Z. Pan, Algebraic reflexivity of linear transformations, Proc. Amer. Math. Soc. 135 (2007) 1695–1699.
[7] R. Megginson, An Introduction to Banach Space Theory, Graduate Texts in Mathematics, vol. 183, Springer-Verlag, 1991.
[8] L. Molnar, Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces, Lecture Notes in Mathematics, vol. 1895, Springer-Verlag, Berlin, Heidelberg, 2006.
[9] T.S.S.R.K. Rao, Weakly continuous functions of Baire class 1, Extracta Math. 15 (2000) 207–212.
[10] T.S.S.R.K. Rao, A note on the algebraic reflexivity of the isometry group of $\mathcal{K}(X, C(K))$, Expo. Math. 26 (2008) 79–83.