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Note

Compact affine manifolds with precompact holonomy are geodesically complete

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ABSTRACT

This note proves the geodesic completeness of any compact manifold endowed with a linear connection such that the closure of its holonomy group is compact.

1. Introduction

The purpose of the present note is to prove the following result:

Theorem 1.1. *Let M be a (Hausdorff, connected, smooth) compact m-manifold endowed with a linear connection ∇ and let $p \in M$. If the holonomy group $Hol_p(\bigtriangledown)$ (regarded as a subgroup of the group $Gl(T_pM)$ of all the linear automorphisms of the tangent space at p, T_pM) has compact closure, then $(M, \bigtriangledown)$ is geodesically complete.*

Some comments on the completeness of compact affine manifolds are in order. There are several results when ∇ is *flat* and, therefore, there exists an atlas $\mathcal{A}$ whose transition functions are affine maps of $\mathbb{R}^m$; in this case, the linear parts will lie in some subgroup G of the (real) general linear group $\mathrm{Gl}(m)$. In fact, a well-known conjecture by Markus states that compact affine flat manifolds which are unimodular (i.e.,

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G can be chosen in the special linear group $\mathrm{Sl}(m)$) must be complete. Carrière [2] introduced an invariant for linear groups, the *discompacity*, which measures the non-compactness of the group by analyzing the degeneration of images of the unit sphere under the action of sequences of its elements. When the closure $\bar{G}$ is compact, the discompacity is equal to 0; Markus conjecture was proven in [2] under the assumption that the discompactness of G is at most 1. Other results on the structure of unimodular manifolds (see, for example, [3]) can be also regarded as partial answers to that conjecture.

When ∇ is the Levi-Civita connection of a Riemannian metric g, the (geodesic) completeness of ∇, which follows directly from Hopf–Rinow theorem, can be reobtained from Theorem 1.1; indeed, $\mathrm{Hol}_p(\bigtriangledown)$ becomes a subgroup of the (orthogonal) group of linear isometries $\mathrm{O}(T_pM)$, which is compact. However, when g is an indefinite semi-Riemannian metric of index ν $(0 < \nu < m)$, the corresponding orthogonal group $\mathrm{O}_\nu(T_pM)$ is non-compact and completeness may not hold; the Clifton–Pohl torus (see for instance [9, Example 7.16]) is a well-known example. There are some results that assure the completeness of a compact semi-Riemannian manifold, among them either to admit ν pointwise independent conformal Killing vector fields which span a negative definite subbundle of TM [12,11], or to be homogeneous [8] (local homogeneity is also enough in dimension 3 [4], and to be conformal to a homogeneous manifold is enough in any dimension [11]; see [13] for a review).

In the particular case that the semi-Riemannian manifold (M, g) is Lorentzian $(\nu = 1 < m)$, compactness implies completeness in other relevant cases, such as when g is flat. Indeed, taking if necessary a finite covering, this is a particular case of Markus' conjecture where G can be regarded as the restricted Lorentz group $\mathrm{SO}_1^\uparrow(m)$ (i.e., the connected component of the identity of the Lorentz group $\mathrm{O}_1(m)$) and, as also proven by Carrière in [2], the discompactness of $\mathrm{SO}_1^\uparrow(m)$ is equal to 1. It is worth pointing out that, when the group G determined by a compact flat affine manifold lies in the group of Lorentzian similarities (generated by homotheties and $\mathrm{O}_1(m)$) but not in the Lorentz group, then the connection is incomplete [1]. Moreover, Klinger [6] extended Carrière's result by showing that any compact Lorentzian manifold of constant curvature is complete, and Leistner and Schliebner [7] proved that completeness also holds in the case of Abelian holonomy (compact pp-waves).

These semi-Riemannian results are independent of Theorem 1.1; in fact, Gutiérrez and Müller [5] have proven recently that, for a Lorentzian metric g, the compactness of $\overline{\mathrm{Hol}(g)}$ implies the existence of a timelike parallel vector field in a finite covering. This conclusion (combined with the cited one in [12]) also gives an alternative proof of Theorem 1.1 in the particular case that ∇ comes from a Lorentzian metric. In any case, the proof of our theorem is very simple and extends or complements the previous results.

2. Proof of Theorem 1.1

Assume that there exists an incomplete geodesic $\gamma : [0, b) \to M$, $b < \infty$. By using the compactness of M, choose any sequence $\{t_n\}_n \nearrow b$ such that $\{\gamma(t_n)\}_n$ converges to some $p \in M$. It is well-known then that the sequence of velocities $\{\gamma'(t_n)\}_n$ cannot converge in TM as γ' is the integral curve of the geodesic vector field on TM (see for example Prop. 3.28 and Lemma 1.56 in [9] or [13, Section 3]). Consider a normal (starshaped) neighborhood U of p (see for example [10,14] for background results on linear connections). With no loss of generality, we will assume that $\{\gamma(t_n)\}_n \subset U$ and will arrive at a contradiction with the compactness of $\overline{\mathrm{Hol}_p(\nabla)}$.

Consider the loops at p given by $\alpha_n = \rho_n^{-1} \star \gamma_{[t_1,t_n]} \star \rho_1$, where $\star$ denotes the concatenation of the corresponding curves and $\rho_n : [0, 1] \to U$ is the radial geodesic from p to $\gamma(t_n)$ for all $n = 1, 2, \ldots$. Put $v_p = \tau_{\rho_1^{-1}}(\gamma'(t_1)) \in T_pM$. For any curve $\alpha : [a, b] \to$, let τ_α be the parallel transport between its endpoints. As γ is a geodesic:

$$v_n := \tau_{\alpha_n}(v_p) = \tau_{\rho_n^{-1}} \circ \tau_{\gamma_{[t_1,t_n]}} \circ \tau_{\rho_1}(\tau_{\rho_1^{-1}}(\gamma'(t_1))) = \tau_{\rho_n^{-1}}(\gamma'(t_n))$$

(in particular, $v_1 = v_p$). The compactness of $\overline{\mathrm{Hol}_p(\nabla)}$ implies that the sequence $\{v_n\}_n$ is contained in a compact subset $K \subset T_pM$. So, it is enough to check that this property also implies that $\{\gamma'(t_n)\}_n$ is included in a compact subset of TM.

With this aim, let $\tilde{K} \subset \exp^{-1}(U)$ be a starshaped compact neighborhood of $0 \in T_pM$ and, for each $u \in \tilde{K}$, let $\rho_u : [0,1] \to U$ be the radial geodesic segment with initial velocity u. As the map

$$\xi : \tilde{K} \times K \to TM, \qquad (u, v) \mapsto \xi(u, v) = \tau_{\rho_u}(v)$$

is continuous, its image $\xi(\tilde{K}, K)$ is compact. This set contains all $\{\gamma'(t_n)\}_n$ up to a finite number (recall that $\gamma(t_n)$ lies in $\exp_p(\tilde{K})$ and $\rho_{u_n} = \rho_n$ for $u_n = \exp^{-1}(\gamma(t_n))$, i.e., $\gamma'(t_n) = \xi(u_n, v_n)$), as required.

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References

[1] T. Aristide, Closed similarity Lorentzian affine manifolds, Proc. Amer. Math. Soc. 132 (1) (2004) 3697–3702.
[2] Y. Carrière, Autour de la conjecture de L. Markus sur les variétés affines, Invent. Math. 95 (3) (1989) 615–628.
[3] Y. Carrière, F. Dal'bo, G. Meigniez, Inexistence de structures affines sur les fibrés de Seifert, Math. Ann. 296 (4) (1993) 743–753.
[4] S. Dumitrescu, A. Zeghib, Géométries Lorentziennes de dimension 3: classification et complétude, Geom. Dedicata 149 (1) (2010) 243–273.
[5] M. Gutiérrez, O. Müller, Compact Lorentzian holonomy, preprint, available at arXiv:1502.05289v2, 2015.
[6] B. Klinger, Complétude des variétés Lorentziennes à courbure constante, Math. Ann. 306 (2) (1996) 353–370.
[7] T. Leistner, D. Schliebner, Completeness of compact Lorentzian manifolds with Abelian holonomy, Math. Ann. (2015), http://dx.doi.org/10.1007/s00208-015-1270-4.
[8] J.E. Marsden, On completeness of homogeneous pseudo-Riemannian manifolds, Indiana Univ. J. 22 (11) (1972–1973) 1065–1066.
[9] B. O'Neill, Semi-Riemannian Geometry, with Applications to Relativity, Pure Appl. Math., vol. 103, Academic Press Inc., NY, 1983.
[10] M.M. Postnikov, Geometry VI: Riemannian Geometry, Encyclopaedia Math. Sci., vol. 91, Springer, Berlin, 2001.
[11] A. Romero, M. Sánchez, On completeness of certain families of semi-Riemannian manifolds, Geom. Dedicata 53 (1) (1994) 103–117.
[12] A. Romero, M. Sánchez, Completeness of compact Lorentzian manifolds admitting a timelike conformal vector field, Proc. Amer. Math. Soc. 123 (9) (1995) 2831–2833.
[13] M. Sánchez, On the completeness of trajectories for some mechanical systems, Geom. Mech. Dyn. 73 (2015) 343–372.
[14] J.H.C. Whitehead, Convex regions in the geometry of paths, Quart. J. Math. Oxford Ser. 3 (1) (1932) 33–42.