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Rademacher functions in Morrey spaces [☆]

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A R T I C L E   I N F O

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A B S T R A C T

The Rademacher sums are investigated in the Morrey spaces $M_{p,w}$ on $[0,1]$ for $1 \le p < \infty$ and weight w being a quasi-concave function. They span l_2 space in $M_{p,w}$ if and only if the weight w is smaller than $\log_2^{-1/2} \frac{2}{t}$ on $(0,1)$. Moreover, if $1 < p < \infty$ the Rademacher subspace $\mathcal{R}_{p,w}$ is complemented in $M_{p,w}$ if and only if it is isomorphic to l_2. However, the Rademacher subspace $\mathcal{R}_{1,w}$ is not complemented in $M_{1,w}$ for any quasi-concave weight w. In the last part of the paper geometric structure of Rademacher subspaces in Morrey spaces $M_{p,w}$ is described. It turns out that for any infinite-dimensional subspace X of $\mathcal{R}_{p,w}$ the following alternative holds: either X is isomorphic to l_2 or X contains a subspace which is isomorphic to c_0 and is complemented in $\mathcal{R}_{p,w}$.

1. Introduction and preliminaries

The well-known Morrey spaces introduced by Morrey in 1938 [20] in relation to the study of partial differential equations were widely investigated during last decades, including the study of classical operators of harmonic analysis: maximal, singular and potential operators—in various generalizations of these spaces. In the theory of partial differential equations, along with the weighted Lebesgue spaces, Morrey-type spaces also play an important role. They appeared to be quite useful in the study of the local behavior of the solutions of partial differential equations, a priori estimates and other topics.

Let $0 < p < \infty$, w be a non-negative non-decreasing function on $[0, \infty)$, and Ω a domain in $\mathbb{R}^n$. The *Morrey space* $M_{p,w} = M_{p,w}(\Omega)$ is the class of Lebesgue measurable real functions f on Ω such that

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$$\|f\|_{M_{p,w}} = \sup_{0<r<\operatorname{diam}(\Omega),\, x_0\in\Omega} w(r) \left(\frac{1}{|B_r(x_0)|} \int\limits_{B_r(x_0)\cap\Omega} |f(t)|^p \, dt \right)^{1/p} < \infty, \tag{1}$$

where $B_r(x_0)$ is a ball with the center at x_0 and radius r. It is a quasi-Banach ideal space on Ω. The so-called ideal property means that if $|f| \le |g|$ a.e. on Ω and $g \in M_{p,w}$, then $f \in M_{p,w}$ and $\|f\|_{M_{p,w}} \le \|g\|_{M_{p,w}}$. In particular, if $w(r) = 1$ then $M_{p,w}(\Omega) = L_\infty(\Omega)$, if $w(r) = r^{1/p}$ then $M_{p,w}(\Omega) = L_p(\Omega)$ and in the case when $w(r) = r^{1/q}$ with $0 < p \le q < \infty$ $M_{p,w}(\Omega)$ are the classical Morrey spaces, denoted shortly by $M_{p,q}(\Omega)$ (see [14, Part 4.3], [15,23] and [29]). Moreover, as a consequence of the Hölder–Rogers inequality we obtain monotonicity with respect to p, that is,

$$M_{p_1,w}(\Omega) \overset{1}{\hookrightarrow} M_{p_0,w}(\Omega) \quad \text{if} \quad 0 < p_0 \le p_1 < \infty.$$

For two quasi-Banach spaces X and Y the symbol $X \overset{C}{\hookrightarrow} Y$ means that the embedding $X \subset Y$ is continuous and $\|f\|_Y \le C\|f\|_X$ for all $f \in X$.

It is easy to see that in the case when $\Omega = [0, 1]$ quasi-norm (1) can be defined as follows

$$\|f\|_{M_{p,w}} = \sup_I \ w(|I|) \left(\frac{1}{|I|} \int\limits_I |f(t)|^p \, dt \right)^{1/p}, \tag{2}$$

where the supremum is taken over all intervals I in $[0, 1]$. In what follows $|E|$ is the Lebesgue measure of a set $E \subset \mathbb{R}$.

The main purpose of this paper is the investigation of the behavior of Rademacher sums

$$R_n(t) = \sum_{k=1}^{n} a_k r_k(t), \ a_k \in \mathbb{R} \text{ for } k = 1, 2, ..., n, \text{ and } n \in \mathbb{N}$$

in general Morrey spaces $M_{p,w}$. Recall that the Rademacher functions on $[0, 1]$ are defined by $r_k(t) = \operatorname{sign}(\sin 2^k \pi t)$, $k \in \mathbb{N}, t \in [0, 1]$.

By $\mathcal{R}_{p,w}$ we denote the subspace spanned by the Rademacher functions r_k, $k = 1, 2, \ldots$ in $M_{p,w}$.

The most important tool in studying Rademacher sums in the classical L_p-spaces and in general rearrangement invariant spaces is the so-called *Khintchine inequality* (cf. [11, p. 10], [1, p. 133], [16, p. 66] and [4, p. 743]): if $0 < p < \infty$, then there exist constants $A_p, B_p > 0$ such that for any sequence of real numbers $\{a_k\}_{k=1}^n$ and any $n \in \mathbb{N}$ we have

$$A_p \Big(\sum_{k=1}^{n} |a_k|^2 \Big)^{1/2} \le \|R_n\|_{L_p[0,1]} \le B_p \Big(\sum_{k=1}^{n} |a_k|^2 \Big)^{1/2}. \tag{3}$$

Therefore, for any $1 \le p < \infty$, the Rademacher functions span in L_p an isomorphic copy of l_2. Also, the subspace $[r_n]$ is complemented in L_p for $1 < p < \infty$ and is not complemented in L_1 since no complemented infinite dimensional subspace of L_1 can be reflexive. In L_∞, the Rademacher functions span an isometric copy of l_1, which is uncomplemented.

The only non-trivial estimate for Rademacher sums in a general rearrangement invariant (r.i.) space X on $[0, 1]$ is the inequality

$$\|R_n\|_X \le C \Big(\sum_{k=1}^{n} |a_k|^2 \Big)^{1/2}, \tag{4}$$

where a constant $C > 0$ depends only on X. The reverse inequality to (4) is always true because $X \subset L_1$ and we can apply the left-hand side inequality from (3) for L_1. Already in 1930, Paley and Zygmund [22] proved estimate (4) for $X = G$, where G is the closure of $L_\infty[0,1]$ in the Orlicz space $L_M[0,1]$ generated by the function $M(u) = e^{u^2} - 1$. The proof can be found in Zygmund's classical books (see [30, p. 134] or [31, p. 214]).

Later on Rodin and Semenov [25] showed that estimate (4) holds if and only if $G \subset X$. This inclusion means that X in a certain sense "lies far" from $L_\infty[0,1]$. In particular, G is contained in every $L_p[0,1]$ for $p < \infty$. Moreover, Rodin–Semenov [26] and Lindenstrauss–Tzafriri [17, pp. 134–138] proved that $[r_n]$ is complemented in X if and only if $G \subset X \subset G'$, where G' denotes the Köthe dual space to G.

In contrast, Astashkin [3] studied the Rademacher sums in r.i. spaces which are situated very "close" to L_∞. In such a case a rather precise description of their behavior may be obtained by using the real method of interpolation (cf. [10]). Namely, every interpolation space X between the spaces L_∞ and G can be represented in the form $X = (L_\infty, G)^K_\Phi$, for some parameter Φ of the real interpolation method, and then $\|\sum_{k=1}^\infty a_k r_k\|_X \approx \|\{a_k\}_{k=1}^\infty\|_F$, where $F = (l_1, l_2)^K_\Phi$.

Investigations of Rademacher sums in r.i. spaces are well presented in the books by Lindenstrauss–Tzafriri [17], Krein–Petunin–Semenov [13] and Astashkin [4]. At the same time, a very few papers are devoted to considering Rademacher functions in Banach function spaces, which are not r.i. Recently, Astashkin–Maligranda [6] initiated studying the behavior of Rademacher sums in a weighted Korenblyum–Krein–Levin space $K_{p,w}$, for $0 < p < \infty$ and a quasi-concave function w on $[0,1]$, equipped with the quasi-norm

$$\|f\|_{K_{p,w}} = \sup_{0<x\le 1} w(x) \left(\frac{1}{x} \int_0^x |f(t)|^p \, dt \right)^{1/p} \tag{5}$$

(cf. [12,18], [28, pp. 469–470], where $w(x) = 1$). If the supremum in (2) is taken over all subsets of $[0,1]$ of measure x, then we obtain an r.i. counterpart of the spaces $M_{p,w}$ and $K_{p,w}$, the Marcinkiewicz space $M^{(*)}_{p,w}[0,1]$, with the quasi-norm

$$\|f\|_{M^{(*)}_{p,w}} = \sup_{0<x\le 1} w(x) \left(\frac{1}{x} \int_0^x f^*(t)^p \, dt \right)^{1/p}, \tag{6}$$

where f^* denotes the non-increasing rearrangement of $|f|$.

In what follows we consider only function spaces on $[0,1]$. Therefore, the weight w will be a non-negative non-decreasing function on $[0,1]$ and without loss of generality we will assume in the rest of the paper that $w(1) = 1$. Then, we have

$$L_\infty \overset{1}{\hookrightarrow} M^{(*)}_{p,w} \overset{1}{\hookrightarrow} M_{p,w} \overset{1}{\hookrightarrow} K_{p,w} \overset{1}{\hookrightarrow} L_p \tag{7}$$

because the corresponding suprema in (5), (2) and (6) are taken over larger classes of subsets of $[0,1]$.

Observe that if $\lim_{t\to 0^+} w(t) > 0$, then $M_{p,w} = M^{(*)}_{p,w} = L_\infty$, and if $\sup_{0<t\le 1} w(t)\, t^{-1/p} < \infty$, then $M_{p,w} = M^{(*)}_{p,w} = L_p$ with equivalent quasi-norms. To avoid these trivial cases, throughout the paper we will assume also that

$$\lim_{t\to 0^+} w(t) = \liminf_{t\to 0^+} \frac{t^{1/p}}{w(t)} = 0. \tag{8}$$

In particular, the latter assumptions ensure that the second and the third inclusions in (7) are proper.

Proposition 1. (i) $K_{p,w} \setminus M_{p,w} \neq \emptyset$.

(ii) *If* $w(t)\, t^{-1/p}$ *is a non-increasing function on* $(0,1]$*, then* $M_{p,w} \setminus M^{(*)}_{p,w} \neq \emptyset$.

Proof. (i) In view of (8), there exists a sequence $\{t_k\} \subset (0,1]$ such that $t_k \searrow 0, t_1 \le 1/2$ and $w(t_k)t_k^{-1/p} \nearrow \infty$. Let us denote $v(t) = w(t)\, t^{-1/p}$ and

$$g(s) := \sum_{k=1}^{\infty} \Big(v(t_k)^{-p/2} - v(t_{k+1})^{-p/2} \Big)^{1/p} (t_k - t_{k+1})^{-1/p} \chi_{(t_{k+1},t_k]}(s).$$

By definition, $\operatorname{supp} g \subset [0,1/2]$. Then, for every $k \in \mathbb{N}$

$$\int_0^{t_k} |g(s)|^p \, ds = \sum_{i=k}^{\infty} \int_{t_{i+1}}^{t_i} g(s)^p \, ds$$
$$= \sum_{i=k}^{\infty} \frac{v(t_i)^{-p/2} - v(t_{i+1})^{-p/2}}{t_i - t_{i+1}} (t_i - t_{i+1}) = v(t_k)^{-p/2}.$$

In particular, we see that $g \in L_p$. Let $f(t) := g(t + \frac{1}{2})$ for $0 \le t \le 1$. Then $\|f\|_p = \|g\|_p$, and therefore $f \in L_p$. Moreover, since $\operatorname{supp} f \subset [1/2, 1]$, we obtain $f \in K_{p,w}$. In fact,

$$\|f\|_{K_{p,w}} = \sup_{0<x\le 1} w(x) \left(\frac{1}{x} \int_0^x |f(t)|^p \, dt \right)^{1/p} = \sup_{\frac{1}{2} \le x \le 1} \frac{w(x)}{x^{1/p}} \Big(\int_{1/2}^x |f(t)|^p \, dt \Big)^{1/p}$$
$$\approx \sup_{\frac{1}{2} \le x \le 1} \Big(\int_{1/2}^x |f(t)|^p \, dt \Big)^{1/p} = \|f\|_{L_p} < \infty.$$

At the same time, if $I_k := [\frac{1}{2}, t_k + \frac{1}{2}], k = 1, 2, \ldots,$ we have

$$w(|I_k|) \left(\frac{1}{|I_k|} \int_{I_k} |f(t)|^p \, dt \right)^{1/p} = v(t_k) \Big(\int_0^{t_k} |g(s)|^p \, ds \Big)^{1/p} = v(t_k) \cdot v(t_k)^{-1/2} = v(t_k)^{1/2}.$$

Since $v(t_k) \nearrow \infty$ as $k \to \infty$, we conclude that $f \notin M_{p,w}$.

(ii) By using the conditions of proposition, it is easy to find a function $g \in L_p \setminus M^{(*)}_{p,w}$. Next, by the main result of the paper [2], there exists a function $f \in M_{p,w}$ and constants $c_0 > 0$ and $\lambda_0 > 0$ such that

$$\Big| \{ t \in [0,1] : |f(t)| > \lambda \} \Big| \ge c \Big| \{ t \in [0,1] : |g(t)| > \lambda \} \Big|$$

for all $\lambda \ge \lambda_0$. Clearly, since $g \notin M^{(*)}_{p,w}$, from the last inequality it follows that $f \notin M^{(*)}_{p,w}$. □

In particular, the proof of Proposition 1 (ii) shows that the Morrey space $M_{p,w}$ is not an r.i. space provided that its conditions hold.

For a normed ideal space $X = (X, \|\cdot\|)$ on $[0,1]$ the *Köthe dual* (or *associated space*) X' is the space of all real-valued Lebesgue measurable functions defined on $[0,1]$ such that the *associated norm*

$$\|f\|_{X'} := \sup_{g\in X,\ \|g\|_X\le 1} \int_0^1 |f(x)g(x)|\,dx$$

is finite. The Köthe dual X' is a Banach ideal space. Moreover, $X \overset{1}{\hookrightarrow} X''$ and we have $X = X''$ isometrically if and only if the norm in X has the *Fatou property*, that is, if $0 \le f_n \nearrow f$ a.e. on $[0,1]$ and $\sup_{n\in\mathbf{N}} \|f_n\| < \infty$, then $f \in X$ and $\|f_n\| \nearrow \|f\|$.

Denote by $\mathcal{D}$ the set of all dyadic intervals $I_k^n = [(k-1)\,2^{-n}, k\,2^{-n}]$, where $n = 0,1,2,\dots$ and $k = 1,2,\dots,2^n$. If f and g are nonnegative functions (or quasi-norms), then the symbol $f \approx g$ means that $C^{-1}\,g \le f \le C\,g$ for some $C \ge 1$. Moreover, we write $X \simeq Y$ if Banach spaces X and Y are isomorphic.

The paper is organized as follows. After Introduction, in Section 2 the behavior of Rademacher sums in Morrey spaces is described (see Theorem 1). The main result of Section 3 is Theorem 2, which states that the Rademacher subspace $\mathcal{R}_{p,w}, 1 < p < \infty$, is complemented in the Morrey space $M_{p,w}$ if and only if $\mathcal{R}_{p,w}$ is isomorphic to l_2 or equivalently if $\sup_{0<t\le 1} w(t)\log_2^{1/2}(2/t) < \infty$. In the case when $p = 1$ situation is different, which is the contents of Section 4, where we are proving in Theorem 3 that the subspace $\mathcal{R}_{1,w}$ is not complemented in $M_{1,w}$ for any quasi-concave weight w. Finally, in Section 5, the geometric structure of Rademacher subspaces in Morrey spaces is investigated (see Theorem 4).

2. Rademacher sums in Morrey spaces

We start with the description of behavior of Rademacher sums in the Morrey spaces $M_{p,w}$ defined by quasi-norms (2), where $0 < p < \infty$ and w is a non-decreasing function on $[0,1]$ satisfying the doubling condition $w(2t) \le C_0\, w(t)$ for all $t \in (0,1/2]$ with a certain $C_0 \ge 1$.

Theorem 1. *With constants depending only on p we have*

$$\Big\| \sum_{k=1}^{\infty} a_k\, r_k \Big\|_{M_{p,w}} \approx \|\{a_k\}_{k=1}^{\infty}\|_{l_2} + \sup_{m\in\mathbb{N}} \Big(w(2^{-m}) \sum_{k=1}^{m} |a_k| \Big). \tag{9}$$

Proof. Firstly, let $1 \le p < \infty$. Consider an arbitrary interval $I \in \mathcal{D}$, i.e., $I = I_k^m$, with $m \in \mathbb{N}$ and $k = 1,2,\dots,2^m$. Then, for every $f = \sum_{k=1}^{\infty} a_k\, r_k$, we have

$$\Big(\int_I |f(t)|^p\,dt \Big)^{1/p} = \Big(\int_I \Big| \sum_{k=1}^{m} a_k\varepsilon_k + \sum_{k=m+1}^{\infty} a_k r_k(t) \Big|^p dt \Big)^{1/p},$$

where $\varepsilon_k = \operatorname{sign} r_{k|I}, k = 1,2,\dots,m$. Since the functions

$$\sum_{k=1}^{m} a_k\varepsilon_k + \sum_{k=m+1}^{\infty} a_k r_k(t) \quad \text{and} \quad \sum_{k=1}^{m} a_k\varepsilon_k - \sum_{k=m+1}^{\infty} a_k r_k(t)$$

are equimeasurable on the interval I, it follows that

$$\Big(\int_I |f(t)|^p\,dt \Big)^{1/p} = \frac{1}{2}\Big(\int_I \Big| \sum_{k=1}^{m} a_k\varepsilon_k + \sum_{k=m+1}^{\infty} a_k r_k(t) \Big|^p dt \Big)^{1/p} + \frac{1}{2}\Big(\int_I \Big| \sum_{k=1}^{m} a_k\varepsilon_k - \sum_{k=m+1}^{\infty} a_k r_k(t) \Big|^p dt \Big)^{1/p},$$

whence by the Minkowski inequality we obtain

$$\Big(\int_I |f(t)|^p\,dt\Big)^{1/p} \ge \Big(\int_I \Big|\sum_{k=1}^m a_k\varepsilon_k\Big|^p\,dt\Big)^{1/p}$$
$$= 2^{-m/p}\Big|\sum_{k=1}^m a_k\varepsilon_k\Big|$$

for every $m = 1, 2, \ldots$. Clearly, one may find $i \in \{1, 2, \ldots, 2^m\}$ such that $r_{k|I_i^m} = \operatorname{sign} a_k$, for all $k = 1, 2, \ldots, m$. Therefore, for every $m = 1, 2, \ldots$

$$\left(\frac{1}{|I_i^m|}\int_{I_i^m} |f(t)|^p\,dt\right)^{1/p} \ge \sum_{k=1}^m |a_k|,$$

and so

$$\|f\|_{M_{p,w}} \ge \sup_{m\in\mathbb{N}} w(2^{-m})\sum_{k=1}^m |a_k|. \tag{10}$$

On the other hand, by (7) and (3) we have

$$\|f\|_{M_{p,w}} \ge \|f\|_{L_p} \ge A_p\,\|\{a_k\}_{k=1}^\infty\|_{l_2}.$$

Combining these inequalities, we obtain

$$\|f\|_{M_{p,w}} \ge \frac{A_p}{2}\Big(\|\{a_k\}_{k=1}^\infty\|_{l_2} + \sup_{m\in\mathbb{N}} w(2^{-m})\sum_{k=1}^m |a_k|\Big).$$

Let us prove the reverse inequality. For a given interval $I \subset [0, 1]$ we can find two adjacent dyadic intervals I_1 and I_2 of the same length such that

$$I \subset I_1 \cup I_2 \quad \text{and} \quad \frac{1}{2}\,|I_1| \le |I| \le 2\,|I_1|. \tag{11}$$

If $|I_1| = |I_2| = 2^{-m}$, then by the Minkowski triangle inequality and inequality in (3) we have

$$\Big(\int_{I_1} |f(t)|^p\,dt\Big)^{1/p} = \Big(\int_{I_1} \Big|\sum_{k=1}^m a_k\varepsilon_k + \sum_{k=m+1}^\infty a_k r_k(t)\Big|^p\,dt\Big)^{1/p}$$
$$\le \Big(\int_{I_1} \Big|\sum_{k=1}^m a_k\varepsilon_k\Big|^p\,dt\Big)^{1/p} + \Big(\int_{I_1} \Big|\sum_{k=m+1}^\infty a_k r_k(t)\Big|^p\,dt\Big)^{1/p}$$
$$\le 2^{-m/p}\sum_{k=1}^m |a_k| + 2^{-m/p}\Big(\int_0^1 \Big|\sum_{k=m+1}^\infty a_k r_{k-m}(t)\Big|^p\,dt\Big)^{1/p}$$
$$\le 2^{-m/p}\sum_{k=1}^m |a_k| + 2^{-m/p}\,B_p\,\|\{a_k\}_{k=1}^\infty\|_{l_2}.$$

The same estimate holds also for the integral $\big(\int_{I_2} |f(t)|^p\,dt\big)^{1/p}$. Therefore, by (11),

$$\Big(\frac{1}{|I|}\int_I |f(t)|^p\,dt\Big)^{1/p} \le 2^{1/p}\Big(\frac{1}{|I_1|}\int_{I_1} |f(t)|^p\,dt + \frac{1}{|I_2|}\int_{I_2} |f(t)|^p\,dt\Big)^{1/p}$$
$$\le 4^{1/p}\,B_p\Big(\sum_{k=1}^m |a_k| + \|\{a_k\}_{k=1}^\infty\|_{l_2}\Big)$$

and, by the doubling condition,

$$w(|I|)\Big(\frac{1}{|I|}\int_I |f(t)|^p\,dt\Big)^{1/p} \le w(2\cdot 2^{-m})\,4^{1/p}\,B_p\Big(\sum_{k=1}^m |a_k| + \|\{a_k\}_{k=1}^\infty\|_{l_2}\Big)$$
$$\le C_0\cdot 4^{1/p}\,B_p\,w(2^{-m})\Big(\sum_{k=1}^m |a_k| + \|\{a_k\}_{k=1}^\infty\|_{l_2}\Big).$$

Hence, using definition of the norm in $M_{p,w}$, we obtain

$$\|f\|_{M_{p,w}} \le C_0\cdot 4^{1/p}\,B_p\Big(\sup_{m\in\mathbb{N}} w(2^{-m})\sum_{k=1}^m |a_k| + \|\{a_k\}_{k=1}^\infty\|_{l_2}\Big).$$

The same proof works also in the case when $0<p<1$ with the only change that the corresponding L_p-triangle inequality holds with the constant $2^{1/p-1}$. $\square$

In the rest of the paper, a weight function w is assumed to be *quasi-concave on* $[0,1]$, that is, $w(0)=0$, w is non-decreasing, and $w(t)/t$ is non-increasing on $(0,1]$. Moreover, as above, we assume that $w(1)=1$.

Recall that a basic sequence $\{x_k\}$ in a Banach space X is called *subsymmetric* if it is unconditional and is equivalent in X to any subsequence of $\{x_k\}$.

Corollary 1. *For every $1\le p<\infty$ $\{r_k\}$ is an unconditional and not subsymmetric basic sequence in $M_{p,w}$.*

Corollary 2. *Let $1\le p<\infty$. The Rademacher functions span l_2 space in $M_{p,w}$ if and only if*

$$\sup_{0<t\le 1} w(t)\log_2^{1/2}(2/t) < \infty. \tag{12}$$

Proof. If (12) holds, then for all $m\in\mathbb{N}$ we have $w(2^{-m})\,m^{1/2}\le C$. Using the Hölder–Rogers inequality, we obtain

$$w(2^{-m})\sum_{k=1}^m |a_k| \le w(2^{-m})\Big(\sum_{k=1}^m |a_k|^2\Big)^{1/2} m^{1/2} \le C\Big(\sum_{k=1}^m |a_k|^2\Big)^{1/2}.$$

Therefore, from (9) it follows that $\|\sum_{k=1}^\infty a_k\,r_k\|_{M_{p,w}} \approx \|\{a_k\}\|_{l_2}$.

Conversely, suppose that condition (12) does not hold. Then, by the quasi-concavity of w, there exists a sequence of natural numbers $m_k\to\infty$ such that

$$w(2^{-m_k})\,m_k^{1/2}\to\infty \text{ as } k\to\infty. \tag{13}$$

Consider the Rademacher sums $R_k(t)=\sum_{i=1}^k a_i^k\,r_i(t)$ corresponding to the sequences of coefficients $a^k = (a_i^k)_{i=1}^{m_k}$, where $a_i^k = m_k^{-1/2}$, $1\le i\le m_k$. We have $\|a^k\|_{l_2}=1$ for all $k=1,2,\dots$ However, $\sum_{i=1}^{m_k} a_i^k = m_k^{1/2}$ $(k=1,2,\dots)$, which together with (13) and (9) imply that $\|R_k\|_{M_{p,w}}\to\infty$ as $k\to\infty$. $\square$

Remark 1. The Rademacher functions span l_2 in each of the spaces $M_{p,w}^{(*)}, M_{p,w}$ and $K_{p,w}$, $1 \le p < \infty$ (see embeddings (7)). In fact, the Orlicz space L_M generated by the function $M(u) = e^{u^2} - 1$ coincides with the Marcinkiewicz space $M_{1,v}^{(*)}$ with $v(t) = \log_2^{-1/2}(2/t)$ (cf. [4, Lemma 3.2]). Recalling that G is the closure of L_∞ in $M_{1,v}^{(*)}$ we note that the embedding $G \subset M_{p,w}^{(*)}$ holds if and only if (12) is satisfied. Therefore, by already mentioned Rodin–Semenov theorem (cf. [25]; see also [17, Theorem 2.b.4]), the Rademacher functions span l_2 in $M_{p,w}^{(*)}$ if and only if (12) holds.

Moreover, it is instructive to compare the behavior of Rademacher sums in the spaces $M_{1,w}^{(*)}, M_{1,w}$ and $K_{1,w}$ in the case when $w(t) = \log_2^{-1/q}(2/t)$, where $q > 2$. Then (12) does not hold and

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{M_{1,w}^{(*)}} \approx \|\{a_k\}_{k=1}^{\infty}\|_{l_2} + \sup_{m\in\mathbb{N}} m^{-1/q} \sum_{k=1}^{m} a_k^*,$$

where $\{a_k^*\}$ is the non-increasing rearrangement of $\{|a_k|\}_{k=1}^{\infty}$ (cf. Rodin–Semenov [25, p. 221] and Pisier [24]; see also Marcus–Pisier [19, pp. 277–278]),

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{M_{1,w}} \approx \|\{a_k\}_{k=1}^{\infty}\|_{l_2} + \sup_{m\in\mathbb{N}} m^{-1/q} \sum_{k=1}^{m} |a_k| \quad \text{by (9)}, \quad \text{and}$$

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{K_{1,w}} \approx \|\{a_k\}_{k=1}^{\infty}\|_{l_2} + \sup_{m\in\mathbb{N}} m^{-1/q} \Big| \sum_{k=1}^{m} a_k \Big| \qquad \text{(cf. [6, Theorem 2]).}$$

Now, we pass to studying the problem of complementability of the closed linear span $\mathcal{R}_{p,w} := [r_n]_{n=1}^{\infty}$ in the space $M_{p,w}$. Since the results turn out to be different for $p > 1$ and $p = 1$, we consider these cases separately.

3. Complementability of Rademacher subspaces in Morrey spaces $M_{p,w}$ for $p > 1$

Theorem 2. *Let $1 < p < \infty$. The subspace $\mathcal{R}_{p,w}$ is complemented in the Morrey space $M_{p,w}$ if and only if condition (12) holds.*

To prove this theorem we will need the following auxiliary assertion.

Proposition 2. *If condition (12) does not hold, then the subspace $\mathcal{R}_{p,w}$ contains a complemented (in $\mathcal{R}_{p,w}$) subspace isomorphic to c_0.*

Proof. Since w is quasi-concave, by the assumption, we have

$$\limsup_{n\to\infty} w(2^{-n})\sqrt{n} = \infty. \tag{14}$$

We select an increasing sequence of positive integers as follows. Let n_1 be the least positive integer satisfying the inequality $w(2^{-n_1})\sqrt{n_1} \ge 2$. As it is easy to see $w(2^{-n_1})\sqrt{n_1} < 2^2$. By induction, assume that the numbers $n_1 < n_2 < \ldots < n_{k-1}$ are chosen. Applying (14), we take for n_k the least positive integer such that

$$w(2^{-n_k})\sqrt{n_k - n_{k-1}} \ge 2^k. \tag{15}$$

Then, obviously,

$$w(2^{-n_k})\sqrt{n_k - n_{k-1}} < 2^{k+1}. \tag{16}$$

Thus, we obtain a sequence $0 = n_0 < n_1 < \ldots$ satisfying inequalities (15) and (16) for all $k \in \mathbb{N}$. Let us consider the block basis $\{v_k\}_{k=1}^{\infty}$ of the Rademacher system defined as follows:

$$v_k = \sum_{i=n_{k-1}+1}^{n_k} a_i\, r_i, \quad \text{where} \quad a_i = \frac{1}{(n_k - n_{k-1})\, w(2^{-n_k})} \quad \text{for} \quad n_{k-1} < i \le n_k.$$

Let us recall that, by Theorem 1, if $R = \sum_{k=1}^{\infty} b_k\, r_k$, then $\|R\|_{M_{p,w}} \approx \|R\|_{l_2} + \|R\|_w$, where

$$\|R\|_{l_2} = \Big(\sum_{k=1}^{\infty} b_k^2\Big)^{1/2} \quad \text{and} \quad \|R\|_w = \sup_{m \in \mathbb{N}} \Big(w(2^{-m}) \sum_{k=1}^{m} |b_k|\Big).$$

Now, we estimate the norm of v_k, $k = 1, 2, \ldots$, in $M_{p,w}$. At first, by (15),

$$\Big(\sum_{i=n_{k-1}+1}^{n_k} a_i^2\Big)^{1/2} = \frac{1}{\sqrt{n_k - n_{k-1}}\, w(2^{-n_k})} \le 2^{-k}, \ k = 1, 2, \ldots \tag{17}$$

Moreover, taking into account (15), (16) and the choice of n_k, for every $k \in \mathbb{N}$ and $n_{k-1} < i \le n_k$ we have

$$w(2^{-i}) \sum_{j=n_{k-1}+1}^{i} a_j = \frac{w(2^{-i})(i - n_{k-1})}{(n_k - n_{k-1}) w(2^{-n_k})} \le \frac{2^{k+1}\sqrt{i - n_{k-1}}}{2^k \sqrt{n_k - n_{k-1}}} \le 2.$$

As a result, from preceding estimates and Theorem 1 (see also inequality (10) from its proof) it follows that

$$1 = w(2^{-n_k}) \sum_{i=n_{k-1}+1}^{n_k} a_i \le \|v_k\|_{M_{w,p}} \le C \tag{18}$$

for some constant $C > 0$ and every $k \in \mathbb{N}$. Thus, $\{v_k\}_{k=1}^{\infty}$ is a semi-normalized block basis of $\{r_k\}_{k=1}^{\infty}$ in $M_{p,w}$.

Further, let us select a subsequence $\{m_i\} \subset \{n_k\}$ such that

$$w(2^{-m_{i+1}}) \le \frac{1}{2} w(2^{-m_i}), \quad i = 1, 2, \ldots \tag{19}$$

and denote by $\{u_i\}_{i=1}^{\infty}$ the corresponding subsequence of $\{v_k\}_{k=1}^{\infty}$. Then, u_i can be represented as follows:

$$u_i = \sum_{k=l_i}^{m_i} a_k\, r_k, \quad \text{where} \quad l_i = n_{j_i - 1} + 1, \ m_i = n_{j_i}, \ j_1 < j_2 < \ldots.$$

We show that the sequence $\{u_i\}_{i=1}^{\infty}$ is equivalent in $M_{p,w}$ to the unit vector basis of c_0.

Let $f = \sum_{i=1}^{\infty} \beta_i\, u_i, \beta_i \in \mathbb{R}$. Then, we have

$$f = \sum_{i=1}^{\infty} \beta_i \sum_{k=l_i}^{m_i} a_k\, r_k = \sum_{k=1}^{\infty} \gamma_k\, r_k,$$

where $\gamma_k = \beta_i\, a_k, l_i \le k \le m_i, i = 1, 2, \ldots$ and $\gamma_k = 0$ if $k \notin \cup_{i=1}^{\infty} [l_i, m_i]$. To estimate $\|f\|_w$, assume, at first, that $m_s \le q < l_{s+1}$ for some $s \in \mathbb{N}$. Then,

$$\sum_{k=1}^{q} |\gamma_k| = \sum_{i=1}^{s} |\beta_i| \sum_{k=l_i}^{m_i} a_k = \sum_{i=1}^{s} |\beta_i| \frac{1}{w(2^{-m_i})} \le \|(\beta_i)\|_{c_0} \sum_{i=1}^{s} \frac{1}{w(2^{-m_i})},$$

and from (19) together with the fact that w increases it follows that

$$w(2^{-q})\sum_{k=1}^{q}|\gamma_k| \le \|(\beta_i)\|_{c_0}\sum_{i=1}^{s}\frac{w(2^{-m_s})}{w(2^{-m_i})} \le \|(\beta_i)\|_{c_0}\sum_{i=0}^{\infty}2^{-i} = 2\,\|(\beta_i)\|_{c_0}.$$

Otherwise, we have $l_s \le q < m_s, s \in \mathbb{N}$. Then, similarly,

$$\begin{aligned}\sum_{k=1}^{q}|\gamma_k| &\le \left(\sum_{i=1}^{s-1}\frac{1}{w(2^{-m_i})} + \sum_{k=l_s}^{q}a_k\right)\|(\beta_i)\|_{c_0}\\ &= \left(\sum_{i=1}^{s-1}\frac{1}{w(2^{-m_i})} + \frac{q-l_s+1}{(m_s-l_s+1)\,w(2^{-m_s})}\right)\|(\beta_i)\|_{c_0}.\end{aligned}$$

Since $m_s = n_{j_s}$ and $l_s = n_{j_s-1}+1$ for some $j_s \in \mathbb{N}$, in view of (15), (19) and the choice of n_{j_s}, we obtain

$$\begin{aligned}w(2^{-q})\sum_{k=1}^{q}|\gamma_k| &\le \left(\sum_{i=1}^{s-1}\frac{w(2^{-m_{s-1}})}{w(2^{-m_i})} + \frac{w(2^{-q})(q-l_s+1)}{(m_s-l_s+1)\,w(2^{-m_s})}\right)\|(\beta_i)\|_{c_0}\\ &\le \left(\sum_{i=0}^{\infty}2^{-i} + \frac{2^{j_s+1}\sqrt{q-l_s+1}}{2^{j_s}\sqrt{m_s-l_s+1}}\right)\|(\beta_i)\|_{c_0} \le 4\,\|(\beta_i)\|_{c_0}.\end{aligned}$$

Combining this with the previous estimate, we obtain that $\|f\|_w \le 4\,\|(\beta_i)\|_{c_0}$. Moreover, by (17),

$$\|f\|_{l_2}^2 \le \sum_{i=1}^{\infty}\beta_i^2\sum_{k=l_i}^{m_i}a_k^2 \le \|(\beta_i)\|_{c_0}^2.$$

Therefore, again by Theorem 1,

$$\|f\|_{M_{w,p}} \le C\left(\|f\|_{l_2} + \|f\|_w\right) \le 5\,C\,\|(\beta_i)\|_{c_0}.$$

In opposite direction, taking into account the fact that $\{u_i\}$ is an unconditional sequence in $M_{p,w}$, by (18), we obtain

$$\|f\|_{M_{w,p}} \ge c\sup_{i\in\mathbb{N}}|\beta_i|\,\|u_i\|_{M_{w,p}} \ge c\,\|(\beta_i)\|_{c_0},$$

for some constant $c > 0$. Thus, we have proved that $E := [u_n]_{M_{p,w}} \simeq c_0$. Since $\mathcal{R}_{p,w}$ is separable, Sobczyk's theorem (see, for example, [1, Corollary 2.5.9]) implies that E is a complemented subspace in $\mathcal{R}_{p,w}$. □

Proof of Theorem 2. At first, let us assume that relation (12) holds. Then, by Corollary 2, $\mathcal{R}_{p,w} \simeq l_2$. Therefore, since $M_{p,w} \overset{1}{\hookrightarrow} L_p$, by the Khintchine inequality, the orthogonal projection P generated by the Rademacher system satisfies the following:

$$\|Pf\|_{M_{p,w}} \approx \|Pf\|_{L_p} \le \|P\|_{L_p\to L_p}\|f\|_{L_p} \le \|P\|_{L_p\to L_p}\|f\|_{M_{p,w}},$$

because P is bounded in $L_p, 1 < p < \infty$. Hence, $P : M_{p,w} \to M_{p,w}$ is bounded.

Conversely, we argue in a similar way as in the proof of Theorem 4 in [5]. Suppose that the subspace $\mathcal{R}_{p,w} = [r_n]_{n=1}^{\infty}$ is complemented in $M_{p,w}$ and let $P_1 : M_{p,w} \to M_{p,w}$ be a bounded linear projection whose range is $\mathcal{R}_{p,w}$. By Proposition 2, there is a subspace E complemented in $\mathcal{R}_{p,w}$ and such that $E \simeq c_0$. Let

$P_2 : \mathcal{R}_{p,w} \to E$ be a bounded linear projection. Then $P := P_2 \circ P_1$ is a linear projection bounded in $M_{p,w}$ whose image coincides with E. Thus, $M_{p,w}$ contains a complemented subspace $E \simeq c_0$.

Since $M_{p,w}$ is a conjugate space (more precisely, $M_{p,w} = (H^{q,u})^*$, where $H^{q,u}$ is the "block space" and $1/p + 1/q = 1$—see, for example, [29, Proposition 5]; see also [9] and [21]), this contradicts the well-known result due to Bessaga–Pełczyński saying that arbitrary conjugate space cannot contain a complemented subspace isomorphic to c_0 (see [8, Corollary 4] and [7, Theorem 4 with its proof]). This contradiction proves the theorem. □

4. Rademacher subspace $\mathcal{R}_{1,w}$ is not complemented in Morrey space $M_{1,w}$

Theorem 3. *For every quasi-concave weight w the subspace $\mathcal{R}_{1,w}$ is not complemented in the Morrey space $M_{1,w}$.*

In the proof we consider two cases separately, depending if the condition (12) is satisfied or not.

Proof of Theorem 3: the case when (12) does not hold. On the contrary, we suppose that $\mathcal{R}_{1,w}$ is complemented in $M_{1,w}$. Then, if Q is a bounded linear projection from $M_{1,w}$ onto $\mathcal{R}_{1,w}$, by Theorem 1, for every $p \in (1, \infty)$ and $f \in M_{p,w}$, we have

$$\|Qf\|_{M_{p,w}} \approx \|Qf\|_{M_{1,w}} \le \|Q\| \, \|f\|_{M_{1,w}} \le \|Q\| \, \|f\|_{M_{p,w}}.$$

Thus, Q is a bounded projection from $M_{p,w}$ onto $\mathcal{R}_{p,w}$, which contradicts Theorem 2. □

To prove the assertion in the case when (12) holds, we will need auxiliary results.

Let $M^d_{p,w}$ be the dyadic version of the space $M_{p,w}$, $1 \le p < \infty$, consisting of all measurable functions $f : [0,1] \to \mathbb{R}$ such that

$$\|f\|_{M^d_{p,w}} = \sup_{I \in \mathcal{D}} w(|I|) \left(\frac{1}{|I|} \int_I |f(t)|^p \, dt \right)^{1/p} < \infty.$$

Lemma 1. *For every $1 \le p < \infty$ $M_{p,w} = M^d_{p,w}$ and*

$$\|f\|_{M^d_{p,w}} \le \|f\|_{M_{p,w}} \le 4 \, \|f\|_{M^d_{p,w}}. \tag{20}$$

Proof. The left-hand side inequality in (20) is obvious. To prove the right-hand side one, we observe that for any interval $I \subset [0,1]$ we can find adjacent dyadic intervals I_1 and I_2 of the same length such that $I \subset I_1 \cup I_2$ and $\frac{1}{2}|I_1| \le |I| \le 2|I_1|$. Then, by the quasi-concavity of w,

$$\begin{aligned}
w(|I|) \left(\frac{1}{|I|} \int_I |f(t)|^p \, dt \right)^{1/p} &= \frac{w(|I|)}{|I|} \Big(|I|^{p-1} \int_I |f(t)|^p \, dt \Big)^{1/p} \\
&\le \frac{w(\frac{1}{2}|I_1|)}{\frac{1}{2}|I_1|} \Big[2^{p-1} |I_1|^{p-1} \Big(\int_{I_1} |f(t)|^p \, dt + \int_{I_2} |f(t)|^p \, dt \Big) \Big]^{1/p} \\
&\le 2^{2-1/p} \, w(|I_1|) \left(\frac{1}{|I_1|} \int_{I_1} |f(t)|^p \, dt + \frac{1}{|I_2|} \int_{I_2} |f(t)|^p \, dt \right)^{1/p}
\end{aligned}$$

$$\leq 4 \sup_{I \in \mathcal{D}} w(|I|) \left(\frac{1}{|I|} \int_I |f(t)|^p \, dt \right)^{1/p} = 4 \, \|f\|_{M^d_{p,w}}.$$

Taking the supremum over all intervals $I \subset [0,1]$, we obtain the right-hand side inequality in (20). □

Let P be the orthogonal projection generated by the Rademacher sequence, i.e.,

$$Pf(t) := \sum_{k=1}^{\infty} \int_0^1 f(s) r_k(s) \, ds \cdot r_k(t).$$

Proposition 3. *Let $1 \leq p < \infty$. If $\mathcal{R}_{p,w}$ is a complemented subspace in $M_{p,w}$, then the projection P is bounded in $M_{p,w}$.*

Proof. By Lemma 1, it is sufficient to prove the same assertion for the dyadic space $M^d_{p,w}$. We almost repeat the arguments from the proof of the similar result for r.i. function spaces (see [26] or [4, Theorem 3.4]).

Let $t = \sum_{i=1}^{\infty} \alpha_i 2^{-i}$ and $u = \sum_{i=1}^{\infty} \beta_i 2^{-i}$ ($\alpha_i, \beta_i = 0, 1$) be binary expansions of the numbers $t, u \in [0,1]$. Define the following operation:

$$t \oplus u = \sum_{i=1}^{\infty} 2^{-i} [(\alpha_i + \beta_i) \bmod 2].$$

One can easily verify that this operation transforms the segment $[0,1]$ into a compact Abelian group. For every $u \in [0,1]$, the transformation $w_u(s) = s \oplus u$ preserves the Lebesgue measure on $[0,1]$, i.e., for any measurable $E \subset [0,1]$, its inverse image $w_u^{-1}(E)$ is measurable and $|w_u^{-1}(E)| = |E|$. Moreover, w_u maps any dyadic interval onto some dyadic interval. Hence, the operators $T_u f = f \circ w_u$ ($0 \leq u \leq 1$) act isometrically in $M^d_{p,w}$. From the definition of the Rademacher functions it follows that the subspace $\mathcal{R}_{p,w}$ is invariant with respect to these operators. Therefore, by the Rudin theorem (see [27, Theorem 5.18, pp. 134–135]), there exists a bounded linear projection Q acting from $M^d_{p,w}$ onto $\mathcal{R}_{p,w}$ and commuting with all operators T_u ($0 \leq u \leq 1$). We show that $Q = P$.

First of all, the projection Q has the representation

$$Qf(t) = \sum_{i=1}^{\infty} Q_i(f) \, r_i(t), \tag{21}$$

where by Theorem 1, Q_i ($i = 1, 2, \ldots$) are linear bounded functionals on $M^d_{p,w}$. It is obvious that

$$Q_i(r_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{22}$$

Consider the sets

$$U_i = \left\{ u \in [0,1] : u = \sum_{j=1}^{\infty} \alpha_j 2^{-j}, \;\; \alpha_i = 0 \right\}, \quad U_i^c = [0,1] \backslash U_i.$$

One can check that

$$r_i(t \oplus u) = \begin{cases} \;\; r_i(t) & \text{if } u \in U_i, \\ -r_i(t) & \text{if } u \in U_i^c. \end{cases}$$

Due to the relation $T_u Q = Q T_u$ $(0 \le u \le 1)$ this implies

$$Q_i(T_u f) = \begin{cases} Q_i(f) & \text{if } u \in U_i, \\ -Q_i(f) & \text{if } u \in U_i^c. \end{cases}$$

Taking into account that $|U_i| = |U_i^c| = 1/2$, we find that

$$\int\limits_{U_i} Q_i(T_u f)\, du = \frac{1}{2} Q_i(f) \quad \text{and} \quad \int\limits_{U_i^c} Q_i(T_u f)\, du = -\frac{1}{2} Q_i(f).$$

Thanks to the boundedness of Q_i, this functional can be moved outside the integral; therefore, we obtain

$$Q_i(f) = Q_i\Big(\int\limits_{U_i} T_u f\, du - \int\limits_{U_i^c} T_u f\, du\Big). \tag{23}$$

Since

$$\{s \in [0,1] :\ s = t \oplus u,\ \ u \in U_i\} = \begin{cases} U_i & \text{if } t \in U_i, \\ U_i^c & \text{if } t \in U_i^c, \end{cases}$$

$$\{s \in [0,1] :\ s = t \oplus u,\ \ u \in U_i^c\} = \begin{cases} U_i^c & \text{if } t \in U_i, \\ U_i & \text{if } t \in U_i^c, \end{cases}$$

and the transformation ω_u preserves Lebesgue measure on $[0,1]$, we have

$$\int\limits_{U_i} T_u f(t) du \;=\; \int\limits_{U_i} f(s)\, ds \cdot \chi_{U_i}(t) \;+\; \int\limits_{U_i^c} f(s)\, ds \cdot \chi_{U_i^c}(t)$$

and

$$\int\limits_{U_i^c} T_u f(t)\, du \;=\; \int\limits_{U_i^c} f(s)\, ds \cdot \chi_{U_i}(t) \;+\; \int\limits_{U_i} f(s)\, ds \cdot \chi_{U_i^c}(t).$$

It is easy to see that $r_i(t) = \chi_{U_i}(t) - \chi_{U_i^c}(t)$. Therefore, from the last two relations it follows that

$$\int\limits_{U_i} T_u f(t)\, du - \int\limits_{U_i^c} T_u f(t)\, du = \int\limits_0^1 f(s) r_i(s)\, ds \cdot r_i(t).$$

This and (21)–(23) yield

$$Q_i(f) = \int\limits_0^1 f(s) r_i(s)\, ds, \quad i = 1, 2, \ldots,$$

i.e., $Q = P$, and Proposition 3 is proved. $\square$

The following result, in fact, is known. However, we provide its proof for completeness.

Lemma 2. *Suppose that the Rademacher sequence is equivalent in a Banach function lattice X on $[0,1]$ to the unit vector basis in l_2, i.e., for some constant $C>0$ and all $a=(a_k)_{k=1}^{\infty}\in l_2$*

$$C^{-1}\,\|a\|_{l_2}\le \Big\|\sum_{k=1}^{\infty}a_k\,r_k\Big\|_X\le C\,\|a\|_{l_2}. \tag{24}$$

Moreover, let $\{r_k\}\subset X'$, where X' is the Köthe dual space for X. Then, the orthogonal projection P is bounded in X if and only if there exists a constant $C_1>0$ such that for every $a=(a_k)_{k=1}^{\infty}\in l_2$

$$\Big\|\sum_{k=1}^{\infty}a_k\,r_k\Big\|_{X'}\le C_1\,\|a\|_{l_2}. \tag{25}$$

Proof. First, suppose that (25) holds. For arbitrary $f\in X$, we set

$$c_k(f)=\int_0^1 f(s)r_k(s)\,ds,\;\; k=1,2,\dots$$

By (25), for every $n\in\mathbb{N}$, we have

$$\begin{aligned}\sum_{k=1}^{n}c_k(f)^2&=\int_0^1 f(s)\sum_{k=1}^{n}c_k(f)r_k(s)\,ds\le\|f\|_X\Big\|\sum_{k=1}^{n}c_k(f)r_k\Big\|_{X'}\\&\le C_1\,\|f\|_X\Big(\sum_{k=1}^{n}c_k(f)^2\Big)^{1/2},\end{aligned}$$

and therefore, taking into account (24), we obtain

$$\|Pf\|_X\le C\Big(\sum_{k=1}^{\infty}c_k(f)^2\Big)^{1/2}\le C\cdot C_1\|f\|_X.$$

Thus, P is bounded in X.

Conversely, if P is a bounded projection in X, then from (24) it follows that

$$\begin{aligned}\int_0^1 f(t)\sum_{k=1}^{n}a_kr_k(t)\,dt&=\sum_{k=1}^{n}a_k\cdot c_k(f)\le\|a\|_{l_2}\Big(\sum_{k=1}^{n}c_k(f)^2\Big)^{1/2}\\&\le C\,\|a\|_{l_2}\,\|Pf\|_X\le C\,\|P\|_{X\to X}\,\|a\|_{l_2}\,\|f\|_X\end{aligned}$$

for each $n\in\mathbb{N}$, all $a=(a_k)_{k=1}^{\infty}\in l_2$ and $f\in X$. Hence,

$$\Big\|\sum_{k=1}^{\infty}a_k\,r_k\Big\|_{X'}\le C\,\|P\|_{X\to X}\,\|a\|_{l_2},$$

and (25) is proved. □

Proof of Theorem 3: the case when (12) holds. In view of Lemmas 1, 2 and Proposition 2 it is sufficient to prove that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \Big\| \sum_{k=1}^{n} r_k \Big\|_{(M^d_{1,w})'} = \infty. \tag{26}$$

For every $m \in \mathbb{N}$ such that $\sqrt{m/2} \in \mathbb{N}$ we consider the set

$$E_m := \{t \in [0,1] : 0 \le \sum_{k=1}^{2m} r_k(t) \le \sqrt{m/2}\,\}.$$

Clearly, $E_m = \bigcup_{k\in S_m} I_k^{2m}$, where $S_m \subset \{1, 2, \dots, 2^{2m}\}$. Also, it is easy to see that $|E_m| \to 0$ as $m \to \infty$. Denoting

$$f_m := \frac{1}{w(|E_m|)}\, \chi_{E_m},\ \ m \in \mathbb{N},$$

we show that

$$\|f_m\|_{M^d_{1,w}} \le 1 \quad \text{for all} \quad m \in \mathbb{N}. \tag{27}$$

In fact, let I be a dyadic interval from $[0, 1]$. Clearly, we can assume that $I \cap E_m \ne \varnothing$. Then, by using the quasi-concavity of w, we have

$$\frac{w(|I|)}{|I|} \int_I |f_m(t)|\, dt = \frac{w(|I|)}{|I|} \cdot \frac{|I \cap E_m|)}{w(|E_m|)} \le \frac{w(|I|)}{|I|} \cdot \frac{|I \cap E_m|}{w(|I \cap E_m|)} \le 1,$$

and (27) is proved.

From (27) it follows that

$$\begin{aligned}\Big\| \sum_{k=1}^{2m} r_k \Big\|_{(M^d_{1,w})'} &\ge \int_0^1 \Big| \sum_{k=1}^{2m} r_k(t) \Big| \cdot f_m(t)\, dt = \frac{1}{w(|E_m|)} \int_{E_m} \Big| \sum_{k=1}^{2m} r_k(t) \Big|\, dt \\ &= \frac{1}{w(|E_m|)}\, 2^{-2m} \sum_{i\in S_m} \Big| \sum_{k=1}^{2m} \varepsilon_k^i \Big|,\end{aligned}$$

where $\varepsilon_k^i = \operatorname{sign} r_k|_{\Delta_i^{2m}}$, $k = 1, 2, \dots, 2m$, $i \in S_m$. Denoting $\sigma_m := \sum_{i\in S_m} |\sum_{k=1}^{2m} \varepsilon_k^i|$, by the definition of E_m, we obtain

$$\sigma_m = 2 \cdot \sum_{m-\sqrt{m/2}\le k\le m} C_k^{2m}(m-k) = 2 \cdot \sum_{k=1}^{\sqrt{m/2}} C_{m-k}^{2m} \cdot k, \tag{28}$$

where $C_i^n = \frac{n!}{i!(n-i)!}$, $n = 1, 2, \dots$, $i = 0, 1, \dots, n$. Let us estimate the ratio C_{m-k}^{2m}/C_m^{2m} for $1 \le k \le \sqrt{m/2}$ from below. At first,

$$\begin{aligned}\frac{C_{m-k}^{2m}}{C_m^{2m}} &= \frac{(m!)^2}{(m-k)!(m+k)!} = \frac{(m-k+1)\cdot \ldots \cdot (m-1)\cdot m}{(m+1)\cdot \ldots \cdot (m+k-1)\cdot (m+k)} \\ &= \frac{m}{m+k} \cdot \frac{(m-k+1)\cdot \ldots \cdot (m-1)}{(m+1)\cdot \ldots \cdot (m+k-1)} = \frac{m}{m+k} \cdot \prod_{j=1}^{k-1} \frac{1-\frac{j}{m}}{1+\frac{j}{m}} \\ &= \frac{m}{m+k} \cdot \exp\Big(\sum_{j=1}^{k-1} \log \frac{1-\frac{j}{m}}{1+\frac{j}{m}} \Big).\end{aligned}$$

Next, we will need the following elementary inequality

$$\log \frac{1-t}{1+t} + 2\,t + 2\,t^3 \geq 0 \quad \text{for all} \quad 0 \leq t \leq \frac{1}{2}. \tag{29}$$

Indeed, we set

$$\varphi(t) := \log \frac{1-t}{1+t} + 2\,t + 2\,t^3.$$

Then, $\varphi(0) = 0$. Moreover, for all $t \in [0, 1/2]$ we have

$$\varphi'(t) = -\frac{2}{1-t^2} + 2 + 6t^2 = \frac{2t^2(2-3t^2)}{1-t^2} \geq 0.$$

Thus, $\varphi(t)$ increases on the interval $[0, 1/2]$, and (29) is proved.

From the above formula, inequality (29) and the condition $1 \leq k \leq \sqrt{m/2}$ we obtain

$$\begin{aligned} \frac{C^{2m}_{m-k}}{C^{2m}_m} &\geq \frac{m}{m+k} \exp\Big(-\frac{2}{m}\sum_{j=1}^{k-1} j - \frac{2}{m^3}\sum_{j=1}^{k-1} j^3\Big) \\ &= \frac{m}{m+k} \exp\Big(\frac{-k(k-1)}{m}\Big) \exp\Big(\frac{-(k-1)^2k^2}{2m^3}\Big) \\ &\geq \frac{1}{2} \exp\Big(-\frac{k^2}{m} - \frac{1}{m}\Big). \end{aligned}$$

Combining this estimate with equation (28), we infer

$$\sigma_m = 2 \cdot \sum_{k=1}^{\sqrt{m/2}} \frac{C^{2m}_{m-k}}{C^{2m}_m} \cdot k \cdot C^{2m}_m \geq C^{2m}_m \cdot e^{-1/m} \cdot \sum_{k=1}^{\sqrt{m/2}} e^{-\frac{k^2}{m}} \cdot k. \tag{30}$$

The function $\psi(u) = e^{-\frac{u^2}{m}} \cdot u$ increases on the interval $[0, \sqrt{m/2}]$ because of

$$\psi'(u) = e^{-\frac{u^2}{m}} + ue^{-\frac{u^2}{m}}(-2u/m) = e^{-\frac{u^2}{m}}(1 - 2u^2/m) \geq 0$$

for $0 \leq u \leq \sqrt{m/2}$. Therefore,

$$\sum_{k=1}^{\sqrt{m/2}} e^{-\frac{k^2}{m}} \cdot k > \sum_{k=1}^{\sqrt{m/2}} \int_{k-1}^{k} e^{-\frac{u^2}{m}} \cdot u\, du = \frac{m}{2}(1 - \frac{1}{\sqrt{e}}) \geq \frac{1}{3}m.$$

Moreover, an easy calculation, by using the Stirling formula, shows that

$$\lim_{m\to\infty} C^{2m}_m 4^{-m}\sqrt{\pi m} = 1.$$

Thus, from the above and (30) it follows that

$$\Big\| \sum_{k=1}^{2m} r_k \Big\|_{(M^d_{1,w})'} \geq \frac{1}{w(|E_m|)}\, 2^{-2m} \sum_{i\in S_m} \Big| \sum_{k=1}^{2m} \varepsilon^i_k \Big| = \frac{1}{w(|E_m|)}\, 2^{-2m}\, \sigma_m$$

$$\geq \frac{1}{w(|E_m|)} 2^{-2m} \cdot C_m^{2m} \cdot e^{-1/m} \cdot \sum_{k=1}^{\sqrt{m/2}} e^{-\frac{k^2}{m}} \cdot k$$
$$\geq \frac{1}{w(|E_m|)} 4^{-m} \cdot C_m^{2m} \cdot e^{-1/m} \cdot \frac{1}{3} m \approx \frac{\sqrt{m}}{3\sqrt{\pi}\, w(|E_m|)}$$

for all $m \in \mathbb{N}$ such that $\sqrt{m/2} \in \mathbb{N}$. Since $|E_m| \to 0$, then by (12) $w(|E_m|) \to 0$ as $m \to \infty$. Hence, the preceding inequality implies (26) and the proof is complete. $\square$

5. Structure of Rademacher subspaces in Morrey spaces

Applying Theorem 1 allows us also to study the geometric structure of Rademacher subspaces in Morrey spaces $M_{p,w}$.

Theorem 4. *Let $1 \leq p < \infty$ and $\lim_{t\to 0^+} w(t) = 0$. Then every infinite-dimensional subspace of $\mathcal{R}_{p,w}$ is either isomorphic to l_2 or contains a subspace, which is isomorphic to c_0 and is complemented in $\mathcal{R}_{p,w}$.*

The following two propositions are main tools in the proof of the above theorem.

Proposition 4. *Suppose that $1 \leq p < \infty$ and $\lim_{t\to 0^+} w(t) = 0$. Then the Rademacher functions form a shrinking basis in $\mathcal{R}_{p,w}$.*

Proof. To prove the shrinking property of $\{r_n\}_{n=1}^\infty$ we need to show that for every $\varphi \in (M_{p,w})^*$ we have

$$\|\varphi_{|[r_n]_{n=m}^\infty}\|_{(M_{p,w})^*} \to 0 \text{ as } m \to \infty. \tag{31}$$

Assume that (31) does not hold. Then there exist $\varepsilon \in (0,1), \varphi \in (M_{p,w})^*$ with $\|\varphi\|_{(M_{p,w})^*} = 1$, and a sequence of functions

$$f_n = \sum_{k=m_n}^\infty a_k^{m_n} r_k, \quad \text{where} \quad m_1 < m_2 < \dots,$$

such that $\|f_n\|_{M_{p,w}} = 1$, $n = 1, 2, \dots$ and

$$\varphi(f_n) \geq \varepsilon \quad \text{for all} \quad n = 1, 2, \dots. \tag{32}$$

Let us construct two sequences of positive integers $\{q_i\}_{i=1}^\infty$ and $\{p_i\}_{i=1}^\infty, 1 \leq q_1 < p_1 < q_2 < p_2 < \dots$ as follows. Setting $q_1 = m_1$, we can find $p_1 > q_1$, so that $\| \sum_{n=p_1+1}^\infty a_k^{q_1} r_k\|_{M_{p,w}} \leq \varepsilon/2$. Now, if the numbers $1 \leq q_1 < p_1 < q_2 < p_2 < \dots q_{i-1} < p_{i-1}$, $i \geq 2$, are chosen, we take for q_i the smallest of numbers m_n, which is larger than p_{i-1} such that

$$w(2^{-q_i}) \leq \frac{1}{2} w(2^{-q_{i-1}}). \tag{33}$$

Moreover, let $p_i > q_i$ be such that

$$\Big\| \sum_{n=p_i+1}^\infty a_k^{q_i} r_k \Big\|_{M_{p,w}} \leq \varepsilon/2. \tag{34}$$

We set $\alpha_k^i := a_k^{q_i}$ if $q_i \leq k \leq p_i$, and $\alpha_k^i := 0$ if $p_i < k < q_{i+1}$, $i = 1, 2, \dots$. Then, the sequence

$$u_i := \sum_{k=q_i}^{q_{i+1}-1} \alpha_k^i\, r_k, \quad i = 1, 2, \dots$$

is a block basis of the Rademacher sequence. Moreover, by the definition of u_i,

$$\sup_{i=1,2,\dots} \|u_i\|_{M_{p,w}} \le 2, \tag{35}$$

and from the choice of the functional φ and (34) it follows that

$$\varphi(u_i) = \varphi\Big(\sum_{k=q_i}^{p_i} a_k^{q_i}\, r_k\Big) = \varphi(f_i) - \varphi\Big(\sum_{k=p_i+1}^{\infty} a_k^{q_i}\, r_k\Big) \ge \varphi(f_i) - \Big\|\sum_{k=p_i+1}^{\infty} a_k^{q_i}\, r_k\Big\|_{M_{p,w}} \ge \frac{\varepsilon}{2}. \tag{36}$$

Let $\{\gamma_n\}_{n=1}^{\infty}$ be an arbitrary sequence of positive numbers such that

$$\sum_{n=1}^{\infty} \gamma_n^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n = \infty. \tag{37}$$

We show that the series $\sum_{n=1}^{\infty} \gamma_n\, u_n$ converges in $M_{p,w}$. To this end, we set $b_k := \alpha_k^i \cdot \gamma_i$ if $q_i \le k < q_{i+1}$. For every $m \in \mathbb{N}$, by Theorem 1,

$$\Big\|\sum_{n=m}^{\infty} \gamma_n\, u_n\Big\|_{M_{p,w}} = \Big\|\sum_{k=q_m}^{\infty} b_k r_k\Big\|_{M_{p,w}} \approx \Big(\sum_{k=q_m}^{\infty} b_k^2\Big)^{1/2} + \sup_{l \ge q_m} w(2^{-l}) \cdot \sum_{k=q_m}^{l} |b_k|. \tag{38}$$

Let us estimate both summands from the right-hand side of (38). At first, from (35) and Theorem 1 it follows that

$$\sum_{k=q_m}^{\infty} b_k^2 = \sum_{i=m}^{\infty} \gamma_i^2 \sum_{k=q_i}^{q_{i+1}-1} (\alpha_k^i)^2 \le C_1 \sum_{i=m}^{\infty} \gamma_i^2. \tag{39}$$

Similarly, if $q_m < \ldots < q_{m+r} \le l < q_{m+r+1}$ for some $r = 1, 2, \ldots$, then

$$\begin{aligned}\sum_{k=q_m}^{l} |b_k| &= \sum_{i=m}^{m+r-1} |\gamma_i| \sum_{k=q_i}^{q_{i+1}-1} |\alpha_k^i| + |\gamma_{m+r}| \sum_{k=q_{m+r}}^{l} |\alpha_k^{m+r}| \\ &\le C_2 \Big(\sum_{i=m}^{m+r-1} \frac{|\gamma_i|}{w(2^{-q_{i+1}})} + \frac{|\gamma_{m+r}|}{w(2^{-l})}\Big).\end{aligned}$$

Combining this inequality together with (33), we obtain

$$\begin{aligned}w(2^{-l}) \sum_{k=q_m}^{l} |b_k| &\le C_2 \Big(\sum_{i=m}^{m+r-1} |\gamma_i| \frac{w(2^{-q_{m+r}})}{w(2^{-q_{i+1}})} + |\gamma_{m+r}|\Big) \\ &\le C_2 \Big(\sum_{i=m}^{m+r-1} |\gamma_i|\, 2^{-m-r+i+1} + |\gamma_{m+r}|\Big) \\ &\le C_2 \max_{i \ge m} |\gamma_i| \Big(\sum_{j=0}^{r-1} 2^{1+j-r} + 1\Big) < 3\, C_2 \max_{i \ge m} |\gamma_i|.\end{aligned}$$

Clearly, the latter estimate holds also in the simpler case when $q_m \le l < q_{m+1}$. Thus, for every $m \in \mathbb{N}$,

$$\sup_{l \ge q_m} w(2^{-l}) \sum_{k=q_m}^{l} |b_k| \le 3\, C_2 \max_{i \ge m} |\gamma_i|. \tag{40}$$

From (37)–(40) it follows that the series $\sum_{n=1}^{\infty} \gamma_n\, u_n$ converges in $M_{p,w}$. At the same time, since $\varphi \in (M_{p,w})^*$, by (36) and (37), we have

$$\varphi\Big(\sum_{n=1}^{\infty} \gamma_n\, u_n\Big) = \sum_{n=1}^{\infty} \gamma_n\, \varphi(u_n) \ge \frac{\varepsilon}{2} \sum_{n=1}^{\infty} \gamma_n = \infty,$$

and so (31) is proved. □

Corollary 3. *Under assumptions of Proposition 4:*

(i) $r_k \to 0$ *weakly in* $M_{p,w}$.
(ii) *The Rademacher functions form a basis in the dual space* $(\mathcal{R}_{p,w})^*$.

Proof. Since $\{r_n\}_{n=1}^{\infty}$ is the biorthogonal system to $\{r_n\}$ itself, (ii) follows from Proposition 4 and Proposition 1.b.1 in [16]. □

Proposition 5. *Let* $1 \le p < \infty$ *and* $\lim_{t \to 0^+} w(t) = 0$. *Suppose that*

$$u_n = \sum_{k=m_n}^{m_{n+1}-1} a_k\, r_k, \quad 1 = m_1 < m_2 < \ldots$$

is a block basis such that $\|u_n\|_{M_{p,w}} = 1$ *for all* $n \in \mathbb{N}$ *and* $\sum_{k=m_n}^{m_{n+1}-1} a_k^2 \to 0$ *as* $n \to \infty$. *Moreover, let*

$$w(2^{-m_{n+1}}) \le \frac{1}{2}\, w(2^{-m_n}), \ \ n = 1, 2, \ldots. \tag{41}$$

Then the sequence $\{u_n\}_{n=1}^{\infty}$ *contains a subsequence equivalent in* $M_{p,w}$ *to the unit vector basis of* c_0.

Proof. Passing to a subsequence if it is needed, without loss of generality we may assume that

$$\sum_{k=m_n}^{m_{n+1}-1} a_k^2 \le 2^{-n}, \quad n = 1, 2, \ldots. \tag{42}$$

Suppose that $f = \sum_{n=1}^{\infty} \beta_n\, u_n \in \mathcal{R}_{p,w}$. Setting $b_k = a_k \beta_i$ if $m_i \le k < m_{i+1}, i = 1, 2, \ldots$, by Theorem 1, we obtain

$$\|f\|_{M_{p,w}} = \Big\| \sum_{k=1}^{\infty} b_k\, r_k \Big\|_{M_{p,w}} \approx \Big(\sum_{k=1}^{\infty} b_k^2\Big)^{1/2} + \sup_{l \in \mathbb{N}} w(2^{-l}) \sum_{k=1}^{l} |b_k|. \tag{43}$$

At first, by (42),

$$\sum_{k=1}^{\infty} b_k^2 = \sum_{i=1}^{\infty} \beta_i^2 \sum_{k=m_i}^{m_{i+1}-1} a_k^2 \le (\sup_{i=1,2,\ldots} |\beta_i|)^2 \cdot \sum_{i=1}^{\infty} 2^{-i} \le \|(\beta_i)\|_{c_0}^2.$$

Moreover, precisely in the same way as in the proof of Proposition 4 from (41) and the equations $\|u_n\|_{M_{p,w}} = 1$, $n = 1, 2, \ldots$ it follows that for some constant $C' > 0$

$$\sup_{l=1,2,\ldots} w(2^{-l}) \sum_{k=1}^{l} |b_k| \le C' \, \|(\beta_i)\|_{c_0}.$$

Combining the last two inequalities together with (43), we conclude that $\|f\|_{M_{p,w}} \le C \, \|(\beta_i)\|_{c_0}$ for some constant $C > 0$.

Conversely, since $\{u_n\}$ is an unconditional sequence in $M_{p,w}$ and $\|u_n\|_{M_{p,w}} = 1$, $n = 1, 2, \ldots$, by Theorem 1, $\|f\|_{M_{p,w}} \ge c|\beta_i|$. $i = 1, 2, \ldots$, with some constant $c > 0$. Hence, $\|f\|_{M_{p,w}} \ge c \, \|(\beta_i)\|_{c_0}$, and the proof is complete. □

Proof of Theorem 4. Assume that X is an infinite-dimensional subspace of $\mathcal{R}_{p,w}$ such that for every $f = \sum_{k=1}^{\infty} b_k r_k \in X$ we have

$$\|f\|_{M_{p,w}} \approx \Big(\sum_{k=1}^{\infty} b_k^2\Big)^{1/2},$$

with a constant independent of b_k, $k = 1, 2, \ldots$. Then, X is isomorphic to some subspace of l_2 and so to l_2 itself.

Therefore, if X is not isomorphic to l_2, then there is a sequence $\{f_n\}_{n=1}^{\infty} \subset X$, $f_n = \sum_{k=1}^{\infty} b_{n,k} r_k$, such that $\|f_n\|_{M_{p,w}} = 1$ and

$$\sum_{k=1}^{\infty} b_{n,k}^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{44}$$

Observe that $\{f_n\}_{n=1}^{\infty}$ does not contain any subsequence converging in $M_{p,w}$-norm. In fact, if $\|f_{n_k} - f\|_{M_{p,w}} \to 0$ for some $\{f_{n_k}\} \subset \{f_n\}$ and $f \in X$, then from Theorem 1 and (44) it follows that $f = \sum_{k=1}^{\infty} b_k r_k$, where $b_k = 0$ for all $k = 1, 2, \ldots$. Hence, $f = 0$. On the other hand, obviously, $\|f\|_{M_{p,w}} = 1$, and we come to a contradiction.

Thus, passing if it is needed to a subsequence, we can assume that

$$\|f_n - f_m\|_{M_{p,w}} \ge \varepsilon > 0 \quad \text{for all} \quad n \ne m. \tag{45}$$

Recall that, by Corollary 3, the sequence $\{r_k\}_{k=1}^{\infty}$ is a basis of the space $(\mathcal{R}_{p,w})^*$. Applying the diagonal process, we can find the sequence $\{n_k\}_{k=1}^{\infty}, n_1 < n_2 < \ldots$, such that for every $i = 1, 2, \ldots$ there exists $\lim_{k\to\infty} \int_0^1 r_i(s) f_{n_k}(s)\, ds$. Then,

$$\lim_{k\to\infty} \int_0^1 r_i(s)(f_{n_{2k+1}}(s) - f_{n_{2k}}(s))\, ds = 0 \quad \text{for all} \quad i = 1, 2, \ldots.$$

Hence, since the sequence $\{f_{n_{2k+1}} - f_{n_{2k}}\}_{k=1}^{\infty}$ is bounded in $M_{p,w}$ we infer that $f_{n_{2k+1}} - f_{n_{2k}} \to 0$ weakly in $\mathcal{R}_{p,w}$ (with respect to the norm of $M_{p,w}$). Now, taking into account (45) and applying the well-known Bessaga–Pełczyński Selection Principle (cf. [1, Proposition 1.3.10, p. 14]), we may construct a subsequence of the sequence $\{f_{n_{2k+1}} - f_{n_{2k}}\}_{k=1}^{\infty}$ (we keep for it the same notation) and a block basis

$$u_k = \sum_{j=m_k}^{m_{k+1}-1} a_j r_j, \; 1 = m_1 < m_2 < \ldots,$$

such that

$$\|u_k - (f_{n_{2k+1}} - f_{n_{2k}})\|_{M_{p,w}} \le B_0^{-1} \cdot 2^{-k-1}, \quad k = 1, 2, \ldots, \tag{46}$$

where B_0 is the basis constant of $\{r_k\}$ in $\mathcal{R}_{p,w}$, and

$$w(2^{-m_{k+1}}) \le \frac{1}{2} \cdot w(2^{-m_k}), \quad k = 1, 2, \ldots.$$

From (46) it follows that the sequences $\{u_k\}_{k=1}^{\infty}$ and $\{f_{n_{2k+1}} - f_{n_{2k}}\}_{k=1}^{\infty}$ are equivalent in $M_{p,w}$ (cf. [16, Proposition 1.a.9]). Moreover, by Theorem 1 and (44),

$$\sum_{j=m_k}^{m_{k+1}-1} a_j^2 \to 0 \quad \text{as} \quad k \to \infty.$$

According to the latter relations we can apply Proposition 5, which implies that the sequence $\{u_k\}_{k=1}^{\infty}$ (and so $\{f_{n_{2k+1}} - f_{n_{2k}}\}_{k=1}^{\infty}$) contains a subsequence equivalent to the unit vector basis of c_0. Since $\{f_{n_{2k+1}} - f_{n_{2k}}\}_{k=1}^{\infty} \subset X$, then X contains a subspace isomorphic to c_0. Complementability of this subspace in $\mathcal{R}_{p,w}$ is an immediate consequence of Sobczyk's theorem (see [1, Corollary 2.5.9]). $\square$

Remark 2. If $\lim_{t\to 0^+} w(t) > 0$, then $M_{p,w} = L_\infty$ and $\{r_k\}$ is equivalent in $M_{p,w}$ to the unit vector basis of l_1 (cf. Theorem 1). Observe also that if $\sup_{0<t\le 1} w(t) \log_2^{1/2}(2/t) < \infty$, then we get another trivial situation: $\mathcal{R}_{p,w} \simeq l_2$ (see Corollary 2).

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