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The index of weighted singular integral operators with shifts and slowly oscillating data

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A B S T R A C T

Let α and β be orientation-preserving diffeomorphism (shifts) of $\mathbb{R}_+ = (0, \infty)$ onto itself with the only fixed points 0 and ∞. We establish a Fredholm criterion and calculate the index of the weighted singular integral operator with shifts

$$(aI - bU_\alpha)P_\gamma^+ + (cI - dU_\beta)P_\gamma^-,$$

acting on the space $L^p(\mathbb{R}_+)$, where $P_\gamma^\pm = (I \pm S_\gamma)/2$ are the operators associated to the weighted Cauchy singular integral operator S_γ given by

$$(S_\gamma f)(t) = \frac{1}{\pi i} \int_{\mathbb{R}_+} \left(\frac{t}{\tau}\right)^\gamma \frac{f(\tau)}{\tau - t} d\tau$$

with $\gamma \in \mathbb{C}$ satisfying $0 < 1/p + \Re\gamma < 1$, and U_α, U_β are the isometric shift operators given by

$$U_\alpha f = (\alpha')^{1/p}(f \circ \alpha), \quad U_\beta f = (\beta')^{1/p}(f \circ \beta),$$

under the assumptions that the coefficients a, b, c, d and the derivatives α', β' of the shifts are bounded and continuous on $\mathbb{R}_+$ and admit discontinuities of slowly oscillating type at 0 and ∞.

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1. Introduction

Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators acting on a Banach space X and let $\mathcal{K}(X)$ be the ideal of all compact operators in $\mathcal{B}(X)$. An operator $A \in \mathcal{B}(X)$ is called *Fredholm* if its image is closed and the spaces $\ker A$ and $\ker A^*$ are finite-dimensional. In that case the number

$$\mathrm{Ind}\, A := \dim\ker A - \dim\ker A^*$$

is referred to as the *index* of A (see, e.g., [5, Chap. 4]). For $A, B \in \mathcal{B}(X)$, we will write $A \simeq B$ if $A - B \in \mathcal{K}(X)$.

Following Sarason [26, p. 820], a bounded continuous function f on $\mathbb{R}_+ = (0, \infty)$ is called slowly oscillating (at 0 and ∞) if

$$\lim_{r\to s} \sup_{t,\tau\in[r,2r]} |f(t) - f(\tau)| = 0 \quad \text{for} \quad s \in \{0, \infty\}.$$

The set $SO(\mathbb{R}_+)$ of all slowly oscillating functions forms a C^*-algebra. This algebra properly contains $C(\overline{\mathbb{R}}_+)$, the C^*-algebra of all continuous functions on $\overline{\mathbb{R}}_+ := [0, +\infty]$.

Suppose α is an orientation-preserving diffeomorphism of $\mathbb{R}_+$ onto itself, which has only two fixed points 0 and ∞. We say that α is a slowly oscillating shift if $\log\alpha'$ is bounded and $\alpha' \in SO(\mathbb{R}_+)$. The set of all slowly oscillating shifts is denoted by $SOS(\mathbb{R}_+)$. By [7, Lemma 2.2], an orientation-preserving diffeomorphism $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $SOS(\mathbb{R}_+)$ if and only if $\alpha(t) = te^{\omega(t)}$, $t \in \mathbb{R}_+$, for some real-valued function $\omega \in SO(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ such that $\psi(t) := t\omega'(t)$ also belongs to $SO(\mathbb{R}_+)$ and $\inf_{t\in\mathbb{R}_+} \big(1 + t\omega'(t)\big) > 0$. The real-valued slowly oscillating function

$$\omega(t) := \log[\alpha(t)/t], \quad t \in \mathbb{R}_+,$$

is called the exponent function of $\alpha \in SOS(\mathbb{R}_+)$.

Through the paper, we will suppose that $1 < p < \infty$. It is easily seen that if $\alpha \in SOS(\mathbb{R}_+)$, then the weighted shift operator defined by

$$U_\alpha f := (\alpha')^{1/p}(f \circ \alpha)$$

is an isometric isomorphism of the Lebesgue space $L^p(\mathbb{R}_+)$ onto itself. It is clear that $U_\alpha^{-1} = U_{\alpha_{-1}}$. Let $a, b \in SO(\mathbb{R}_+)$. We say that a dominates b and write $a \gg b$ if

$$\inf_{t\in\mathbb{R}_+} |a(t)| > 0, \quad \liminf_{t\to 0}(|a(t)| - |b(t)|) > 0, \quad \liminf_{t\to\infty}(|a(t)| - |b(t)|) > 0.$$

Theorem 1.1 *([13, Theorem 1.1]). Suppose $a, b \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$. The binomial functional operator $aI - bU_\alpha$ is invertible on the Lebesgue space $L^p(\mathbb{R}_+)$ if and only if either $a \gg b$ or $b \gg a$.*

(a) *If $a \gg b$, then* $(aI - bU_\alpha)^{-1} = \sum_{n=0}^{\infty} (a^{-1}bU_\alpha)^n a^{-1}I$.

(b) *If $b \gg a$, then* $(aI - bU_\alpha)^{-1} = -U_\alpha^{-1} \sum_{n=0}^{\infty} (b^{-1}aU_\alpha^{-1})^n b^{-1}I$.

Let $\Re\gamma$ and $\Im\gamma$ denote the real and imaginary part of $\gamma \in \mathbb{C}$, respectively. As usual, $\overline{\gamma} = \Re\gamma - i\Im\gamma$ denotes the complex conjugate of γ. If $\gamma \in \mathbb{C}$ satisfies

$$0 < 1/p + \Re\gamma < 1, \tag{1.1}$$

then the operators

$$(S_\gamma f)(t) := \frac{1}{\pi i}\int\limits_{\mathbb{R}_+}\left(\frac{t}{\tau}\right)^\gamma \frac{f(\tau)}{\tau - t}d\tau, \quad (R_\gamma f)(t) := \frac{1}{\pi i}\int\limits_{\mathbb{R}_+}\left(\frac{t}{\tau}\right)^\gamma \frac{f(\tau)}{\tau + t}d\tau,$$

where the integrals are understood in the principal value sense, are bounded on the Lebesgue space $L^p(\mathbb{R}_+)$ (see, e.g., [3] or [25, Propositions 4.2.11 and 4.2.15]). Put

$$P_\gamma^\pm := (I \pm S_\gamma)/2.$$

This paper is a continuation of our works [6,11,14] (see also [7–9]), where we studied Fredholm properties of the weighted singular integral operator with two slowly oscillating shifts of the form

$$N := (aI - bU_\alpha)P_\gamma^+ + (cI - dU_\beta)P_\gamma^-, \tag{1.2}$$

where $a, b, c, d \in SO(\mathbb{R}_+)$, $\alpha, \beta \in SOS(\mathbb{R}_+)$, and $\gamma \in \mathbb{C}$ satisfies (1.1). In two particular cases it is known that this operator is Fredholm and its index is available. More precisely, if $a, c = 0$ and $b, d = -1$, then by [6, Theorem 1.1], the operator N is Fredholm and its index is equal to zero provided that $|\Im\gamma|$ is sufficiently small (for $\gamma = 0$ this result was obtained in [9]). Further, if $a = c = 1$, $1 \gg b$, $1 \gg d$, and $\gamma = \gamma_*$ with

$$\gamma_* := 1/2 - 1/p, \tag{1.3}$$

then N is again a Fredholm operator of index zero in view of [11, Theorem 1.1]. For general coefficients $a, b, c, d \in SO(\mathbb{R}_+)$, Fredholm criteria for the operator N were obtained only under the assumption $\alpha = \beta$ and only in the non-weighted case $\gamma = 0$ (see [7,8]). However, the index of N even in this less general case was not available. We should also note that the proof of the necessity portion of that result [7] contains a gap, which was filled in recently (see [14]). In the latter paper, necessary conditions for the Fredholmness were proved for the operator N with two possibly different shifts α, β and for all parameters γ satisfying (1.1).

The first aim of the present paper is to prove that the above mentioned necessary conditions for Fredholmness for the operator N are also sufficient, and the second (main) aim of this work is to provide a formula for the index of N in case of its Fredholmness. To formulate our first main result, we need a little bit more notation. By $M(\mathfrak{A})$ we denote the maximal ideal space of a unital commutative Banach algebra $\mathfrak{A}$. Identifying the points $t \in \overline{\mathbb{R}}_+$ with the evaluation functionals $t(f) = f(t)$ for $f \in C(\overline{\mathbb{R}}_+)$, we get $M(C(\overline{\mathbb{R}}_+)) = \overline{\mathbb{R}}_+$. Consider the fibers

$$M_s(SO(\mathbb{R}_+)) := \left\{\xi \in M(SO(\mathbb{R}_+)) : \xi|_{C(\overline{\mathbb{R}}_+)} = s\right\}$$

of the maximal ideal space $M(SO(\mathbb{R}_+))$ over the points $s \in \{0, \infty\}$. By [17, Proposition 2.1], the set

$$\Delta := M_0(SO(\mathbb{R}_+)) \cup M_\infty(SO(\mathbb{R}_+))$$

coincides with $\mathrm{clos}_{SO^*}\mathbb{R}_+ \setminus \mathbb{R}_+$, where $\mathrm{clos}_{SO^*}\mathbb{R}_+$ is the weak-star closure of $\mathbb{R}_+$ in the dual space of $SO(\mathbb{R}_+)$. Then $M(SO(\mathbb{R}_+)) = \Delta \cup \mathbb{R}_+$. In what follows we write $a(\xi) := \xi(a)$ for every $a \in SO(\mathbb{R}_+)$ and every $\xi \in \Delta$.

With the operator N we associate the function

$$n(t, x) = (a(t) - b(t)e^{i\omega(t)x})p_\gamma^+(x) + (c(t) - d(t)e^{i\eta(t)x})p_\gamma^-(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},$$

where $\omega, \eta \in SO(\mathbb{R}_+)$ are the exponent functions of α, β, respectively, and

$$p_\gamma^\pm(x) := (1 \pm s_\gamma(x))/2, \quad s_\gamma(x) := \coth[\pi(x + i/p + i\gamma)], \quad x \in \mathbb{R}. \tag{1.4}$$

Since $n(\cdot, x) \in SO(\mathbb{R}_+)$ for every $x \in \mathbb{R}$, taking the Gelfand transform of $n(\cdot, x)$, we obtain

$$n(\xi, x) := (a(\xi) - b(\xi)e^{i\omega(\xi)x})p_\gamma^+(x) + (c(\xi) - d(\xi)e^{i\eta(\xi)x})p_\gamma^-(x), \quad (\xi, x) \in (\Delta \cup \mathbb{R}_+) \times \mathbb{R}, \tag{1.5}$$

which gives extensions of the functions $n(\cdot, x)$ to $M(SO(\mathbb{R}_+))$.

Theorem 1.2 *([14, Theorem 1.2]). Let $1 < p < \infty$ and $\gamma \in \mathbb{C}$ satisfy (1.1). Suppose a, b, c, d belong to $SO(\mathbb{R}_+)$ and α, β belong to $SOS(\mathbb{R}_+)$. If the operator N given by (1.2) is Fredholm on the space $L^p(\mathbb{R}_+)$, then the following two conditions are fulfilled:*

(i) *the binomial functional operators $A_+ := aI - bU_\alpha$ and $A_- := cI - dU_\beta$ are invertible on the space $L^p(\mathbb{R}_+)$;*
(ii) *for every $\xi \in \Delta$, the function n given by (1.5) satisfies $\inf_{x\in\mathbb{R}} |n(\xi, x)| > 0$.*

Now our first main result reads as follows.

Theorem 1.3 *(Main result 1). Let $1 < p < \infty$ and $\gamma \in \mathbb{C}$ satisfy (1.1). Suppose a, b, c, d belong to $SO(\mathbb{R}_+)$ and α, β belong to $SOS(\mathbb{R}_+)$. If conditions* (i)–(ii) *of Theorem 1.2 are fulfilled, then the operator N given by (1.2) is Fredholm on the space $L^p(\mathbb{R}_+)$.*

Note that Theorem 1.3 can be proved by the methods of our previous work [7]. However, those methods do not allow us to prove an index formula for the operator N. In this paper we present another proof of Theorem 1.3. This proof relies on the reduction of the Fredholm study of the operator N to the study of a Mellin pseudodifferential operator $\mathrm{Op}(\mathfrak{h})$, for which a Fredholm criterion and an index formula are available. Hence this new approach shed light on the problem of calculation of the index.

Further we obtain an index formula for the operator N given by (1.2) in the case of its Fredholmness. In a sense this formula is a combination of the index formula for Mellin pseudodifferential operators with slowly oscillating symbols (see, e.g., [19, Theorem 4.3]) and the index formula for singular integral operators with shifts and piecewise continuous data (see [20,21] and also [22, Chap. 4, Section 2.4]). We also mention [18, Section 6], where the index of the operator $T = W_\alpha P_0^+ + GP_0^-$ related to the Haseman boundary problem with slowly oscillating data was calculated.

By $C(\overline{\mathbb{R}})$ we denote the C^*-algebra of all functions continuous on the two-point compactification $\overline{\mathbb{R}} = [-\infty, +\infty]$ of the real line. Let AP denote the C^*-algebra of almost periodic functions generated in $L^\infty(\mathbb{R})$ by all exponents $e_\lambda(x) := e^{i\lambda x}$, where $\lambda, x \in \mathbb{R}$. Let SAP stand for the C^*-algebra of all semi-almost periodic functions generated in $L^\infty(\mathbb{R})$ by $\{AP, C(\overline{\mathbb{R}})\}$ (see, e.g., [2, Section 1.5]). By $\mathcal{G}SAP$ we denote the set of all functions $f \in SAP$, which are invertible in $L^\infty(\mathbb{R})$, that is, such that $\inf_{x\in\mathbb{R}} |f(x)| > 0$. Then we use the following definition of their index introduced by V.G. Kravchenko and the second author in [20,21] (see also [22, pp. 194–195]). Given a function $f \in \mathcal{G}SAP$, its generalized Cauchy index $\mathrm{ind}_\mathbb{R} f$ is defined by

$$\mathrm{ind}_\mathbb{R} f := \frac{1}{2\pi}[M_+(\varphi_+) - M_-(\varphi_-)] \tag{1.6}$$

where

$$M_\pm(\varphi) := \lim_{x\to\pm\infty} \frac{1}{x}\int_0^x \varphi(y)\, dy, \quad \varphi_\pm(x) := \arg f(x) - x \lim_{y\to\pm\infty} \frac{\arg f(y)}{y}. \tag{1.7}$$

The generalized Cauchy index exists and is finite for every function $f \in \mathcal{G}SAP$.

By $\mathcal{G}SO(\mathbb{R}_+)$ we denote the set of all functions $f \in SO(\mathbb{R}_+)$, which are invertible in $L^\infty(\mathbb{R}_+)$, that is, such that $\inf_{t\in\mathbb{R}_+} |f(t)| > 0$. For a function $f \in \mathcal{G}SO(\mathbb{R}_+)$, let $\{\arg f(t)\}_{t\in[\tau^{-1},\tau]}$ denote the increment of any continuous branch of the argument of f when the point t traces the segment $[\tau^{-1}, \tau]$ for $\tau > 1$.

Theorem 1.4 *(Main result 2). Let $1 < p < \infty$ and $\gamma \in \mathbb{C}$ satisfy (1.1). Suppose a, b, c, d belong to $SO(\mathbb{R}_+)$, α, β belong to $SOS(\mathbb{R}_+)$, and $\omega, \eta \in SO(\mathbb{R}_+)$ are the exponent functions of α, β, respectively. Let the operator N and the function n be given by (1.2) and (1.5), respectively. If the operator N is Fredholm on the space $L^p(\mathbb{R}_+)$, then*

(a) *$n(\tau, \cdot) \in \mathcal{G}SAP$ for all $\tau \in \mathbb{R}_+$ sufficiently close to 0 and ∞;*
(b) *the functions*

$$\nu_{a,b} := \begin{cases} a, & \text{if} \quad a \gg b, \\ b, & \text{if} \quad b \gg a, \end{cases} \qquad \nu_{c,d} := \begin{cases} c, & \text{if} \quad c \gg d, \\ d, & \text{if} \quad d \gg c, \end{cases} \tag{1.8}$$

are well defined and belong to $\mathcal{G}SO(\mathbb{R}_+)$;
(c) *the index of the operator N is calculated by the formula*

$$\operatorname{Ind} N = \lim_{\tau\to+\infty} \left(\frac{1}{2\pi} \Big(\big\{ \arg \nu_{c,d}(t) \big\}_{t\in[\tau^{-1},\tau]} - \big\{ \arg \nu_{a,b}(t) \big\}_{t\in[\tau^{-1},\tau]} \Big) + \operatorname{ind}_{\mathbb{R}} n(\tau, \cdot) - \operatorname{ind}_{\mathbb{R}} n(\tau^{-1}, \cdot) \right).$$

The paper is organized as follows. Section 2 deals with Mellin pseudodifferential operators $\operatorname{Op}(\mathfrak{a})$. Here we recall properties of the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ of slowly oscillating functions of limited smoothness on $\mathbb{R}_+$ with values in the algebra $V(\mathbb{R})$ of absolutely continuous functions of finite total variation (see, e.g., [9,10,15,19]). Further we explain how a function $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ can be extended from $\mathbb{R}_+ \times \mathbb{R}$ to its "boundary" $(\mathbb{R}_+ \times \{\pm\infty\}) \cup (\Delta \times \overline{\mathbb{R}})$ and recall that a Mellin pseudodifferential operator $\operatorname{Op}(\mathfrak{a})$ with $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is Fredholm if and only if its symbol $\mathfrak{a}$ does not vanish on this "boundary". Moreover, an index formula for a Fredholm Mellin pseudodifferential operator $\operatorname{Op}(\mathfrak{a})$ is stated. The latter results were obtained by the second author in [15,19]. They extend previous results by V. Rabinovich [23] (see also [24, Sections 4.5–4.6] and [10]) obtained for Mellin pseudodifferential operators with infinitely smooth slowly oscillating symbols in $C^\infty(\mathbb{R}_+ \times \mathbb{R})$. Finally, we observe that if $\alpha \in SOS(\mathbb{R}_+)$, then the operators $U_\alpha^{\pm 1} R_\gamma$ are similar (up to a compact operator) to Mellin pseudodifferential operators $\operatorname{Op}(\mathfrak{c}_\pm)$ with $\mathfrak{c}_\pm \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ given by $\mathfrak{c}_\pm(t, x) = e^{\pm i\omega(t)x} / \sinh[\pi(x + i/p + i\gamma)]$. This observation, made in [9] for $\gamma = 0$ and then extended in [6] for arbitrary $\gamma \in \mathbb{C}$ satisfying (1.1), is crucial for our analysis.

Assuming that the operators $A_+ = aI - bU_\alpha$ and $A_- = cI - dU_\beta$ are invertible, in Section 3 we prove with the aid of results of Section 2 and [6, Theorem 1.1], [11, Theorem 7.1] that $NF \simeq HV$, where F is a Fredholm operator of zero index of the form

$$F := U_\alpha^{\varepsilon_1} P_\gamma^+ + U_\beta^{\varepsilon_2} P_\gamma^-, \quad \varepsilon_1, \varepsilon_2 \in \{-1, 0, 1\}, \tag{1.9}$$

V is a Fredholm operator of zero index of the form

$$V := (I - fU_\alpha^{\varepsilon_1}) P_{\gamma_*}^+ + (I - gU_\beta^{\varepsilon_2}) P_{\gamma_*}^-, \quad \varepsilon_1, \varepsilon_2 \in \{-1, 1\}, \tag{1.10}$$

and H is an operator similar to a Mellin pseudodifferential operator $\operatorname{Op}(\mathfrak{h})$ with $\mathfrak{h} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Hence the operators N and $\operatorname{Op}(\mathfrak{h})$ are Fredholm only simultaneously and $\operatorname{Ind} N = \operatorname{Ind} \operatorname{Op}(\mathfrak{h})$. The operators F and V are of the form of the operator N, whence we can associate with them by (1.5) the functions f and v, respectively. Further, we prove in Section 3 that $n(\xi, x) f(\xi, x) = \mathfrak{h}(\xi, x) v(\xi, x)$ for all $(\xi, x) \in \Delta \times \mathbb{R}$. From this identity and Theorem 1.2 applied to the Fredholm operators F and V it follows that $\mathfrak{h}$ does not vanish for $(\xi, x) \in \Delta \times \mathbb{R}$. Finally, we show that the invertibility of the operators $A_\pm$ implies that $\mathfrak{h}$ does not vanish for $(\xi, x) \in (\Delta \cup \mathbb{R}) \times \{\pm\infty\}$. Hence the operator $\mathrm{Op}(\mathfrak{h})$ is Fredholm. Thus, the operator N is Fredholm, too. This completes the proof of Theorem 1.3.

Section 4 is devoted to the proof of Theorem 1.4. In Subsection 4.1 we collect properties of the index of semi-almost periodic functions, known from [21, Section 2.1] and [22, pp. 194–195], and recall that the indices of two important semi-almost periodic functions are equal to zero [12]. In Subsection 4.2 we show that $n(t,x)f(t,x) = \widetilde{\mathfrak{h}}(t,x)v(t,x)$ for all sufficiently large t and all $x \in \mathbb{R}$ with a function $\widetilde{\mathfrak{h}}$ such that $\widetilde{\mathfrak{h}}(\xi, x) = \mathfrak{h}(\xi, x)$ for all (ξ, x) on the "boundary" part $\Delta \times \overline{\mathbb{R}}$. From the former equality it follows that $\mathrm{ind}_{\mathbb{R}}\, n(t, \cdot)$ coincides with the Cauchy index of the function $\widetilde{\mathfrak{h}} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. These facts are the main ingredients in the proof of the index formula. Subsection 4.3 contains the proof of Theorem 1.4 by using the index formula from [19, Theorem 4.3] for the Fredholm Mellin pseudodifferential operator $\mathrm{Op}(\mathfrak{h})$ and the indices of suitable semi-almost periodic functions.

2. Mellin pseudodifferential operators and their symbols

2.1. Boundedness of Mellin PDO's

The second author [15] (see also [16]) developed a Fredholm theory of Fourier pseudodifferential operators with slowly oscillating symbols of limited smoothness in the spirit of Sarason's definition [26, p. 820] of slow oscillation adopted in the present paper (much less restrictive than that in [23] and in the works mentioned in [24]). Results of [15,16] were translated to the Mellin pseudodifferential operators setting in [16,17,19] and [9] with the aid of the transformation defined by $A \mapsto E^{-1}AE$, where E is the isometric isomorphism

$$E : L^p(\mathbb{R}_+, d\mu) \to L^p(\mathbb{R}), \quad (Ef)(x) := f(e^x), \quad x \in \mathbb{R},$$

and $d\mu(t) = dt/t$ is the normalized invariant measure on $\mathbb{R}_+$. For the convenience of readers, we here reproduce necessary results for Mellin pseudodifferential operators exactly in the same form as they were stated in [9], where more details on their proofs can be found.

Let $V(\mathbb{R})$ be the Banach algebra of all absolutely continuous functions of finite total variation $a : \mathbb{R} \to \mathbb{C}$ equipped with the norm

$$\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + \int_{\mathbb{R}} |a'(x)|\, dx.$$

Let $C_b(\mathbb{R}_+, V(\mathbb{R}))$ denote the Banach algebra of all bounded continuous $V(\mathbb{R})$-valued functions on $\mathbb{R}_+$ with the norm

$$\|\mathfrak{a}(\cdot,\cdot)\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))} := \sup_{t \in \mathbb{R}_+} \|\mathfrak{a}(t,\cdot)\|_V.$$

Let $\mathfrak{a} \in C_b(\mathbb{R}_+, V(\mathbb{R}))$. For every $t \in \mathbb{R}_+$, the function $\mathfrak{a}(t,\cdot)$ belongs to $V(\mathbb{R})$ and, therefore, has finite limits at $\pm\infty$, which will be denoted by $\mathfrak{a}(t, \pm\infty)$. Let $C_0^\infty(\mathbb{R}_+)$ denote the set of all infinitely differentiable functions of compact support on $\mathbb{R}_+$.

Theorem 2.1 *([17, Theorem 3.1]). If $\mathfrak{a} \in C_b(\mathbb{R}_+, V(\mathbb{R}))$, then the Mellin pseudodifferential operator* $\mathrm{Op}(\mathfrak{a})$, *defined for functions $f \in C_0^\infty(\mathbb{R}_+)$ by the iterated integral*

$$(\mathrm{Op}(\mathfrak{a})f)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} dx \int_{\mathbb{R}_+} \mathfrak{a}(t,x) \left(\frac{t}{\tau}\right)^{ix} f(\tau) \frac{d\tau}{\tau} \quad \textit{for} \quad t \in \mathbb{R}_+,$$

extends to a bounded linear operator on the space $L^p(\mathbb{R}_+, d\mu)$ *and there exists a constant* $C_p \in (0,\infty)$ *depending only on* p *such that*

$$\|\operatorname{Op}(\mathfrak{a})\|_{\mathcal{B}(L^p(\mathbb{R}_+,d\mu))} \le C_p \|\mathfrak{a}\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))}.$$

2.2. Algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ *and products of Mellin PDO's*

Consider the Banach subalgebra $SO(\mathbb{R}_+, V(\mathbb{R}))$ of the algebra $C_b(\mathbb{R}_+, V(\mathbb{R}))$ consisting of all $V(\mathbb{R})$-valued functions $\mathfrak{a}$ on $\mathbb{R}_+$ that slowly oscillate at 0 and ∞, that is,

$$\lim_{r\to s} \max_{t,\tau\in[r,2r]} \|\mathfrak{a}(t,\cdot) - \mathfrak{a}(\tau,\cdot)\|_{L^\infty(\mathbb{R})} = 0, \quad s \in \{0,\infty\}.$$

Since $V(\mathbb{R}) \subset C(\overline{\mathbb{R}})$, the latter equality implies that for every $\mathfrak{a} \in SO(\mathbb{R}_+, V(\mathbb{R}))$ and every $x \in \overline{\mathbb{R}}$ the function $\mathfrak{a}(\cdot, x)$ belongs to the C^*-algebra $SO(\mathbb{R}_+)$, which allows us to define the values $\mathfrak{a}(\xi, x)$ for all $(\xi, x) \in (\Delta \cup \mathbb{R}_+) \times \overline{\mathbb{R}}$ by applying the Gelfand transform of $SO(\mathbb{R}_+)$ to $\mathfrak{a}(\cdot, x)$.

Let $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ be the Banach algebra of all $V(\mathbb{R})$-valued functions $\mathfrak{a}$ in the algebra $SO(\mathbb{R}_+, V(\mathbb{R}))$ such that

$$\lim_{|h|\to 0} \sup_{t\in\mathbb{R}_+} \left\|\mathfrak{a}(t,\cdot) - \mathfrak{a}^h(t,\cdot)\right\|_V = 0$$

where $\mathfrak{a}^h(t,x) := \mathfrak{a}(t, x+h)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

Let $\mathfrak{a} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. Since $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) \subset SO(\mathbb{R}_+, V(\mathbb{R}))$, we conclude that for every $x \in \overline{\mathbb{R}}$ the function $\mathfrak{a}(\cdot, x)$ belongs to the C^*-algebra $SO(\mathbb{R}_+)$, and therefore the function $\mathfrak{a}$ can be extended to $\Delta \times \overline{\mathbb{R}}$ by applying the Gelfand transform of $SO(\mathbb{R}_+)$ to $\mathfrak{a}(\cdot, x)$.

Similarly to [1, Proposition 4.2, Corollary 4.3] we have the following.

Lemma 2.2 *([17, Proposition 2.2]). Suppose* $\{a_k\}_{k\in\mathbb{N}}$ *is a countable subset of the space* $SO(\mathbb{R}_+)$ *and* $s \in \{0,\infty\}$. *For each* $\xi \in M_s(SO(\mathbb{R}_+))$ *there exists a sequence* $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ *such that* $t_n \to s$ *as* $n \to \infty$ *and*

$$\xi(a_k) = a_k(\xi) = \lim_{n\to\infty} a_k(t_n) \quad \text{for all} \quad k \in \mathbb{N}. \tag{2.1}$$

Conversely, if $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ *is a sequence such that* $t_n \to s$ *as* $n \to \infty$ *and the limits* $\lim_{n\to\infty} a_k(t_n)$ *exist for all* $k \in \mathbb{N}$, *then there exists a functional* $\xi \in M_s(SO(\mathbb{R}_+))$ *such that* (2.1) *holds.*

By analogy with [15, Lemma 2.7] with the aid of Lemma 2.2 one can prove the following assertion.

Lemma 2.3. *Let* $s \in \{0,\infty\}$ *and* $\{\mathfrak{a}_k\}_{k\in\mathbb{N}}$ *be a countable subset of the algebra* $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. *For each* $\xi \in M_s(SO(\mathbb{R}_+))$ *there is a sequence* $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_+$ *and functions* $\mathfrak{a}_k(\xi,\cdot) \in V(\mathbb{R})$ *such that* $t_j \to s$ *as* $j \to \infty$ *and*

$$\mathfrak{a}_k(\xi, x) = \lim_{j\to\infty} \mathfrak{a}_k(t_j, x)$$

for every $x \in \overline{\mathbb{R}}$ *and every* $k \in \mathbb{N}$.

This lemma gives another possibility to define the values $\mathfrak{a}(\xi, x)$ of $\mathfrak{a} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ for all $(\xi, x) \in \Delta \times \overline{\mathbb{R}}$. Note that the latter approach was used in our works [6,9–11]. From the proof of [15, Lemma 2.7] one can see that $\mathfrak{a}(\xi, x)$ defined as the Gelfand transform of $\mathfrak{a}(\cdot, x) \in SO(\mathbb{R}_+)$ coincides with $\mathfrak{a}(\xi, x)$ calculated by Lemma 2.3, that is, both these definitions of $\mathfrak{a}(\xi, x)$ for $(\xi, x) \in \Delta \times \overline{\mathbb{R}}$ are equivalent.

Theorem 2.4 *([9, Theorem 3.3]). If $\mathfrak{a}, \mathfrak{b} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$, then* $\operatorname{Op}(\mathfrak{a}) \operatorname{Op}(\mathfrak{b}) \simeq \operatorname{Op}(\mathfrak{a}\mathfrak{b})$.

Lemma 2.5 *([9, Lemma 3.4]). If $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ are such that $\mathfrak{a}$ depends only on the first variable and $\mathfrak{c}$ depends only on the second variable, then*

$$\operatorname{Op}(\mathfrak{a}) \operatorname{Op}(\mathfrak{b}) \operatorname{Op}(\mathfrak{c}) = \operatorname{Op}(\mathfrak{a}\mathfrak{b}\mathfrak{c}).$$

2.3. Algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ and the Fredholmness of Mellin PDO's

Consider the Banach algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ consisting of all functions $\mathfrak{a}$ belonging to $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ and such that

$$\lim_{m\to\infty} \sup_{t\in\mathbb{R}_+} \int\limits_{\mathbb{R}\setminus[-m,m]} |\partial_x \mathfrak{a}(t,x)|\, dx = 0.$$

Now we state two results on the inversion of functions in the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

Lemma 2.6 *([10, Lemma 4.2]). If $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is such that*

$$\inf_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} |\mathfrak{a}(t,x)| > 0,$$

then $1/\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

For $\ell > 1$, put $T_\ell := (0, \ell^{-1}] \cup [\ell, \infty)$.

Lemma 2.7 *([10, Lemma 5.4]). If $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is such that*

$$\mathfrak{a}(t, \pm\infty) \neq 0 \ \textit{ for all } \ t \in \mathbb{R}_+, \quad \mathfrak{a}(\xi, x) \neq 0 \ \textit{ for all } \ (\xi, x) \in \Delta \times \overline{\mathbb{R}}, \tag{2.2}$$

then there exists an $\ell > 1$ such that

$$\inf_{(t,x)\in T_\ell\times\overline{\mathbb{R}}} |\mathfrak{a}(t,x)| > 0.$$

By analogy with [10, Lemma 5.4] one can also prove the following.

Lemma 2.8. *If $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is such that $\mathfrak{a}(\xi, x) = 0$ for all $(\xi, x) \in \Delta \times \overline{\mathbb{R}}$, then for every $\delta > 0$ there exists an $\ell(\delta) > 1$ such that*

$$\sup_{(t,x)\in T_{\ell(\delta)}\times\overline{\mathbb{R}}} |\mathfrak{a}(t,x)| < \delta.$$

The following theorem is the key ingredient in our analysis.

Theorem 2.9 *([9, Theorem 3.6]). If $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$, then the operator $\operatorname{Op}(\mathfrak{a})$ is Fredholm on the space $L^p(\mathbb{R}_+, d\mu)$ if and only if (2.2) is fulfilled. In the case of Fredholmness, $\mathfrak{a}(t, x) \neq 0$ whenever $(t, x) \in \partial\Pi_\tau$ for all sufficiently large τ, and*

$$\operatorname{Ind} \operatorname{Op}(\mathfrak{a}) = \lim_{\tau\to+\infty} \frac{1}{2\pi} \big\{ \arg \mathfrak{a}(t,x) \big\}_{(t,x)\in\partial\Pi_\tau},$$

where $\Pi_\tau = [\tau^{-1}, \tau] \times \overline{\mathbb{R}}$ *and* $\{\arg \mathfrak{a}(t,x)\}_{(t,x)\in\partial\Pi_\tau}$ *denotes the increment of the function* $\arg \mathfrak{a}(t,x)$ *when the point* (t,x) *traces the boundary* $\partial\Pi_\tau$ *of* Π_τ *counter-clockwise.*

This result follows from [19, Theorem 4.3]. Note that for infinitely differentiable slowly oscillating symbols such result was obtained earlier in [23, Theorem 2.6].

2.4. Singular integral operators as Mellin pseudodifferential operators

Along with functions s_γ and $p_\gamma^\pm$ given by (1.4), consider the function r_γ defined by

$$r_\gamma(x) := 1/\sinh[\pi(x + i/p + i\gamma)], \quad x \in \mathbb{R}. \tag{2.3}$$

Lemma 2.10 *([6, Lemma 4.1]). Suppose* $f \in SO(\mathbb{R}_+)$ *and* $\gamma \in \mathbb{C}$ *satisfies (1.1). Then the functions*

$$\mathfrak{f}(t,x) := f(t), \quad \mathfrak{s}_\gamma(t,x) := s_\gamma(x), \quad \mathfrak{r}_\gamma(t,x) := r_\gamma(x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

belong to the Banach algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$*.*

Consider the isometric isomorphism

$$\Phi : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+, d\mu), \quad (\Phi f)(t) := t^{1/p} f(t), \quad t \in \mathbb{R}_+.$$

From [25, Propositions 4.2.11 and 4.2.15] (see also [3,4,23,27]) we can get the following important and well known fact.

Theorem 2.11. *Let* $1 < p < \infty$ *and* $\gamma \in \mathbb{C}$ *be such that* $0 < 1/p + \Re\gamma < 1$*. Then*

$$S_\gamma = \Phi^{-1} \operatorname{Op}(\mathfrak{s}_\gamma)\Phi, \quad R_\gamma = \Phi^{-1} \operatorname{Op}(\mathfrak{r}_\gamma)\Phi.$$

This result together with Lemmas 2.5 and 2.10 implies, in particular, that the operators $P_\gamma^\pm$, $P_\delta^\pm$, R_γ, and R_δ are pairwise commuting whenever $\gamma, \delta \in \mathbb{C}$ satisfy $0 < 1/p + \Re\gamma, 1/p + \Re\delta < 1$. The next lemma provides more relations for these operators.

Lemma 2.12 *([6, Lemma 2.4]). Let* $1 < p < \infty$ *and* $\gamma, \delta \in \mathbb{C}$ *be such that* $0 < 1/p + \Re\gamma < 1$ *and* $0 < 1/p + \Re\delta < 1$*. Then for every* $x \in \mathbb{R}$*,*

$$p_\gamma^\pm(x)p_\delta^\pm(x) = \frac{1}{2}p_\gamma^\pm(x) + \frac{1}{2}p_\delta^\pm(x) + \frac{\cos[\pi(\gamma-\delta)]}{4} r_\gamma(x) r_\delta(x), \quad p_\gamma^-(x)p_\delta^+(x) = -\frac{e^{i\pi(\delta-\gamma)}}{4} r_\gamma(x) r_\delta(x) \tag{2.4}$$

and

$$P_\gamma^\pm P_\delta^\pm = \frac{1}{2}P_\gamma^\pm + \frac{1}{2}P_\delta^\pm + \frac{\cos[\pi(\gamma-\delta)]}{4} R_\gamma R_\delta, \quad P_\gamma^- P_\delta^+ = -\frac{e^{i\pi(\delta-\gamma)}}{4} R_\gamma R_\delta. \tag{2.5}$$

2.5. Some important functions in the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$

By analogy with [9, Lemma 4.3], one can prove the following.

Lemma 2.13. *Let* $\gamma \in \mathbb{C}$ *satisfy (1.1) and* $\omega \in SO(\mathbb{R}_+)$ *be a real-valued function. Then the function*

$$\mathfrak{b}(t,x) := e^{i\omega(t)x} r_\gamma(x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

belongs to the Banach algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ *and there is a constant* $C(p,\gamma) \in (0,\infty)$ *depending only on* p *and* γ *such that*

$$\|\mathfrak{b}\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))} \le C(p,\gamma)\left(1 + \sup_{t\in\mathbb{R}_+} |\omega(t)|\right).$$

Recall that $C_b(\mathbb{R}_+)$ denotes the space of all bounded continuous functions on $\mathbb{R}_+$ with the supremum norm.

Lemma 2.14. *Let* $\gamma \in \mathbb{C}$ *satisfy* (1.1) *and* $f, \omega \in SO(\mathbb{R}_+)$. *If* $\|f\|_{C_b(\mathbb{R}_+)} < 1$ *and* ω *is a real-valued function, then the functions*

$$\mathfrak{a}(t,x) := (1 - f(t)e^{i\omega(t)x})r_\gamma(x), \quad \mathfrak{c}(t,x) := (1 - f(t)e^{i\omega(t)x})^{-1} r_\gamma(x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

belong to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

Proof. From Lemmas 2.10 and 2.13 we immediately get $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Since $\|f\|_{C_b(\mathbb{R})} < 1$, we have

$$\mathfrak{c}(t,x) = \sum_{n=0}^{\infty} (f(t))^n e^{in\omega(t)x} r_\gamma(x) = \sum_{n=0}^{\infty} \mathfrak{c}_n(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

where $\mathfrak{c}_n(t,x) := (f(t))^n e^{in\omega(t)x} r_\gamma(x)$. From Lemmas 2.10 and 2.13 it follows that $\mathfrak{c}_n \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ for all n, and

$$\|\mathfrak{c}_n\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))} \le C(p,\gamma)\|f\|^n_{C_b(\mathbb{R}_+)}\left(1 + n\sup_{t\in\mathbb{R}_+} |\omega(t)|\right).$$

Taking into account the fact that $\|f\|_{C_b(\mathbb{R}_+)} < 1$, we have

$$\begin{aligned}\limsup_{n\to\infty} \|\mathfrak{c}_n\|^{1/n}_{C_b(\mathbb{R}_+,V(\mathbb{R}))} &\le \|f\|_{C_b(\mathbb{R}_+)} \limsup_{n\to\infty} C(p,\gamma)^{1/n} \limsup_{n\to\infty}\left(1 + n\sup_{t\in\mathbb{R}_+}|\omega(t)|\right)^{1/n} \\ &= \|f\|_{C_b(\mathbb{R}_+)} < 1.\end{aligned}$$

Thus the series $\sum_{n=0}^{\infty} \mathfrak{c}_n$ is absolutely convergent in the norm of $C_b(\mathbb{R}_+, V(\mathbb{R}))$, whence the function $\mathfrak{c} = \sum_{n=0}^{\infty} \mathfrak{c}_n$ belongs to the Banach subalgebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ of $C_b(\mathbb{R}_+, V(\mathbb{R}))$ along with each function $\mathfrak{c}_n$. □

2.6. Operators $U_\alpha R_\gamma$ *and* $U_\alpha U_\beta R_\gamma$ *as Mellin pseudodifferential operators*

It was observed in [9] that the product of U_α and R_0 can be realized as a Mellin pseudodifferential operator with symbol in the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. In this subsection we formulate a slight generalization of that result and some of its consequences.

Lemma 2.15 *([6, Lemma 4.4]). Let* $\gamma \in \mathbb{C}$ *satisfy* (1.1). *Suppose* $\alpha \in SOS(\mathbb{R}_+)$, ω *is its exponent function, and* U_α *is the associated isometric shift operator on* $L^p(\mathbb{R}_+)$. *Then the operator* $U_\alpha R_\gamma$ *can be realized as the Mellin pseudodifferential operator:*

$$U_\alpha R_\gamma = \Phi^{-1} \operatorname{Op}(\mathfrak{c}_{\omega,\gamma})\Phi,$$

where the function $\mathfrak{c}_{\omega,\gamma}$*, given by*

$$\mathfrak{c}_{\omega,\gamma}(t,x) := (1+t\omega'(t))^{1/p} e^{i\omega(t)x} r_\gamma(x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

belongs to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

From the above lemma, making minor modifications in the proof of [9, Lemma 4.5], we can get the following two lemmas.

Lemma 2.16. *Let* $\gamma \in \mathbb{C}$ *satisfy* (1.1)*. Suppose* $\alpha \in SOS(\mathbb{R}_+)$*,* ω *is its exponent function, and* U_α *is the associated isometric shift operator on* $L^p(\mathbb{R}_+)$*. Then the operators* $U_\alpha R_\gamma$ *and* $U_\alpha^{-1} R_\gamma$ *can be realized as the Mellin pseudodifferential operators up to compact operators:*

$$U_\alpha^{\pm 1} R_\gamma \simeq \Phi^{-1} \operatorname{Op}(\mathfrak{c}_\pm)\Phi,$$

where the functions $\mathfrak{c}^\pm$*, given by*

$$\mathfrak{c}_\pm(t,x) := e^{\pm i\omega(t)x} r_\gamma(x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

belong to the Banach algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

Lemma 2.17. *Let* $\gamma \in \mathbb{C}$ *satisfy* (1.1)*. Suppose* $\alpha, \beta \in SOS(\mathbb{R}_+)$*,* ω, η *are their exponent functions, respectively, and* U_α, U_β *are the associated isometric shift operators on* $L^p(\mathbb{R}_+)$*. Then the operators* $U_\alpha U_\beta^{\pm 1} R_\gamma$ *and* $U_\alpha^{-1} U_\beta^{\pm 1} R_\gamma$ *can be realized as the Mellin pseudodifferential operators up to compact operators:*

$$\begin{aligned} U_\alpha U_\beta R_\gamma &\simeq \Phi^{-1} \operatorname{Op}(\mathfrak{c}_{++})\Phi, & U_\alpha U_\beta^{-1} R_\gamma &\simeq \Phi^{-1} \operatorname{Op}(\mathfrak{c}_{+-})\Phi, \\ U_\alpha^{-1} U_\beta R_\gamma &\simeq \Phi^{-1} \operatorname{Op}(\mathfrak{c}_{-+})\Phi, & U_\alpha^{-1} U_\beta^{-1} R_\gamma &\simeq \Phi^{-1} \operatorname{Op}(\mathfrak{c}_{--})\Phi, \end{aligned}$$

where the functions $\mathfrak{c}_{++}, \mathfrak{c}_{+-}, \mathfrak{c}_{-+}, \mathfrak{c}_{--}$*, given for* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ *by*

$$\begin{aligned} \mathfrak{c}_{++}(t,x) &:= e^{i\omega(t)x} e^{i\eta(t)x} r_\gamma(x), & \mathfrak{c}_{+-}(t,x) &:= e^{i\omega(t)x} e^{-i\eta(t)x} r_\gamma(x), \\ \mathfrak{c}_{-+}(t,x) &:= e^{-i\omega(t)x} e^{i\eta(t)x} r_\gamma(x), & \mathfrak{c}_{--}(t,x) &:= e^{-i\omega(t)x} e^{-i\eta(t)x} r_\gamma(x), \end{aligned}$$

belong to the Banach algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

3. Sufficient conditions for the Fredholmness of the operator N

3.1. Notation

We suppose that $\gamma \in \mathbb{C}$ satisfies (1.1), γ_* is defined by (1.3), a, b, c, d belong to $SO(\mathbb{R}_+)$, α, β belong to $SOS(\mathbb{R}_+)$, and ω, η are the exponent functions of α, β, respectively. Suppose the functional operators $A_+ = aI - bU_\alpha$ and $A_- = cI - dU_\beta$ are invertible on the space $L^p(\mathbb{R}_+)$. Then, according to Theorem 1.1, we have four mutually exclusive possibilities:

(A) $a \gg b$ and $c \gg d$; (B) $b \gg a$ and $c \gg d$; (C) $a \gg b$ and $d \gg c$; (D) $b \gg a$ and $d \gg c$.

In this case we put

$$\nu_{a,b}^+ := \begin{cases} a, & \text{if } a \gg b, \\ b, & \text{if } b \gg a, \end{cases} \quad \nu_{a,b}^- := \begin{cases} b, & \text{if } a \gg b, \\ a, & \text{if } b \gg a, \end{cases} \quad \varepsilon_{a,b} := \begin{cases} 1, & \text{if } a \gg b, \\ -1, & \text{if } b \gg a, \end{cases} \quad \theta_{a,b} := \begin{cases} 0, & \text{if } a \gg b, \\ -1, & \text{if } b \gg a. \end{cases} \tag{3.1}$$

The functions $\nu_{c,d}^\pm$ and the numbers $\varepsilon_{c,d}, \theta_{c,d}$ are defined analogously. Note that $\nu_{a,b}^+, \nu_{c,d}^+$ coincide, respectively, with $\nu_{a,b}, \nu_{c,d}$ given by (1.8). Further, we set

$$\theta_1 := \theta_{a,b}, \quad \theta_2 := \theta_{c,d}, \quad \varepsilon_1 := \varepsilon_{a,b}, \quad \varepsilon_2 := \varepsilon_{c,d}, \quad \nu_1^\pm := \nu_{a,b}^\pm, \quad \nu_2^\pm := \nu_{c,d}^\pm. \tag{3.2}$$

3.2. First operator relation

Recall that multiplication operators and shift operators commute with singular integral operators up to compact operators. More precisely, from [11, Lemmas 2.8 and 3.1] we immediately obtain the following.

Lemma 3.1. *Let $\gamma \in \mathbb{C}$ satisfy (1.1). If $a \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$, then*

$$aP_\gamma^\pm \simeq P_\gamma^\pm aI, \quad U_\alpha P_\gamma^\pm \simeq P_\gamma^\pm U_\alpha, \quad U_\alpha^{-1} P_\gamma^\pm \simeq P_\gamma^\pm U_\alpha^{-1}.$$

Now we establish a relation between the operator N given by (1.2), an operator F of the form (1.9), a weighted singular integral operator without shifts B, an operator V of the form (1.10), some operator M, and the operator R_{γ_*}.

Lemma 3.2. *If the functional operators $A_+ = aI - bU_\alpha$ and $A_- = cI - dU_\beta$ are invertible on the space $L^p(\mathbb{R}_+)$, then*

$$NF \simeq BV + MR_{\gamma_*}, \tag{3.3}$$

where

$$F := U_\alpha^{\theta_1} P_{\gamma_*}^+ + U_\beta^{\theta_2} P_{\gamma_*}^-, \tag{3.4}$$

$$B := \varepsilon_1 \nu_1^+ P_\gamma^+ + \varepsilon_2 \nu_2^+ P_\gamma^-, \tag{3.5}$$

$$V := \left(I - \frac{\nu_1^-}{\nu_1^+} U_\alpha^{\varepsilon_1}\right) P_{\gamma_*}^+ + \left(I - \frac{\nu_2^-}{\nu_2^+} U_\beta^{\varepsilon_2}\right) P_{\gamma_*}^-, \tag{3.6}$$

and

$$\begin{aligned} M := {} & \frac{e^{i\pi(\gamma-\gamma_*)}}{4} \left[\varepsilon_1 \nu_1^+ \left(I - \frac{\nu_2^-}{\nu_2^+} U_\beta^{\varepsilon_2}\right) - (aI - bU_\alpha) U_\beta^{\theta_2}\right] R_\gamma \\ & + \frac{e^{i\pi(\gamma_*-\gamma)}}{4} \left[\varepsilon_2 \nu_2^+ \left(I - \frac{\nu_1^-}{\nu_1^+} U_\alpha^{\varepsilon_1}\right) - (cI - dU_\beta) U_\alpha^{\theta_1}\right] R_\gamma. \end{aligned} \tag{3.7}$$

Proof. From Lemma 3.1 it follows that

$$\begin{aligned} NF \simeq {} & (aI - bU_\alpha) U_\alpha^{\theta_1} P_\gamma^+ P_{\gamma_*}^+ + (cI - dU_\beta) U_\beta^{\theta_2} P_\gamma^- P_{\gamma_*}^- \\ & + (aI - bU_\alpha) U_\beta^{\theta_2} P_\gamma^+ P_{\gamma_*}^- + (cI - dU_\beta) U_\alpha^{\theta_1} P_\gamma^- P_{\gamma_*}^+ \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} BV \simeq {} & \varepsilon_1 (\nu_1^+ - \nu_1^- U_\alpha^{\varepsilon_1}) P_\gamma^+ P_{\gamma_*}^+ + \varepsilon_2 (\nu_2^+ - \nu_2^- U_\beta^{\varepsilon_2}) P_\gamma^- P_{\gamma_*}^- \\ & + \varepsilon_1 \nu_1^+ \left(I - \frac{\nu_2^-}{\nu_2^+} U_\beta^{\varepsilon_2}\right) P_\gamma^+ P_{\gamma_*}^- + \varepsilon_2 \nu_2^+ \left(I - \frac{\nu_1^-}{\nu_1^+} U_\alpha^{\varepsilon_1}\right) P_\gamma^- P_{\gamma_*}^+. \end{aligned} \tag{3.9}$$

Taking into account (3.1)–(3.2), it is easy to see that

$$(aI - bU_\alpha)U_\alpha^{\theta_1} = \varepsilon_1(\nu_1^+ - \nu_1^- U_\alpha^{\varepsilon_1}), \quad (cI - dU_\beta)U_\beta^{\theta_2} = \varepsilon_2(\nu_2^+ - \nu_2^- U_\beta^{\varepsilon_2}). \tag{3.10}$$

Combining (3.8)–(3.10) with the second identity in (2.5), we obtain

$$\begin{aligned} NF - BV &\simeq \left[(aI - bU_\alpha)U_\beta^{\theta_2} - \varepsilon_1\nu_1^+\left(I - \frac{\nu_2^-}{\nu_2^+}U_\beta^{\varepsilon_2}\right)\right] P_\gamma^+ P_{\gamma_*}^- \\ &\quad + \left[(cI - dU_\beta)U_\alpha^{\theta_1} - \varepsilon_2\nu_2^+\left(I - \frac{\nu_1^-}{\nu_1^+}U_\alpha^{\varepsilon_1}\right)\right] P_\gamma^- P_{\gamma_*}^+ \\ &= \frac{e^{i\pi(\gamma-\gamma_*)}}{4}\left[\varepsilon_1\nu_1^+\left(I - \frac{\nu_2^-}{\nu_2^+}U_\beta^{\varepsilon_2}\right) - (aI - bU_\alpha)U_\beta^{\theta_2}\right] R_\gamma R_{\gamma_*} \\ &\quad + \frac{e^{i\pi(\gamma_*-\gamma)}}{4}\left[\varepsilon_2\nu_2^+\left(I - \frac{\nu_1^-}{\nu_1^+}U_\alpha^{\varepsilon_1}\right) - (cI - dU_\beta)U_\alpha^{\theta_1}\right] R_\gamma R_{\gamma_*}. \end{aligned}$$

From the above relation and (3.7) we immediately get (3.3). □

3.3. Main operator relation

The following theorem is the heart of the proof of Theorem 1.3 and the index formula in Theorem 1.4.

Theorem 3.3. *Let $\gamma \in \mathbb{C}$ satisfy (1.1) and γ_* be given by (1.3). Suppose a, b, c, d belong to $SO(\mathbb{R}_+)$, α, β belong to $SOS(\mathbb{R}_+)$, and $\omega, \eta \in SO(\mathbb{R}_+)$ are the exponent functions of α, β, respectively. If the functional operators $A_+ = aI - bU_\alpha$ and $A_- = cI - dU_\beta$ are invertible on the space $L^p(\mathbb{R}_+)$, then*

$$NF \simeq HV, \tag{3.11}$$

where

(a) *for $i = 1, 2$, the functions ν_i and the constants ε_i, θ_i are defined by (3.1)–(3.2);*
(b) *the operator F defined by (3.4) is Fredholm on the space $L^p(\mathbb{R}_+)$ and its index is equal to zero;*
(c) *the operator V defined by (3.6) is Fredholm on the space $L^p(\mathbb{R}_+)$ and its index is equal to zero;*
(d) *the operator H is of the form*

$$H = \Phi^{-1}\,\mathrm{Op}(\mathfrak{h})\Phi,$$

where $\mathfrak{h} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is given by

$$\mathfrak{h}(t, x) := \mathfrak{b}(t, x) + \mathfrak{m}(t, x)\mathfrak{g}(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{3.12}$$

with the functions $\mathfrak{b}, \mathfrak{m} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ given by

$$\mathfrak{b}(t, x) := \varepsilon_1\nu_1^+(t)p_\gamma^+(x) + \varepsilon_2\nu_2^+(t)p_\gamma^-(x) \tag{3.13}$$

and

$$\begin{aligned} \mathfrak{m}(t, x) &:= \frac{e^{i\pi(\gamma-\gamma_*)}}{4}\left[\varepsilon_1\nu_1^+(t)\left(1 - \frac{\nu_2^-(t)}{\nu_2^+(t)}e^{i\varepsilon_2\eta(t)x}\right) - (a(t) - b(t)e^{i\omega(t)x})e^{i\theta_2\eta(t)x}\right] r_\gamma(x) \\ &\quad + \frac{e^{i\pi(\gamma_*-\gamma)}}{4}\left[\varepsilon_2\nu_2^+(t)\left(1 - \frac{\nu_1^-(t)}{\nu_1^+(t)}e^{i\varepsilon_1\omega(t)x}\right) - (c(t) - d(t)e^{i\eta(t)x})e^{i\theta_1\omega(t)x}\right] r_\gamma(x), \end{aligned} \tag{3.14}$$

and some function $\mathfrak{g} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ *satisfying*

$$\mathfrak{g}(\xi, x) = \begin{cases} \dfrac{r_{\gamma_*}(x)}{v(\xi, x)}, & \text{if} \quad (\xi, x) \in \Delta \times \mathbb{R}, \\ 0, & \text{if} \quad (\xi, x) \in (\Delta \cup \mathbb{R}_+) \times \{\pm\infty\}, \end{cases} \tag{3.15}$$

where

$$v(\xi, x) := \left(1 - \frac{\nu_1^-(\xi)}{\nu_1^+(\xi)} e^{i\varepsilon_1 \omega(\xi) x}\right) p_{\gamma_*}^+(x) + \left(1 - \frac{\nu_2^-(\xi)}{\nu_2^+(\xi)} e^{i\varepsilon_2 \eta(\xi) x}\right) p_{\gamma_*}^-(x) \neq 0 \tag{3.16}$$

for $(\xi, x) \in \Delta \times \mathbb{R}$.

Proof. Part (a) is trivial. Parts (b) and (c) follow from [6, Theorem 1.1] and [11, Theorem 7.1], respectively. Let us establish relation (3.11) and part (d). From Theorem 2.11 and Lemmas 2.5, 2.10 we immediately get the following description of the operator B given by (3.5) via a Mellin pseudodifferential operator:

$$B = \Phi^{-1} \operatorname{Op}(\mathfrak{b}) \Phi, \tag{3.17}$$

where the function $\mathfrak{b}$ given by (3.13) belongs to the Banach algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. From [11, Theorem 7.2] it follows that there is a function $\mathfrak{g} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ such that

$$(\Phi^{-1} \operatorname{Op}(\mathfrak{g}) \Phi) V \simeq R_{\gamma_*} \tag{3.18}$$

and (3.15)–(3.16) are fulfilled.

From Lemmas 2.5, 2.10, and 2.16 we obtain

$$\varepsilon_1 \nu_1^+ \left(I - \frac{\nu_2^-}{\nu_2^+} U_\beta^{\varepsilon_2}\right) R_\gamma \simeq \Phi^{-1} \operatorname{Op}(\mathfrak{m}_1) \Phi, \quad \varepsilon_2 \nu_2^+ \left(I - \frac{\nu_1^-}{\nu_1^+} U_\alpha^{\varepsilon_1}\right) R_\gamma \simeq \Phi^{-1} \operatorname{Op}(\mathfrak{m}_2) \Phi, \tag{3.19}$$

where the functions $\mathfrak{m}_1, \mathfrak{m}_2$, given for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\mathfrak{m}_1(t, x) := \varepsilon_1 \nu_1^+(t) \left(1 - \frac{\nu_2^-(t)}{\nu_2^+(t)} e^{i\varepsilon_2 \eta(t) x}\right) r_\gamma(x), \tag{3.20}$$

$$\mathfrak{m}_2(t, x) := \varepsilon_2 \nu_2^+(t) \left(1 - \frac{\nu_1^-(t)}{\nu_1^+(t)} e^{i\varepsilon_1 \omega(t) x}\right) r_\gamma(x), \tag{3.21}$$

belong to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Analogously, applying the above mentioned lemmas and Lemma 2.17, we get

$$(aI - bU_\alpha) U_\beta^{\theta_2} R_\gamma \simeq \Phi^{-1} \operatorname{Op}(\mathfrak{m}_3) \Phi, \quad (cI - dU_\beta) U_\alpha^{\theta_1} R_\gamma \simeq \Phi^{-1} \operatorname{Op}(\mathfrak{m}_4) \Phi, \tag{3.22}$$

where the functions $\mathfrak{m}_3, \mathfrak{m}_4$, given for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\mathfrak{m}_3(t, x) := (a(t) - b(t) e^{i\omega(t) x}) e^{i\theta_2 \eta(t) x} r_\gamma(x), \tag{3.23}$$

$$\mathfrak{m}_4(t, x) := (c(t) - d(t) e^{i\eta(t) x}) e^{i\theta_1 \omega(t) x} r_\gamma(x), \tag{3.24}$$

belong to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Combining (3.7) and (3.14) with (3.19)–(3.24), we arrive at

$$M \simeq \Phi^{-1} \operatorname{Op}(\mathfrak{m}) \Phi. \tag{3.25}$$

From Lemma 3.2, relations (3.17)–(3.18) and (3.25) we get

$$\begin{aligned} NF &\simeq BV + MR_{\gamma_*} \simeq BV + M(\Phi^{-1}\operatorname{Op}(\mathfrak{g})\Phi)V \\ &\simeq [\Phi^{-1}\operatorname{Op}(\mathfrak{b})\Phi + (\Phi^{-1}\operatorname{Op}(\mathfrak{m})\Phi)(\Phi^{-1}\operatorname{Op}(\mathfrak{g})\Phi)]V. \end{aligned} \tag{3.26}$$

By Theorem 2.4,

$$\operatorname{Op}(\mathfrak{m})\operatorname{Op}(\mathfrak{g}) \simeq \operatorname{Op}(\mathfrak{m}\mathfrak{g}). \tag{3.27}$$

From (3.26) and (3.27) we obtain

$$NF \simeq (\Phi^{-1}\operatorname{Op}(\mathfrak{b}+\mathfrak{m}\mathfrak{g})\Phi)V = (\Phi^{-1}\operatorname{Op}(\mathfrak{h})\Phi)V,$$

where $\mathfrak{h} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is given by (3.12), which completes the proof of (3.11) and part (d). $\square$

Let X be a Banach space. Recall that an operator $B_r \in \mathcal{B}(X)$ (resp. $B_l \in \mathcal{B}(X)$) is said to be a right (resp. left) regularizer for A if

$$AB_r \simeq I \quad (\text{resp.} \quad B_l A \simeq I).$$

It is well known that an operator A is Fredholm on X if and only if it admits simultaneously a right and a left regularizers. Moreover, each right regularizer differs from each left regularizer by a compact operator (see, e.g., [5, Chap. 4, Section 7]). Therefore, we may speak of a regularizer $B = B_r = B_l$ of a Fredholm operator A.

Corollary 3.4. *Suppose a, b, c, d belong to $SO(\mathbb{R}_+)$, α, β belong to $SOS(\mathbb{R}_+)$, and the functional operators $A_+ = aI - bU_\alpha$ and $A_- = cI - dU_\beta$ are invertible on the space $L^p(\mathbb{R}_+)$. Then the operator N is Fredholm on the space $L^p(\mathbb{R}_+)$ if and only if the operator $\operatorname{Op}(\mathfrak{h})$ is Fredholm on the space $L^p(\mathbb{R}_+, d\mu)$, where $\mathfrak{h} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is defined in Theorem 3.3(d). In the case of Fredholmness,*

$$\operatorname{Ind} N = \operatorname{Ind}\operatorname{Op}(\mathfrak{h}).$$

Proof. By Theorem 3.3, the operators F and V are Fredholm and have zero indices. If the operator N is Fredholm, then the operator NF is Fredholm. Hence there exist operators R_1 and R_2 such that

$$R_1(NF) \simeq (NF)R_1 \simeq I, \quad R_2V \simeq VR_2 \simeq I.$$

From these relations and (3.11) it follows that

$$(VR_1)H \simeq (VR_1)(NF)R_2 \simeq VR_2 \simeq I, \quad H(VR_1) \simeq (NF)R_2(VR_1) \simeq (NF)R_1 \simeq I,$$

that is, the operator VR_1 is a regularizer of H. Therefore, the operator $H = \Phi^{-1}\operatorname{Op}(\mathfrak{h})\Phi$ is Fredholm, whence the operator $\operatorname{Op}(\mathfrak{h})$ is Fredholm. Analogously it can be shown, that if the operator $\operatorname{Op}(\mathfrak{h})$ is Fredholm, then the operator N is Fredholm. From the relations $NF \simeq HV$ and $\operatorname{Ind} F = \operatorname{Ind} V = 0$ we deduce that

$$\operatorname{Ind} N = \operatorname{Ind} N + \operatorname{Ind} F = \operatorname{Ind}(NF) = \operatorname{Ind}(HV) = \operatorname{Ind} H + \operatorname{Ind} V = \operatorname{Ind} H.$$

It remains to observe that $\operatorname{Ind} H = \operatorname{Ind}\operatorname{Op}(\mathfrak{h})$. $\square$

3.4. Functional identities

With the operators F and V defined by (3.4) and (3.6), respectively, we associate the functions

$$f(\xi,x) := e^{i\theta_1\omega(\xi)x}p^+_{\gamma_*}(x) + e^{i\theta_2\eta(\xi)x}p^-_{\gamma_*}(x), \tag{3.28}$$

$$v(\xi,x) := \left(1 - \frac{\nu_1^-(\xi)}{\nu_1^+(\xi)}e^{i\varepsilon_1\omega(\xi)x}\right)p^+_{\gamma_*}(x) + \left(1 - \frac{\nu_2^-(\xi)}{\nu_2^+(\xi)}e^{i\varepsilon_2\eta(\xi)x}\right)p^-_{\gamma_*}(x), \tag{3.29}$$

defined for all $(\xi,x) \in (\Delta\cup\mathbb{R}_+)\times\mathbb{R}$. Notice that both F and V are particular cases of the operator N given by (1.2) and the functions f and v are particular cases of the function n defined by (1.5).

Lemma 3.5. *Suppose a,b,c,d belong to $SO(\mathbb{R}_+)$, α,β belong to $SOS(\mathbb{R}_+)$, and the functional operators $A_+ = aI - bU_\alpha$ and $A_- = cI - dU_\beta$ are invertible on the space $L^p(\mathbb{R}_+)$. For every $(t,x) \in \mathbb{R}_+\times\mathbb{R}$, we have*

$$n(t,x)f(t,x) = \mathfrak{b}(t,x)v(t,x) + \mathfrak{m}(t,x)r_{\gamma_*}(x), \tag{3.30}$$

where the functions $n, f, \mathfrak{b}, v, \mathfrak{m}$, and r_{γ_*} are defined by (1.5), (3.28), (3.13), (3.29), (3.14), and (2.3), respectively.

Proof. This proof resembles the proof of Lemma 3.2. We suppose that $(t,x) \in \mathbb{R}_+\times\mathbb{R}$. Obviously,

$$\begin{aligned} n(t,x)f(t,x) &= (a(t) - b(t)e^{i\omega(t)x})e^{i\theta_1\omega(t)x}p^+_\gamma(x)p^+_{\gamma_*}(x) + (c(t) - d(t)e^{i\eta(t)x})e^{i\theta_2\eta(t)x}p^-_\gamma(x)p^-_{\gamma_*}(x) \\ &\quad + (a(t) - b(t)e^{i\omega(t)x})e^{i\theta_2\eta(t)x}p^+_\gamma(x)p^-_{\gamma_*}(x) + (c(t) - d(t)e^{i\eta(t)x})e^{i\theta_1\omega(t)x}p^-_\gamma(x)p^+_{\gamma_*}(x) \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} \mathfrak{b}(t,x)v(t,x) &= \varepsilon_1(\nu_1^+(t) - \nu_1^-(t)e^{i\varepsilon_1\omega(t)x})p^+_\gamma(x)p^+_{\gamma_*}(x) + \varepsilon_2(\nu_2^+(t) - \nu_2^-(t)e^{i\eta(t)x})p^-_\gamma(x)p^-_{\gamma_*}(x) \\ &\quad + \varepsilon_1\nu_1^+(t)\left(1 - \frac{\nu_2^-(t)}{\nu_2^+(t)}e^{i\varepsilon_2\eta(t)x}\right)p^+_\gamma(x)p^-_{\gamma_*}(x) \\ &\quad + \varepsilon_2\nu_2^+(t)\left(1 - \frac{\nu_1^-(t)}{\nu_1^+(t)}e^{i\varepsilon_1\omega(t)x}\right)p^-_\gamma(x)p^+_{\gamma_*}(x). \end{aligned} \tag{3.32}$$

Taking into account (3.1)–(3.2), it is easy to see that

$$(a(t) - b(t)e^{i\omega(t)x})e^{i\theta_1\omega(t)x} = \varepsilon_1(\nu_1^+(t) - \nu_1^-(t)e^{i\varepsilon_1\omega(t)x}), \tag{3.33}$$

$$(c(t) - d(t)e^{i\eta(t)x})e^{i\theta_2\eta(t)x} = \varepsilon_2(\nu_2^+(t) - \nu_2^-(t)e^{i\varepsilon_2\eta(t)x}). \tag{3.34}$$

From (3.31)–(3.34), (2.4), and (3.14) we obtain

$$\begin{aligned} n(t,x)f(t,x) - \mathfrak{b}(t,x)v(t,x) &= \left[(a(t) - b(t)e^{i\omega(t)x})e^{i\theta_2\eta(t)x} - \varepsilon_1\nu_1^+(t)\left(1 - \frac{\nu_2^-(t)}{\nu_2^+(t)}e^{i\varepsilon_2\eta(t)x}\right)\right]p^+_\gamma(x)p^-_{\gamma_*}(x) \\ &\quad + \left[(c(t) - d(t)e^{i\eta(t)x})e^{i\theta_1\omega(t)x} - \varepsilon_2\nu_2^+(t)\left(1 - \frac{\nu_1^-(t)}{\nu_1^+(t)}e^{i\varepsilon_1\omega(t)x}\right)\right]p^-_\gamma(x)p^+_{\gamma_*}(x) \\ &= \mathfrak{m}(t,x)r_{\gamma_*}(x), \end{aligned}$$

which completes the proof. $\square$

Lemma 3.6. *Suppose a, b, c, d belong to $SO(\mathbb{R}_+)$, α, β belong to $SOS(\mathbb{R}_+)$, and the functional operators $A_+ = aI - bU_\alpha$ and $A_- = cI - dU_\beta$ are invertible on the space $L^p(\mathbb{R}_+)$. For every $(\xi, x) \in \Delta \times \mathbb{R}$, we have*

$$n(\xi, x)f(\xi, x) = \mathfrak{h}(\xi, x)v(\xi, x),$$

where the functions $n, f, \mathfrak{h}$, and v are defined by (1.5), (3.28), (3.12), and (3.29), respectively.

Proof. For each fixed $x \in \mathbb{R}$, the functions $n(\cdot, x)$, $f(\cdot, x)$, $\mathfrak{b}(\cdot, x)$, $v(\cdot, x)$ and $\mathfrak{m}(\cdot, x)$ belong to $SO(\mathbb{R}_+)$. Then, for every $x \in \mathbb{R}$ and every functional $\xi \in \Delta$, we immediately infer from (3.30) that

$$n(\xi, x)f(\xi, x) = \mathfrak{b}(\xi, x)v(\xi, x) + \mathfrak{m}(\xi, x)r_{\gamma_*}(x). \tag{3.35}$$

Analogously, from (3.12) we obtain

$$\mathfrak{h}(\xi, x) = \mathfrak{b}(\xi, x) + \mathfrak{m}(\xi, x)\mathfrak{g}(\xi, x), \quad (\xi, x) \in \Delta \times \mathbb{R}, \tag{3.36}$$

where $\mathfrak{g} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ satisfies

$$\mathfrak{g}(\xi, x)v(\xi, x) = r_{\gamma_*}(x), \quad (\xi, x) \in \Delta \times \mathbb{R}, \tag{3.37}$$

in view of (3.15). Combining (3.35)–(3.37), we obtain

$$n(\xi, x)f(\xi, x) = \mathfrak{b}(\xi, x)v(\xi, x) + \mathfrak{m}(\xi, x)\mathfrak{g}(\xi, x)v(\xi, x) = \mathfrak{h}(\xi, x)v(\xi, x),$$

which completes the proof. □

Lemma 3.7. *Suppose a, b, c, d belong to $SO(\mathbb{R}_+)$, α, β belong to $SOS(\mathbb{R}_+)$, and the functional operators $A_+ = aI - bU_\alpha$ and $A_- = cI - dU_\beta$ are invertible on the space $L^p(\mathbb{R}_+)$. For every $\xi \in \mathbb{R}_+ \cup \Delta$, we have*

$$\mathfrak{h}(\xi, -\infty) = \varepsilon_{c,d}\nu_{c,d}(\xi), \quad \mathfrak{h}(\xi, +\infty) = \varepsilon_{a,b}\nu_{a,b}(\xi), \tag{3.38}$$

where the function $\mathfrak{h}$ is defined by (3.12), the functions $\nu_{a,b}, \nu_{c,d} \in SO(\mathbb{R}_+)$ are defined by (1.8), and the numbers $\varepsilon_{a,b}, \varepsilon_{c,d}$ are given by (3.1).

Proof. Since $r_\gamma(\pm\infty) = 0$ and the functions in the brackets in (3.14) are bounded, we have $\mathfrak{m}(t, \pm\infty) = 0$ for every $t \in \mathbb{R}_+$. From this observation and (3.12) we obtain

$$\mathfrak{h}(t, \pm\infty) = \mathfrak{b}(t, \pm\infty), \quad t \in \mathbb{R}_+. \tag{3.39}$$

On the other hand, $p_\gamma^\pm(\mp\infty) = 0$ and $p_\gamma^\pm(\pm\infty) = 1$. Then from (3.13), (3.1)–(3.2) and (1.8) it follows that for $t \in \mathbb{R}_+$,

$$\mathfrak{b}(t, -\infty) = \varepsilon_2\nu_2^+(t) = \varepsilon_{c,d}\nu_{c,d}^+(t) = \varepsilon_{c,d}\nu_{c,d}(t), \tag{3.40}$$

$$\mathfrak{b}(t, +\infty) = \varepsilon_1\nu_1^+(t) = \varepsilon_{a,b}\nu_{a,b}^+(t) = \varepsilon_{a,b}\nu_{a,b}(t). \tag{3.41}$$

Combining (3.39)–(3.41), we arrive at

$$\mathfrak{h}(t, -\infty) = \varepsilon_{c,d}\nu_{c,d}(t), \quad \mathfrak{h}(t, +\infty) = \varepsilon_{a,b}\nu_{a,b}(t), \quad t \in \mathbb{R}_+.$$

Since the functions $\mathfrak{h}(\cdot, \pm\infty), \nu_{a,b}, \nu_{c,d}$ belong to $SO(\mathbb{R}_+)$, we immediately conclude from the above equalities that (3.38) holds for all $\xi \in \mathbb{R}_+ \cup \Delta$, which completes the proof. □

3.5. Proof of Theorem 1.3

If the functional operator $A_+ = aI - bU_\alpha$ (resp. $A_- = cI - dU_\beta$) is invertible, then, by Theorem 1.1, either $a \gg b$ or $b \gg a$ (resp. either $c \gg d$ or $d \gg b$). Hence the function $\nu_{a,b}$ (resp. $\nu_{c,d}$) is well defined by (1.8). Moreover, $\nu_{a,b}, \nu_{c,d} \in \mathcal{G}SO(\mathbb{R}_+)$. Then from Lemma 3.7 it follows that

$$\mathfrak{h}(\xi, \pm\infty) \neq 0 \quad \text{for all} \quad \xi \in \mathbb{R}_+ \cup \Delta. \tag{3.42}$$

From Theorem 3.3(b)–(c) we know that the operators F and V given by (3.4) and (3.6), respectively, are Fredholm. Then from Theorem 1.2 we conclude that

$$\inf_{x\in\mathbb{R}} |f(\xi, x)| > 0, \quad \inf_{x\in\mathbb{R}} |v(\xi, x)| > 0 \quad \text{for all} \quad \xi \in \Delta, \tag{3.43}$$

where the functions f and v are given for $(\xi, x) \in \Delta \times \mathbb{R}$ by (3.28) and (3.29), respectively. By the hypothesis,

$$\inf_{x\in\mathbb{R}} |n(\xi, x)| > 0 \quad \text{for all} \quad \xi \in \Delta. \tag{3.44}$$

From (3.43)–(3.44) and Lemma 3.6 it follows that

$$\mathfrak{h}(\xi, x) \neq 0 \quad \text{for all} \quad (\xi, x) \in \Delta \times \mathbb{R}. \tag{3.45}$$

Combining conditions (3.42) and (3.45) with Theorem 2.9, we deduce that the pseudodifferential operator $\mathrm{Op}(\mathfrak{h})$ is Fredholm on the space $L^p(\mathbb{R}_+, d\mu)$. Hence, in view of Corollary 3.4, the operator N is Fredholm on the space $L^p(\mathbb{R}_+)$. The proof of Theorem 1.3 is completed. □

4. Index of the operator *N*

4.1. On indices of semi-almost periodic functions

First we collect basic properties of the index of semi-almost periodic functions.

Lemma 4.1 *([21, Section 2.1], [22, pp. 194–195]).*

(a) *If $g \in C(\overline{\mathbb{R}})$ and $\inf_{x\in\mathbb{R}} |g(x)| > 0$, then*

$$\mathrm{ind}_{\mathbb{R}}\, g = \frac{1}{2\pi}\{\arg g(x)\}_{x\in\overline{\mathbb{R}}}.$$

(b) *If $g \in AP$ and $\inf_{x\in\mathbb{R}} |g(x)| > 0$, then $\mathrm{ind}_{\mathbb{R}}\, g = 0$.*

(c) *If $f, g \in \mathcal{G}SAP$, then $fg \in \mathcal{G}SAP$ and*

$$\mathrm{ind}_{\mathbb{R}}(fg) = \mathrm{ind}_{\mathbb{R}}\, f + \mathrm{ind}_{\mathbb{R}}\, g.$$

(d) *If $f \in \mathcal{G}SAP$, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all functions $g \in SAP$ such that $\sup_{x\in\mathbb{R}} |f(x) - g(x)| < \delta$ one has $g \in \mathcal{G}SAP$ and $|\mathrm{ind}_{\mathbb{R}}\, f - \mathrm{ind}_{\mathbb{R}}\, g| < \varepsilon$.*

From [12, Theorems 1.1–1.2] we immediately get the following results.

Lemma 4.2. *Let γ_* be defined by (1.3). If $\omega, \eta \in \mathbb{R}$, then the semi-almost periodic function*

$$h(x) := e^{i\omega x} p_{\gamma_*}^{+}(x) + e^{i\eta x} p_{\gamma_*}^{-}(x), \quad x \in \mathbb{R},$$

belongs to $\mathcal{G}SAP$ *and* $\operatorname{ind}_{\mathbb{R}} h = 0$.

Lemma 4.3. *Let* γ_* *be defined by* (1.3). *If* $\omega, \eta \in \mathbb{R} \setminus \{0\}$ *and* $z_1, z_2 \in \mathbb{C}$ *with* $|z_1| < 1$, $|z_2| < 1$, *then the semi-almost periodic function*

$$w(x) := (1 - z_1 e^{i\omega x}) p_{\gamma_*}^{+}(x) + (1 - z_2 e^{i\eta x}) p_{\gamma_*}^{-}(x), \quad x \in \mathbb{R},$$

belongs to $\mathcal{G}SAP$ *and* $\operatorname{ind}_{\mathbb{R}} w = 0$.

4.2. Indices of the functions $n(t, \cdot)$ *for sufficiently small and sufficiently large* $t \in \mathbb{R}_+$

Given $a \in \mathbb{C}$ and $r > 0$, let $\mathbb{D}(a, r) := \{z \in \mathbb{C} : |z - a| \le r\}$.

Lemma 4.4 *([11, Lemmas 2.9–2.10]). Let* γ_* *be defined by* (1.3), $\omega, \eta \in \mathbb{R}$, *and* $z_1, z_2 \in \mathbb{C}$ *be such that* $|z_1| < 1$, $|z_2| < 1$. *If the semi-almost periodic functions* w *and* u *are given by*

$$\begin{aligned} w(x) &:= (1 - z_1 e^{i\omega x}) p_{\gamma_*}^{+}(x) + (1 - z_2 e^{i\eta x}) p_{\gamma_*}^{-}(x), \\ u(x) &:= (1 - z_1 e^{i\omega x})^{-1} p_{\gamma_*}^{+}(x) + (1 - z_2 e^{i\eta x})^{-1} p_{\gamma_*}^{-}(x) \end{aligned}$$

for $x \in \mathbb{R}$, *then* $w(\mathbb{R}) \subset \mathbb{D}(1, r)$ *and* $u(\mathbb{R}) \subset \mathbb{D}((1 - r^2)^{-1}, (1 - r^2)^{-1} r)$, *where* $r = \max(|z_1|, |z_2|)$.

Now we prove two auxiliary results.

Lemma 4.5. *Suppose* γ_* *is defined by* (1.3) *and* $c, d, \omega, \eta \in SO(\mathbb{R}_+)$. *If* $\|c\|_{C_b(\mathbb{R}_+)} < 1$, $\|d\|_{C_b(\mathbb{R}_+)} < 1$ *and* ω, η *are real-valued functions, then the functions*

$$\begin{aligned} w(t, x) &:= (1 - c(t) e^{i\omega(t) x}) p_{\gamma_*}^{+}(x) + (1 - d(t) e^{i\eta(t) x}) p_{\gamma_*}^{-}(x), \\ u(t, x) &:= (1 - c(t) e^{i\omega(t) x})^{-1} p_{\gamma_*}^{+}(x) + (1 - d(t) e^{i\eta(t) x})^{-1} p_{\gamma_*}^{-}(x), \end{aligned}$$

defined for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, *satisfy*

$$\inf_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}} |w(t, x)| > 0, \qquad \inf_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}} |u(t, x)| > 0, \tag{4.1}$$

and the function

$$\mathfrak{w}(t, x) := \frac{r_{\gamma_*}(x)}{w(t, x)}, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{4.2}$$

belongs to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

Proof. From Lemma 4.4 we immediately get

$$w(\mathbb{R}_+ \times \mathbb{R}) \subset \mathbb{D}(1, r), \quad u(\mathbb{R}_+ \times \mathbb{R}) \subset \mathbb{D}((1 - r^2)^{-1}, (1 - r^2)^{-1} r)$$

with $r = \max(\|c\|_{C_b(\mathbb{R}_+)}, \|d\|_{C_b(\mathbb{R}_+)}) < 1$, whence (4.1) holds. By Lemma 2.14, the functions

$$\mathfrak{a}_+(t,x) := (1 - c(t)e^{i\omega(t)x})r_{\gamma_*}(x), \quad \mathfrak{c}_+(t,x) := (1 - c(t)e^{i\omega(t)x})^{-1}r_{\gamma_*}(x),$$
$$\mathfrak{a}_-(t,x) := (1 - d(t)e^{i\eta(t)x})r_{\gamma_*}(x), \quad \mathfrak{c}_-(t,x) := (1 - d(t)e^{i\eta(t)x})^{-1}r_{\gamma_*}(x),$$

defined for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, belong to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Taking into account identities (2.4), we easily get for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\begin{aligned} w(t,x)u(t,x) &= (p^+_{\gamma_*}(x))^2 + (1 - c(t)e^{i\omega(t)x})(1 - d(t)e^{i\eta(t)x})^{-1}p^+_{\gamma_*}(x)p^-_{\gamma_*}(x) \\ &\quad + (p^-_{\gamma_*}(x))^2 + (1 - c(t)e^{i\omega(t)x})^{-1}(1 - d(t)e^{i\eta(t)x})p^+_{\gamma_*}(x)p^-_{\gamma_*}(x) \\ &= p^+_{\gamma_*}(x) + \frac{1}{4}(r_{\gamma_*}(x))^2 - \frac{1}{4}(1 - c(t)e^{i\omega(t)x})(1 - d(t)e^{i\eta(t)x})^{-1}(r_{\gamma_*}(x))^2 \\ &\quad + p^-_{\gamma_*}(x) + \frac{1}{4}(r_{\gamma_*}(x))^2 - (1 - c(t)e^{i\omega(t)x})^{-1}(1 - d(t)e^{i\eta(t)x})(r_{\gamma_*}(x))^2 \\ &= 1 + \frac{1}{4}[2(r_{\gamma_*}(x))^2 - \mathfrak{a}_+(t,x)\mathfrak{c}_-(t,x) - \mathfrak{c}_+(t,x)\mathfrak{a}_-(t,x)]. \end{aligned}$$

From Lemmas 2.10 and 2.14 it follows that $wu, ur_{\gamma_*} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. From Lemma 2.6 and (4.1) we deduce that $1/(wu) \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Hence $\mathfrak{w} = r_{\gamma_*}/w = (ur_{\gamma_*})/(wu) \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. $\square$

For $\ell > 1$, put $T_\ell := (0, \ell^{-1}] \cup [\ell, \infty)$ and

$$L_\pm(t) := \frac{\ln \ell \pm \ln t}{2 \ln \ell}, \quad t \in [\ell^{-1}, \ell].$$

Lemma 4.6. *If $f \in SO(\mathbb{R}_+)$ is such that $1 \gg f$, then there exists an $\ell > 1$ such that the function*

$$f_\ell(t) := \begin{cases} f(t), & t \in T_\ell, \\ f(\ell^{-1})L_-(t) + f(\ell)L_+(t), & t \in [\ell^{-1}, \ell], \end{cases} \tag{4.3}$$

belongs to $SO(\mathbb{R}_+)$ and $\|f_\ell\|_{C_b(\mathbb{R}_+)} < 1$.

Proof. It is clear that $f_\ell \in SO(\mathbb{R}_+)$ for every $\ell > 1$. Since $1 \gg f$, we have

$$\limsup_{t \to s} |f(t)| < 1 \quad \text{for} \quad s \in \{0, \infty\}.$$

Hence there exists an $\ell > 1$ such that $\sup_{t \in T_\ell} |f(t)| < 1$. Then we have $\sup_{t \in T_\ell} |f_\ell(t)| < 1$. If $t \in [\ell^{-1}, \ell]$, then

$$|f_\ell(t)| \le |f(\ell^{-1})|L_-(t) + |f(\ell)|L_+(t) \le \sup_{t \in T_\ell} |f_\ell(t)|(L_-(t) + L_+(t)) = \sup_{t \in T_\ell} |f_\ell(t)| < 1.$$

Thus $\|f_\ell\|_{C_b(\mathbb{R}_+)} < 1$. $\square$

Taking into account Theorem 1.1, from Lemmas 4.5 and 4.6 we immediately get the following.

Lemma 4.7. *Let γ_* be given by (1.3). Suppose a, b, c, d belong to $SO(\mathbb{R}_+)$, α, β belong to $SOS(\mathbb{R}_+)$, and $\omega, \eta \in SO(\mathbb{R}_+)$ are the exponent functions of α, β, respectively. If the functional operators $A_+ = aI - bU_\alpha$ and $A_- = cI - dU_\beta$ are invertible on the space $L^p(\mathbb{R}_+)$, then there exists an $\ell > 1$ such that the function v_ℓ, defined for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ by*

$$v_\ell(t,x) := \left(1 - \left(\frac{\nu_1^-}{\nu_1^+}\right)_\ell (t) e^{i\varepsilon_1 \omega(t)x}\right) p_{\gamma_*}^+(x) + \left(1 - \left(\frac{\nu_2^-}{\nu_2^+}\right)_\ell (t) e^{i\varepsilon_2 \eta(t)x}\right) p_{\gamma_*}^-(x), \tag{4.4}$$

where ε_i, ν_i are defined by (3.1)–(3.2) and $(\nu_i^-/\nu_i^+)_\ell$ are defined according to (4.3) for $i = 1, 2$, satisfies

$$\inf_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} |v_\ell(t,x)| > 0, \tag{4.5}$$

and the function

$$\widetilde{\mathfrak{g}}(t,x) := \frac{r_{\gamma_*}(x)}{v_\ell(t,x)}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{4.6}$$

belongs to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

Now we are in a position to construct a required substitute for the function $\mathfrak{h}$ given by (3.12).

Lemma 4.8. *Let γ satisfy (1.1) and γ_* be given by (1.3). Suppose a, b, c, d belong to $SO(\mathbb{R}_+)$, α, β belong to $SOS(\mathbb{R}_+)$, and $\omega, \eta \in SO(\mathbb{R}_+)$ are the exponent functions of α, β, respectively. If the operator N given by (1.2) is Fredholm on the space $L^p(\mathbb{R}_+)$, then there exist numbers $\ell_1 \geq \ell > 1$ such that the function v_ℓ given by (4.4) satisfies (4.5); the function*

$$\widetilde{\mathfrak{h}}(t,x) := \mathfrak{b}(t,x) + \mathfrak{m}(t,x)\widetilde{\mathfrak{g}}(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{4.7}$$

where $\mathfrak{b}$, $\mathfrak{m}$, and $\widetilde{\mathfrak{g}}$ are given by (3.13), (3.14) and (4.6), respectively, belongs to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ and satisfies

$$\mathfrak{h}(\xi,x) = \widetilde{\mathfrak{h}}(\xi,x) \quad \textit{for all} \quad (\xi,x) \in \Delta \times \overline{\mathbb{R}}, \tag{4.8}$$

where $\mathfrak{h}$ is given by (3.12), and

$$\inf_{(t,x)\in T_{\ell_1}\times\overline{\mathbb{R}}} |\widetilde{\mathfrak{h}}(t,x)| > 0. \tag{4.9}$$

Proof. From Theorem 1.2 it follows that Theorem 3.3 and Lemma 4.7 are applicable. Hence we have $\mathfrak{b}, \mathfrak{m}, \widetilde{\mathfrak{g}} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Thus the function $\widetilde{\mathfrak{h}}$ given by (4.7) belongs to the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Since $r_{\gamma_*}(\pm\infty) = 0$, from (3.15)–(3.16) and (4.6) we obtain

$$\mathfrak{g}(t,\pm\infty) = \widetilde{\mathfrak{g}}(t,\pm\infty) \quad \text{for all} \quad t \in \mathbb{R}_+, \quad \mathfrak{g}(\xi,x) = \widetilde{\mathfrak{g}}(\xi,x) \quad \text{for all} \quad (\xi,x) \in \Delta \times \overline{\mathbb{R}}.$$

From these equalities and (3.12), (4.7) we get (4.8) and

$$\mathfrak{h}(t,\pm\infty) = \widetilde{\mathfrak{h}}(t,\pm\infty) \quad \text{for all} \quad t \in \mathbb{R}_+. \tag{4.10}$$

Since the operator N is Fredholm, from Corollary 3.4 we deduce that the pseudodifferential operator $\mathrm{Op}(\mathfrak{h})$ is also Fredholm. Then from (4.8), (4.10) and Theorem 2.9 it follows that

$$\widetilde{\mathfrak{h}}(t,\pm\infty) \neq 0 \quad \text{for all} \quad t \in \mathbb{R}_+, \quad \widetilde{\mathfrak{h}}(\xi,x) \neq 0 \quad \text{for all} \quad (\xi,x) \in \Delta \times \overline{\mathbb{R}}. \tag{4.11}$$

From (4.11) and Lemma 2.7 we deduce that there exists an $\ell_1 > 1$ such that (4.9) is fulfilled. Obviously, ℓ_1 can be chosen so that $\ell_1 \geq \ell$. $\square$

The following lemma is the main result of this subsection.

Lemma 4.9. *If the operator N is Fredholm on the space $L^p(\mathbb{R}_+)$, then*

(a) *there exist an $\ell > 1$ such that*

$$n(t,x)f(t,x) = \widetilde{\mathfrak{h}}(t,x)v(t,x) \quad \textit{for all} \quad (t,x) \in T_\ell \times \mathbb{R}, \tag{4.12}$$

where the functions n, f, $\widetilde{\mathfrak{h}}$ and v are defined by (1.5), (3.28), (4.7) and (3.29), respectively;

(b) *there exists an $\ell_1 \geq \ell$ such that $n(t,\cdot) \in \mathcal{G}SAP$ and*

$$\operatorname{ind}_{\mathbb{R}} n(t,\cdot) = \frac{1}{2\pi}\{\arg \widetilde{\mathfrak{h}}(t,x)\}_{x\in\overline{\mathbb{R}}} \quad \textit{for all} \quad t \in T_{\ell_1}. \tag{4.13}$$

Proof. (a) From Theorem 1.2 we deduce that Lemmas 3.5 and 4.7–4.8 are applicable. From Lemma 4.7, the construction in Lemma 4.6, and (4.6)–(4.7) we deduce that there exists an $\ell > 1$ such that

$$\sup_{t\in T_\ell} \max\left(\left|\frac{\nu_1^-(t)}{\nu_1^+(t)}\right|, \left|\frac{\nu_2^-(t)}{\nu_2^+(t)}\right|\right) < 1, \tag{4.14}$$

$$\inf_{(t,x)\in T_\ell\times\mathbb{R}} |v(t,x)| > 0, \tag{4.15}$$

and

$$\widetilde{\mathfrak{h}}(t,x) = \mathfrak{b}(t,x) + \mathfrak{m}(t,x)\frac{r_{\gamma_*}(x)}{v(t,x)} \quad \text{for all} \quad (t,x) \in T_\ell \times \mathbb{R}.$$

Combining this equality with (3.30), we arrive at (4.12). Part (a) is proved.

(b) Taking into account that $\widetilde{\mathfrak{h}} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$, we deduce that $\widetilde{\mathfrak{h}}(t,\cdot) \in V(\mathbb{R}) \subset C(\overline{\mathbb{R}}) \subset SAP$ for all $t \in \mathbb{R}_+$. By Lemma 4.8, there exists an $\ell_1 \geq \ell$ such that (4.9) holds, whence the functions $\widetilde{\mathfrak{h}}(t,\cdot)$ belong to $\mathcal{G}SAP$ for all $t \in T_{\ell_1}$. By Lemma 4.1(a),

$$\operatorname{ind}_{\mathbb{R}} \widetilde{\mathfrak{h}}(t,\cdot) = \frac{1}{2\pi}\{\arg \widetilde{\mathfrak{h}}(t,x)\}_{x\in\overline{\mathbb{R}}}, \quad t \in T_{\ell_1}. \tag{4.16}$$

By Lemma 4.2, $f(t,\cdot) \in \mathcal{G}SAP$ for all $t \in \mathbb{R}_+$ and

$$\operatorname{ind}_{\mathbb{R}} f(t,\cdot) = 0, \quad t \in \mathbb{R}_+. \tag{4.17}$$

Since the slowly oscillating shifts $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$ have only two fixed points at zero and infinity, we see that

$$\omega(t) = \log[\alpha(t)/t] \neq 0, \quad \eta(t) = \log[\beta(t)/t] \neq 0 \quad \text{for all} \quad t \in \mathbb{R}_+.$$

Then from (4.14)–(4.15) and Lemma 4.3 it follows that $v(t,\cdot) \in \mathcal{G}SAP$ for every $t \in T_{\ell_1}$ and

$$\operatorname{ind}_{\mathbb{R}} v(t,\cdot) = 0, \quad t \in T_{\ell_1}. \tag{4.18}$$

Since $\widetilde{\mathfrak{h}}(t,\cdot), f(t,\cdot), v(t,\cdot) \in \mathcal{G}SAP$ for all $t \in T_{\ell_1}$, from (4.12) we deduce that $n(t,\cdot) \in \mathcal{G}SAP$ for $t \in T_{\ell_1}$. Moreover, from (4.12), (4.16)–(4.18) and Lemma 4.1(c) we obtain (4.13). $\square$

4.3. Proof of Theorem 1.4

Part (a) of Theorem 1.4 follows from Lemma 4.9(b).

(b) We note that if the operator N is Fredholm, then the operators $A_+ = aI - bU_\alpha$ and $A_- = cI - dU_\beta$ are invertible in view of Theorem 1.2. In turn, the invertibility of A_+ (resp. A_-) implies by Theorem 1.1 that either $a \gg b$, or $b \gg a$ (resp., either $c \gg d$, or $d \gg c$). Hence the functions $\nu_{a,b}$ and $\nu_{c,d}$ are well defined by (1.8). It is obvious that $\nu_{a,b}, \nu_{c,d} \in \mathcal{G}SO(\mathbb{R}_+)$. The proof of part (b) of Theorem 1.4 is completed.

(c) If the operator N is Fredholm, then from Corollary 3.4, Theorem 2.9 and Lemma 2.7 it follows that there exists an $\ell > 1$ such that

$$\inf_{(t,x)\in T_\ell \times \overline{\mathbb{R}}} |\mathfrak{h}(t,x)| > 0 \tag{4.19}$$

and

$$\operatorname{Ind} N = \lim_{\tau\to+\infty} \frac{1}{2\pi} \{\arg \mathfrak{h}(t,x)\}_{(t,x)\in\partial\Pi_\tau}, \tag{4.20}$$

where $\Pi_\tau = [\tau^{-1}, \tau] \times \overline{\mathbb{R}}$ and the function $\mathfrak{h} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is defined by (3.12). Let $\tau \in (\ell, \infty)$. Then, obviously,

$$\begin{aligned}\{\arg \mathfrak{h}(t,x)\}_{(t,x)\in\partial\Pi_\tau} &= \{\arg \mathfrak{h}(\tau,x)\}_{x\in\overline{\mathbb{R}}} - \{\arg \mathfrak{h}(\tau^{-1},x)\}_{x\in\overline{\mathbb{R}}} \\ &\quad + \{\arg \mathfrak{h}(t,-\infty)\}_{t\in[\tau^{-1},\tau]} - \{\arg \mathfrak{h}(t,+\infty)\}_{t\in[\tau^{-1},\tau]}.\end{aligned} \tag{4.21}$$

From Lemma 3.7 we obtain

$$\{\arg \mathfrak{h}(t,-\infty)\}_{t\in[\tau^{-1},\tau]} = \{\arg(\varepsilon_{c,d}\nu_{c,d}(t))\}_{t\in[\tau^{-1},\tau]} = \{\arg \nu_{c,d}(t)\}_{t\in[\tau^{-1},\tau]}, \tag{4.22}$$

$$\{\arg \mathfrak{h}(t,+\infty)\}_{t\in[\tau^{-1},\tau]} = \{\arg(\varepsilon_{a,b}\nu_{a,b}(t))\}_{t\in[\tau^{-1},\tau]} = \{\arg \nu_{a,b}(t)\}_{t\in[\tau^{-1},\tau]}. \tag{4.23}$$

On the other hand, by Lemmas 4.8 and 4.9(b), there exists an $\ell_1 \geq \ell > 1$ such that for all $t \in T_{\ell_1}$ we have

$$\inf_{(t,x)\in T_{\ell_1} \times \overline{\mathbb{R}}} |\widetilde{\mathfrak{h}}(t,x)| > 0 \tag{4.24}$$

and

$$\operatorname{ind}_{\mathbb{R}} n(t,\cdot) = \frac{1}{2\pi} \{\arg \widetilde{\mathfrak{h}}(t,x)\}_{x\in\overline{\mathbb{R}}} \quad \text{for all} \quad t \in T_{\ell_1}, \tag{4.25}$$

where $n(t,\cdot) \in \mathcal{G}SAP$ for all $t \in T_{\ell_1}$ and the function $\widetilde{\mathfrak{h}} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is defined by (4.7). Moreover,

$$\mathfrak{h}(\xi,x) = \widetilde{\mathfrak{h}}(\xi,x), \quad (\xi,x) \in \Delta \times \overline{\mathbb{R}}. \tag{4.26}$$

From (4.19) and (4.24) we conclude that $\mathfrak{h}(t,\cdot), \widetilde{\mathfrak{h}}(t,\cdot) \in C(\overline{\mathbb{R}}) \cap \mathcal{G}SAP$ for $t \in T_{\ell_1}$. Fix $\varepsilon > 0$. Then in view of Lemma 4.1(d), (a) one can choose a $\delta = \delta(\varepsilon) > 0$ such that for all $t \in T_{\ell_1}$ the inequality

$$\|\mathfrak{h}(t,\cdot) - \widetilde{\mathfrak{h}}(t,\cdot)\|_{L^\infty(\mathbb{R})} < \delta \tag{4.27}$$

implies that

$$|\operatorname{ind}_{\mathbb{R}} \mathfrak{h}(t,\cdot) - \operatorname{ind}_{\mathbb{R}} \widetilde{\mathfrak{h}}(t,\cdot)| = \frac{1}{2\pi} \left| \{\arg \mathfrak{h}(t,\cdot)\}_{x\in\overline{\mathbb{R}}} - \{\arg \widetilde{\mathfrak{h}}(t,\cdot)\}_{x\in\overline{\mathbb{R}}} \right| < \varepsilon. \tag{4.28}$$

Since $\mathfrak{h} - \widetilde{\mathfrak{h}} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$, from (4.26) and Lemma 2.8 we deduce that for the chosen $\delta > 0$ there exists an $\ell_2 = \ell_2(\delta) > \ell_1 > 1$ such that (4.27) holds for all $t \in T_{\ell_2}$. Hence, for every $\varepsilon > 0$ there is an $\ell_2 > 1$ such that if $t \in T_{\ell_2}$, then (4.28) holds. Thus

$$\lim_{t \to s} \frac{1}{2\pi} \Big(\{\arg \mathfrak{h}(t,x)\}_{x \in \overline{\mathbb{R}}} - \{\arg \widetilde{\mathfrak{h}}(t,x)\}_{x \in \overline{\mathbb{R}}} \Big) = 0, \quad s \in \{0, \infty\}.$$

From these equalities and (4.25) it follows that

$$\lim_{\tau \to +\infty} \left(\frac{1}{2\pi} \{\arg \mathfrak{h}(\tau^{\pm 1}, x)\}_{x \in \overline{\mathbb{R}}} - \mathrm{ind}_{\mathbb{R}}\, n(\tau^{\pm 1}, \cdot) \right) = 0. \tag{4.29}$$

Finally, applying (4.20)–(4.23) and then (4.29), we get

$$\begin{aligned}
&\mathrm{Ind}\, N = \\
&= \lim_{\tau \to +\infty} \frac{1}{2\pi} \Big(\{\arg \mathfrak{h}(\tau, x)\}_{x \in \overline{\mathbb{R}}} - \{\arg \mathfrak{h}(\tau^{-1}, x)\}_{x \in \overline{\mathbb{R}}} + \{\arg \mathfrak{h}(t, -\infty)\}_{t \in [\tau^{-1}, \tau]} - \{\arg \mathfrak{h}(t, +\infty)\}_{t \in [\tau^{-1}, \tau]} \Big) \\
&= \lim_{\tau \to +\infty} \frac{1}{2\pi} \Big(\{\arg \mathfrak{h}(\tau, x)\}_{x \in \overline{\mathbb{R}}} - \{\arg \mathfrak{h}(\tau^{-1}, x)\}_{x \in \overline{\mathbb{R}}} + \{\arg \nu_{c,d}(t)\}_{t \in [\tau^{-1}, \tau]} - \{\arg \nu_{a,b}(t)\}_{t \in [\tau^{-1}, \tau]} \Big) \\
&\quad - \lim_{\tau \to +\infty} \left(\frac{1}{2\pi} \{\arg \mathfrak{h}(\tau, x)\}_{x \in \overline{\mathbb{R}}} - \mathrm{ind}_{\mathbb{R}}\, n(\tau, \cdot) \right) + \lim_{\tau \to +\infty} \left(\frac{1}{2\pi} \{\arg \mathfrak{h}(\tau^{-1}, x)\}_{x \in \overline{\mathbb{R}}} - \mathrm{ind}_{\mathbb{R}}\, n(\tau^{-1}, \cdot) \right) \\
&= \lim_{\tau \to +\infty} \left(\frac{1}{2\pi} \Big(\big\{ \arg \nu_{c,d}(t) \big\}_{t \in [\tau^{-1}, \tau]} - \big\{ \arg \nu_{a,b}(t) \big\}_{t \in [\tau^{-1}, \tau]} \Big) + \mathrm{ind}_{\mathbb{R}}\, n(\tau, \cdot) - \mathrm{ind}_{\mathbb{R}}\, n(\tau^{-1}, \cdot) \right),
\end{aligned}$$

which completes the proof of Theorem 1.4. □

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