



# A variant of Gromov's problem on Hölder equivalence of Carnot groups

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## Abstract

It is unknown if there exists a locally  $\alpha$ -Hölder homeomorphism  $f : \mathbb{R}^3 \rightarrow \mathbb{H}^1$  for any  $\frac{1}{2} < \alpha \leq \frac{2}{3}$ , although the identity map  $\mathbb{R}^3 \rightarrow \mathbb{H}^1$  is locally  $\frac{1}{2}$ -Hölder. More generally, Gromov asked: Given  $k$  and a Carnot group  $G$ , for which  $\alpha$  does there exist a locally  $\alpha$ -Hölder homeomorphism  $f : \mathbb{R}^k \rightarrow G$ ? Here, we equip a Carnot group  $G$  with the Carnot-Carathéodory metric. In 2014, Balogh, Hajlasz, and Wildrick considered a variant of this problem. These authors proved that if  $k > n$ , there does not exist an injective,  $(\frac{1}{2}+)$ -Hölder mapping  $f : \mathbb{R}^k \rightarrow \mathbb{H}^n$  that is also locally Lipschitz as a mapping into  $\mathbb{R}^{2n+1}$ . For their proof, they use the fact that  $\mathbb{H}^n$  is purely  $k$ -unrectifiable for  $k > n$ . In this paper, we will extend their result from the Heisenberg group to model filiform groups and Carnot groups of step at most three. We will now require that the Carnot group is purely  $k$ -unrectifiable. The main key to our proof will be showing that  $(\frac{1}{2}+)$ -Hölder maps  $f : \mathbb{R}^k \rightarrow G$  that are locally Lipschitz into Euclidean space, are weakly contact. Proving weak contactness in these two settings requires understanding the relationship between the algebraic and metric structures of the Carnot group. We will use coordinates of the first and second kind for Carnot groups.

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## 1. Introduction

A Lie algebra  $\mathfrak{g}$  is said to have an  $r$ -step **stratification** if

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r,$$

where  $\mathfrak{g}_1 \subseteq \mathfrak{g}$  is a subspace,  $\mathfrak{g}_{j+1} = [\mathfrak{g}_1, \mathfrak{g}_j]$  for all  $j = 1, \dots, r-1$ , and  $[\mathfrak{g}, \mathfrak{g}_r] = 0$ . A **Carnot group** is a connected, simply-connected, nilpotent Lie group with a stratified Lie algebra. If the Lie algebra of a Carnot group  $G$  admits an  $r$ -step stratification, then we will say  $G$  is step  $r$ . Each Carnot group can be identified with a Euclidean space equipped with a metric structure and a group operation arising from its Lie algebra structure.

It is natural to ask the following general question:

When are two Carnot groups equivalent?

In [1], Pansu proved that two Carnot groups are biLipschitz homeomorphic if and only if they are isomorphic. With the problem of biLipschitz equivalence somewhat well-understood, we can go on to ask when two Carnot groups are Hölder equivalent.

In [2], Gromov considered the problem of Hölder equivalence of Carnot groups: If a Carnot group  $G$  is identified with  $\mathbb{R}^n$  equipped with a group operation, for which  $\alpha$  does there exist a locally  $\alpha$ -Hölder homeomorphism  $f : \mathbb{R}^n \rightarrow G$ ? If such  $\alpha$  exist, what is the supremum of the set of such  $\alpha$ ? Here, we do not require any regularity of  $f^{-1}$  beyond continuity.

Before we discuss past work on this problem, we will comment on the notation that will be used throughout this paper. We will simply write  $\mathbb{R}^n$  to denote Euclidean space equipped with addition and the standard Euclidean metric. We will write  $(\mathbb{R}^n, \cdot)$  to denote a Carnot group equipped with coordinates of the first or second kind and with the Carnot-Carathéodory metric. When we equip a Carnot group with coordinates of the first or second kind, it is implied that we are taking coordinates with respect to a basis compatible with the stratification of its Lie algebra. We will introduce these two systems of coordinates and the Carnot-Carathéodory metric for Carnot groups in section 2. In section 3, we will discuss coordinates of the second kind for a class of jet spaces: the model filiform groups. We will begin section 4 by looking at the geometry of Carnot groups of step at most three.

Nagel, Stein, and Wenger [3, Proposition 1.1] proved the existence of  $\alpha$  as above:

**Proposition 1.** *Let  $(\mathbb{R}^n, \cdot)$  be a step  $r$  Carnot group. Then  $\text{id} : \mathbb{R}^n \rightarrow (\mathbb{R}^n, \cdot)$  is locally  $\frac{1}{r}$ -Hölder and  $\text{id} : (\mathbb{R}^n, \cdot) \rightarrow \mathbb{R}^n$  is locally Lipschitz.*

On the other hand, Gromov [2, Section 4] used an isoperimetric inequality for Carnot groups [4] to prove that if there exists a locally  $\alpha$ -Hölder homeomorphism  $f : \mathbb{R}^n \rightarrow (\mathbb{R}^n, \cdot)$ , then

$$\alpha \leq \frac{n-1}{Q-1}.$$

Here,  $Q$  denotes the Hausdorff dimension of  $(\mathbb{R}^n, \cdot)$  with respect to its cc-metric.

Beyond these results, little is known about this problem. For example, in the case of the first Heisenberg group, the supremum of  $\alpha$  for which there exists a locally  $\alpha$ -Hölder homeomorphism  $f : \mathbb{R}^3 \rightarrow \mathbb{H}^1$  is only known to lie between  $1/2$  and  $2/3$  [5, Page 3].

In this paper, we will consider a related problem. We first define a class of maps related to the class  $C^{0,\alpha}(X; Y)$  of  $\alpha$ -Hölder maps  $f : X \rightarrow Y$ .

**Definition 2.** Fix metric spaces  $(X, d_X), (Y, d_Y)$  and  $\alpha > 0$ . We say a map  $f : X \rightarrow Y$  is of class  $C^{0,\alpha+}(X; Y)$  if there exists a homeomorphism  $\beta : [0, \infty) \rightarrow [0, \infty)$  such that

$$d_Y(f(a), f(b)) \leq d_X(a, b)^\alpha \beta(d_X(a, b)) \quad \text{for all } a, b \in X. \quad (1)$$

We will sometimes simply write  $C^{0,\alpha+}$  if the domain and target are clear.

**Remark 3.** Suppose  $X, Y$  are metric spaces with  $X$  bounded. It is easy to check that

$$C^{0,\eta}(X; Y) \subseteq C^{0,\alpha+}(X; Y) \subseteq C^{0,\alpha}(X; Y).$$

whenever  $0 < \alpha < \eta$ . Thus,  $C^{0,\alpha+}(X; Y)$  can thought of as a right limit of Hölder spaces.

For certain models of model filiform groups and Carnot groups of small step, we will prove that there do not exist  $(\alpha+)$ -Hölder equivalences for  $\alpha \geq 1/2$ . Before stating our paper's two main results, we make the following definition.

**Definition 4.** A Carnot group  $(\mathbb{R}^n, \cdot)$  is said to be **purely  $k$ -unrectifiable** if for every  $A \subseteq \mathbb{R}^k$  and Lipschitz map  $f : A \rightarrow (\mathbb{R}^n, \cdot)$ , we have

$$\mathcal{H}_{cc}^k(f(A)) = 0.$$

Here, we endow  $(\mathbb{R}^n, \cdot)$  with the Carnot-Carathéodory metric to be described in subsection 2.2.

Ambrosio and Kirchheim proved that  $\mathbb{H}^1$  is purely  $k$ -unrectifiable for  $k = 2, 3, 4$  [6, Theorem 7.2]. More generally, Magnani proved that a Carnot group is purely  $k$ -unrectifiable if and only if its horizontal layer does not contain a Lie subalgebra of dimension  $k$  [7, Theorem 1.1]. In particular,  $\mathbb{H}^n$  is purely  $k$ -unrectifiable for all  $k > n$ . In 2014, Balogh, Hajlasz, and Wildrick provided a different proof of this last result by using approximate derivatives and a weak contact condition [8, Theorem 1.1]. In the process, they prove that a Lipschitz mapping of an open subset of  $\mathbb{R}^k$ ,  $k > n$ , into  $\mathbb{H}^n$  has an approximate derivative that is horizontal almost everywhere.

Motivated by Gromov's Hölder equivalence problem, Balogh, Hajlasz, and Wildrick go on to prove that one cannot embed  $\mathbb{R}^k$ ,  $k > n$ , into  $\mathbb{H}^n$  via a sufficiently regular  $(\alpha+)$ -Hölder mapping. More specifically, they prove that if  $k > n$  and  $\Omega \subseteq \mathbb{R}^k$  is open, then there is no injective mapping of class  $C^{0,\frac{1}{2}+}(\Omega, \mathbb{H}^n)$  that is locally Lipschitz as a mapping into  $\mathbb{R}^{2n+1}$  [8, Theorem 1.11]. The main key to their proof is showing that if such a map existed, then it would have to be horizontal almost everywhere. Notice that Remark 3 combined

with the identity map  $\text{id} : \mathbb{R}^3 \rightarrow \mathbb{H}^1$  being locally  $\frac{1}{2}$ -Hölder suggest that this result is sharp except for the extra local Lipschitz assumption.

In this paper, we will extend the result in the previous paragraph to more general Carnot groups, specifically model filiform groups and Carnot groups of step at most three. The model filiform groups can be realized as the class of jet spaces  $J^k(\mathbb{R})$ . In these groups, there are few nontrivial bracket relations relative to the step. For Carnot groups of small step, the Baker-Campbell-Hausdorff has a simple form; this allows one to describe the structure (e.g., left-invariant vector fields and contact forms) of the Carnot group in coordinates and perform computations. The Lie algebraic properties of these two classes of Carnot group make them ideal settings to generalize the result from the previous paragraph. The proofs for these Carnot groups will again boil down to showing the almost everywhere horizontality of certain  $C^{0, \frac{1}{2}+}$  mappings into these groups.

The standard basis  $\{e^{(k)}, e_k, \dots, e_0\}$  of  $\text{Lie}(J^k(\mathbb{R}))$  is such that  $[e_j, e^{(k)}] = e_{j-1}$ ,  $j \geq 1$ , are the only nontrivial bracket relations. We will equip  $J^k(\mathbb{R})$  with coordinates of the first and second kind with respect to this basis. For example,  $J^1(\mathbb{R})$  is isomorphic to  $\mathbb{H}^1$ . This will be discussed further in subsection 3.1. It is implied that  $J^k(\mathbb{R})$  is equipped with either one of the two systems of coordinates in the following result, the first of our two main theorems.

**Theorem 5.** *Fix  $\alpha \geq \frac{1}{2}$  and positive integers  $n, k$  with  $n > 1$ . Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ . Then there is no injective mapping in the class  $C^{0, \alpha+}(\Omega; J^k(\mathbb{R}))$  that is also locally Lipschitz when considered as a map into  $\mathbb{R}^{k+2}$ .*

We will prove this result in the case  $\alpha = \frac{1}{2}$ , and the cases for  $\alpha > \frac{1}{2}$  will follow from the fact

$$C^{0, \alpha+}(\Omega; J^k(\mathbb{R})) \subset C^{0, \frac{1}{2}+}(\Omega; J^k(\mathbb{R})).$$

The identity map  $\mathbb{R}^{k+2} \rightarrow J^k(\mathbb{R})$  is locally  $\frac{1}{k+1}$ -Hölder. From the Heisenberg case, one may expect for it to be unknown whether there exist locally  $\alpha$ -Hölder, injective maps  $f : \mathbb{R}^n \rightarrow J^k(\mathbb{R})$  for  $\alpha > \frac{1}{k+1}$ . However, we will give an example of a locally  $\frac{1}{2}$ -Hölder, injective map  $f : \mathbb{R}^2 \rightarrow J^k(\mathbb{R})$  that is locally Lipschitz as a map into  $\mathbb{R}^{k+2}$  (Example 21). Comparing with Remark 3, this suggests that our result is sharp, at least in the case  $n = 2$ .

We will first prove Theorem 5 for when  $J^k(\mathbb{R})$  is equipped with coordinates of the second kind. We will then prove at end of the subsection 2.4 that this implies the theorem holds for first kind coordinates as well. We will use Warhurst's model for jet spaces equipped with coordinates of the second kind (see [9, Section 3]). Rigot, Wenger, and Young have used this model to investigate extendability of Lipschitz maps into jet spaces [10, 11].

For the next result, we can choose coordinates with respect to any basis compatible with the stratification of  $\mathfrak{g}$ , but the metric on  $(\mathbb{R}^n, \cdot)$  will be induced by this choice.

**Theorem 6.** Fix  $\alpha \geq \frac{1}{2}$  and an open subset  $\Omega \subseteq \mathbb{R}^k$ . Suppose  $(\mathbb{R}^n, \cdot)$  is a Carnot group of step at most three that is purely  $k$ -unrectifiable. Then there is no injective mapping in the class  $C^{0,\alpha+}(\Omega; (\mathbb{R}^n, \cdot))$  that is also locally Lipschitz when considered as a map into  $\mathbb{R}^n$ .

As for  $J^k(\mathbb{R})$ , we will only explicitly prove this for  $\alpha = \frac{1}{2}$ . These two theorems will be proven in a similar fashion, implied by the following result:

**Proposition 7.** Fix an open subset  $\Omega \subseteq \mathbb{R}^k$ . Let  $(\mathbb{R}^n, \cdot)$  be a Carnot group that is purely  $k$ -unrectifiable. Then there is no injective mapping  $f : \Omega \rightarrow (\mathbb{R}^n, \cdot)$  that is weakly contact and locally Lipschitz when considered as a map into  $\mathbb{R}^n$ .

Thus, to prove Theorems 5 and 6, it suffices to show that if a map in  $C^{0,\frac{1}{2}+}(\Omega, (\mathbb{R}^n, \cdot))$  is locally Lipschitz as a map into  $\mathbb{R}^n$ , then it is weakly contact. We prove this for the class of model filiform jet spaces,  $J^k(\mathbb{R})$ , in Proposition 19, for step 2 Carnot groups in Lemma 24, and for step 3 Carnot groups in Lemma 26. We will discuss weakly contact maps further in subsection 2.3.

Proposition 19 follows from considering the group structure on  $J^k(\mathbb{R})$ , specifically Lemma 17. The proofs of Lemmas 24 and 26 are a bit technical and requires one to carefully work with group structures, bounding terms via the Ball-Box Theorem (Theorem 9) and the modulating homeomorphism. It is expected that Theorem 5 and 6 should generalize to all Carnot groups if one attains a better understanding of the group structure arising from coordinates of the first kind. We will discuss this more at the end of this paper.

## 2. Background on Carnot groups

In this section, we will review the basics of Carnot groups, discussing two systems of coordinates, the Carnot-Carathéodory metric, and weakly contact maps.

For some  $r$ , the Lie algebra  $\mathfrak{g}$  of a Carnot group  $G$  admits an  $r$ -step stratification:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r,$$

where  $\mathfrak{g}_1 \subseteq \mathfrak{g}$  is a subspace,  $\mathfrak{g}_{j+1} = [\mathfrak{g}_1, \mathfrak{g}_j]$  for all  $j = 1, \dots, r-1$ , and  $[\mathfrak{g}, \mathfrak{g}_r] = 0$ . We write  $[\mathfrak{g}_1, \mathfrak{g}_j]$  to denote the subspace generated by commutators of elements of  $\mathfrak{g}_1$  with elements of  $\mathfrak{g}_j$ , and similarly with  $[\mathfrak{g}, \mathfrak{g}_r]$ . The subspaces  $\mathfrak{g}_j$  are commonly referred to as the **layers** of  $\mathfrak{g}$ , with  $\mathfrak{g}_1$  referred to as the **horizontal layer**. We define the step of  $G$  to be  $r$ , and this is well-defined [12, Proposition 2.2.8]. Throughout this paper, we will implicitly fix a stratification for each Carnot group. In other words, we will view the stratification of  $\mathfrak{g}$  as data of a Carnot group  $G$ .

After combining bases of the subspaces  $\mathfrak{g}_j$  to obtain a basis of  $\mathfrak{g}$ , we can define an inner product  $g = \langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$  by declaring the combined basis to be orthonormal. Thus, we say that a basis  $\mathcal{B} = \{X^1, \dots, X^n\}$  of  $\mathfrak{g}$  is **compatible with the stratification of  $\mathfrak{g}$**  if

$$\{X^{h_{j-1}+1}, \dots, X^{h_j}\}$$

is a basis of  $\mathfrak{g}_j$  for each  $j$ , where  $h_j = \sum_{i=1}^j \dim(\mathfrak{g}_i)$ . As we discuss coordinates of the first and second kind, it will be implied that coordinates are being taken with respect to a basis compatible with the stratification of  $\mathfrak{g}$ . While choosing different bases may technically result in different group structures, we will see that the resulting Carnot groups are all isomorphic to  $G$ .

### 2.1. Coordinates of the first kind

For Carnot groups, the exponential map  $\exp : \mathfrak{g} \rightarrow G$  is a diffeomorphism [13, Page 13]. Hence we can define  $\star : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  by

$$X \star Y = \exp^{-1}(\exp(X) \exp(Y)).$$

The Baker-Campbell-Hausdorff formula gives us an explicit formula for  $X \star Y$ :

$$X \star Y = \sum_{n \geq 0} \frac{(-1)^{n+1}}{n} \sum_{0 < p_i + q_i} \frac{1}{C_{p,q}} (\text{ad} X)^{p_1} (\text{ad} Y)^{q_1} \cdots (\text{ad} Y)^{q_{n-1}} W(p_n, q_n),$$

where

$$C_{p,q} = p_1! q_1! \cdots p_n! q_n! \sum_{i=1}^n (p_i + q_i)$$

and

$$W(p_n, q_n) = \begin{cases} (\text{ad} X)^{p_n} (\text{ad} Y)^{q_n-1} Y, & \text{if } q_n \geq 1, \\ (\text{ad} X)^{p_n-1} X, & \text{if } q_n = 0. \end{cases}$$

The expansion of  $X \star Y$  up to order 3 is given by

$$X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]).$$

Set  $n$  equal to the topological dimension of  $G$ , and let  $\mathcal{B} \subset \mathfrak{g}$  be a basis compatible with the stratification of  $\mathfrak{g}$ . We can identify  $\mathfrak{g}$  with  $\mathbb{R}^n$  via coordinates of  $\mathcal{B}$ , and then  $\star$  on  $\mathfrak{g}$  translates into an operation on  $\mathbb{R}^n$ . With a slight abuse of notation, we will also denote this operation on  $\mathbb{R}^n$  by  $\star$ . Then  $(\mathbb{R}^n, \star)$  is a Carnot group isomorphic to  $G$  via  $\exp$  [12, Proposition 2.2.22]. We say that  $(\mathbb{R}^n, \star)$  is a **normal model of the first kind** of  $G$  and that  $(\mathbb{R}^n, \star)$  is  $G$  equipped with **coordinates of the first kind with respect to  $\mathcal{B}$** . Observe that if  $G$  is of step  $r$ , each coordinate of  $X \star Y$  is a polynomial of homogeneous degree at most  $r$  in the coordinates of  $X$  and  $Y$ .

### 2.2. Path metric on Carnot groups

Let  $(\mathbb{R}^n, \cdot)$  be a Carnot group, and set  $m_j = \dim(\mathfrak{g}_j)$  for each  $j$ . Fix a basis  $\mathcal{B}_1 = \{X^1, \dots, X^{m_1}\}$  for the horizontal layer  $\mathfrak{g}_1$ . The **horizontal bundle** of  $(\mathbb{R}^n, \cdot)$  is defined fiberwise by

$$H_p(\mathbb{R}^n, \cdot) := \text{span}\{X_p^1, \dots, X_p^{m_1}\}.$$

Note the horizontal bundle is left-invariant:

$$H_p(\mathbb{R}^n, \cdot) = dL_p H_0(\mathbb{R}^n, \cdot).$$

Declaring  $(\mathcal{B}_1)_p$  to be orthonormal, we obtain an inner product on each fiber  $H_p(\mathbb{R}^n, \cdot)$ .

Recall that we just write  $\mathbb{R}^n$  to denote Euclidean space equipped with the standard Euclidean metric.

**Definition 8.** We say a path  $\gamma : [a, b] \rightarrow (\mathbb{R}^n, \cdot)$  is **horizontal** if it is absolutely continuous as a map into  $\mathbb{R}^n$  and

$$\gamma'(t) \in H_{\gamma(t)}(\mathbb{R}^n, \cdot) \quad \text{for a.e. } t \in [a, b].$$

We define the **length** of a horizontal path to be

$$l_H(\gamma) := \int_a^b |\gamma'(t)| \, dt.$$

Here,  $|\gamma'(t)| := \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}}$  whenever  $\gamma$  is differentiable at  $t$  with  $\gamma'(t) \in H_{\gamma(t)}(\mathbb{R}^n, \cdot)$ .

Note the length of a horizontal path is finite (see 3.35, [14]).

A theorem by Chow [15] states that  $(\mathbb{R}^n, \cdot)$  is horizontally path-connected. This enables us to define the **Carnot-Carathéodory metric** on  $(\mathbb{R}^n, \cdot)$ :

$$d_{cc}(x, y) := \inf_{\gamma: [a, b] \rightarrow (\mathbb{R}^n, \cdot)} \{l_H(\gamma) : \gamma \text{ is horizontal, } \gamma(a) = x, \gamma(b) = y\}.$$

Another common name for this metric is *cc-metric*. It is well-known that the Carnot-Carathéodory metric defines a geodesic metric on  $(\mathbb{R}^n, \cdot)$ , i.e., for every  $x, y \in (\mathbb{R}^n, \cdot)$ , there exists a horizontal path  $\gamma$  connecting  $x$  to  $y$  with  $d_{cc}(x, y) = l_H(\gamma)$  [12, Theorem 5.15.5].

Suppose  $(\mathbb{R}^n, \cdot)$  is step  $r$ . From the previous two sections, a point  $x \in (\mathbb{R}^n, \cdot)$  is of the form  $(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_r)$ , where each  $\vec{x}_j$  lies in  $\mathbb{R}^{m_j}$  and corresponds to the coefficients of the elements of  $\mathfrak{g}_j$ . For each  $\epsilon > 0$ , we can define a **dilation**  $\delta_\epsilon : (\mathbb{R}^n, \cdot) \rightarrow (\mathbb{R}^n, \cdot)$  by

$$\delta_\epsilon(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_r) := (\epsilon \vec{x}_1, \epsilon^2 \vec{x}_2, \dots, \epsilon^r \vec{x}_r).$$

The Carnot-Carathéodory metric is left-invariant and one-homogeneous with respect to these dilations:

For all  $\epsilon > 0$  and  $x, y, z \in (\mathbb{R}^n, \cdot)$ ,

- $d_{cc}(z \cdot x, z \cdot y) = d_{cc}(x, y)$
- $d_{cc}(\delta_\epsilon(x), \delta_\epsilon(y)) = \epsilon \cdot d_{cc}(x, y)$ .

One may wonder how the Carnot-Carathéodory metric on  $(\mathbb{R}^n, \cdot)$  relates to the standard Euclidean metric on  $\mathbb{R}^n$ . From Proposition 1,  $(\mathbb{R}^n, \cdot)$  and  $\mathbb{R}^n$  have the same topologies. Furthermore, Proposition 1 (combined with left-invariance and homogeneity) implies the following version of the Ball-Box Theorem:

**Theorem 9.** (*Ball-Box Theorem*) Suppose  $(\mathbb{R}^n, \cdot)$  is a step  $r$  Carnot group. For  $\epsilon > 0$  and  $p \in (\mathbb{R}^n, \cdot)$ , define

$$Box(\epsilon) := \prod_{j=1}^r [-\epsilon^j, \epsilon^j]^{m_j}$$

and

$$B_{cc}(p, \epsilon) := \{q \in (\mathbb{R}^n, \cdot) : d_{cc}(p, q) \leq \epsilon\}.$$

There exists  $C > 0$  such that for all  $\epsilon > 0$  and  $p \in (\mathbb{R}^n, \cdot)$ ,

$$B_{cc}(p, \epsilon/C) \subseteq p \cdot Box(\epsilon) \subseteq B_{cc}(p, C\epsilon).$$



We obtain an important corollary which allows us to estimate the cc-metric:

**Corollary 10.** *Suppose  $(\mathbb{R}^n, \cdot)$  is a step  $r$  Carnot group. There exists  $C > 0$  such that for all  $p = (a_1^1, \dots, a_{m_1}^1, a_1^2, \dots, a_{m_r}^r) \in (\mathbb{R}^n, \cdot)$ ,*

$$\frac{1}{C} \cdot d_{cc}(0, p) \leq \max\{|a_k^j|^{1/j} : 1 \leq j \leq r, 1 \leq k \leq m_j\} \leq C d_{cc}(0, p).$$

### 2.3. Weakly contact Lipschitz mappings

Fix an open set  $\Omega \subseteq \mathbb{R}^k$  and a Carnot group  $(\mathbb{R}^n, \cdot)$ . If  $f : \Omega \rightarrow (\mathbb{R}^n, \cdot)$  is Lipschitz,  $f$  is locally Lipschitz as a map into  $\mathbb{R}^n$  by Proposition 1. By Rademacher's Theorem, then  $f$  is differentiable almost everywhere in  $\Omega$ . We say a locally Lipschitz map  $f : \Omega \rightarrow \mathbb{R}^n$  is **weakly contact** if

$$\text{im } df_x \subset H_{f(x)}(\mathbb{R}^n, \cdot) \quad \text{for } \mathcal{H}^k - \text{almost every } x \in \Omega.$$

Here, we write  $df_x$  to denote the **differential** or **total derivative** of  $f$  at  $x$ . Observe that by Theorem 9.18 of [16], if  $f$  is differentiable at  $x \in \Omega$ , then

$$\text{im } df_x \subset H_{f(x)}(\mathbb{R}^n, \cdot) \quad \Longleftrightarrow \quad \partial_i f(x) \in H_{f(x)}(\mathbb{R}^n, \cdot) \text{ for all } i = 1, \dots, k.$$

Balogh, Hajlasz, and Wildrick proved for the  $n^{\text{th}}$  Heisenberg group  $\mathbb{H}^n$  that if a Lipschitz map  $f : [0, 1]^k \rightarrow \mathbb{R}^{2n+1}$  is weakly contact, then it is actually Lipschitz as a map into  $\mathbb{H}^n$  [8, Proposition 8.2]. Their proof easily converts into a statement for all Carnot groups. To keep this paper as self-contained as possible, we will repeat the argument here.

**Proposition 11.** *Let  $k$  be a positive integer. If  $f : [0, 1]^k \rightarrow \mathbb{R}^n$  is Lipschitz and weakly contact, then  $f : [0, 1]^k \rightarrow (\mathbb{R}^n, \cdot)$  is Lipschitz.*

*Proof.* Fix a weakly contact map  $f : [0, 1]^k \rightarrow \mathbb{R}^n$  that is  $L$ -Lipschitz. Fubini's Theorem implies the restriction of  $f$  to almost every line segment parallel to a coordinate axis is horizontal. On bounded sets, the lengths with respect to the sub-Riemannian metrics and to the Euclidean metrics are equivalent for horizontal vectors. As  $f[0, 1]^k$  is bounded and the Euclidean speed of  $f$  is bounded by  $L$  on line segments, it follows that the restriction of  $f$  on almost every line segment parallel to a coordinate axis is  $CL$ -Lipschitz as a map into  $(\mathbb{R}^n, \cdot)$ . Hence the restriction of  $f$  on *each* line segment parallel to a coordinate axis is  $CL$ -Lipschitz as a map into  $(\mathbb{R}^n, \cdot)$ , and the result follows.  $\square$

This enables us to prove Proposition 7, a result fundamental to our paper. The proof of Theorem 1.11 in [8] for the Heisenberg group translates into a result for all Carnot groups.

*Proof of Proposition 7.* Assume that there is an injective map  $f : \Omega \rightarrow (\mathbb{R}^n, \cdot)$  that is locally Lipschitz as a map into  $\mathbb{R}^n$ . Restricting  $f$ , we may assume  $\Omega$  is a closed cube and  $f$  is Lipschitz as a map into  $\mathbb{R}^n$ . If  $f$  is weakly contact,  $f : \Omega \rightarrow (\mathbb{R}^n, \cdot)$  is Lipschitz, which implies  $\mathcal{H}_{(\mathbb{R}^n, \cdot)}^k(f(\Omega)) = 0$ . As the identity map from  $(\mathbb{R}^n, \cdot)$  to  $\mathbb{R}^n$  is locally Lipschitz (by Proposition 1),  $\mathcal{H}_{\mathbb{R}^n}^k(f(\Omega)) = 0$ . It follows from Theorem 8.15 of [17]

that the topological dimension of  $f(\Omega)$  is at most  $k - 1$ . Since  $f|_\Omega$  is a homeomorphism,  $f(\Omega)$  is of the same topological dimension as  $\Omega$ , which is a contradiction.  $\square$

The main theorems of this paper thus reduce to showing locally Lipschitz maps  $f$  into  $\mathbb{R}^n$  that are of class  $C^{0, \frac{1}{2}+}(\Omega, (\mathbb{R}^n, \cdot))$ , are weakly contact. Balogh, Hajlasz, and Wildrick proved this for the Heisenberg group [8, Proposition 8.1]. In this paper, we will prove it for models of jet spaces and models of Carnot groups of step at most three.

#### 2.4. Strata-preserving isomorphisms

Suppose  $G$  is a Carnot group with stratification

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r.$$

We define the family of dilations  $\{\mathfrak{d}_\epsilon\}_{\epsilon>0}$  to be the collection of isomorphism of  $\mathfrak{g}$  induced by  $\mathfrak{d}_\epsilon(X_j) = \epsilon^j X_j$ ,  $X_j \in \mathfrak{g}_j$ . Each  $\mathfrak{d}_\epsilon$  is a Lie group automorphism of  $(\mathfrak{g}, \star)$  [12, Remark 1.3.32], i.e.,

$$\mathfrak{d}_\epsilon(X \star Y) = (\mathfrak{d}_\epsilon(X)) \star (\mathfrak{d}_\epsilon(Y)) \quad \text{for all } X, Y \in \mathfrak{g}. \quad (2)$$

These dilations on  $\mathfrak{g}$  are also commonly notated as  $\delta_\epsilon$ , but we will not do so here to avoid confusion with the dilations on  $G$ .

As the exponential map  $\exp_G : \mathfrak{g} \rightarrow G$  is a diffeomorphism, this induces a family of dilations  $\delta_\epsilon$  on  $G$ :

$$\delta_\epsilon := \exp_G \circ \mathfrak{d}_\epsilon \circ \exp_G^{-1}. \quad (3)$$

This aligns with our earlier definition of dilations in subsection 2.2.

Suppose  $H$  is a Carnot group isomorphic to  $G$ , with stratification

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r.$$

A Lie group isomorphism  $\varphi : G \rightarrow H$  induces a Lie algebra isomorphism  $\varphi_* : \mathfrak{g} \rightarrow \mathfrak{h}$  that satisfies the following identity:

$$\exp_H \circ \varphi_* = \varphi \circ \exp_G. \quad (4)$$

We say that a Lie group isomorphism  $\varphi : G \rightarrow H$  **commutes with dilation** if

$$\varphi(\delta_\epsilon^G g) = \delta_\epsilon^H \varphi(g) \quad \text{for all } g \in G, \epsilon > 0,$$

where  $\delta_\epsilon^G, \delta_\epsilon^H$  denote the dilations on  $G, H$ , respectively. If we say that a Lie algebra isomorphism  $f : \mathfrak{g} \rightarrow \mathfrak{h}$  commutes with dilation if

$$f(\mathfrak{d}_\epsilon^G X) = \mathfrak{d}_\epsilon^H f(X) \quad \text{for all } X \in \mathfrak{g}, \epsilon > 0,$$

it is easy to check using (3) and (4) that an isomorphism  $\varphi : G \rightarrow H$  commutes with dilations if and only if  $\varphi_* : \mathfrak{g} \rightarrow \mathfrak{h}$  commutes with dilations.

**Example 12.** Let  $G$  be a Carnot group. Suppose  $\mathcal{B} \subset \mathfrak{g}$  is a basis compatible with the stratification of  $\mathfrak{g}$ . Let  $(\mathbb{R}^n, \odot)$  and  $(\mathbb{R}^n, \star)$  be  $G$  equipped with coordinates of the second and first kind, respectively, with respect to  $\mathcal{B}$ . Then  $(\mathbb{R}^n, \odot)$  is isomorphic to  $(\mathbb{R}^n, \star)$  via  $\exp^{-1} \circ \Phi$  and coordinates. Moreover, this isomorphism commutes with dilations.

We say that an isomorphism  $\varphi : G \rightarrow H$  is **strata-preserving** if

$$\varphi_*(\mathfrak{g}_j) = \mathfrak{h}_j \quad \text{for all } j = 1, \dots, r.$$

Note that  $\varphi$  is strata-preserving if and only if  $\varphi^{-1}$  is strata-preserving.

The next result follows from the use of dilations:

**Lemma 13.** *Let  $G, H$  be isomorphic Carnot groups. An isomorphism  $\varphi : G \rightarrow H$  commutes with dilations if and only if  $\varphi$  is strata-preserving.*

In fact, if we say that an isomorphism  $\varphi : G \rightarrow H$  is *contact* if  $\varphi_*(\mathfrak{g}_1) = \mathfrak{h}_1$ , it's easy to check from the stratifications of  $\mathfrak{g}$  and  $\mathfrak{h}$  that  $\varphi$  is a contact map if and only if it is strata-preserving.

We will show weakly contact mappings are invariant under isomorphisms that commute with dilations. We first prove that such isomorphisms are biLipschitz.

**Proposition 14.** *Let  $\varphi : (\mathbb{R}^n, \cdot) \rightarrow (\mathbb{R}^n, *)$  be an isomorphism between Carnot groups, that commutes with dilations. Then  $\varphi$  is biLipschitz, i.e., there exists a constant  $C$  such that*

$$\frac{1}{C} d_{cc}^{(\mathbb{R}^n, \cdot)}(g, h) \leq d_{cc}^{(\mathbb{R}^n, *)}(\varphi(g), \varphi(h)) \leq C d_{cc}^{(\mathbb{R}^n, \cdot)}(g, h) \quad \text{for all } g, h \in (\mathbb{R}^n, \cdot).$$

*Proof.* As  $\varphi$  commutes with dilations and the cc-metrics on  $(\mathbb{R}^n, \cdot)$  and  $(\mathbb{R}^n, *)$  are one-homogeneous, it suffices to show  $\varphi$  is biLipschitz when restricted to  $B_{cc}(e, 1)$ .

Let  $\{X^1, \dots, X^{m_1}\}, \{Y^1, \dots, Y^{m_1}\}$  be left-invariant frames for  $H(\mathbb{R}^n, \cdot), H(\mathbb{R}^n, *)$ , respectively. For each  $g \in (\mathbb{R}^n, \cdot)$ , define the linear isomorphism  $S_g : H_{\varphi(g)}(\mathbb{R}^n, *) \rightarrow H_{\varphi(g)}(\mathbb{R}^n, \cdot)$  induced by  $(\varphi_* X^j)_{\varphi(g)} \mapsto Y_{\varphi(g)}^j$ . The function  $g \mapsto \|S_g\|$  is continuous, and hence, is bounded on  $B_{cc}(e, 2)$ , say by  $C$ . This implies for all  $g \in B_{cc}(e, 2)$  and  $v \in H_g(\mathbb{R}^n, \cdot)$ , we have  $|d\varphi_g(v)|_{\varphi(g)} \leq C|v|_g$ . It then follows from Lemma 13 that

$$d_{cc}^{(\mathbb{R}^n, *)}(\varphi(g), \varphi(h)) \leq C d_{cc}^{(\mathbb{R}^n, \cdot)}(g, h)$$

for all  $g, h \in B_{cc}(e, 1)$ . Applying this argument to  $\varphi^{-1}$ , the lemma follows.  $\square$

It follows from the chain rule that weak contactness is preserved by strata-preserving isomorphisms.

**Corollary 15.** *Fix  $\Omega \subseteq \mathbb{R}^k$  an open subset. Let  $\varphi : (\mathbb{R}^n, \cdot) \rightarrow (\mathbb{R}^n, *)$  be an isomorphism between Carnot groups, that commutes with dilations. If  $f : \Omega \rightarrow (\mathbb{R}^n, \cdot)$  is locally Lipschitz and weakly contact, then  $\varphi \circ f : \Omega \rightarrow (\mathbb{R}^n, *)$  is also locally Lipschitz and weakly contact.*

### 2.5. Coordinates of the second kind

Now that we have defined dilations on  $G$  and  $\mathfrak{g}$ , we can introduce coordinates of the second kind, another model for Carnot groups. The Carnot group that we obtain via this construction will be isomorphic to the coordinates of the first kind model we described in subsection 2.1. We will first state a result that will allow us to define our other model.

**Theorem 16.** ([18, Theorem 2.10.1]) *Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ . Suppose  $\mathfrak{g}$  is the direct sum of linear subspaces  $\mathfrak{h}_1, \dots, \mathfrak{h}_s$ . Then there are open neighborhoods  $B_i$  of 0 in  $\mathfrak{h}_i$  ( $1 \leq i \leq s$ ) and  $U$  of 1 in  $G$ , such that the map*

$$\Psi : (Z_1, \dots, Z_s) \mapsto \exp Z_1 \cdots \exp Z_s$$

*is an analytic diffeomorphism of  $B_1 \times \cdots \times B_s$  onto  $U$ .*

Fix a basis  $\mathcal{B} = \{X^1, \dots, X^n\}$  of  $\mathfrak{g}$  compatible with the stratification, and define  $\Phi : \mathfrak{g} \rightarrow G$  by

$$\Phi(a_1 X^1 + \cdots + a_n X^n) = \exp(a_1 X^1) \cdots \exp(a_n X^n).$$

By Theorem 16, the restriction  $\Phi|_V : V \rightarrow U$  is a diffeomorphism for some open neighborhoods  $V \subset \mathfrak{g}$  of 0 and  $U \subset G$  of  $e$ . After noticing  $\Phi(a_1 X^1 + \cdots + a_n X^n) = \exp(a_1 X^1 \star \cdots \star a_n X^n)$ , it follows from (2) and (3) that  $\Phi$  is a global diffeomorphism.

We can then define  $\odot : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  by

$$X \odot Y = \Phi^{-1}(\Phi(X)\Phi(Y)).$$

As in subsection 2.1, we can identify  $\mathfrak{g}$  with  $\mathbb{R}^n$  and define a corresponding operation  $\odot$  on  $\mathbb{R}^n$ , with a slight abuse of notation. We say that  $(\mathbb{R}^n, \odot)$  is a **normal model of the second kind** of  $\mathfrak{g}$ , and  $(\mathbb{R}^n, \odot)$  is  $G$  equipped with **coordinates of the second kind with respect to  $\mathcal{B}$** . Identifying  $(\mathbb{R}^n, \star)$  with  $\mathfrak{g}$  via the same basis, observe that  $\exp^{-1} \circ \Phi : (\mathbb{R}^n, \odot) \rightarrow (\mathbb{R}^n, \star)$  is a Lie group isomorphism. In particular,  $(\mathbb{R}^n, \odot)$  is isomorphic to  $G$ . It then follows from Corollary 15 that it suffices to prove each of Theorems 5 and 6 for a single system of coordinates.

## 3. Result for $J^k(\mathbb{R})$

### 3.1. $J^k(\mathbb{R})$ as Carnot groups

We will only do our discussion in this section for jet spaces  $J^k(\mathbb{R}) = J^k(\mathbb{R}, \mathbb{R})$  (for  $k \geq 1$ ) to make things clearer. Similar constructions can be used to define more general jet spaces  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  (see [9, Section 4]). The results in this paper concerning model filiform groups translate into results for general jet spaces, and I will note the more general results.

Given  $f, g \in C^k(\mathbb{R})$ , we say  $f$  is equivalent to  $g$  at  $x \in \mathbb{R}$ , and write  $f \sim_x g$ , if their  $k^{th}$ -order Taylor polynomials agree at  $x$ . Define

$$J^k(\mathbb{R}) = \bigcup_{x \in \mathbb{R}} C^k(\mathbb{R}) / \sim_x,$$

and observe we have global coordinates on  $J^k(\mathbb{R})$  by

$$J^k(\mathbb{R}) \ni [f]_{\sim_x} \mapsto (x, u_k, \dots, u_0) \in \mathbb{R}^{k+2},$$

where  $u_j := f^{(j)}(x)$ .

The horizontal bundle  $HJ^k(\mathbb{R})$  is defined pointwise by

$$H_p J^k(\mathbb{R}) = \{v \in T_p J^k(\mathbb{R}) \mid \omega_i(v) = 0, \ i = 0, \dots, k-1\},$$

where

$$\omega_i := du_i - u_{i+1} dx.$$

In coordinates,  $HJ^k(\mathbb{R})$  is a 2-dimensional tangent distribution on  $J^k(\mathbb{R})$  with global frame  $\{X^{(k)}, \frac{\partial}{\partial u_k}\}$ , where

$$X^{(k)} = \frac{\partial}{\partial x} + u_k \frac{\partial}{\partial u_{k-1}} + \dots + u_1 \frac{\partial}{\partial u_0}.$$

The nontrivial bracket relations are

$$\left[ \frac{\partial}{\partial u_j}, X^{(k)} \right] = \frac{\partial}{\partial u_{j-1}}, \quad j = 1, \dots, k.$$

It follows that

$$\text{Lie}(J^k(\mathbb{R})) = HJ^k(\mathbb{R}) \oplus \text{span} \left\{ \frac{\partial}{\partial u_{k-1}} \right\} \oplus \dots \oplus \text{span} \left\{ \frac{\partial}{\partial u_0} \right\}$$

is a  $(k+1)$ -step stratified Lie algebra.

One can use coordinates of the second kind to turn  $J^k(\mathbb{R})$  into a Carnot group with the following group operation:

$$(x, u_k, \dots, u_0) \odot (y, v_k, \dots, v_0) = (z, w_k, \dots, w_0),$$

where  $z = x + y$ ,  $w_k = u_k + v_k$ , and

$$w_s = u_s + v_s + \sum_{j=s+1}^k u_j \frac{y^{j-s}}{(j-s)!}, \quad s = 0, \dots, k-1$$

(see [9, Example 4.3]). For  $(x, u_k, \dots, u_0) \in J^k(\mathbb{R})$ , it is easy to show

$$((x, u_k, \dots, u_0)^{-1})_s = - \sum_{j=s}^k \frac{(-x)^{j-s}}{(j-s)!} u_j, \quad s = 0, \dots, k.$$

### 3.2. A horizontality result for $J^k(\mathbb{R})$

In this section, we will prove a horizontality condition for  $J^k(\mathbb{R})$ , from which Theorem 5 will follow. We begin with a crucial lemma concerning the group structure of  $J^k(\mathbb{R})$ , similar to Corollary 1.3.18 of [12].

**Lemma 17.** For  $(x, u_k, \dots, u_0), (y, v_k, \dots, v_0) \in J^k(\mathbb{R})$ ,

$$((x, u_k, \dots, u_0)^{-1} \odot (y, v_k, \dots, v_0))_0 = v_0 - u_0 - \sum_{j=1}^k \frac{u_j}{j!} (y - x)^j.$$

*Proof.* Recall

$$((x, u_k, \dots, u_0)^{-1})_s = - \sum_{j=s}^k \frac{(-x)^{j-s}}{(j-s)!} u_j, \quad s = 0, \dots, k,$$

and the last coordinate of  $(x, u_k, \dots, u_0) \odot (y, v_k, \dots, v_0)$  is

$$((x, u_k, \dots, u_0) \odot (y, v_k, \dots, v_0))_0 = v_0 + \sum_{s=0}^k \frac{y^s}{s!} u_s.$$

Thus,

$$\begin{aligned} ((x, u_k, \dots, u_0)^{-1} \odot (y, v_k, \dots, v_0))_0 &= v_0 - \sum_{s=0}^k \sum_{j=s}^n \frac{y^s}{s!} \cdot \frac{(-x)^{j-s}}{(j-s)!} \cdot u_j \\ &= v_0 - \sum_{j=0}^k \sum_{s=0}^j \binom{j}{s} y^s (-x)^{j-s} \cdot \frac{u_j}{j!} \\ &= v_0 - \sum_{j=0}^k \frac{1}{j!} \cdot (y-x)^j u_j, \end{aligned}$$

where the last equality comes from the Binomial Theorem.  $\square$

**Remark 18.** The same reasoning using the Multinomial Theorem gives us the following generalization for all jet spaces:

Fix positive integers  $k, m, n$ . Let the notation for  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  be as in Warhurst (see [9, Subsection 4.4]), and equip  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  with the group operation arising from coordinates of the second kind (see [9, subsection 4.4]). Given  $(x, u^{(k)}), (y, v^{(k)}) \in J^k(\mathbb{R}^m, \mathbb{R}^n)$ ,

$$((x, u^{(k)})^{-1} \odot (y, v^{(k)}))_0^l = v_0^l - \sum_{I \in \tilde{I}(m)} \frac{u_I^l}{I!} (y-x)^I, \quad l = 1, \dots, n.$$

Here, for  $I = (i_1, \dots, i_m) \in \tilde{I}(m)$  and  $z = (z_1, \dots, z_m) \in \mathbb{R}^m$ , we define  $I! = i_1! \dots i_m!$  and  $z^I = z_1^{i_1} \dots z_m^{i_m}$ .

**Proposition 19.** Let  $k, n$  be positive integers with  $\Omega \subseteq \mathbb{R}^n$  an open set. Suppose that  $f = (f^x, f^{u_k}, \dots, f^{u_0}) : \Omega \rightarrow J^k(\mathbb{R})$  is of class  $C^{0, \frac{1}{2}+}$ . If the component  $f^x$  is differentiable at a point  $p_0 \in \Omega$ , then the components  $f^{u_{k-1}}, f^{u_{k-2}}, \dots, f^{u_0}$  are also differentiable at  $p_0$  with

$$df_{p_0}^{u_j} = f^{u_{j+1}}(p_0) df_{p_0}^x$$

for all  $j = 0, \dots, k-1$ . In particular, if  $f^{u_k}$  is also differentiable at  $p_0$ , then the image of  $df_{p_0}$  lies in the horizontal space  $H_{f(p_0)} J^k(\mathbb{R})$ .

*Proof.* We prove this result by induction on  $k \geq 1$ . Below,  $p$  is a point in  $\Omega$ .

Let  $f = (f^x, f^{u_1}, f^{u_0}) : \Omega \rightarrow J^1(\mathbb{R})$  be given of class  $C^{0, \frac{1}{2}+}$ . Choose a map  $\beta$  for  $f$  satisfying (1). By Lemma 17,

$$(f(p_0)^{-1} f(p))_0 = f^{u_0}(p) - f^{u_0}(p_0) - f^{u_1}(p_0)(f^x(p) - f^x(p_0)).$$

Thus by Corollary 10, there exists  $C > 0$  such that

$$\begin{aligned} |f^{u_0}(p) - f^{u_0}(p_0) - f^{u_1}(p_0)(f^x(p) - f^x(p_0))|^{1/2} &\leq C d_{cc}(f(p), f(p_0)) \\ &\leq C \tilde{\beta}(|p - p_0|) \cdot |p - p_0|^{1/2}. \end{aligned}$$

We have

$$\begin{aligned} &|f^{u_0}(p) - f^{u_0}(p_0) - f^{u_1}(p_0)df_{p_0}^x(p - p_0)| \\ &\leq C^2 \tilde{\beta}^2(|p - p_0|) \cdot |p - p_0| + |f^{u_1}(p_0)(f^x(p) - f^x(p_0)) - f^{u_1}(p_0)df_{p_0}^x(p - p_0)| \\ &= o(|p - p_0|), \end{aligned}$$

where we used the differentiability of  $f^x$  at  $p_0$  for the last equality.

Suppose we have proven the result up to  $k$ . Let  $f = (f^x, f^{u_{k+1}}, \dots, f^{u_0}) : \Omega \rightarrow J^{k+1}(\mathbb{R})$  be given of class  $C^{0, \frac{1}{2}+}$  with  $f^x$  differentiable at  $p_0$ . Let  $\tilde{\beta}$  be a map satisfying (1) for  $f$ . Define the projection  $\pi : J^{k+1}(\mathbb{R}) \rightarrow J^k(\mathbb{R})$  by

$$\pi(x, u_{k+1}, \dots, u_0) = (x, u_{k+1}, \dots, u_1).$$

As  $\pi$  maps horizontal curves to horizontal curves of the same length, it's not hard to see that  $\pi$  is a contraction. This implies  $\pi \circ f = (f^x, f^{u_{k+1}}, \dots, f^{u_1})$  is of class  $C^{0, \frac{1}{2}+}(\Omega, J^k(\mathbb{R}))$ . By induction,  $f^{u_k}, \dots, f^{u_1}$  are differentiable at  $p_0$  with

$$df_{p_0}^{u_j} = f^{u_{j+1}}(p_0)df_{p_0}^x$$

for all  $j = 1, \dots, k$ .

It remains to show  $f^{u_0}$  is also differentiable at  $p_0$  with

$$df_{p_0}^{u_0} = f^{u_1}(p_0)df_{p_0}^x.$$

Lemma 17 and Corollary 10 combine to imply

$$\left| f^{u_0}(p) - f^{u_0}(p_0) - \sum_{j=1}^{k+1} \frac{f^{u_j}(p_0)}{j!} (f^x(p) - f^x(p_0))^j \right|^{1/(k+1)} \leq C \tilde{\beta}(|p - p_0|) \cdot |p - p_0|^{1/2}.$$

Moreover, as  $f^x$  is differentiable at  $p_0$ ,

$$f^x(p) - f^x(p_0) = O(|p - p_0|),$$

and hence

$$|f^x(p) - f^x(p_0)|^j = o(|p - p_0|) \quad \text{for all } j \geq 2.$$

It follows

$$\begin{aligned} &|f^{u_0}(p) - f^{u_0}(p_0) - f^{u_1}(p_0)df_{p_0}^x(p - p_0)| \\ &\leq C^{k+1} \tilde{\beta}^{k+1}(|p - p_0|) \cdot |p - p_0|^{\frac{k+1}{2}} + |f^{u_1}(p_0)(f^x(p) - f^x(p_0)) - f^{u_1}(p_0)df_{p_0}^x(x - x_0)| \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=2}^{k+1} \left| \frac{f^{u_j}(p_0)}{j!} (f^x(p) - f^x(p_0)) \right|^j \\
 & = o(|x - x_0|).
 \end{aligned}$$

This proves  $f^{u_0}$  is differentiable at  $p_0$  with

$$df_{p_0}^{u_0} = f^{u_1}(p_0) df_{p_0}^x,$$

and the proposition follows.  $\square$

**Remark 20.** In the above proof, we needed  $f$  to lie in  $C^{0, \frac{1}{2}+}(\Omega, J^k(\mathbb{R}))$  in order to ensure  $f^{u_{k-1}}$  was differentiable at the point with the desired form. To prove the differentiability of the components of  $f$  corresponding to higher layers, one can assume lower regularity. In fact, the above proof shows the following:

Assume  $\Omega \subseteq \mathbb{R}^n$  is open and  $j \geq 2$ . Suppose  $f = (f^x, f^{u_k}, \dots, f^{u_0}) : \Omega \rightarrow J^k(\mathbb{R})$  is of class  $C^{0, \frac{1}{j}+}$ . If  $f^x$  is differentiable at a point  $p_0 \in \Omega$ , then  $f^{u_{k+1-j}}, f^{u_{k-j}}, \dots, f^{u_0}$  are also differentiable at  $p_0$  with

$$df_{p_0}^{u_l} = f^{u_{l+1}}(p_0) df_{p_0}^x, \quad l = k+1-j, \dots, 0.$$

### 3.3. Proof of Theorem 5

Before we prove Theorem 5, we will give an example of a locally  $\frac{1}{2}$ -Hölder map  $f : \mathbb{R}^2 \rightarrow J^k(\mathbb{R})$  that is Lipschitz as a map into  $\mathbb{R}^{k+2}$ . Comparing with Remark 3, this suggests that our result is sharp in the case  $n = 2$ .

**Example 21.** Define  $f : \mathbb{R}^2 \rightarrow J^k(\mathbb{R})$  by

$$f(x, y) = (0, x, y, 0, \dots, 0).$$

Then  $f$  is Lipschitz (in fact, is an isometry) as a map into  $\mathbb{R}^{k+2}$ .

To show  $f$  is locally  $\frac{1}{2}$ -Hölder, first note in  $J^k(\mathbb{R})$ ,

$$(0, -x_1, -y_1, 0, \dots, 0) \odot (0, x_2, y_2, 0, \dots, 0) = (0, x_2 - x_1, y_2 - y_1, 0, \dots, 0).$$

By Corollary 10, there exists a constant  $C$  such that

$$d_{cc}(f(x_1, y_1), f(x_2, y_2)) \leq C \max\{|x_2 - x_1|, |y_2 - y_1|^{1/2}\}$$

for all  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ . By considering cases, one can then show

$$d_{cc}(f(x_1, y_1), f(x_2, y_2)) \leq \sqrt{2MC} |(x_1, y_1) - (x_2, y_2)|^{1/2}$$

whenever  $(x_1, y_1), (x_2, y_2) \in [-M, M]^2$  with  $M > 1$ .



*Proof of Theorem 5.* Fix positive integers  $n, k$  with  $n \geq 2$ . Suppose  $f : \Omega \rightarrow J^k(\mathbb{R})$  is of class  $C^{0, \frac{1}{2}+}$  and is locally Lipschitz as a map into  $\mathbb{R}^{k+2}$ . By Rademacher's Theorem, each of the components of  $f$  is differentiable almost everywhere, and in particular,  $f^x$  is differentiable almost everywhere. Proposition 19 then implies that  $f$  is weakly contact. Since  $J^k(\mathbb{R})$  is purely  $n$ -unrectifiable [7, Theorem 1.1], Theorem 5 in the case of second kind coordinates follows from Proposition 11. The discussion at the end of subsection 2.5 then proves the result for coordinates of the first kind.  $\square$

**Remark 22.** Observe that  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  is purely  $j$ -unrectifiable if  $j > \binom{m+k-1}{k}$  [7, Theorem 1.1]. Hence, from Remark 18, one can use similar reasoning to show the following generalization:

Fix a jet space  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  and equip it with the group structure from Subsection 4.4 of [9]. Suppose  $j > \binom{m+k-1}{k}$  and  $\Omega$  is an open subset of  $\mathbb{R}^j$ . If  $N$  is the topological dimension of  $J^k(\mathbb{R}^m, \mathbb{R}^n)$ , there is no injective mapping in the class  $C^{0, \frac{1}{2}+}(\Omega; J^k(\mathbb{R}^m, \mathbb{R}^n))$  that is also locally Lipschitz when considered as a map into  $\mathbb{R}^N$ .

**Remark 23.** Theorem 5 has an easier proof if we assume  $n < \frac{1}{2} \left( 1 + \frac{(k+1)(k+2)}{2} \right)$ . Making this assumption, suppose that  $f : \Omega \rightarrow J^k(\mathbb{R})$  is injective and of class  $C^{0, \frac{1}{2}+}$ . Let  $B(x, r)$  be an open ball with  $\overline{B(x, r)} \subseteq \Omega$ . Then the restriction  $f|_{\overline{B(x, r)}}$  is injective and of class  $C^{0, \frac{1}{2}+}(\overline{B(x, r)}, J^k(\mathbb{R}))$ . Since  $\overline{B(x, r)}$  is bounded, it follows that  $f|_{\overline{B(x, r)}}$  is a  $\frac{1}{2}$ -Hölder homeomorphism. In particular,  $f(B(x, r))$  is open in  $J^k(\mathbb{R})$ , which implies

$$\dim_{\text{Hau}} f(B(x, r)) = \dim_{\text{Hau}} J^k(\mathbb{R}) = 1 + \frac{(k+1)(k+2)}{2}.$$

But as  $f$  is  $\frac{1}{2}$ -Hölder,

$$\dim_{\text{Hau}} f(B(x, r)) \leq 2 \cdot \dim_{\text{Hau}} B(x, r) = 2n,$$

which is a contradiction.

## 4. Result for Carnot groups of step at most three

### 4.1. Geometry of step two Carnot groups

In this section, we will consider the geometry of Carnot groups of step two and equip these groups with coordinates of the first kind.

Fix a step two Carnot group  $G$  with Lie algebra  $\mathfrak{g}$ . Writing  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ , let  $d_1, \dots, d_r$  be a basis for  $\mathfrak{g}_1$  and  $e_1, \dots, e_s$  be a basis for  $\mathfrak{g}_2$ . We can write

$$[d_i, d_j] = \sum_{k=1}^s \alpha_k^{ij} e_k$$

for some structural constants  $\alpha_k^{ij}$ , with all other bracket relations trivial. By antisymmetry,  $\alpha_k^{ij} = -\alpha_k^{ji}$  for all  $i, j$ , and  $k$ . In fact, Bonfiglioli, Lanconelli, and Uguzzoni prove that there exists a Carnot group of step two with these bracket relations if and only if the skew-symmetric matrices  $(\alpha_k^{ij})$ ,  $k = 1, \dots, s$ , are linearly independent [12, Proposition 3.2.1].

Using the procedure described in subsection 2.1, we can identify  $G$  with  $\mathbb{R}^{r+s}$  equipped with the following multiplication via coordinates of the first kind:

$$(A_1, \dots, A_r, B_1, \dots, B_s) \star (a_1, \dots, a_r, b_1, \dots, b_s) = (A_1, \dots, A_r, B_1, \dots, B_s),$$

where

$$\mathcal{A}_i = A_i + a_i, \quad \mathcal{B}_k = B_k + b_k + \frac{1}{2} \sum_{1 \leq i < j \leq r} \alpha_k^{ij} (A_i a_j - a_i A_j).$$

We write  $(A_1, \dots, A_r, B_1, \dots, B_s) = (A_i, B_k)$  henceforth, and we will use similar notation for step 3 Carnot groups.

By translating the canonical basis at the basis, we obtain the left-invariant vector fields

$$\begin{aligned} X^i &:= \frac{\partial}{\partial A_i} + \frac{1}{2} \sum_{k=1}^s \left( \sum_{j < i} \alpha_k^{ji} A_j - \sum_{j > i} \alpha_k^{ij} A_j \right) \frac{\partial}{\partial B_k}, \quad i = 1, \dots, r, \\ Y^k &:= \frac{\partial}{\partial B_k}, \quad k = 1, \dots, s. \end{aligned}$$

We obtain the stratification

$$\text{Lie}(\mathbb{R}^{r+s}, \star) = \langle X^i \rangle_{1 \leq i \leq r} \oplus \langle Y^k \rangle_{1 \leq k \leq s}$$

[12, Remark 1.4.8]. In fact, it is easy to check that the linear map  $\varphi : \text{Lie}(\mathbb{R}^{r+s}, \star) \rightarrow \mathfrak{g}$  induced by  $X^i \mapsto d_i$ ,  $Y^k \mapsto e_k$  is a Lie algebra isomorphism.

The contact forms, satisfying

$$H(\mathbb{R}^{r+s}, \star) = \bigcap_{k=1}^s \ker \omega^k,$$

are given by

$$\omega^k := dB_k - \frac{1}{2} \sum_{i=1}^r \left( \sum_{j < i} \alpha_k^{ji} A_j - \sum_{j > i} \alpha_k^{ij} A_j \right) dA_i.$$

In other words, if  $v \in T_p(\mathbb{R}^{r+s}, \star)$ , then

$$v \in H_p(\mathbb{R}^{r+s}, \star) \iff \omega_p^k(v) = 0 \text{ for all } k = 1, \dots, s.$$

#### 4.2. Geometry of step three Carnot groups

Let  $G$  be a step three Carnot group. Let  $d_1, \dots, d_r$  be a basis for  $\mathfrak{g}_1$ ,  $e_1, \dots, e_s$  a basis for  $\mathfrak{g}_2$ , and  $f_1, \dots, f_t$  a basis for  $\mathfrak{g}_3$ . We write

$$\begin{aligned} [d_i, d_j] &= \sum_{k=1}^s \alpha_k^{ij} e_k \\ [d_i, e_k] &= \sum_{m=1}^t \beta_m^{ik} f_m \end{aligned}$$

with all other bracket relations trivial.

As in the step two case, we can identify  $G$  with  $\mathbb{R}^{r+s+t}$  equipped with the following operation via coordinates of the first kind:

$$(A_i, B_k, C_m) \star (a_i, b_k, c_m) = (\mathcal{A}_i, \mathcal{B}_k, \mathcal{C}_m),$$

where

$$\begin{aligned}\mathcal{A}_i &= A_i + a_i \\ \mathcal{B}_k &= B_k + b_k + \frac{1}{2} \sum_{i < j} \alpha_k^{ij} (A_i a_j - a_i A_j) \\ \mathcal{C}_m &= C_m + c_m + \frac{1}{2} \sum_{i,j} \beta_m^{ij} (A_i b_j - B_j a_i) + \frac{1}{12} \sum_{l,k} \sum_{i < j} (A_l - a_l) \alpha_k^{ij} (A_i a_j - a_i A_j) \beta_m^{lk}.\end{aligned}$$

Observe  $(A_i, B_k, C_m)^{-1} = (-A_i, -B_k, -C_m)$  and

$$(A_i, B_k, C_m)^{-1} \star (a_i, b_k, c_m) = (\tilde{\mathcal{A}}_i, \tilde{\mathcal{B}}_k, \tilde{\mathcal{C}}_m),$$

where

$$\begin{aligned}\tilde{\mathcal{A}}_i &= a_i - A_i \\ \tilde{\mathcal{B}}_k &= b_k - B_k - \frac{1}{2} \sum_{i < j} \alpha_k^{ij} (A_i a_j - a_i A_j) \\ \tilde{\mathcal{C}}_m &= c_m - C_m - \frac{1}{2} \sum_{i,j} \beta_m^{ij} (A_i b_j - B_j a_i) + \frac{1}{12} \sum_{l,k} \sum_{i < j} (A_l + a_l) \alpha_k^{ij} (A_i a_j - a_i A_j) \beta_m^{lk}.\end{aligned}\tag{5}$$

Left-translating the canonical basis at the origin, we obtain the left-invariant vector fields

$$\begin{aligned}X^i &= \frac{\partial}{\partial A_i} + \sum_{k=1}^s \frac{1}{2} \left( \sum_{j < i} \alpha_k^{ji} A_j - \sum_{j > i} \alpha_k^{ij} A_j \right) \frac{\partial}{\partial B_k} \\ &\quad + \sum_{m=1}^t \left[ -\frac{1}{2} \sum_{j=1}^s B_j \beta_m^{ij} + \frac{1}{12} \sum_{l=1}^r \sum_{k=1}^s A_l \left( \sum_{j < i} \alpha_k^{ji} A_j - \sum_{j > i} \alpha_k^{ij} A_j \right) \beta_m^{lk} \right] \frac{\partial}{\partial C_m}, \\ Y^k &= \frac{\partial}{\partial B_k} + \sum_{m=1}^t \left( \frac{1}{2} \sum_{i=1}^r \beta_m^{ik} A_i \right) \frac{\partial}{\partial C_m}, \\ Z^m &= \frac{\partial}{\partial C_m}.\end{aligned}$$

It is clear that  $\{X^i\}_i \cup \{Y^k\}_k \cup \{Z^m\}_m$  forms a basis for  $Lie(\mathbb{R}^n, \star)$ . Moreover, we have the expected step three stratification of  $Lie(\mathbb{R}^n, \star)$  [12, Remark 1.4.8]:

$$Lie(\mathbb{R}^{r+s+t}, \star) = \langle X^i \rangle_{1 \leq i \leq r} \oplus \langle Y^k \rangle_{1 \leq k \leq s} \oplus \langle Z^m \rangle_{1 \leq m \leq t}\tag{6}$$

In fact, one can show using the Jacobi identity that the linear map  $\varphi : Lie(\mathbb{R}^n, \star) \rightarrow \mathfrak{g}$  induced by

$$X^i \mapsto d_i, \quad Y^k \mapsto e_k, \quad Z^m \mapsto f_m$$

is a Lie algebra isomorphism.

The contact forms are given by

$$\begin{aligned}\omega_1^k &:= dB_k - \sum_{i=1}^r \frac{1}{2} \left( \sum_{j<i} \alpha_k^{ji} A_i - \sum_{j>i} \alpha_k^{ij} A_j \right) dA_i \\ \omega_2^m &:= dC_m - \sum_{i=1}^r \left[ -\frac{1}{2} \sum_{j=1}^s B_j \beta_m^{ij} + \frac{1}{12} \sum_{l=1}^r \sum_{k=1}^s A_l \left( \sum_{j<i} \alpha_k^{ji} A_j - \sum_{j>i} \alpha_k^{ij} A_j \right) \beta_m^{lk} \right] dA_i.\end{aligned}$$

We have

$$H(\mathbb{R}^{r+s+t}, \star) = \bigcap_{k=1}^s \ker \omega_1^k \cap \bigcap_{m=1}^t \ker \omega_2^m,$$

so that a tangent vector  $v$  lies in  $H_p(\mathbb{R}^{r+s+t}, \star)$  if and only if  $(\omega_1^k)_p(v) = (\omega_2^m)_p(v) = 0$  for all  $k$  and  $m$ .

#### 4.3. Result for step two Carnot groups

In this subsection, we will consider step two Carnot groups  $G$  using the notation from subsection 4.1. Recall that we identify  $G$  with  $\mathbb{R}^{r+s}$  equipped with an operation arising from coordinates of the first kind:

$$(A_i, B_k) \star (a_i, b_k) = (\mathcal{A}_i, \mathcal{B}_k),$$

where

$$\mathcal{A}_i = A_i + a_i, \quad \mathcal{B}_k = B_k + b_k + \frac{1}{2} \sum_{1 \leq i < j \leq r} \alpha_k^{ij} (A_i a_j - a_i A_j).$$

The contact forms defining the horizontal bundle of  $(\mathbb{R}^{r+s}, \star)$  are given by

$$\omega^k = dB_k - \frac{1}{2} \sum_{i=1}^r \left( \sum_{j<i} \alpha_k^{ji} A_j - \sum_{j>i} \alpha_k^{ij} A_j \right) dA_i, \quad k = 1, \dots, r.$$

Here, the constants  $\alpha_k^{ij}$  come from the bracket relations on  $\mathfrak{g}_1$ .

The goal of this section will be to prove Theorem 6 by proving a result similar to Proposition 19. Theorem 6 will then follow from this result, in the same way that Theorem 5 followed from Proposition 19. We show the following:

**Lemma 24.** *Fix a step two Carnot group  $G$  and an open set  $\Omega \subseteq \mathbb{R}^n$ . Let  $f = (f^{A_1}, \dots, f^{A_r}, f^{B_1}, \dots, f^{B_s}) : \Omega \rightarrow G$  be of class  $C^{0, \frac{1}{2}+}$ , where  $f^{A_1}, \dots, f^{A_r}$  are the horizontal components of  $f$ . If each  $f^{A_i}$  is differentiable at a point  $x_0 \in \Omega$ , then  $f$  is differentiable at  $x_0$  with the image of  $df_{x_0}$  contained in  $H_{f(x_0)}G$ .*

*Proof.* We need to show for all  $k$ , the component  $f^{B_k}$  is differentiable at  $x_0$  with

$$df_{x_0}^{B_k} = \frac{1}{2} \sum_{i=1}^r \left( \sum_{j<i} \alpha_k^{ji} f^{A_j}(x_0) - \sum_{j>i} \alpha_k^{ij} f^{A_j}(x_0) \right) df_{x_0}^{A_i}.$$

Fix  $k$ . By Corollary 10, there exists a constant  $C$  such that

$$|f^{B_k}(x) - f^{B_k}(x_0) - \frac{1}{2} \sum_{1 \leq i < j \leq r} \alpha_k^{ij} (f^{A_i}(x_0) f^{A_j}(x) - f^{A_i}(x) f^{A_j}(x_0))| \leq C d_{cc}(f(x), f(x_0))^2 \quad (7)$$

for all  $x \in \Omega$ .

Choose a function  $\beta$  so that (1) holds for  $f$ . From (7),

$$|f^{B_k}(x) - f^{B_k}(x_0) - \frac{1}{2} \sum_{1 \leq i < j \leq r} \alpha_k^{ij} (f^{A_i}(x_0) f^{A_j}(x) - f^{A_i}(x) f^{A_j}(x_0))| \leq \beta^2(|x - x_0|) \cdot |x - x_0|,$$

absorbing a constant into  $\beta$ . Thus,

$$\begin{aligned} & \left| f^{B_k}(x) - f^{B_k}(x_0) - \frac{1}{2} \sum_{i=1}^r \left( \sum_{j < i} \alpha_k^{ji} f^{A_j}(x_0) - \sum_{j > i} \alpha_k^{ij} f^{A_j}(x_0) \right) df_{x_0}^{A_i}(x - x_0) \right| \\ & \leq \beta^2(|x - x_0|) \cdot |x - x_0| \\ & \quad + \frac{1}{2} \left| \sum_{i < j} \alpha_k^{ij} (f^{A_i}(x_0) f^{A_j}(x) - f^{A_i}(x) f^{A_j}(x_0)) - \alpha_k^{ij} f^{A_i}(x_0) df_{x_0}^{A_j}(x - x_0) \right| \\ & \quad + \frac{1}{2} \left| \sum_{i < j} \alpha_k^{ij} (f^{A_i}(x) f^{A_j}(x_0) - f^{A_i}(x_0) f^{A_j}(x_0)) - \alpha_k^{ij} f^{A_j}(x_0) df_{x_0}^{A_i}(x - x_0) \right| \\ & = o(|x - x_0|), \end{aligned}$$

where we used the differentiability of each  $f^{A_i}$  at  $x_0$  for the last estimate.  $\square$

Theorem 6 for step two Carnot groups now follows from Lemma 24, using the same reasoning as in the proof of Theorem 5.

**Remark 25.** Lemma 24 was proven in the case  $G = \mathbb{H}^n$  by Balogh, Hajlasz, and Wildrick [8, Proposition 8.1]. The proof of Lemma 24 above was directly obtained from their proof by taking into account structural constants.

#### 4.4. Result for step three Carnot groups

In this section, we will prove Theorem 6 for step three Carnot groups using similar reasoning as in subsection 4.3. We begin by reviewing notation:

Let  $G$  be a step 3 Carnot group. We identify  $G$  with  $\mathbb{R}^{r+s+t}$  equipped with an operation arising from coordinates of the first kind:

$$(A_i, B_k, C_m) \star (a_i, b_k, c_m) = (\mathcal{A}_i, \mathcal{B}_k, \mathcal{C}_m)$$

where

$$\begin{aligned} \mathcal{A}_i &= A_i + a_i, & \mathcal{B}_k &= B_k + b_k + \frac{1}{2} \sum_{i < j} \alpha_k^{ij} (A_i a_j - a_i A_j), \\ \mathcal{C}_m &= C_m + c_m + \frac{1}{2} \sum_{i,j} \beta_m^{ij} (A_i b_j - B_j a_i) + \frac{1}{12} \sum_{l=1}^r \sum_{k=1}^s \sum_{i < j} (A_l - a_l) \alpha_k^{ij} (A_i a_j - a_i A_j) \beta_m^{lk}. \end{aligned}$$

The 1-forms defining  $HG$  are given by

$$\omega_1^k := dB_k - \sum_{i=1}^r \frac{1}{2} \left( \sum_{j < i} \alpha_k^{ji} A_i - \sum_{j > i} \alpha_k^{ij} A_j \right) dA_i,$$

$$\omega_2^m := dC_m - \sum_{i=1}^r \left[ -\frac{1}{2} \sum_{j=1}^s B_j \beta_m^{ij} + \frac{1}{12} \sum_{l=1}^r \sum_{k=1}^s A_l \left( \sum_{j<i} \alpha_k^{ji} A_j - \sum_{j>i} \alpha_k^{ij} A_j \right) \beta_m^{lk} \right] dA_i.$$

As in Section 4.3, to prove Theorem 6 for step three Carnot groups, it suffices to prove the following:

**Lemma 26.** *Fix a step three Carnot group  $G$  and an open set  $\Omega \subseteq \mathbb{R}^n$ . Let  $f = (f^{A_1}, \dots, f^{A_r}, f^{B_1}, \dots, f^{B_s}, f^{C_1}, \dots, f^{C_t}) : \Omega \rightarrow G$  be given of class  $C^{0, \frac{1}{2}+}$ , where  $f^{A_1}, \dots, f^{A_r}$  are the horizontal components of  $f$ . If each  $f^{A_i}$  is differentiable at a point  $x_0 \in \Omega$ , then  $f$  is differentiable at  $x_0$  with the image of  $df_{x_0}$  lying in  $H_{f(x_0)}G$ .*

*Proof.* By the proof of Lemma 24, each component  $f^{B_k}$  is differentiable at  $x_0$  with

$$df_{x_0}^{B_k} = \frac{1}{2} \sum_i \left( \sum_{j<i} \alpha_k^{ji} f^{A_j}(x_0) - \sum_{j>i} \alpha_k^{ij} f^{A_j}(x_0) \right) df_{x_0}^{A_i}.$$

It remains to show that each component  $f^{C_m}$  is differentiable at  $x_0$  with

$$df_{x_0}^{C_m} = \sum_i \left[ -\frac{1}{2} \sum_j \beta_m^{ij} f^{B_j}(x_0) + \frac{1}{12} \sum_{l,k} f^{A_l}(x_0) \left( \sum_{j<i} \alpha_k^{ji} f^{A_j}(x_0) - \sum_{j>i} \alpha_k^{ij} f^{A_j}(x_0) \right) \beta_m^{lk} \right] df_{x_0}^{A_i}.$$

Fix  $m$ . Choose  $\beta$  so that (1) holds. By the calculations in (5) and Corollary 10, we have

$$\begin{aligned} |f^{C_m}(x) - f^{C_m}(x_0) - \frac{1}{2} \sum_{i,j} \beta_m^{ij} (f^{A_i}(x_0) f^{B_j}(x) - f^{B_j}(x_0) f^{A_i}(x)) + \frac{1}{12} \sum_{l,k} \sum_{i<j} (f^{A_l}(x_0) \\ + f^{A_l}(x)) \alpha_k^{ij} (f^{A_i}(x_0) f^{A_j}(x) - f^{A_i}(x) f^{A_j}(x_0)) \beta_m^{lk}|^{1/3} \leq \beta(|x - x_0|) \cdot |x - x_0|^{1/2}, \\ |f^{B_k}(x) - f^{B_k}(x_0) - \frac{1}{2} \sum_{i<j} \alpha_k^{ij} (f^{A_i}(x_0) f^{A_j}(x) - f^{A_i}(x) f^{A_j}(x_0))|^{1/2} \leq \beta(|x - x_0|) \cdot |x - x_0|^{1/2} \end{aligned} \quad (8)$$

for each  $k$ , absorbing constants into  $\beta$ .

From (8), we have

$$\begin{aligned} & \left| f^{C_m}(x) - f^{C_m}(x_0) - \sum_i \left[ -\frac{1}{2} \sum_j \beta_m^{ij} f^{B_j}(x_0) \right. \right. \\ & \quad \left. \left. + \frac{1}{12} \sum_{l,k} f^{A_l}(x_0) \left( \sum_{j<i} \alpha_k^{ji} f^{A_j}(x_0) - \sum_{j>i} \alpha_k^{ij} f^{A_j}(x_0) \right) \beta_m^{lk} \right] df_{x_0}^{A_i} (x - x_0) \right| \\ & \leq \beta(|x - x_0|)^3 \cdot |x - x_0|^{3/2} \\ & + \left| \frac{1}{2} \sum_{i,j} \beta_m^{ij} (f^{A_i}(x_0) f^{B_j}(x) - f^{B_j}(x_0) f^{A_i}(x)) - \frac{1}{12} \sum_{l,k} \sum_{i<j} \left[ (f^{A_l}(x_0) + f^{A_l}(x)) \cdot \right. \right. \\ & \quad \left. \left. \alpha_k^{ij} (f^{A_i}(x_0) f^{A_j}(x) - f^{A_i}(x) f^{A_j}(x_0)) \beta_m^{lk} \right] + \frac{1}{2} \sum_{i,j} \beta_m^{ij} f^{B_j}(x_0) df_{x_0}^{A_i} (x - x_0) \right. \\ & \quad \left. - \frac{1}{12} \sum_{l=1}^r \sum_{k=1}^s f^{A_l}(x_0) \left( \sum_{j<i} \alpha_k^{ji} f^{A_j}(x_0) - \sum_{j>i} \alpha_k^{ij} f^{A_j}(x_0) \right) \beta_m^{lk} df_{x_0}^{A_i} (x - x_0) \right| \\ & \leq \beta(|x - x_0|)^3 \cdot |x - x_0|^{3/2} \\ & + \left| -\frac{1}{2} \sum_{i,j} \beta_m^{ij} f^{B_j}(x_0) f^{A_i}(x) + \frac{1}{2} \sum_{i,j} \beta_m^{ij} f^{B_j}(x_0) f^{A_i}(x_0) + \frac{1}{2} \sum_{i,j} \beta_m^{ij} f^{B_j}(x_0) df_{x_0}^{A_i} (x - x_0) \right| \end{aligned}$$

$$\begin{aligned}
 & + \left| \frac{1}{12} \sum_{l,k} \sum_{i < j} (f^{A_l}(x_0) + f^{A_l}(x)) \alpha_k^{ij} (f^{A_i}(x) f^{A_j}(x_0) - f^{A_i}(x_0) f^{A_j}(x)) \beta_m^{lk} \right. \\
 & \quad + \frac{1}{2} \sum_{i,j} \beta_m^{ij} f^{A_i}(x_0) (f^{B_j}(x) - f^{B_j}(x_0)) \\
 & \quad \left. - \frac{1}{12} \sum_{i,l,k} f^{A_l}(x_0) \left( \sum_{j < i} \alpha_k^{ji} f^{A_j}(x_0) - \sum_{j > i} \alpha_k^{ij} f^{A_j}(x_0) \right) \beta_m^{lk} df_{x_0}^{A_i}(x - x_0) \right|,
 \end{aligned}$$

where we regrouped terms in the second inequality. As the  $f^{A_i}$  are differentiable at  $x_0$ , we can then estimate

$$\begin{aligned}
 & \left| f^{C_m}(x) - f^{C_m}(x_0) - \sum_i \left[ -\frac{1}{2} \sum_j \beta_m^{ij} f^{B_j}(x_0) \right. \right. \\
 & \quad \left. \left. + \frac{1}{12} \sum_{l,k} f^{A_l}(x_0) \left( \sum_{j < i} \alpha_k^{ji} f^{A_j}(x_0) - \sum_{j > i} \alpha_k^{ij} f^{A_j}(x_0) \right) \beta_m^{lk} \right] df_{x_0}^{A_i}(x - x_0) \right| \\
 & \leq \beta(|x - x_0|)^3 \cdot |x - x_0|^{3/2} + o(|x - x_0|) \\
 & + \left| -\frac{1}{12} \sum_{l,k} \sum_{i < j} f^{A_l}(x_0) \alpha_k^{ij} f^{A_i}(x) f^{A_j}(x_0) \beta_m^{lk} + \frac{1}{12} \sum_{l,k} \sum_{i < j} f^{A_l}(x_0) \alpha_k^{ij} f^{A_i}(x_0) f^{A_j}(x) \beta_m^{lk} \right. \\
 & \quad \left. + \frac{1}{12} \sum_{l,k} \sum_{i < j} f^{A_l}(x_0) \alpha_k^{ij} f^{A_j}(x_0) \beta_m^{lk} df_{x_0}^{A_i}(x - x_0) \right| \\
 & + \left| \frac{1}{12} \sum_{l,k} \sum_{j < i} f^{A_l}(x_0) f^{A_j}(x_0) \alpha_k^{ji} f^{A_i}(x) \beta_m^{lk} - \frac{1}{12} \sum_{l,k} \sum_{j < i} f^{A_l}(x_0) f^{A_i}(x_0) \alpha_k^{ji} f^{A_j}(x) \beta_m^{lk} \right. \\
 & \quad \left. - \frac{1}{12} \sum_{l,k} \sum_{j < i} f^{A_l}(x_0) \alpha_k^{ji} f^{A_j}(x_0) \beta_m^{lk} df_{x_0}^{A_i}(x - x_0) \right| \\
 & + \left| \frac{1}{6} \sum_{l,k} \sum_{i < j} f^{A_l}(x_0) \alpha_k^{ij} f^{A_j}(x_0) \beta_m^{lk} - \frac{1}{6} \sum_{l,k} \sum_{j < i} f^{A_l}(x_0) f^{A_j}(x_0) \alpha_k^{ji} f^{A_i}(x) \beta_m^{lk} \right. \\
 & \quad + \frac{1}{12} \sum_{l,k} \sum_{i < j} f^{A_l}(x_0) \alpha_k^{ij} (f^{A_i}(x) f^{A_j}(x_0) - f^{A_i}(x_0) f^{A_j}(x)) \beta_m^{lk} \\
 & \quad \left. + \frac{1}{2} \sum_{i,j} \beta_m^{ij} f^{A_i}(x_0) f^{B_j}(x) - \frac{1}{2} \sum_{i,j} \beta_m^{ij} f^{B_j}(x_0) f^{A_i}(x_0) \right| \\
 & \leq o(|x - x_0|) \\
 & + \left| \frac{1}{2} \sum_{l,k} \beta_m^{lk} f^{A_l}(x_0) f^{B_k}(x) - \frac{1}{2} \sum_{l,k} \beta_m^{lk} f^{A_l}(x_0) f^{B_k}(x_0) \right. \\
 & \quad \left. + \frac{1}{4} \sum_{l,k} \beta_m^{lk} f^{A_l}(x_0) \sum_{i < j} \alpha_k^{ij} (f^{A_i}(x) f^{A_j}(x_0) - f^{A_j}(x) f^{A_i}(x_0)) \right| \\
 & + \left| -\frac{1}{12} \sum_{l,k} \beta_m^{lk} f^{A_l}(x_0) \sum_{i < j} \alpha_k^{ij} (f^{A_i}(x) f^{A_j}(x_0) - f^{A_j}(x) f^{A_i}(x_0)) \right. \\
 & \quad \left. + \frac{1}{12} \sum_{l,k} \sum_{i < j} \beta_m^{lk} f^{A_l}(x_0) \alpha_k^{ij} (f^{A_i}(x) f^{A_j}(x_0) - f^{A_i}(x_0) f^{A_j}(x)) \right|,
 \end{aligned}$$

where we used the differentiability of the  $f^{A_i}$  for both inequalities. Since the second term in the last expression is bounded by  $|x - x_0| \cdot \beta(|x - x_0|)^2$  by (8) (up to a constant factor depending only on  $G$ ), we can

bound the last expression by

$$\begin{aligned}
 & o(|x - x_0|) + \frac{1}{12} \left| \sum_{l,k} \beta_m^{lk} (f^{A_l}(x) - f^{A_l}(x_0)) \sum_{i < j} \alpha_k^{ij} (f^{A_i}(x) f^{A_j}(x_0) - f^{A_j}(x) f^{A_i}(x_0)) \right| \\
 & \leq o(|x - x_0|) + \frac{1}{12} \left| \sum_{l,k} \beta_m^{lk} (f^{A_l}(x) - f^{A_l}(x_0)) \sum_{i < j} \alpha_k^{ij} (f^{A_i}(x) f^{A_j}(x_0) - f^{A_i}(x_0) f^{A_j}(x_0)) \right| \\
 & \quad + \frac{1}{12} \left| \sum_{l,k} \beta_m^{lk} (f^{A_l}(x) - f^{A_l}(x_0)) \sum_{i < j} \alpha_k^{ij} (f^{A_i}(x_0) f^{A_j}(x) - f^{A_j}(x) f^{A_i}(x_0)) \right| \\
 & \leq o(|x - x_0|) + \frac{1}{12} \sum_{l,k} \sum_{i < j} |\beta_m^{lk} \alpha_k^{ij} f^{A_j}(x_0)| \cdot |(f^{A_l}(x) - f^{A_l}(x_0))(f^{A_i}(x) - f^{A_i}(x_0))| \\
 & \quad + \frac{1}{12} \sum_{l,k} \sum_{i < j} |\beta_m^{lk} \alpha_k^{ij} f^{A_i}(x_0)| \cdot |(f^{A_l}(x) - f^{A_l}(x_0))(f^{A_j}(x) - f^{A_j}(x_0))| \\
 & = o(|x - x_0|),
 \end{aligned}$$

where for the last equality we noted

$$f^{A_i}(x) - f^{A_i}(x_0) = df_{x_0}^{A_i}(x - x_0) + o(|x - x_0|) = O(|x - x_0|)$$

for all  $i$ . This proves that each component  $f^{C_m}$  is differentiable at  $x_0$  with  $df_{x_0}^{C_m}$  of the desired form. The lemma follows.  $\square$

Theorem 6 follows for step three Carnot groups from initial remarks.

## 5. Future work

We would like to generalize Theorems 5 and 6 to all Carnot groups. By the work in this paper, it would suffice to prove a result of the form:

*Let  $(\mathbb{R}^n, \cdot)$  be a Carnot group and  $\Omega \subseteq \mathbb{R}^k$  an open subset. Suppose  $f : \Omega \rightarrow (\mathbb{R}^n, \cdot)$  is of class  $C^{0, \frac{1}{2}+}(\Omega, (\mathbb{R}^n, \cdot))$ . If each of the components of  $f$  are differentiable at a point  $x_0 \in \Omega$ , then the image of  $df_{x_0}$  lies in  $H_{f(x_0)}(\mathbb{R}^n, \cdot)$ .*

Lemmas 24 and 26 were first proved for model filiform groups and then proved in general by repeating calculations with additional structural constants. If one attempts this strategy for higher step Carnot groups, one could run into issues. For example, there may be a nontrivial bracket relation of elements in the second layer of a general stratification, while such a relation for a model filiform group must be trivial. In addition, proving weak contactness (Lemmas 24 and 26) became much more computationally difficult as one moved from the step two case to the step three case; one would expect this increasing difficulty to continue. Thus generalizing these two lemmas may require a deeper understanding of the polynomials arising from the Baker-Campbell-Hausdorff formula.



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