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Bilinear integral operators with certain hypersingularities ☆

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In this paper, we establish the sharp L^p boundedness for the bilinear integral operators with certain hypersingularities that generalize the bilinear Hilbert transform.

1. Introduction and statement of results

Let $a \neq 0, 1$. For $0 < \gamma < 1$ and Schwartz functions $f, g \in \mathcal{S}(\mathbb{R})$ we consider the hypersingular bilinear integral operators H^γ given by

$$H^\gamma(f,g)(x) := \lim_{\epsilon \to 0} \int\limits_{|y|>\epsilon} \frac{f(x-y)g(x-ay)}{y|y|^\gamma}\, dy. \tag{1.1}$$

In case $\gamma = 0$, H^0 is the bilinear Hilbert transform, and Lacey and Thiele [11,12] proved that

$$\|H^0(f,g)\|_{L^r} \le C\,\|f\|_{L^p}\|g\|_{L^q} \tag{1.2}$$

for all $1 < p, q \le \infty$, $1/r = 1/p + 1/q$ and $2/3 < r < \infty$. Also it is conjectured that this result is true for $r > 1/2$. If $a = 1$, then by taking $f = g$ we see that the inequality (1.2) holds only for $r > 1$. For $0 < \gamma < 1$, one can easily check that

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$$H^\gamma(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} m^\gamma(\xi + a\eta)\,\widehat{f}(\xi)\,\widehat{g}(\eta)\,\mathrm{e}^{2\pi \mathrm{i}x(\xi+\eta)}\,d\xi d\eta \tag{1.3}$$

where $\widehat{f}$ denotes the Fourier transform of f and $m^\gamma(t) = sgn(t)\,|t|^\gamma \int_{\mathbb{R}} \frac{\mathrm{e}^{-2\pi \mathrm{i}y}-1}{y|y|^\gamma}\,dy$ (see Appendix A.1 for the proof). By (1.3), H^γ is reduced to the study of the bilinear operator T_m given by

$$T_m(f,g)(x) := \int_{\mathbb{R}}\int_{\mathbb{R}} m(\xi,\eta)\,\widehat{f}(\xi)\,\widehat{g}(\eta)\,\mathrm{e}^{2\pi \mathrm{i}x(\xi+\eta)}\,d\xi d\eta,$$

for some appropriate m. The purpose of this paper is to obtain some optimal estimates for the bilinear operator T_m under certain assumptions on m. We now give some historical remarks on this type of bilinear operators T_m according to the singularity conditions of m.

The classical Coifman–Meyer theorem: If m satisfies the following Marcinkiewicz–Mikhlin–Hörmander condition

$$|\partial^\beta m(\xi,\eta)| \le C_\beta(|\xi| + |\eta|)^{-|\beta|}$$

for sufficiently many multi-indices β, then by the classical multiplier result of Coifman–Meyer [4] T_m maps $L^p \times L^q \to L^r$ as long as $1 < p, q \le \infty$, $1/p + 1/q = 1/r$ and $0 < r < \infty$. See also [3,8,10,13] for the classical Coifman–Meyer theorem on multilinear singular integrals.

Bilinear pseudodifferential operators: If m is replaced by the classes of symbols in bilinear pseudodifferential operators, there have been some interesting results on the boundedness on Sobolev spaces. In [1] the boundedness on Sobolev spaces of any multiplier bounded on Lebesgue spaces is established, though no Leibniz-type estimates are proved in such general situation. After then V. Naibo [19] obtained also the boundedness on Besov spaces of L^p bounded multipliers, this time with Leibniz-type estimates.

When the singularity of the symbol m is not at the origin but on a line, classical Littlewood–Paley theory, and Calderón–Zygmund techniques do not suffice to study the operators T_m. The breakthrough in this direction is due to Lacey and Thiele [11,12] on the bilinear Hilbert transform, for which the symbol is $m(\xi,\eta) = \mathrm{sgn}(\xi - \eta)$. Later the results for the bilinear Hilbert transform were generalized for bilinear operators with nonsmooth symbols having singularities on a line.

Bilinear operators with nonsmooth symbols: More generally, let Γ be a closed one-sided cone with vertex at the origin and $m(\xi,\eta)$ a function having derivatives of all orders inside Γ such that

$$|\partial^\beta(m(\xi,\eta))| \le C(\beta)\;\mathrm{dist}\big((\xi,\eta),\Gamma\big)^{-|\beta|}$$

for every $(\xi,\eta) \in \mathbb{R}^2 \setminus \Gamma$ and sufficiently many multi-indices β. Then the bilinear operator $T_\Gamma(f,g)$ that is defined by

$$T_\Gamma(f,g)(x) := \int_{\mathbb{R}^2} m(\xi,\eta)\,\widehat{f}(\xi)\,\widehat{g}(\eta)\,\mathrm{e}^{2\pi \mathrm{i}x(\xi+\eta)} d\xi d\eta, \tag{1.4}$$

maps $L^p \times L^q \to L^r$, provided that $1 < p, q < \infty$, $1/r = 1/p + 1/q$, and for $r > 2/3$, so long as no edge of Γ lies on the diagonal $\xi + \eta = 0$ or on a coordinate axis. See [6].

The Kato–Ponce inequality (the Leibniz rules): For Schwartz function h defined on the real line, the fractional derivative $D^\gamma h$ is defined for every $\gamma > 0$ by $\widehat{D^\gamma h}(\xi) = (2\pi|\xi|)^\gamma\,\widehat{h}(\xi)$. Then

$$D^\gamma(fg)(x) = \int_{\mathbb{R}^2} (2\pi)^\gamma |\xi+\eta|^\gamma\, \widehat{f}(\xi)\,\widehat{g}(\eta)\, \mathrm{e}^{2\pi i x(\xi+\eta)}\, d\xi d\eta.$$

The following Kato–Ponce type inequality is well-known (see [7,9,14–17]):

$$\|D^\gamma(fg)\|_{L^r} \le C(\gamma)\left(\|D^\gamma f\|_{L^{p_1}}\|g\|_{L^{q_1}} + \|f\|_{L^{p_2}}\|D^\gamma g\|_{L^{q_2}}\right) \tag{1.5}$$

for every $\gamma > 0$, for every $1 < p_i, q_i < \infty$ satisfying $1/r = 1/p_i + 1/q_i$ for $i = 1, 2$, and for $r > 1/(1+\gamma)$. This result is sharp.

Hyper bilinear Hilbert transform: For the hyper bilinear Hilbert transform H^γ $(0 < \gamma < 1)$ in (1.3)

$$H^\gamma(f,g)(x) = d_\gamma \int_{\mathbb{R}}\int_{\mathbb{R}} sgn(\xi + a\eta)\, |\xi + a\eta|^\gamma\, \widehat{f}(\xi)\,\widehat{g}(\eta)\, \mathrm{e}^{2\pi i x(\xi+\eta)}\, d\xi d\eta,$$

where $d_\gamma = \int_{\mathbb{R}} \frac{\mathrm{e}^{-2\pi i y}-1}{y|y|^\gamma}\, dy$, in [2] it was proved that

$$\|H^\gamma(f,g)\|_{L^r} \le C\left(\|D^\gamma f\|_{L^p}\|g\|_{L^q} + \|f\|_{L^p}\|D^\gamma g\|_{L^q}\right) \tag{1.6}$$

for every $1 < p, q < \infty$, $1/r = 1/p + 1/q$ with $r > 2/3$ which is based on the arguments for bilinear operators with nonsmooth symbols in [6]. The range $r > 2/3$ is not optimal and in this paper we extend this result up to sharp range $r > 1/2$.

Our model example: Consider the bilinear operators

$$T_\Gamma^\gamma(f,g)(x) = \int_{\mathbb{R}^2} |\xi + a\eta|^\gamma\, \widehat{f}(\xi)\,\widehat{g}(\eta)\, \mathrm{e}^{2\pi i x(\xi+\eta)}\, d\xi d\eta, \tag{1.7}$$

where $\gamma > 0$ and $a \ne 0, 1$. Theses bilinear operators intervene between the classical Kato–Ponce inequality and the bilinear Hilbert transform. For these bilinear operators one can combine arguments for Coifman–Meyer theorem and bilinear operators with nonsmooth symbols in [6] to obtain that for $\gamma > 0$ if $a \ne 0, 1$, then

$$\|T_\Gamma^\gamma(f,g)\|_{L^r} \le C(\gamma, a)\left(\|D^\gamma f\|_{L^{p_1}}\|g\|_{L^{q_1}} + \|f\|_{L^{p_2}}\|D^\gamma g\|_{L^{q_2}}\right) \tag{1.8}$$

provided that $1 < p_i, q_i \le \infty$, $1/p_i + 1/q_i = 1/r$, $1 \le i \le 2$, and $2/3 < r < \infty$ (see Appendix A.2 for the proof). However, in this paper we extend the result to the optimal range of r in a more general context, which is stated in Theorem 1. The model example above is a special case of operators in Theorem 1 by which one can immediately obtain that (1.8) holds for all $1 < p_i, q_i < \infty$, $1/p_i + 1/q_i = 1/r$, $i = 1, 2$, and $1/2 < r < \infty$.

We are now ready to state our main result. We let $\mathbf{m} \in C^{\mathcal{N}}(\mathbb{R} \setminus \{0\})$ with an integer $\mathcal{N}$ which will be more specified in the statement of the theorem below. We assume that for a real number $\gamma > 0$, $\mathbf{m}$ satisfies the condition

$$|\partial_t^k \mathbf{m}(t)| \le D_k\, |t|^{\gamma-k} \quad \text{for all } t \ne 0 \text{ and for all } k \le \mathcal{N}. \tag{1.9}$$

We now define the bilinear operator T by

$$T(f,g)(x) = \int_{\mathbb{R}^2} \mathbf{m}(\xi + a\eta)\, \widehat{f}(\xi)\, \widehat{g}(\eta)\, \mathrm{e}^{2\pi \mathrm{i} x(\xi+\eta)}\, d\xi d\eta. \tag{1.10}$$

It would be interesting to observe that the bilinear operators T in (1.10) convey singularities along two distinct hyperplane $\xi + \eta = 0$ and $\xi + a\eta = 0$ which are from $\mathrm{e}^{2\pi \mathrm{i} x(\xi+\eta)}$ and $\mathbf{m}(\xi + a\eta)$, respectively. When $a = 1$, the operator becomes the Leibnitz rule which gives the worst scenario for operators in this direction because two hyperplanes are overlapped to maximize singularities along just one hyperplane $\xi + \eta = 0$ in this case, which gives the natural restriction on $r > \frac{1}{1+\gamma}$.

In our case $a \neq 1$, however, the singularities along $\xi + \eta = 0$ and $\xi + a\eta = 0$ are concentrated only on the origin, which allows us to make use of arguments of suitable change of variables in kernel side and to extend the result to the full range of $r > 1/2$. It is also worthy of noticing that when $r > \frac{1}{1+\gamma}$, it is likely that one can adapt the arguments for Coifman–Meyer theorem on the classical paraproducts with a small modification. However the complementary case leads us to totally different situation, which has forced us to develop new idea to prove Theorem 1. This is the reason why we separately consider two cases: $r > 1$ and $r < 1$ during the proof. For the case $r > 1$ we use the standard arguments such as the shifted square function and the shifted Hardy–Littlewood maximal function estimates (Theorem 4.6 in [17]). For the case $r < 1$, in discretization we do not use the usual uncertainty principle, instead we decompose the space variable x according to the size of y variable in kernel side, and then use averages in x, y variables and associated maximal operators together with Lemma 4.2 in Section 4.

Theorem 1. *Let $\gamma > 0$, and $\mathcal{N}$ be the smallest integer so that $\mathcal{N} > \max(2, 1+\gamma)$. We assume that $\mathbf{m} \in C^{\mathcal{N}}(\mathbb{R} \setminus \{0\})$ and satisfies the derivative conditions (1.9). Let $a \neq 0, 1$, then the bilinear operator T in (1.10) satisfies*

$$\|T(f,g)\|_{L^r} \le C(\gamma, a) \Big(\|D^\gamma f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|D^\gamma g\|_{L^{q_2}} \Big)$$

for every $1 < p_i, q_i < \infty$, $1/r = 1/p_i + 1/q_i$, $1 \le i \le 2$, and for $r > 1/2$.

Remark.

(1) The power $\gamma > 0$ in $|\xi + a\eta|^\gamma$ gives some smoothing effect near the singularity line $\xi + a\eta = 0$, and this is why we could obtain the results up to full range $r > 1/2$. By applying the arguments for bilinear operators with nonsmooth symbols in [6] we can only prove the results up to $r > 2/3$.

(2) If D_k, $k = 0, 1, \cdots, \mathcal{N}$, are as in (1.9), then for any positive integer $s_0 > 1/\gamma$ we have

$$C(\gamma, a) := C\,(1 + |\gamma|)^{10}\,\big(1 - 2^{-\gamma + \frac{1}{s_0}}\big)^{-1} \Big(\frac{(1+|a|)^2 (1 + |a-1|)^2}{|a||a-1|}\, \big(|a| + |a|^{-1}\big)^{3+\gamma+\frac{1}{s_0}} \Big),$$

where the constant C depends only on $r, p_1, q_1, p_2, q_2, s_0$, and $D_0, \cdots, D_{\mathcal{N}}, \mathcal{N}$ in (1.9). Note that $C(\gamma, a) \to \infty$ as $\gamma \to 0$ or $a \to 1$.

Corollary 1.1. *For $0 < \gamma < 1$, let H^γ be as in (1.1) then*

$$\|H^\gamma(f,g)\|_{L^r} \le C \Big(\|D^\gamma f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|D^\gamma g\|_{L^{q_2}} \Big)$$

for every $1 < p_i, q_i < \infty$, $1/r = 1/p_i + 1/q_i$, $1 \le i \le 2$, and for $r > 1/2$.

For each positive integer ℓ and Schwartz function $f \in \mathcal{S}(\mathbb{R})$ define

$$\triangle_h^\ell f(x) := (I - \tau_h)^\ell f(x) = \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} f(x - kh),$$

where $\tau_h f(x) = f(x - h)$ and I denotes identity operator. Then for $0 < \gamma < \ell$, the Marchaud fractional derivative is given by

$$\mathbb{D}^\gamma f(x) := \int \frac{\triangle_y^\ell f(x)}{|y|^{1+\gamma}}\, dy = d_\ell(\gamma) \int |\xi|^\gamma \widehat{f}(\xi) \mathrm{e}^{2\pi \mathrm{i} x\xi}\, d\xi,$$

where

$$d_\ell(\gamma) = \int (1 - \mathrm{e}^{-2\pi \mathrm{i} y})^\ell \frac{dy}{|y|^{1+\gamma}}. \tag{1.11}$$

We refer to [20,21] for details about this derivative and its applications. For Schwartz functions $f, g \in \mathcal{S}(\mathbb{R})$ we define

$$\widetilde{\triangle}_h^\ell (f, g)(x) := \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} f(x - kh) g(x + kh).$$

Then for $0 < \gamma < \ell$ we consider the hypersingular bilinear integral operator

$$\widetilde{\mathbb{D}}^\gamma (f, g)(x) := \lim_{\epsilon \to 0} \int\limits_{|y|>\epsilon} \frac{\widetilde{\triangle}_y^\ell (f, g)(x)}{|y|^{1+\gamma}}\, dy. \tag{1.12}$$

Lemma 1.1. *For $0 < \gamma < \ell$ we have*

$$\widetilde{\mathbb{D}}^\gamma (f, g)(x) = d_\ell(\gamma) \iint |\xi - \eta|^\gamma\, \widehat{f}(\xi)\, \widehat{g}(\eta)\, \mathrm{e}^{2\pi \mathrm{i} x(\xi+\eta)}\, d\xi d\eta$$

where $d_\ell(\gamma)$ is as in (1.11).

Proof of Lemma 1.1. For $0 < \gamma < \ell$, we have

$$\begin{aligned}
\widetilde{\mathbb{D}}^\gamma (f, g)(x) &= \lim_{\epsilon \to 0} \int\limits_{|y|>\epsilon} \frac{\widetilde{\triangle}_y^\ell (f, g)(x)}{|y|^{1+\gamma}}\, dy \\
&= \lim_{\epsilon \to 0} \left(\sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \int\limits_{|y|>\epsilon} f(x - ky) g(x + ky) \frac{dy}{|y|^{1+\gamma}} \right) \\
&= \lim_{\epsilon \to 0} \left(\iint \left(\int\limits_{|y|>\epsilon} \frac{\sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \mathrm{e}^{-2\pi \mathrm{i} k y(\xi - \eta)}}{|y|^{1+\gamma}} dy \right) \widehat{f}(\xi)\, \widehat{g}(\eta)\, \mathrm{e}^{2\pi \mathrm{i} x(\xi+\eta)}\, d\xi d\eta \right) \\
&= \iint \lim_{\epsilon \to 0} \left(\int\limits_{|y|>\epsilon} \frac{\left(1 - \mathrm{e}^{-2\pi \mathrm{i} y(\xi - \eta)}\right)^\ell}{|y|^{1+\gamma}} dy \right) \widehat{f}(\xi)\, \widehat{g}(\eta)\, \mathrm{e}^{2\pi \mathrm{i} x(\xi+\eta)}\, d\xi d\eta,
\end{aligned}$$

the result follows by change of variable $y \to y(\xi - \eta)$ for $\xi - \eta \neq 0$. $\square$

Corollary 1.2. *For $0 < \gamma < \ell$, let $\widetilde{\mathbb{D}}^\gamma$ be as in (1.12), then*

$$\|\widetilde{\mathbb{D}}^\gamma(f,g)\|_{L^r} \le C\left(\|D^\gamma f\|_{L^{p_1}}\|g\|_{L^{q_1}} + \|f\|_{L^{p_2}}\|D^\gamma g\|_{L^{q_2}}\right)$$

for every $1 < p_i, q_i < \infty$, $1/r = 1/p_i + 1/q_i$, $1 \le i \le 2$, and for $r > 1/2$.

2. Preliminaries

Notation. Throughout this paper, we denote by

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) := \int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi \mathrm{i} x\xi}\, dx$$

the Fourier transform of a Schwartz function f. We also denote by $\mathcal{F}^{-1}(f)(\xi) := \mathcal{F}(f)(-\xi)$ the inverse Fourier transform of f. M denotes the Hardy–Littlewood maximal function, and $\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx$. For two quantities A and B, we shall write $A \lesssim B$ if $A \le CB$ for some positive constant C, depending on the dimension and possibly other parameters apparent from the context. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. For a measurable set E, $|E|$ denotes the measure of E. For each interval I in $\mathbb{R}$ and $b > 0$, bI denotes the interval having the same center as I with $|bI| = b\,|I|$.

Lemma 2.1 *(cf. [17,18]). Let $0 < r \le 1$ and $A > 0$. Then the following are equivalent:*

(1) $\|f\|_{L^{r,\infty}} \le A$;
(2) for every set E with $0 < |E| < \infty$, there exists a subset $E' \subseteq E$ with $|E'| \simeq |E|$ and $|\langle f, \chi_{E'}\rangle| \lesssim A|E|^{1/r'}$, where $1/r + 1/r' = 1$.

Lemma 2.2 *(cf. Theorem 2.7 in [17]). Let $\mathcal{J}$ be a finite family of dyadic intervals. For any $0 < r < \infty$ and any complex sequence $(a_I)_{I\in\mathcal{J}}$ one defines*

$$\left\|(a_I)_{I\in\mathcal{J}}\right\|_{BMO(r)} := \sup_{I_0\in\mathcal{J}} \frac{1}{|I_0|^{1/r}} \left\|\left(\sum_{I\subseteq I_0,\, I\in\mathcal{J}} |a_I|^2\chi_I(x)\right)^{1/2}\right\|_{L^r(\mathbb{R})},$$

where χ denotes the characteristic function. Then for any $0 < p < q < \infty$, one has

$$\left\|(a_I)_{I\in\mathcal{J}}\right\|_{BMO(p)} \le C(p,q)\left\|(a_I)_{I\in\mathcal{J}}\right\|_{BMO(q)} \le C'(p,q)\left\|(a_I)_{I\in\mathcal{J}}\right\|_{BMO(p)}, \tag{2.1}$$

for some positive constants $C'(p,q)$ and $C(p,q)$ only depending on p and q. We write (2.1) as

$$\left\|(a_I)_{I\in\mathcal{J}}\right\|_{BMO(p)} \simeq \left\|(a_I)_{I\in\mathcal{J}}\right\|_{BMO(q)}.$$

Let $\mathcal{J}$ be a family of dyadic intervals. For any $0 < p < \infty$, and any complex sequence $(a_I)_{I\in\mathcal{J}}$ define

$$Size_{\mathcal{J},\,p}\big((a_I)_{I\in\mathcal{J}}\big) := \left\|(a_I)_{I\in\mathcal{J}}\right\|_{BMO(p)}. \tag{2.2}$$

Then by Lemma 2.2, for any $0 < p_0 < \infty$

$$Size_{\mathcal{J},\,p}\big((a_I)_{I\in\mathcal{J}}\big) \simeq Size_{\mathcal{J},\,p_0}\big((a_I)_{I\in\mathcal{J}}\big). \tag{2.3}$$

Now for $0 < p < \infty$ we define

$$Energy_{\mathcal{J},\,p}\big((a_I)_{I\in\mathcal{J}}\big) := \sup_{\nu\in\mathbb{Z}} 2^\nu \sup_{\mathbb{D}_\nu} \Big(\sum_{I_0\in\mathbb{D}_\nu} |I_0|\Big)^{1/p}, \tag{2.4}$$

where $\mathbb{D}_\nu$ ranges over all collections of disjoint dyadic intervals $I_0 \in \mathcal{J}$ having the property that

$$\frac{1}{|I_0|^{1/p}} \Big\| \Big(\sum_{I\subseteq I_0,\, I\in\mathcal{J}} |a_I|^2 \chi_I \Big)^{1/2} \Big\|_{L^p} \geq 2^\nu.$$

Lemma 2.3 *(cf. Corollary 2.11 in [17]). Let $\mathcal{J}$ be a family of dyadic intervals. Let $Size_{\mathcal{J},\,p}\big((a_I)_{I\in\mathcal{J}}\big)$ and $Energy_{\mathcal{J},\,p}\big((a_I)_{I\in\mathcal{J}}\big)$ be as in (2.2) and (2.4). Then there exists a partition*

$$\mathcal{J} = \bigcup_{\nu\in\mathbb{Z}} \mathcal{J}^\nu$$

such that for any $\nu \in \mathbb{Z}$ one has

$$2^{-\nu-1} Energy_{\mathcal{J},\,p}\big((a_I)_{I\in\mathcal{J}}\big) \leq Size_{\mathcal{J}^\nu,\,p}\big((a_I)_{I\in\mathcal{J}_\nu}\big) \leq \min\big(2^{-\nu} Energy_{\mathcal{J},\,p}\big((a_I)_{I\in\mathcal{J}}\big),\, Size_{\mathcal{J},\,p}\big((a_I)_{I\in\mathcal{J}}\big)\big).$$

Also one can write each $\mathcal{J}^\nu$ as a disjoint union of subsets $T \in \mathbb{T}^\nu$ having the properties such that for every $T \in \mathbb{T}^\nu$ there exists a dyadic interval I_T in T having the properties that every $I \in T$ satisfies $I \subseteq I_T$ and also

$$\sum_{T\in\mathbb{T}^\nu} |I_T| \lesssim 2^{\nu p}.$$

On the other hand, let $\mathcal{J}$ be a family of dyadic intervals. For any $0 < s < \infty$, and any complex sequence $(c_I)_{I\in\mathcal{J}}$ we define

$$S\text{-}Size_{\mathcal{J},\,s}\big((c_I)_{I\in\mathcal{J}}\big) := \sup_{I_0\in\mathcal{J}} \frac{1}{|I_0|^{1/s}} \Big\| \sup_{I\subseteq I_0,\, I\in\mathcal{J}} \big(|c_I|\, \chi_I(x)\big) \Big\|_{L^s}. \tag{2.5}$$

And if we define

$$S\text{-}Energy_{\mathcal{J},\,s}\big((c_I)_{I\in\mathcal{J}}\big) := \sup_{\nu\in\mathbb{Z}} 2^\nu \sup_{\mathbb{D}_\nu} \Big(\sum_{I_0\in\mathbb{D}_\nu} |I_0|\Big)^{1/s}, \tag{2.6}$$

where $\mathbb{D}_\nu$ ranges over all collections of disjoint dyadic intervals $I_0 \in \mathcal{J}$ having the property that

$$\frac{1}{|I_0|^{1/s}} \Big\| \sup_{I\subseteq I_0,\, I\in\mathcal{J}} \big(|c_I|\, \chi_I\big) \Big\|_{L^s} \geq 2^\nu.$$

Lemma 2.4 *(cf. Lemma 2.10 in [17]). Let $\mathcal{J}$ be a family of dyadic intervals and let $\mathcal{J}' \subseteq \mathcal{J}$ such that*

$$S\text{-}Size_{\mathcal{J}',\,s}\big((c_I)_{I\in\mathcal{J}'}\big) \leq 2^{-n_0} S\text{-}Energy_{\mathcal{J},\,s}\big((c_I)_{I\in\mathcal{J}}\big)$$

for a certain fixed integer n_0. Then there is a decomposition $\mathcal{J}' = \mathcal{J}'' \bigcup \mathcal{J}'''$ such that

$$S\text{-}Size_{\mathcal{J}'',\,s}\big((c_I)_{I\in\mathcal{J}}\big) \leq 2^{-n_0-1} S\text{-}Energy_{\mathcal{J},\,s}\big((c_I)_{I\in\mathcal{J}}\big) \tag{2.7}$$

and such that $\mathcal{J}'''$ can be written as a disjoint union of subsets $T \in \mathbb{T}$ such that for every $T \in \mathbb{T}$ there exists a dyadic interval I_T in T having the properties that every $I \in T$ satisfies $I \subseteq I_T$ and so

$$\sum_{T\in\mathbb{T}} |I_T| \lesssim 2^{n_0 s}. \tag{2.8}$$

Proof of Lemma 2.4. Choose an interval $I_0 \in \mathcal{J}'$ such that $|I_0|$ is as large as possible and such that

$$\frac{1}{|I_0|^{1/s}} \left\| \sup_{I\subseteq I_0,\, I\in\mathcal{J}'} \big(|c_I|\, \chi_I(x)\big) \right\|_{L^s} > 2^{-n_0-1} S\text{-}Energy_{\mathcal{J},\, s}\big((c_I)_{I\in\mathcal{J}}\big). \tag{2.9}$$

Now collect all intervals $I \in \mathcal{J}'$ with $I \subseteq I_0$ in a set T. Then define $I_T := I_0$, look at the remaining intervals in $\mathcal{J}' \setminus T$, and repeat the procedure. Since the cardinality of $\mathcal{J}$ is finite, this algorithm ends after finitely many steps, producing the subsets $T \in \mathbb{T}$. Since $\mathcal{J}$ is a collection of dyadic intervals, all the intervals $(I_T)_{T\in\mathbb{T}}$ are disjoint by construction. Define

$$\mathcal{J}''' := \bigcup_{T\in\mathbb{T}} T, \quad \mathcal{J}'' := \mathcal{J}' \setminus \mathcal{J}'''.$$

By construction (2.7) is automatically satisfied, and it remains to check (2.8). Let ν be an integer such that

$$2^{\nu} \le 2^{-n_0-1}\, S\text{-}Energy_{\mathcal{J},\, s}\big((c_I)_{I\in\mathcal{J}}\big) < 2^{\nu+1}. \tag{2.10}$$

Since all the intervals $(I_T)_{T\in\mathbb{T}}$ are disjoint and

$$\frac{1}{|I_T|^{1/s}} \left\| \sup_{I\subseteq I_T,\, I\in\mathcal{J}'} \Big(|c_I|\, \chi_I\Big) \right\|_{L^s} > 2^{-n_0-1} S\text{-}Energy_{\mathcal{J},\, s}\big((c_I)_{I\in\mathcal{J}}\big) \ge 2^{\nu},$$

by definition (2.6) of $S\text{-}Energy_{\mathcal{J}',\, r}\big((c_I)_{I\in\mathcal{J}'}\big)$ and (2.10) we have

$$S\text{-}Energy_{\mathcal{J}',\, s}\big((c_I)_{I\in\mathcal{J}'}\big) \ge 2^{\nu}\Big(\sum_{T\in\mathbb{T}} |I_T|\Big)^{1/s} > \Big(2^{-n_0-2} S\text{-}Energy_{\mathcal{J},\, s}\Big((c_I)_{I\in\mathcal{J}}\Big)\Big)\Big(\sum_{T\in\mathbb{T}} |I_T|\Big)^{1/s}.$$

Thus from $S\text{-}Energy_{\mathcal{J},\, s}\big((c_I)_{I\in\mathcal{J}}\big) \ge S\text{-}Energy_{\mathcal{J}',\, s}\big((c_I)_{I\in\mathcal{J}'}\big)$ we obtain

$$\Big(\sum_{T\in\mathbb{T}} |I_T|\Big)^{1/s} < 2^{n_0+2}. \quad \square$$

Now, if one iterates the above Lemma 2.4, one obtains the following.

Lemma 2.5 *(cf. Corollary 2.11 in [17]). Let $S\text{-}Size_{\mathcal{J},\, s}\big((c_I)_{I\in\mathcal{J}}\big)$ and $S\text{-}Energy_{\mathcal{J},\, s}\big((c_I)_{I\in\mathcal{J}}\big)$ be as in (2.5) and (2.6). Let $\mathcal{J}$ be a family of dyadic intervals. Then there exists a partition*

$$\mathcal{J} = \bigcup_{\nu\in\mathbb{Z}} \mathcal{J}^{\nu}$$

such that for any $\nu \in \mathbb{Z}$ one has

$$2^{-\nu-1} S\text{-}Energy_{\mathcal{J},\, s} \le S\text{-}Size_{\mathcal{J}^{\nu},\, s} \le \min(2^{-\nu} S\text{-}Energy_{\mathcal{J},\, s},\, S\text{-}Size_{\mathcal{J},\, s}).$$

Also one can write each $\mathcal{J}^\nu$ as a disjoint union of subsets having the properties such that for every $T \in \mathbb{T}^\nu$ there exists a dyadic interval I_T in T having the properties that every $I \in T$ satisfies $I \subseteq I_T$ and also

$$\sum_{T \in \mathbb{T}^\nu} |I_T| \lesssim 2^{\nu s}.$$

Lemma 2.6 *(Fefferman–Stein [5]). Let* M *denote the Hardy–Littlewood maximal operator. Then for $1 < p < \infty$*

$$\left\| \left(\sum_{k \in \mathbb{Z}} |\mathrm{M}(f_k)|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left(\sum_{k \in \mathbb{Z}} |f_k|^2 \right)^{1/2} \right\|_{L^p}.$$

3. Discretization

Let $\varphi \in \mathcal{S}(\mathbb{R})$ be a Schwartz function such that

$$\mathrm{supp}(\widehat{\varphi}) \subseteq [-2, 2] \quad \text{and} \quad \widehat{\varphi}(\xi) = 1 \text{ on } [-1, 1].$$

Then define $\psi \in \mathcal{S}(\mathbb{R})$ so that $\widehat{\psi}(\xi) := \widehat{\varphi}(\xi) - \widehat{\varphi}(2\xi)$. Note that $\mathrm{supp}(\widehat{\psi}) \subseteq \{\xi : 1/2 \le |\xi| \le 2\}$. For each $k \in \mathbb{Z}$, define $\widehat{\psi}_k(\xi) := \widehat{\psi}(2^{-k}\xi)$, then $\mathrm{supp}(\widehat{\psi}_k) \subseteq \{\xi : 2^{k-1} \le |\xi| \le 2^{k+1}\}$ and

$$\sum_{k \in \mathbb{Z}} \widehat{\psi}_k(\xi) = 1 \quad \text{if } \xi \neq 0.$$

Let $|a| \sim 2^i$ for some $i \in \mathbb{Z}$ and $a \neq 0, 1$. Since the bilinear operator T have singularities along the two lines $\xi + a\eta = 0$ and $\xi + \eta = 0$, we consider (ξ, η) in the following three cases:

(1) $|\xi| > 2^{|i|+5}|\eta|$, in this case $|\xi + a\eta| \sim |\xi|$ and $|\xi + \eta| \sim |\xi|$.
(2) $|\xi| < 2^{-|i|-5}|\eta|$, in this case $|\xi + a\eta| \sim 2^i|\eta|$ and $|\xi + \eta| \sim |\eta|$.
(3) $2^{-|i|-5}|\eta| \le |\xi| \le 2^{|i|+5}|\eta|$, in this case $|\xi + a\eta| \le 2^{|i|+10}|\eta|$ and $|\xi + \eta| \le 2^{|i|+10}|\eta|$.

Let $T(f, g)$ be as in (1.10), then by using $\sum_{k_1, k_2 \in \mathbb{Z}} \widehat{\psi}_{k_1}(\xi)\widehat{\psi}_{k_2}(\eta) = 1$ for all $\xi \neq 0$ and $\eta \neq 0$, one obtains

$$T(f, g)(x) = \mathrm{I}(f, g)(x) + \mathrm{II}(f, g)(x) + \mathrm{III}(f, g)(x),$$

where

$$\begin{aligned} \mathrm{I}(f, g)(x) &:= \sum_{k_1 - k_2 \ge |i|+5} \int_{\mathbb{R}^2} \mathbf{m}(\xi + a\eta) \left(\widehat{f * \psi_{k_1}}(\xi)\right) \left(\widehat{g * \psi_{k_2}}(\eta)\right) \mathrm{e}^{2\pi \mathrm{i} x(\xi+\eta)}\, d\xi d\eta, \\ \mathrm{II}(f, g)(x) &:= \sum_{k_1 - k_2 \le -|i|-5} \int_{\mathbb{R}^2} \mathbf{m}(\xi + a\eta) \left(\widehat{f * \psi_{k_1}}(\xi)\right) \left(\widehat{g * \psi_{k_2}}(\eta)\right) \mathrm{e}^{2\pi \mathrm{i} x(\xi+\eta)}\, d\xi d\eta, \\ \mathrm{III}(f, g)(x) &:= \sum_{|k_1 - k_2| < |i|+5} \int_{\mathbb{R}^2} \mathbf{m}(\xi + a\eta) \left(\widehat{f * \psi_{k_1}}(\xi)\right) \left(\widehat{g * \psi_{k_2}}(\eta)\right) \mathrm{e}^{2\pi \mathrm{i} x(\xi+\eta)}\, d\xi d\eta. \end{aligned} \tag{3.1}$$

The estimates for the first two terms I and II in (3.1) are very similar. III is the main term. Note that

$$\mathrm{III}(f, g)(x) = \sum_{j=-|i|-4}^{|i|+4} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^2} \mathbf{m}(\xi + a\eta) \left(\widehat{f * \psi_{k+j}}(\xi)\right) \left(\widehat{g * \psi_k}(\eta)\right) d\xi d\eta. \tag{3.2}$$

Let Ψ and Φ be Schwartz functions such that

$$\begin{aligned} &\operatorname{supp}(\widehat{\Psi}) \subseteq \{\xi : 2^{-10} \leq |\xi| \leq 2^{10}\}, \quad \widehat{\Psi}(\xi) = 1 \quad \text{if} \quad 2^{-9} \leq |\xi| \leq 2^{9}, \\ &\operatorname{supp}(\widehat{\Phi}) \subseteq \{\xi : |\xi| \leq 2^{10}\}, \quad \widehat{\Phi}(\xi) = 1 \quad \text{if } |\xi| < 2^{9}. \end{aligned} \tag{3.3}$$

For each $k \in \mathbb{Z}$, define $\widehat{\Psi}_k(\cdot) = \widehat{\Psi}(2^{-k}\cdot)$ and $\widehat{\Phi}_k(\cdot) = \widehat{\Phi}(2^{-k}\cdot)$. Then if $|j| \leq |i| + 4$ and (ξ, η) lies on the support of $\widehat{\psi}_{k+j}(\xi)\widehat{\psi}_k(\eta)$, then

$$\widehat{\Phi}_{k+|i|}(\xi + a\eta)\, \widehat{\Phi}_{k+|i|}(\xi + \eta) = 1.$$

Thus we can insert the function $\widehat{\Phi}_{k+|i|}(\xi + a\eta)\, \widehat{\Phi}_{k+|i|}(\xi + \eta)$ inside of the integral in (3.2), and then by using the identity

$$\int_{\mathbb{R}} \mathcal{F}\big[\mathbf{m}(\cdot)\widehat{\Phi}_{k+|i|}(\cdot)\big](y)\, \mathrm{e}^{2\pi \mathrm{i}(\xi+a\eta)y}\, dy = \mathbf{m}(\xi + a\eta)\, \widehat{\Phi}_{k+|i|}(\xi + a\eta),$$

one gets

$$\begin{aligned} \mathrm{III}(f,g)(x) = \sum_{j=-|i|-4}^{|i|+4} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} & \mathcal{F}\big[\mathbf{m}(\cdot)\widehat{\Phi}_{k+|i|}(\cdot)\big](y) \\ & \left(\int_{\mathbb{R}^2} \mathrm{e}^{2\pi \mathrm{i}[\xi(x+y)+\eta(x+ay)]} \left(\widehat{f * \psi_{k+j}}(\xi)\right) \left(\widehat{g * \psi_k}(\eta)\right) \widehat{\Phi}_{k+|i|}(\xi + \eta)\, d\xi d\eta \right) dy. \end{aligned}$$

Lemma 3.1. *Let* $\mathbf{m}$ *be satisfied with the conditions* (1.9)*, and let* $\widehat{\Psi}$ *and* $\widehat{\Phi}$ *be as in* (3.3)*, then*

(1) $|\mathcal{F}\big[\mathbf{m}(\cdot)\widehat{\Psi}_k(\cdot)\big](y)| \lesssim \frac{(2^k)^{1+\gamma}}{(1+2^k|y|)^{\mathcal{N}}}$,
(2) $|\mathcal{F}\big[\mathbf{m}(\cdot)\widehat{\Phi}_k(\cdot)\big](y)| \lesssim \frac{(2^k)^{1+\gamma}}{(1+2^k|y|)^{1+\gamma}}$.

Proof of Lemma 3.1. (1) follows by integrating by parts via $\left(\frac{d}{dt}\right)^{\mathcal{N}} \left(\mathrm{e}^{-2\pi \mathrm{i}ty}\right) = (-2\pi \mathrm{i}y)^{\mathcal{N}}\, \mathrm{e}^{-2\pi \mathrm{i}ty}$ together with the conditions (1.9). For (2), if we integrate by parts by using the conditions (1.9), then since $\mathcal{N} > 1+\gamma$ we obtain

$$\begin{aligned} \left|\mathcal{F}\big[\mathbf{m}(\cdot)\widehat{\Phi}_k(\cdot)\big](y)\right| &= \left| \sum_{l\geq -12} \int \mathbf{m}(t) \left(\widehat{\Phi}(2^{-k}t)\widehat{\psi}(2^{-k+l}t)\right) \mathrm{e}^{-2\pi \mathrm{i}ty}\, dt \right| \\ &\lesssim \sum_{l\geq -12} \frac{(2^{k-l})^{1+\gamma}}{(1 + 2^{k-l}|y|)^{\mathcal{N}}} \lesssim \frac{(2^k)^{1+\gamma}}{(1 + 2^k|y|)^{1+\gamma}}. \quad \square \end{aligned}$$

4. Estimates for the main term III

Recall that $|a| \sim 2^i$ and $a \neq 0, 1$. We claim that

$$\|\mathrm{III}(f,g)\|_{L^{r,\infty}} \leq C(s_0)\, \frac{(1 + |a-1|)^2\, 2^{|i|(3+\gamma+\frac{1}{s_0})+2|j|}}{|a-1|\left(1 - 2^{-\gamma+\frac{1}{s_0}}\right)}\, \|f\|_{L^p}\, \|D^\gamma g\|_{L^q} \tag{4.1}$$

for any positive integer $s_0 > 1/\gamma$, where $1/r = 1/p + 1/q$, $1 < p, q < \infty$. For $-|i| - 4 \leq j \leq |i| + 4$, let us define

$$\mathrm{III}^j(f,g)(x) := \sum_k 2^{-k\gamma} \int_{\mathbb{R}} \mathcal{F}\big[\mathbf{m}(\cdot)\widehat{\Phi}_{k+|i|}(\cdot)\big](y) \left(\int_{\mathbb{R}^2} \mathrm{e}^{2\pi\mathrm{i}[\xi(x+y)+\eta(x+ay)]} \left(\widehat{f*\psi_{k+j}}(\xi)\right) \left(\widehat{g*\psi_k}(\eta)\right) \widehat{\Phi}_{k+|i|}(\xi+\eta)\, d\xi d\eta \right) dy. \tag{4.2}$$

Then by using

$$\widehat{g}(\eta)\widehat{\psi}_k(\eta) = 2^{-k\gamma}\left(|\eta|^\gamma \widehat{g}(\eta)\right)\widehat{\psi}(2^{-k}\eta)\left(2^{-k}|\eta|\right)^{-\gamma} := 2^{-k\gamma}\left(|\eta|^\gamma\widehat{g}(\eta)\right)\widehat{\psi}_{k,\gamma}(\eta)$$

where $\widehat{\psi}_{k,\gamma}(\eta) := \widehat{\psi}(2^{-k}\eta)\left(2^{-k}|\eta|\right)^{-\gamma}$, (4.1) will follow from the estimate

$$\|\mathrm{III}^j(f,g)\|_{L^{r,\infty}} \le C(s_0)\frac{(1+|a-1|)^2\, 2^{|i|(2+\gamma+\frac{1}{s_0})+2|j|}}{|a-1|\left(1-2^{-\gamma+\frac{1}{s_0}}\right)}\|f\|_{L^p}\,\|g\|_{L^q} \tag{4.3}$$

for any positive integer $s_0 > 1/\gamma$, where $1/r = 1/p + 1/q$, $1 < p, q < \infty$. In proving (4.3) we may assume that $\|f\|_{L^p} \neq 0 \neq \|g\|_{L^q}$. Define an “exceptional set” Ω by

$$\Omega := \left\{x : \mathrm{M}_p(f)(x) > t\,\|f\|_{L^p}\,|E|^{-1/p}\right\} \bigcup \left\{x : \mathrm{M}_q(g)(x) > t\,\|g\|_{L^q}\,|E|^{-1/q}\right\} \tag{4.4}$$

where $\mathrm{M}_p(f)(x) := \left(\sup_{x\in I}|I|^{-1}\int_I |f(y)|^p\,dy\right)^{1/p}$. Since $\left|\{x : \mathrm{M}_p f(x) > t\}\right| \le Ct^{-p}\int|f(x)|^p\,dx$, if t is large enough then we have $|\Omega| \le C\left(t^{-p}+t^{-q}\right)|E| \le 1/100|E|$. Let $E' = E\setminus\Omega$. Let $h = \chi_{E'}$, then by Lemma 2.1 it suffices to show that

$$\langle \mathrm{III}^j(f,g),h\rangle \le C(s_0)\frac{(1+|a-1|)^2\, 2^{|i|(2+\gamma+\frac{1}{s_0})+2|j|}}{|a-1|\left(1-2^{-\gamma+\frac{1}{s_0}}\right)}\|f\|_p\,\|g\|_q\,|E|^{1-1/p-1/q}. \tag{4.5}$$

4.1. The case $1 < p, q < \infty$, $1/p + 1/q < 1$

The estimate (4.5) for this case is easy compared to the case $1/p+1/q \ge 1$. This is because if $1/p+1/q < 1$, then there exists $1 < s < \infty$ such that $1/p + 1/q + 1/s = 1$, and we can use the Minkowski's inequality

$$\|f\,g\,h\|_{L^1} \le \|f\|_{L^p}\,\|g\|_{L^q}\,\|h\|_{L^s}.$$

Then the result follows by using the standard arguments such as the shifted square function and the shifted Hardy–Littlewood maximal function estimates (Theorem 4.6 in [17]). By (4.2), we have

$$\begin{aligned}&\langle \mathrm{III}^j(f,g),h\rangle\\ &= \sum_k 2^{-k\gamma}\int_{\mathbb{R}^2}\mathcal{F}\Big[\mathbf{m}(\cdot)\widehat{\Phi}_{k+|i|}(\cdot)\Big](y)\left(f*\psi_{k+j}(x+y)\right)\left(g*\psi_k(x+ay)\right)\left(h*\Phi_{k+|i|}(x)\right)dxdy.\end{aligned}$$

For each fixed $k \in \mathbb{Z}$, we decompose y variable as:

$$\mathbb{R} = \left\{y : |y| < 2^{-k}\right\} \bigcup \left(\bigcup_{m=1}^{\infty}\left\{y : 2^{-k+m-1} \le |y| < 2^{-k+m}\right\}\right), \tag{4.6}$$

then by Lemma 3.1

$$\left|\mathcal{F}\big[\mathbf{m}(\cdot)\widehat{\Phi}_{k+|i|}(\cdot)\big](y)\right| \lesssim \begin{cases} (2^{k-m})^{1+\gamma}, & \text{if } |y| \sim 2^{-k+m}; \\ (2^{k+|i|})^{1+\gamma}, & \text{if } |y| < 2^{-k}. \end{cases} \tag{4.7}$$

Next we decompose x variable as:

$$\mathbb{R} = \bigcup_{n\in\mathbb{Z}} \{x : 2^{-k}n \le x < 2^{-k}(n+1)\}. \tag{4.8}$$

By using the decompositions (4.6), (4.8) and change of variables, one gets the identity

$$\begin{aligned} \int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)\, dxdy &= \sum_{n\in\mathbb{Z}} 2^{-2k} \int_0^1\int_{-1}^1 K\big(2^{-k}(n+\beta), 2^{-k}y\big)\, dyd\beta \\ &+ \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} 2^{-2k+m} \int_0^1 \int_{1/2\le|y|<1} K\big(2^{-k}(n+\beta), 2^{-k+m}y\big)\, dyd\beta. \end{aligned}$$

Thus together with (4.7), one gets

$$|\langle \mathrm{III}^j(f,g),h\rangle| \lesssim 2^{|i|(1+\gamma)} \sum_{m\ge0}\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} 2^{-m\gamma}2^{-k} \int_0^1\int_{-1}^1 \mathrm{F}_n^k(m,\beta,y)\, \mathrm{G}_n^k(m,\beta,y)\, \mathrm{H}_n^k(\beta)\, dy\, d\beta$$

where

$$\begin{aligned} \mathrm{F}_n^k(m,\beta,y) &:= \big|f * \psi_{k+j}\big(2^{-k}(n+\beta+2^m y)\big)\big|, \\ \mathrm{G}_n^k(m,\beta,y) &:= \big|g * \psi_{k}\big(2^{-k}(n+\beta+a2^m y)\big)\big|, \\ \mathrm{H}_n^k(\beta) &:= \big|h * \Phi_{k+|i|}\big(2^{-k}(n+\beta)\big)\big|. \end{aligned} \tag{4.9}$$

Thus if we set $I_n^k := [2^{-k}n, 2^{-k}(n+1)]$, then

$$|\langle \mathrm{III}^j(f,g),h\rangle| \lesssim 2^{|i|(1+\gamma)} \sum_{m\ge0} 2^{-m\gamma} \int_0^1\int_{-1}^1 \sum_{k,\,n} \left(\int \mathrm{F}_n^k(m,\beta,y)\, \mathrm{G}_n^k(m,\beta,y)\, \mathrm{H}_n^k(\beta)\ \chi_{I_n^k}(x)\, dx\right) dyd\beta, \tag{4.10}$$

where χ denotes the characteristic function. We note that

$$\begin{aligned} &\sum_{k,\,n} \left(\int \mathrm{F}_n^k(m,\beta,y)\, \mathrm{G}_n^k(m,\beta,y)\, \mathrm{H}_n^k(\beta)\ \chi_{I_n^k}(x)\, dx\right) \\ &\lesssim \int \Big[\sum_{k,\,n} |\mathrm{F}_n^k(m,\beta,y)|^2 \chi_{I_n^k}(x)\Big]^{\frac12} \Big[\sum_{k,n} |\mathrm{G}_n^k(m,\beta,y)|^2 \chi_{I_n^k}(x)\Big]^{\frac12} \sup_{k,\,n} \big[|\mathrm{H}_n^k(\beta)|\, \chi_{I_n^k}(x)\big]\, dx. \end{aligned}$$

Lemma 4.1. *Let N be a positive integer. For $1 < p < \infty$, we have*

$$\left\| \left(\sum_{k,\,n} \big|f * \psi_{k+j}\big(2^{-k}(n+2^N)\big)\big|^2\, \chi_{I_n^k}(x) \right)^{1/2} \right\|_{L^p} \lesssim (1+N+2^{|j|})\|f\|_{L^p}.$$

See Theorem 4.6 in [17] for the case $j = 0$. The proof for the case $j \ne 0$ is similar. For reader's convenience we contain its proof in Appendix A.3.

Corollary 4.1. *Let $\mathrm{F}_n^k(m,\beta,y)$, $\mathrm{G}_n^k(m,\beta,y)$ and $\mathrm{H}_n^k(\beta)$ be as in (4.9). For $1<p,q,s<\infty$ we have*

(1) $\left\|\left[\sum_{k,\,n}|\mathrm{F}_n^k(m,\beta,y)|^2\,\chi_{I_n^k}\right]^{\frac{1}{2}}\right\|_{L^p}\lesssim(1+m+2^{|j|})\|f\|_{L^p}$,

(2) $\left\|\left[\sum_{k,\,n}|\mathrm{G}_n^k(m,\beta,y)|^2\,\chi_{I_n^k}\right]^{\frac{1}{2}}\right\|_{L^q}\lesssim(1+m+2^{|j|})\|g\|_{L^q}$,

(3) $\left\|\sup_{k,\,n}\left[|\mathrm{H}_n^k(\beta)|\,\chi_{I_n^k}\right]\right\|_{L^s}\lesssim 2^{|i|}\,\|h\|_{L^s}$,

where the bounds are uniform if $|\beta|\lesssim 1$, $|y|\lesssim 1$.

Proof of Corollary 4.1. (1) and (2) follow from (4.9) and Lemma 4.1. (3) follows from

$$\left|\mathrm{H}_n^k(\beta)\right|\chi_{I_n^k}(x)=\Big(\left|h*\Phi_{k+|i|}\big(2^{-k}(n+\beta)\big)\right|\chi_{I_n^k}(x)\Big)\lesssim 2^{|i|}\,\mathrm{M}h(x).\quad\square$$

If $1/p+1/q<1$, then there exists $s>1$ such that $1/p+1/q+1/s=1$. Thus by (4.10) and Corollary 4.1,

$$\begin{aligned}|\langle \mathrm{III}^j(f,g),h\rangle| &\lesssim 2^{|i|(1+\gamma)}\sum_{m\geq 0}2^{-m\gamma}\left(1+m+2^{|j|}\right)^2 2^{|i|}\,\|f\|_{L^p}\,\|g\|_{L^q}\,\|h\|_{L^s}\\ &\lesssim 2^{|i|(2+\gamma)+2|j|}\,\|f\|_{L^p}\,\|g\|_{L^q}\,|E|^{1/s}.\quad\square\end{aligned}$$

4.2. The case $1<p,q<\infty$, $1/p+1/q\geq 1$

Let III^j and h be as in Subsection 4.1, then

$$\begin{aligned}&\langle \mathrm{III}^j(f,g),h\rangle\\ &=\sum_k 2^{-k\gamma}\int_{\mathbb{R}^2}\mathcal{F}\big[\mathbf{m}(\cdot)\widehat{\Phi}_{k+|i|}(\cdot)\big](y)\Big(f*\psi_{k+j}(x+y)\Big)\Big(g*\psi_k(x+ay)\Big)\Big(h*\Phi_{k+|i|}(x)\Big)\,dxdy\\ &=\sum_k 2^{-k\gamma}\int_{\mathbb{R}^2}\mathcal{F}\big[\mathbf{m}(\cdot)\widehat{\Phi}_{k+|i|}(\cdot)\big](y)\Big(f*\psi_{k+j}(x)\Big)\Big(g*\psi_k(x+(a-1)y)\Big)\Big(h*\Phi_{k+|i|}(x-y)\Big)\,dxdy.\end{aligned}\tag{4.11}$$

For each fixed $k\in\mathbb{Z}$, we decompose y variable as:

$$\mathbb{R}=\{y:|y|<2^{-k}\}\bigcup\bigcup_{m=1}^{\infty}\{y:2^{-k+m-1}\leq|y|<2^{-k+m}\}.\tag{4.12}$$

In discretization, by the uncertainty principle, it is typical to decompose x variable as $[2^{-k}n,2^{-k}(n+1))$, $n\in\mathbb{Z}$. But we decompose x variable according to the size of y variable. That is, if $|y|\sim 2^{-k+m}$, then we decompose x variable as:

$$\mathbb{R}=\bigcup_{n\in\mathbb{Z}}\Big\{x:2^{-k+m}n\leq x<2^{-k+m}(n+1)\Big\}.\tag{4.13}$$

Although we can not directly apply the previous estimates in this discretization, we conquer these difficulties by taking averages and using maximal function estimates. By using the decompositions in (4.12) and (4.13), and change of variables, one obtains the identity

$$\int_{\mathbb{R}}\int_{\mathbb{R}}K(x,y)\,dxdy=\sum_{n\in\mathbb{Z}}2^{-2k}\int_0^1\int_{-1}^1K\big(2^{-k}(n+\beta),2^{-k}y\big)\,dyd\beta$$

$$+\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}2^{-2k+2m}\int_0^1\int_{1/2\le|y|<1}K\big(2^{-k+m}(n+\beta),2^{-k+m}y\big)\,dyd\beta.$$

By applying this identity to (4.11) with (4.7), one gets

$$|\langle \mathrm{III}^j(f,g),h\rangle|\lesssim 2^{|i|(1+\gamma)}\sum_{m\ge0}\sum_k\sum_{n\in\mathbb{Z}}2^{-m\gamma}\,2^{-k+m}\,\mathrm{F}_n^k(m)\,\mathrm{G}_n^k(m)\,\mathrm{H}_n^k(m)$$

where

$$\begin{aligned}\mathrm{F}_n^k(m)&:=\left(\int_0^1\big|f*\psi_{k+j}\big(2^{-k+m}(n+\beta)\big)\big|d\beta\right),\\ \mathrm{G}_n^k(m)&:=\sup_{\beta\in[0,1]}\left(\int_{|y|\sim1}\big|g*\psi_k\big(2^{-k+m}(n+\beta+(a-1)y)\big)\big|\,dy\right),\\ \mathrm{H}_n^k(m)&:=\left(\sup_{|y|\sim1,\ \beta\in[0,1]}\big|h*\Phi_{k+|i|}\big(2^{-k+m}(n+\beta-y)\big)\big|\right).\end{aligned}\tag{4.14}$$

Thus if we set $I_n^{k-m}:=[2^{-k+m}n,2^{-k+m}(n+1)]$, then we have

$$|\langle \mathrm{III}^j(f,g),h\rangle|\lesssim 2^{|i|(1+\gamma)}\sum_{m\ge0}2^{-m\gamma}\sum_{k,\,n}\int \mathrm{F}_n^k(m)\,\mathrm{G}_n^k(m)\,\mathrm{H}_n^k(m)\,\chi_{I_n^{k-m}}(x)dx.\tag{4.15}$$

From now on we fix $m\ge0$, and for any finite collection $\mathcal{J}$ of dyadic intervals of the form $I_n^{k-m}:=[2^{-k+m}n,2^{-k+m}(n+1)]$, $k,n\in\mathbb{Z}$, we define

$$\mathrm{III}_{\mathcal{J}}^{j,m}(f,g,h):=\sum_{I_n^{k-m}\in\mathcal{J}}\int \mathrm{F}_n^k(m)\,\mathrm{G}_n^k(m)\,\mathrm{H}_n^k(m)\,\chi_{I_n^{k-m}}(x)\,dx.\tag{4.16}$$

Then by (4.15), it suffices to show that

$$\mathrm{III}_{\mathcal{J}}^{j,m}(f,g,h)\le C(s_0)\,\frac{(1+|a-1|)^2\,2^{2|j|}}{|a-1|}2^{\frac{m+|i|}{s_0}}\,\|f\|_{L^p}\|g\|_{L^q}|E|^{1-\frac1p-\frac1q}\tag{4.17}$$

for any positive integer s_0 and any finite collection $\mathcal{J}$ of dyadic intervals, where the constant $C(s_0)$ is independent of $\mathcal{J}$ and m. Then if we take s_0 large enough so that $s_0>1/\gamma$, we obtain

$$\begin{aligned}|\langle \mathrm{III}^j(f,g),h\rangle|&\le C(s_0)\,2^{|i|(1+\gamma)}\sum_{m\ge0}2^{-m\gamma}\left(\frac{(1+|a-1|)^2\,2^{2|j|}}{|a-1|}2^{\frac{m+|i|}{s_0}}\,\|f\|_{L^p}\|g\|_{L^q}|E|^{1-\frac1p-\frac1q}\right)\\ &\le C(s_0)\,\frac{(1+|a-1|)^2\,2^{|i|(1+\gamma+\frac{1}{s_0})+2|j|}}{|a-1|\big(1-2^{-\gamma+\frac{1}{s_0}}\big)}\,\|f\|_{L^p}\|g\|_{L^q}|E|^{1-\frac1p-\frac1q}.\end{aligned}$$

By the following well-known lemma, we dominate the supremum of a $C^1(I)$ function $F(t)$ by $\left(s_0\|F\|_{L^{s_0}(I)}^{s_0/s_0'}\|F'\|_{L^{s_0}(I)}\right)^{1/s_0}$, which is one of the main ideas of this paper.

Lemma 4.2. *Suppose that $F\in C^1([-1/2,3/2])$ that is supported in $[-1/4,5/4]$. Then for any positive integer s_0 we have*

$$\sup_{t\in[-1/2,3/2]}|F(t)|\leq\left[s_0\Bigg(\int_{-1/2}^{3/2}|F(\tau)|^{s_0}d\tau\Bigg)^{1/s_0'}\Bigg(\int_{-1/2}^{3/2}|F'(\tau)|^{s_0}\,d\tau\Bigg)^{1/s_0}\right]^{1/s_0}, \tag{4.18}$$

where $1/s_0+1/s_0'=1$.

Proof of Lemma 4.2. Since $F(-1/2)=0$, for each $t\in[-1/2,3/2]$, by Hölder's inequality

$$\begin{aligned}(F(t))^{s_0}&=\int_{-1/2}^{t}\frac{d}{d\tau}\left(F(\tau)^{s_0}\right)d\tau=s_0\int_{-1/2}^{t}F(\tau)^{s_0-1}F'(\tau)\,d\tau\\&\leq s_0\Bigg(\int_{-1/2}^{3/2}|F(\tau)^{s_0-1}|^{s_0'}\,d\tau\Bigg)^{1/s_0'}\Bigg(\int_{-1/2}^{3/2}|F'(\tau)|^{s_0}\,d\tau\Bigg)^{1/s_0}\\&=s_0\Bigg(\int_{-1/2}^{3/2}|F(\tau)|^{s_0}d\tau\Bigg)^{1/s_0'}\Bigg(\int_{-1/2}^{3/2}|F'(\tau)|^{s_0}\,d\tau\Bigg)^{1/s_0}.\quad\square\end{aligned}$$

Lemma 4.3. *Let* $1\leq p<\infty$, *and* $R\neq 0$, *then for any* $x\in I_n^{k-m}=[2^{-k+m}n,2^{-k+m}(n+1)]$

$$\Bigg(\int_{|t|\lesssim 1}|K(2^{-k+m}(n+Rt))|^p\,dt\Bigg)^{1/p}\lesssim\left(\frac{1+|R|}{|R|}\right)^{1/p}\mathrm{M}_p(K)(x).$$

Proof. Let $x\in I_n^{k-m}$ and $x_0:=2^{-k+m}n-x$, then $|x_0|\leq 2^{-k+m}$. By change of variable

$$\begin{aligned}\int_{|t|\lesssim 1}|K(2^{-k+m}(n+Rt))|^p\,dt&=\frac{1}{2^{-k+m}|R|}\int_{|s|\lesssim 2^{-k+m}|R|}|K(x+x_0+s)|^p\,ds\\&=\frac{1}{2^{-k+m}|R|}\int_{|s-x_0|\lesssim 2^{-k+m}|R|}|K(x+s)|^p\,ds\\&\leq\left(\frac{1+|R|}{|R|}\right)\frac{1}{2^{-k+m}(|R|+1)}\int_{|s|\lesssim 2^{-k+m}(|R|+1)}|K(x+s)|^p\,ds\\&\lesssim\left(\frac{1+|R|}{|R|}\right)\Big(\mathrm{M}_p(K)(x)\Big)^p.\quad\square\end{aligned}$$

Lemma 4.4. *Let* $\mathrm{F}_n^k(m)$, $\mathrm{G}_n^k(m)$ *and* $\mathrm{H}_n^k(m)$ *be as in (4.14), then for any* $x\in I_n^{k-m}$ *we have*

(1) $\mathrm{F}_n^k(m)\lesssim\mathrm{M}(f*\psi_{k+j})(x)$,
(2) $\mathrm{G}_n^k(m)\lesssim\left(\frac{|a-1|+1}{|a-1|}\right)\mathrm{M}(g*\psi_k)(x)$,
(3) $\mathrm{H}_n^k(m)\leq C(s_0)\,2^{(m+|i|)/s_0}\,\mathrm{M}_{s_0}(\mathrm{M}(h))(x)$ *for any positive integer* s_0.

Proof of Lemma 4.4. (1) and (2) are clear from (4.14) and Lemma 4.3. For (3), let ϕ be a positive smooth function supported in $[-4,4]$ and that is equal to 1 on $[-3,3]$, then

$$\Bigg(\sup_{|y|\sim 1,\,\beta\in[0,1]}\left|h*\Phi_{k+|i|}\big(2^{-k+m}(n+\beta+y)\big)\right|\Bigg)\leq\Bigg(\sup_{t}\left|h*\Phi_{k+|i|}\big(2^{-k+m}(n+t)\big)\,\phi(t)\right|\Bigg).$$

Thus by applying Lemma 4.2 and 4.3 with

$$F(t) := \Big(h * \Phi_{k+|i|}\big(2^{-k+m}(n+t)\big)\Big)\,\phi(t),$$

for any positive integer s_0 one gets

$$\begin{aligned}\mathrm{H}_n^k(m) &\le C(s_0)\left[\Big(\mathrm{M}_{s_0}\big(h*\Phi_{k+|i|}\big)(x)\Big)^{s_0/s_0'}\Big(\mathrm{M}_{s_0}\big(h*\Phi_{k+|i|}\big)(x)+2^{m+|i|}\mathrm{M}_{s_0}\big(h*(\Phi')_{k+|i|}\big)(x)\Big)\right]^{1/s_0}\\ &\le C(s_0)\,2^{(m+|i|)/s_0}\,\mathrm{M}_{s_0}\big(\mathrm{M}(h)\big)(x). \quad\square\end{aligned}$$

Lemma 4.5. *Let $\mathrm{F}_n^k(m)$, $\mathrm{G}_n^k(m)$ and $\mathrm{H}_n^k(m)$ be as in (4.14), and s_0 be a fixed positive integer. Then for any $1 < p, q < \infty$ and $s_0 < s < \infty$, we have*

(1) $\left\|\Big(\sum_{k,n}|\mathrm{F}_n^k(m)|^2\chi_{I_n^{k-m}}\Big)^{1/2}\right\|_{L^p} \lesssim \|f\|_{L^p}$,
(2) $\left\|\Big(\sum_{k,n}|\mathrm{G}_n^k(m)|^2\chi_{I_n^{k-m}}\Big)^{1/2}\right\|_{L^q} \lesssim \left(\frac{|a-1|+1}{|a-1|}\right)\|g\|_{L^q}$,
(3) $\left\|\sup_{k,n}\left[\mathrm{H}_n^k(m)\right]\chi_{I_n^{k-m}}\right]\right\|_{L^s} \le C(s_0)\,2^{(m+|i|)/s_0}\|h\|_{L^s}$.

Proof of Lemma 4.5. For (1), by Lemma 4.4 and 2.6, if $1 < p < \infty$, then

$$\begin{aligned}\left\|\Big(\sum_{k,\,n}|\mathrm{F}_n^k(m)|^2\chi_{I_n^{k-m}}(x)\Big)^{1/2}\right\|_{L^p} &\lesssim \left\|\Big(\sum_{k,\,n}|\mathrm{M}(f*\psi_{k+j})(x)|^2\chi_{I_n^{k-m}}(x)\Big)^{1/2}\right\|_{L^p}\\ &\lesssim \left\|\Big(\sum_{k}|\mathrm{M}(f*\psi_{k+j})|^2\Big)^{1/2}\right\|_{L^p}\\ &\lesssim \left\|\Big(\sum_{k}|(f*\psi_{k+j})|^2\Big)^{1/2}\right\|_{L^p}\\ &\lesssim \|f\|_{L^p}.\end{aligned}$$

(2) follows as in (1). For (3), if $s_0 < s < \infty$, then by Lemma 4.4

$$\begin{aligned}\left\|\sup_{k,n}\left[\mathrm{H}_n^k(m)\,\chi_{I_n^{k-m}}(x)\right]\right\|_{L^s} &\le C(s_0)\,2^{(m+|i|)/s_0}\,\|\mathrm{M}_{r_0}(\mathrm{M}(h))\|_{L^s}\\ &\le C(s_0)\,2^{(m+|i|)/s_0}\,\|\mathrm{M}(h)\|_{L^s}\\ &\le C(s_0)\,2^{(m+|i|)/s_0}\|h\|_{L^s}. \quad\square\end{aligned}$$

Recall that $m \ge 0$ is fixed. Let $\mathcal{J}$ be any finite family of dyadic intervals, then for $1 < p, q, s < \infty$ with $s > s_0$ we define

$$\begin{aligned}\mathrm{size}^{(1)}_{\mathcal{J},\,p}\Big(\big(\mathrm{F}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}}\Big) &:= \mathit{Size}_{\mathcal{J},\,p}\Big(\big(\mathrm{F}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}}\Big),\\ \mathrm{size}^{(2)}_{\mathcal{J},\,q}\Big(\big(\mathrm{G}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}}\Big) &:= \mathit{Size}_{\mathcal{J},\,q}\Big(\big(\mathrm{G}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}}\Big),\\ \mathrm{size}^{(3)}_{\mathcal{J},\,s}\Big(\big(\mathrm{H}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}}\Big) &:= \mathit{S\text{-}Size}_{\mathcal{J},\,s}\Big(\big(\mathrm{H}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}}\Big),\end{aligned}$$

and

$$\mathrm{energy}^{(1)}_{\mathcal{J},\,p}\Big(\,\big(\mathrm{F}^k_n(m)\big)_{I_n^{k-m}\in\mathcal{J}}\,\Big):=\mathit{Energy}_{\mathcal{J},\,p}\Big(\,\big(\mathrm{F}^k_n(m)\big)_{I_n^{k-m}\in\mathcal{J}}\,\Big),$$
$$\mathrm{energy}^{(2)}_{\mathcal{J},\,q}\Big(\,\big(\mathrm{G}^k_n(m)\big)_{I_n^{k-m}\in\mathcal{J}}\,\Big):=\mathit{Energy}_{\mathcal{J},\,q}\Big(\,\big(\mathrm{G}^k_n(m)\big)_{I_n^{k-m}\in\mathcal{J}}\,\Big),$$
$$\mathrm{energy}^{(3)}_{\mathcal{J},\,s}\Big(\,\big(\mathrm{H}^k_n(m)\big)_{I_n^{k-m}\in\mathcal{J}}\,\Big):=S\text{-}\mathit{Energy}_{\mathcal{J},\,s}\Big(\,\big(\mathrm{H}^k_n(m)\big)_{I_n^{k-m}\in\mathcal{J}}\,\Big).$$

Then by Lemma 2.3 and 2.5 we have the following lemma.

Lemma 4.6 *(cf. Lemma 2.10 in [17]). Let $\mathcal{J}$ be a family of dyadic intervals, let $i=1,2,3$. Then there exists a partition*

$$\mathcal{J}=\bigcup_{\nu\in\mathbb{Z}}\mathcal{J}^{\nu,i}$$

such that for any $\nu\in\mathbb{Z}$ one has

$$2^{-\nu-1}\mathrm{energy}^{(1)}_{\mathcal{J},\,p}\le\mathrm{size}^{(1)}_{\mathcal{J}^{\nu,1},\,p}\le\min\left(2^{-\nu}\mathrm{energy}^{(1)}_{\mathcal{J},\,p},\,\mathrm{size}^{(1)}_{\mathcal{J},\,p}\right),$$
$$2^{-\nu-1}\mathrm{energy}^{(2)}_{\mathcal{J},\,q}\le\mathrm{size}^{(2)}_{\mathcal{J}^{\nu,2},\,q}\le\min\left(2^{-\nu}\mathrm{energy}^{(2)}_{\mathcal{J},\,q},\,\mathrm{size}^{(2)}_{\mathcal{J},\,q}\right),$$
$$2^{-\nu-1}\mathrm{energy}^{(3)}_{\mathcal{J},\,s}\le\mathrm{size}^{(3)}_{\mathcal{J}^{\nu,3},\,s}\le\min\left(2^{-\nu}\mathrm{energy}^{(3)}_{\mathcal{J},\,s},\,\mathrm{size}^{(3)}_{\mathcal{J},\,s}\right).$$

Also one can write each $\mathcal{J}^{\nu,j}$ as a disjoint union of subsets $T\in\mathbb{T}^{\nu,j}$ such that for every $T\in\mathbb{T}^{\nu,j}$ there exists a dyadic interval I_T in T having the properties that every $I\in T$ satisfies $I\subset I_T$ and also

$$\sum_{T\in\mathbb{T}^{\nu,1}}|I_T|\lesssim 2^{\nu p},\qquad\sum_{T\in\mathbb{T}^{\nu,2}}|I_T|\lesssim 2^{\nu q},\qquad\sum_{T\in\mathbb{T}^{\nu,3}}|I_T|\lesssim 2^{\nu s}.$$

Proposition 4.1 *(cf. Proposition 2.12 in [17]). For $1<p,q,s<\infty$, let us denote*

$$S_1:=\mathrm{size}^{(1)}_{\mathcal{J},\,p},\ E_1:=\mathrm{energy}^{(1)}_{\mathcal{J},\,p},\ S_2:=\mathrm{size}^{(2)}_{\mathcal{J},\,q},\ E_2:=\mathrm{energy}^{(2)}_{\mathcal{J},\,q},\ S_3:=\mathrm{size}^{(3)}_{\mathcal{J},\,s},\ E_3:=\mathrm{energy}^{(3)}_{\mathcal{J},\,s}.$$

Then

$$\mathrm{III}^{j,m}_{\mathcal{J}}(f,g,h)\lesssim S_1^{1-p\theta_1}E_1^{p\theta_1}S_2^{1-q\theta_2}E_2^{q\theta_2}S_3^{1-s\theta_3}E_3^{s\theta_3},\tag{4.19}$$

for any $0\le\theta_1,\,\theta_2,\,\theta_3<1$ such that

$$\theta_1+\theta_2+\theta_3=1\ \text{and}\ 1-p\theta_1>0,\ 1-q\theta_2>1,\ 1-s\theta_3>0.\tag{4.20}$$

In particular if $1/p+1/q\ge 1$, then for any $1<s<\infty$, there are $0\le\theta_1,\,\theta_2,\,\theta_3<1$ satisfying (4.20).

Proof of Proposition 4.1. For $i=1,2,3$, let $\mathbb{T}^{n,i}$ be as in Lemma 4.6, and let

$$\mathbb{T}^{n_1,n_2,n_3}:=\Big\{T=T_1\cap T_2\cap T_3\,:\,T_i\in\mathbb{T}^{n_i,i},1\le i\le 3\Big\}.$$

For each $T\in\mathbb{T}^{n_1,n_2,n_3}$, there are finite number of disjoint dyadic intervals $I_1,\cdots,I_{N_T}$ in T so that every $I\in T$ is contained in some unique I_ℓ for some $1\le\ell\le N_T$. For each $1\le\ell\le N_T$, define

$$T(\ell):=\{I\in T\,:\,I\subseteq I_\ell\}.$$

Let $I_T := I_{T_1} \cap I_{T_2} \cap I_{T_3}$, then we have

$$\sum_{\ell=1}^{N_T} |I_\ell| \le |I_T|.$$

Now we have

$$\begin{aligned}
\mathrm{III}^{j,m}_{\mathcal{J}}(f,g,h) &= \sum_{I_n^{k-m}\in\mathcal{J}} \int_{\mathbb{R}} \mathrm{F}_n^k(m)\,\mathrm{G}_n^k(m)\,\mathrm{H}_n^k(m)\,\chi_{I_n^{k-m}}(x)\,dx \\
&= \sum_{n_1,n_2,n_3} \sum_{T\in\mathbb{T}^{n_1,n_2,n_3}} \sum_{\ell=1}^{N_T} \sum_{I_n^{k-m}\in T(\ell)} \int_{\mathbb{R}} \mathrm{F}_n^k(m)\,\mathrm{G}_n^k(m)\,\mathrm{H}_n^k(m)\,\chi_{I_n^{k-m}}(x)\,dx.
\end{aligned}$$

Since $1 < s < \infty$, we can choose $1 < p_0, q_0 < \infty$ so that $1/p_0 + 1/q_0 + 1/s = 1$. Then for each $T(\ell)$, we have

$$\begin{aligned}
&\sum_{I_n^{k-m}\in T(\ell)} \int_{\mathbb{R}} \mathrm{F}_n^k(m)\,\mathrm{G}_n^k(m)\,\mathrm{H}_n^k(m)\,\chi_{I_n^{k-m}}(x)\,dx \\
&\le \Big\| \Big(\sum_{I_n^{k-m}\in T(\ell)} |\mathrm{F}_n^k(m)|^2 \chi_{I_n^{k-m}} \Big)^{\frac12} \Big\|_{L^{p_0}} \Big\| \Big(\sum_{I_n^{k-m}\in T(\ell)} |\mathrm{G}_n^k(m)|^2 \chi_{I_n^{k-m}} \Big)^{\frac12} \Big\|_{L^{q_0}} \Big\| \sup_{I_n^{k-m}\in T(\ell)} |\mathrm{H}_n^k(m)| \chi_{I_n^{k-m}} \Big\|_{L^s} \\
&\le \mathrm{size}^{(1)}_{T,\,p_0}\, \mathrm{size}^{(2)}_{T,\,q_0}\, \mathrm{size}^{(3)}_{T,\,s}\, |I_\ell|.
\end{aligned}$$

By Lemma 2.2

$$\begin{aligned}
\mathrm{size}^{(1)}_{T,\,p_0}\Big(\big(\mathrm{F}_n^k(m)\big)_{I_n^{k-m}\in T} \Big) &\simeq \mathrm{size}^{(1)}_{T,\,p}\Big(\big(\mathrm{F}_n^k(m)\big)_{I_n^{k-m}\in T} \Big), \\
\mathrm{size}^{(2)}_{T,\,q_0}\Big(\big(\mathrm{G}_n^k(m)\big)_{I_n^{k-m}\in T} \Big) &\simeq \mathrm{size}^{(2)}_{T,\,q}\Big(\big(\mathrm{G}_n^k(m)\big)_{I_n^{k-m}\in T} \Big),
\end{aligned}$$

and so we get

$$\mathrm{III}^{j,m}_{\mathcal{J}}(f,g,h) \lesssim \sum_{n_1,n_2,n_3} \sum_{T\in\mathbb{T}^{n_1,n_2,n_3}} \mathrm{size}^{(1)}_{T,\,p}\, \mathrm{size}^{(2)}_{T,\,q}\, \mathrm{size}^{(3)}_{T,\,s}\, |I_T|.$$

Thus by Lemma 4.6

$$\begin{aligned}
\mathrm{III}^{j,m}_{\mathcal{J}}(f,g,h) &\lesssim \sum_{n_1,n_2,n_3} \sum_{T\in\mathbb{T}^{n_1,n_2,n_3}} \mathrm{size}^{(1)}_{T,\,p}\, \mathrm{size}^{(2)}_{T,\,q}\, \mathrm{size}^{(3)}_{T,\,s}\, |I_T| \\
&\lesssim E_1E_2E_3 \sum_{n_1,n_2,n_3} 2^{-n_1}2^{-n_2}2^{-n_3} \sum_{T\in\mathbb{T}^{n_1,n_2,n_3}} |I_T|
\end{aligned}$$

where the summations run over those indices n_1, n_2, n_3 for which

$$2^{-n_i} \lesssim \frac{S_i}{E_i}, \quad j = 1, 2, 3.$$

Lemma 4.6 allows us to estimate $\sum_{T\in\mathbb{T}^{n_1,n_2,n_3}} |I_T|$ in three different ways

$$\sum_{T\in\mathbb{T}^{n_1,n_2,n_3}} |I_T| \lesssim 2^{n_1 p},\ 2^{n_2 q},\ 2^{n_3 s}.$$

In particular

$$\sum_{T\in\mathbb{T}^{n_1,n_2,n_3}} |I_T| \lesssim 2^{n_1p\theta_1+n_2q\theta_2+n_3s\theta_3}$$

where $0 \le \theta_1,\theta_2,\theta_3 < 1$ and $\theta_1+\theta_2+\theta_3 = 1$. Using all this information, if $1-p\theta_1 > 0$, $1-q\theta_2 > 0$, $1-s\theta_3 > 0$, then

$$\begin{aligned}\mathrm{III}^{j,m}_{\mathcal{J}}(f,g,h) &\lesssim E_1E_2E_3 \sum_{n_1,n_2,n_3} 2^{-n_1(1-p\theta_1)}2^{-n_2(1-q\theta_2)}2^{-n_3(1-s\theta_3)}\\ &\lesssim E_1E_2E_3\left(\frac{S_1}{E_1}\right)^{1-p\theta_1}\left(\frac{S_2}{E_2}\right)^{1-q\theta_2}\left(\frac{S_3}{E_3}\right)^{1-s\theta_3}\\ &\lesssim S_1^{1-p\theta_1}E_1^{p\theta_1}S_2^{1-q\theta_2}E_2^{q\theta_2}S_3^{1-s\theta_3}E_3^{s\theta_3}. \quad\square\end{aligned}$$

Lemma 4.7. *Let $\mathrm{F}^k_n(m)$, $\mathrm{G}^k_n(m)$ and $\mathrm{H}^k_n(m)$ be as in (4.14). If $1 < p,q < \infty$ and $1 < s_0 < s < \infty$. Let $\widetilde{I} := 2(1+|a-1|)I$, then for any $N > 0$ we have*

$$\text{(1)}\quad \left\|\left(\sum_{I^{k-m}_n\subseteq I}|\mathrm{F}^k_n(m)|^2\chi_{I^{k-m}_n}\right)^{1/2}\right\|^p_{L^p} \le C_N\,2^{|j|N}\int |f(x)|^p\left(1+2^m\frac{\mathrm{dist}(I,x)}{|I|}\right)^{-N}dx,$$

$$\text{(2)}\quad \left\|\left(\sum_{I^{k-m}_n\subseteq I}|\mathrm{G}^k_n(m)|^2\chi_{I^{k-m}_n}\right)^{1/2}\right\|^q_{L^q} \le C_N\left(\frac{1+|a-1|}{|a-1|}\right)^q\int |g(x)|^q\left(1+2^m\frac{\mathrm{dist}(\widetilde{I},x)}{|\widetilde{I}|}\right)^{-N}dx,$$

$$\text{(3)}\quad \left\|\left(\sup_{I^{k-m}_n\subseteq I}|\mathrm{H}^k_n(m)|\chi_{I^{k-m}_n}\right)\right\|^s_{L^s} \le C_{N,s_0}\,2^{sm/s_0}\int |h(x)|^s\left(1+2^m\frac{\mathrm{dist}(I,x)}{|I|}\right)^{-N}dx.$$

Proof of Lemma 4.7. For (1), by splitting the real line as a disjoint union $(J_\ell)_{\ell\in\mathbb{Z}}$ having the same length as I and $\mathrm{dist}(I,J_\ell)\sim|\ell||I|$, it suffices to show that

$$\begin{aligned}&\left\|\left(\sum_{I^{k-m}_n\subseteq I}\left|\int_0^1 |f\chi_{J_\ell}*\psi_{k+j}(2^{-k+m}(n+\beta))|\,d\beta\right|^2\chi_{I^{k-m}_n}(x)\right)^{1/2}\right\|^p_{L^p}\\ &\lesssim \begin{cases}\int|f(x)\chi_{J_\ell}(x)|^pdx, & \text{if } \ell=0;\\ C_N(2^{m+j}|\ell|)^{-N}\int|f(x)\chi_{J_\ell}(x)|^pdx, & \text{if } \ell\ne0.\end{cases}\end{aligned}$$

The case $\ell=0$ and $1<p<\infty$ follows from Lemma 4.5. For the case $|\ell|>0$, by using $I^{k-m}_n\subseteq I$ and $\mathrm{dist}(I,J_\ell)\sim|\ell||I|$, if $x_0\in I^{k-m}_n$ then we have

$$\begin{aligned}|f\chi_{J_\ell}*\psi_{k+j}(x_0)| &\le C_N\int|(f\chi_{J_\ell})(y)\,|\frac{2^{k+j}}{\left(1+2^{k+j}|x_0-y|\right)^{2N}}dy\\ &\le C_N\left(1+2^{k+j}|\ell||I|\right)^{-N}\int|(f\chi_{J_\ell})(y)|\,\frac{2^{k+j}}{\left(1+2^{k+j}|x_0-y|\right)^N}dy\\ &\le C_N\left(1+2^{k+j}|\ell||I|\right)^{-N}\mathrm{M}(f\chi_{J_\ell})(x_0).\end{aligned}$$

Thus if $x\in I^{k-m}_n$, then by Lemma 4.3

$$\begin{aligned}\int_0^1\left|(f\chi_{J_\ell})*\psi_{k+j}(2^{-k+m}(n+\beta))\right|d\beta &\le C_N\left(1+2^{k+j}|\ell||I|\right)^{-N}\int_0^1\mathrm{M}(f\chi_{J_\ell})(2^{-k+m}(n+\beta))\,d\beta\\ &\le C_N(1+2^{k+j}|\ell||I|)^{-N}\mathrm{MM}(f\chi_{J_\ell})(x).\end{aligned}$$

Hence if $|\ell| > 0$ and $1 < p < \infty$, then

$$\begin{aligned}
&\left\| \left(\sum_{I_n^{k-m} \subseteq I} \left| \int_0^1 \left| f\chi_{J_\ell} * \psi_{k+j}(2^{-k+m}(n+\beta)) \right| d\beta \right|^2 \chi_{I_n^{k-m}}(x) \right)^{1/2} \right\|_{L^p} \\
&\leq C_N \left\| \left(\sum_{k\,:\,2^{-k+m} \leq |I|} \sum_{n\in\mathbb{Z}} (1 + 2^{k+j}|\ell||I|)^{-2N} |\mathrm{MM}(f\chi_{J_\ell})(x)|^2 \chi_{I_n^{k-m}}(x) \right)^{1/2} \right\|_{L^p} \\
&\leq C_N \left\| \left(\sum_{k\,:\,2^{-k+m} \leq |I|} (1 + 2^{k+j}|\ell||I|)^{-2N} |\mathrm{MM}(f\chi_{J_\ell})|^2 \right)^{1/2} \right\|_{L^p} \\
&\leq C_N (2^{m+j}|\ell|)^{-N} \left\| \mathrm{MM}(f\chi_{J_\ell}) \right\|_{L^p} \\
&\leq C_N (2^{m+j}|\ell|)^{-N} \left\| f\chi_{J_\ell} \right\|_{L^p} .
\end{aligned}$$

For (2), let $\widetilde{I} := 2(1 + |a-1|)I$, then by splitting the real line as a disjoint union $(J_\ell)_{\ell\in\mathbb{Z}}$ having the same length as $\widetilde{I}$ and $\mathrm{dist}(\widetilde{I}, J_\ell) \sim |\ell||\widetilde{I}|$, it suffices to show that

$$\begin{aligned}
&\left\| \left(\sum_{I_n^{k-m} \subseteq I} \sup_{\beta\in[0,1]} \left| \int_{|y|\sim 1} |g\chi_{J_\ell} * \psi_k\big(2^{-k+m}(n+\beta+(a-1)y)\big)|\, dy \right|^2 \chi_{I_n^{k-m}}(x) \right)^{1/2} \right\|_{L^q(dx)}^q \\
&\lesssim \begin{cases} \left(\frac{1+|a-1|}{|a-1|}\right)^q \int |g(x)\chi_{J_\ell}(x)|^q\, dx, & \text{if } \ell = 0; \\ C_N \left(\frac{1+|a-1|}{|a-1|}\right)^q (1 + 2^m|\ell|)^{-N} \int |g(x)\chi_{J_\ell}(x)|^q\, dx, & \text{if } \ell \neq 0. \end{cases}
\end{aligned}$$

The case $\ell = 0$ and $1 < q < \infty$ follows from Lemma 4.5. For the case $|\ell| > 0$, by using $\mathrm{dist}(\widetilde{I}, J_\ell) \sim |\ell||\widetilde{I}|$, if $x_0 \in \widetilde{I}$, then $\mathrm{dist}(x_0, J_\ell) \sim |\ell||\widetilde{I}|$ and we have

$$\begin{aligned}
|g\chi_{J_\ell} * \psi_{k+j}(x_0)| &\leq C_N \int |(f\chi_{J_\ell})(y)| \, \frac{2^k}{\big(1 + 2^k|x_0 - y|\big)^{2N}}\, dy \\
&\leq C_N \big(1 + 2^k|\ell||\widetilde{I}|\big)^{-N} \int |(g\chi_{J_\ell})(y)| \, \frac{2^k}{\big(1 + 2^k|x_0 - y|\big)^{N}}\, dy \\
&\leq C_N \big(1 + 2^k|\ell||\widetilde{I}|\big)^{-N} \mathrm{M}(g\chi_{J_\ell})(x_0).
\end{aligned} \tag{4.21}$$

If $I_n^{k-m} \subseteq I$, then $x_0 := 2^{-k+m}(n+\beta+(a-1)y) \in \widetilde{I}$ for $|\beta|, |y| \leq 1$. Thus by (4.21) and Lemma 4.3 if $x \in I_n^{k-m} \subseteq I$, then

$$\begin{aligned}
&\int_{|y|\sim 1} \big|(g\chi_{J_\ell}) * \psi_k(2^{-k+m}\big(n+\beta+(a-1)y)\big)\big|\, dy \\
&\leq C_N \big(1 + 2^k|\ell||\widetilde{I}|\big)^{-N} \int_0^1 \mathrm{M}(g\chi_{J_\ell})\big(2^{-k+m}(n+\beta+(a-1)y)\big)\, dy \\
&\leq C_N (1 + 2^k|\ell||\widetilde{I}|)^{-N} \left(\frac{1+|a-1|}{|a-1|} \right) \mathrm{MM}(g\chi_{J_\ell})(x).
\end{aligned}$$

Hence if $|\ell| > 0$ and $1 < q < \infty$, then

$$
\begin{aligned}
&\left\|\left(\sum_{I_n^{k-m}\subseteq I}\sup_{\beta\in[0,1]}\left|\int_{|y|\lesssim 1}\left|g\chi_{J_\ell}*\psi_k\big(2^{-k+m}(n+\beta+(a-1)y)\big)\right|dy\right|^2\chi_{I_n^{k-m}}(x)\right)^{1/2}\right\|_{L^q(dx)}\\
&\le C_N\left(\frac{1+|a-1|}{|a-1|}\right)\left\|\left(\sum_{k\,:\,2^{-k+m}\le|I|}\sum_{n\in\mathbb{Z}}(1+2^k|\ell||\widetilde{I}|)^{-2N}|\mathrm{MM}(g\chi_{J_\ell})(x)|^2\chi_{I_n^{k-m}}(x)\right)^{1/2}\right\|_{L^q(dx)}\\
&\le C_N\left(\frac{1+|a-1|}{|a-1|}\right)\left\|\left(\sum_{k\,:\,2^{-k+m}\le|I|}(1+2^k|\ell||\widetilde{I}|)^{-2N}|\mathrm{MM}(g\chi_{J_\ell})(x)|^2\right)^{1/2}\right\|_{L^q(dx)}\\
&\le C_N\left(\frac{1+|a-1|}{|a-1|}\right)(2^m|\ell|)^{-N}\left\|\mathrm{MM}(g\chi_{J_\ell})\right\|_{L^q}\\
&\le C_N\left(\frac{1+|a-1|}{|a-1|}\right)(2^m|\ell|)^{-N}\left\|g\chi_{J_\ell}\right\|_{L^q}.
\end{aligned}
$$

To treat (3) we first observe that if $I_n^{k-m}\subseteq I$ and $|\beta|,|y|\le 1$, then

$$
\begin{aligned}
&\left|(h\chi_{J_\ell})*\Phi_k\big(2^{-k+m}(n+\beta-y)\big)\right|\\
&\le C_N\int\left|(h\chi_{J_\ell})(t)\right|\frac{2^k}{\big(1+2^k|2^{-k+m}(n+\beta-y)-t|\big)^{2N}}\,dt\\
&\le C_N(1+2^k|\ell||I|)^{-N}\int\left|(h\chi_{J_\ell})(t)\right|\frac{2^k}{\big(1+2^k|2^{-k+m}(n+\beta-y)-t|\big)^{N}}\,dt.
\end{aligned}
$$

If $x\in I_n^{k-m}$, then we apply the same argument for the proof of Lemma 4.4 (3) to obtain that for any positive integer s_0,

$$
\sup_{|\beta|,|y|\le 1}\left|\int\left|(h\chi_{J_\ell})(t)\right|\frac{2^k}{\big(1+2^k|2^{-k+m}(n+\beta-y)-t|\big)^{N}}\,dt\right|\le C(s_0)2^{m/s_0}\mathrm{M}_{s_0}(\mathrm{M}(h\chi_{J_\ell}))(x),
$$

and so

$$
\sup_{|\beta|,|y|\le 1}\big(|(h\chi_{J_\ell})*\Phi_k(2^{-k+m}(n+\beta-y))|\big)\,\chi_{I_n^{k-m}}(x)\le C_{N,s_0}(1+2^m|\ell|)^{-N}2^{m/s_0}\mathrm{M}_{s_0}(\mathrm{M}(h\chi_{J_\ell}))(x).
$$

We therefore obtain that if $s > s_0$, then

$$
\begin{aligned}
\left\|\sup_{I_n^{k-m}\subseteq I}\left(|\mathrm{H}_n^k(m)|\chi_{I_n^{k-m}}\right)\right\|_{L^s}&\le C_{N,s_0}(1+2^m|\ell|)^{-N}2^{m/s_0}\|\mathrm{M}_{s_0}\mathrm{M}(h\chi_{J_\ell})\|_{L^s}\\
&\le C_{N,s_0}(1+2^m|\ell|)^{-N}2^{m/s_0}\|\mathrm{M}(h\chi_{J_\ell})\|_{L^s}\\
&\le C_{N,s_0}(1+2^m|\ell|)^{-N}2^{m/s_0}\|h\chi_{J_\ell}\|_{L^s}.\quad\square
\end{aligned}
$$

Corollary 4.2. *Let $\mathrm{F}_n^k(m)$, $\mathrm{G}_n^k(m)$ and $\mathrm{H}_n^k(m)$ be as in (4.14). For $1 < p, q < \infty$, and $s_0 < s < \infty$. Let $\widetilde{I} := 2(1+|a-1|)I$, the for any $N > 0$*

(1) $\mathrm{size}^{(1)}_{\mathcal{J},p}\Big(\big(\mathrm{F}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}}\Big)\le C_N 2^{|j|N}\sup_{I\in\mathcal{J}}\left(\frac{1}{|I|}\int|f(x)|^p\left(1+2^m\frac{\mathrm{dist}(I,x)}{|I|}\right)^{-N}dx\right)^{1/p}$,

(2) $\mathrm{size}^{(2)}_{\mathcal{J},\,q}\Big(\big(\mathrm{G}^k_n(m)\big)_{I^{k-m}_n\in\mathcal{J}} \Big) \le C_N \left(\frac{1+|a-1|}{|a-1|}\right) \sup_{I\in\mathcal{J}} \left(\frac{1}{|I|}\int |g(x)|^q \left(1+2^m\frac{\mathrm{dist}(\widetilde{I},x)}{|\widetilde{I}|}\right)^{-N} dx\right)^{1/q}$,

(3) $\mathrm{size}^{(3)}_{\mathcal{J},\,s}\Big(\big(\mathrm{H}^k_n(m)\big)_{I^{k-m}_n\in\mathcal{J}} \Big) \le C_{N,s_0}\, 2^{m/s_0} \sup_{I\in\mathcal{J}} \left(\frac{1}{|I|}\int |h(x)|^s \left(1+2^m\frac{\mathrm{dist}(I,x)}{|I|}\right)^{-N} dx\right)^{1/s}$.

Lemma 4.8. *Let $\mathrm{F}^k_n(m)$, $\mathrm{G}^k_n(m)$ and $\mathrm{H}^k_n(m)$ be as in (4.14). For $1<p,q<\infty$, and $s_0<s<\infty$*

(1) $\mathrm{energy}^{(1)}_{\mathcal{J},\,p}\Big(\big(\mathrm{F}^k_n(m)\big)_{I^{k-m}_n\in\mathcal{J}} \Big) \lesssim 2^{2|j|}\|f\|_{L^p}$,
(2) $\mathrm{energy}^{(2)}_{\mathcal{J},\,q}\Big(\big(\mathrm{G}^k_n(m)\big)_{I^{k-m}_n\in\mathcal{J}} \Big) \lesssim \frac{(1+|a-1|)^{1+1/q}}{|a-1|}\|g\|_{L^q}$,
(3) $\mathrm{energy}^{(3)}_{\mathcal{J},\,s}\Big(\big(\mathrm{H}^k_n(m)\big)_{I^{k-m}_n\in\mathcal{J}} \Big) \le C(s_0)\, 2^{m/s_0}\|h\|_{L^s}$.

Proof of Lemma 4.8. For (1), recall that

$$\mathrm{energy}^{(1)}_{\mathcal{J},\,p}\Big(\big(\mathrm{F}^k_n(m)\big)_{I^{k-m}_n\in\mathcal{J}} \Big) := \sup_{\nu\in\mathbb{Z}}\sup_{\mathbb{D}_\nu}\Big(\sum_{I_0\in\mathbb{D}_\nu}|I_0|\Big)^{1/p}$$

where $\mathbb{D}_\nu$ ranges over all collections of disjoint dyadic intervals $I_0\in\mathcal{J}$ having the property that

$$\frac{1}{|I_0|^{1/p}}\Bigg\|\Bigg(\sum_{I^{k-m}_n\subseteq I_0,\, I^{k-m}_n\in\mathcal{J}} |\mathrm{F}^k_n(m)|^2\chi_{I^{k-m}_n}\Bigg)^{1/2}\Bigg\|_{L^p} \ge 2^\nu.$$

Thus by applying Lemma 4.7 with $N=2$ we have

$$\begin{aligned}
2^{\nu p}\Big(\sum_{I_0\in\mathbb{D}_\nu}|I_0|\Big) &= \Big\|\sum_{I_0\in\mathbb{D}_\nu}2^{\nu p}\chi_{I_0}\Big\|_{L^{1,\infty}}\\
&\le \Bigg\|\sum_{I_0\in\mathbb{D}_\nu}\frac{1}{|I_0|}\Bigg\|\Bigg(\sum_{I^{k-m}_n\subseteq I_0,\, I^{k-m}_n\in\mathcal{J}} |\mathrm{F}^k_n(m)|^2\chi_{I^{k-m}_n}\Bigg)^{1/2}\Bigg\|^p_{L^p}\chi_{I_0}\Bigg\|_{L^{1,\infty}}\\
&\lesssim 2^{2|j|p}\Bigg\|\sum_{I_0\in\mathbb{D}_\nu}\frac{1}{|I_0|}\Bigg(\int |f(y)|^p\Big(1+2^m\frac{\mathrm{dist}(I_0,y)}{|I_0|}\Big)^{-2}dy\Bigg)\chi_{I_0}\Bigg\|_{L^{1,\infty}}\\
&\lesssim 2^{2|j|p}\Big\|\sum_{I_0\in\mathbb{D}_\nu}\mathrm{M}(|f|^p)\,\chi_{I_0}\Big\|_{L^{1,\infty}}\\
&\lesssim 2^{2|j|p}\,\||f|^p\|_{L^1} = 2^{2|j|p}\,\|f\|^p_{L^p}.
\end{aligned}$$

The proof of (2) follows similarly by using Lemma 4.7 and the fact

$$\frac{1}{|I_0|}\int |g(y)|^q\left(1+2^m\frac{\mathrm{dist}(\widetilde{I_0},y)}{|\widetilde{I_0}|}\right)^{-N}dy \lesssim (1+|a-1|)\,\mathrm{M}(|g|^q)(x) \quad \text{if } x\in I_0,$$

where $\widetilde{I_0} := 2(1+|a-1|)I_0$. For (3), recall that

$$\mathrm{energy}^{(3)}_{\mathcal{J},\,s}\Big(\big(\mathrm{H}^k_n(m)\big)_{I^{k-m}_n\in\mathcal{J}} \Big) := \sup_{\nu\in\mathbb{Z}}\sup_{\mathbb{D}_\nu}\Big(\sum_{I_0\in\mathbb{D}_\nu}|I_0|\Big)^{1/s}$$

where $\mathbb{D}_\nu$ ranges over all collections of disjoint dyadic intervals $I_0\in\mathcal{J}$ having the property that

$$\frac{1}{|I_0|^{1/s}}\left\|\sup_{I_n^{k-m}\subseteq I_0,\, I_n^{k-m}\in\mathcal{J}}\left(|\mathrm{H}_n^k(m)|\chi_{I_n^{k-m}}\right)\right\|_{L^s}\geq 2^{\nu}.$$

Thus for $s>s_0$ we have

$$\begin{aligned}
2^{\nu s}\left(\sum_{I_0\in\mathbb{D}_\nu}|I_0|\right) &= \left\|\sum_{I_0\in\mathbb{D}_\nu}2^{\nu s}\chi_{I_0}\right\|_{L^{1,\infty}}\\
&\leq \left\|\sum_{I_0\in\mathbb{D}_\nu}\frac{1}{|I_0|}\left\|\sup_{I_n^{k-m}\subseteq I_0,\, I_n^{k-m}\in\mathcal{J}}\left(|\mathrm{H}_n^k(m)|\chi_{I_n^{k-m}}\right)\right\|_{L^s}^s\chi_{I_0}\right\|_{L^{1,\infty}}\\
&\leq C_{N,s_0}\,2^{sm/s_0}\left\|\sum_{I_0\in\mathbb{D}_\nu}\frac{1}{|I_0|}\left(\int|h(y)|^s\left(1+2^m\frac{\mathrm{dist}(I_0,y)}{|I_0|}\right)^{-N}dy\right)\chi_{I_0}\right\|_{L^{1,\infty}}\\
&\leq C_{N,s_0}\,2^{sm/s_0}\left\|\sum_{I_0\in\mathbb{D}_\nu}\mathrm{M}(|h|^s)\,\chi_{I_0}\right\|_{L^{1,\infty}}\\
&\leq C_{N,s_0}\,2^{sm/s_0}\||h|^s\|_{L^1}=C_{N,s_0}\,2^{sm/s_0}\|h\|_{L^s}^s. \quad\square
\end{aligned}$$

Let Ω be as in (4.4). Split the collection of dyadic intervals $\mathcal{J}$ as $\bigcup_d\mathcal{J}_d$ where

$$\mathcal{J}_d:=\left\{I:1+\mathrm{dist}(I,\Omega^c)/|I|\simeq 2^d\right\}.$$

If $I\in\mathcal{J}_d$, then $2^dI\cap\Omega^c\neq\emptyset$, and so for $1<p,q<\infty$ and $s_0<s<\infty$ one has

$$\begin{aligned}
S_1(d)&:=\mathrm{size}^{(1)}_{\mathcal{J}_d,\,p}\Big(\left(\mathrm{F}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}_d}\Big)\lesssim 2^d2^{2|j|}\|f\|_{L^p}|E|^{-1/p},\\
S_2(d)&:=\mathrm{size}^{(2)}_{\mathcal{J}_d,\,q}\Big(\left(\mathrm{G}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}_d}\Big)\lesssim 2^d\left(\frac{1+|a-1|}{|a-1|}\right)\|g\|_{L^q}|E|^{-1/q},\\
S_3(d)&:=\mathrm{size}^{(3)}_{\mathcal{J}_d,\,s}\Big(\left(\mathrm{H}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}_d}\Big)\leq C_{N,s_0}\,2^{-Nd}2^{m/s_0},
\end{aligned}$$

and

$$\begin{aligned}
E_1(d)&:=\mathrm{energy}^{(1)}_{\mathcal{J}_d,\,p}\Big(\left(\mathrm{F}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}_d}\Big)\lesssim 2^{2|j|}\|f\|_{L^p},\\
E_2(d)&:=\mathrm{energy}^{(2)}_{\mathcal{J}_d,\,q}\Big(\left(\mathrm{G}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}_d}\Big)\lesssim\left(\frac{(1+|a-1|)^{1+1/q}}{|a-1|}\right)\|g\|_{L^q},\\
E_3(d)&:=\mathrm{energy}^{(3)}_{\mathcal{J}_d,\,s}\Big(\left(\mathrm{H}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}_d}\Big)\leq C(s_0)\,2^{m/s_0}|E|^{1/s}.
\end{aligned}$$

Then if $1/p+1/q\geq 1$, by Proposition 4.1, for any positive integer s_0 and any $s_0<s<\infty$, there are $0\leq\theta_1,\theta_2,\theta_3<1$ such that

$$\begin{aligned}
\mathrm{III}^{j,m}_{\mathcal{J}}(f,g,h)&\leq\sum_d\mathrm{III}^{j,m}_{\mathcal{J}_d}(f,g,h)\\
&\lesssim\sum_d S_1(d)^{1-p\theta_1}E_1(d)^{p\theta_1}S_2(d)^{1-q\theta_2}E_2(d)^{q\theta_2}S_3(d)^{1-s\theta_3}E_3(d)^{s\theta_3}\\
&\leq C(s_0)\,2^{2|j|}\left(\frac{(1+|a-1|)^2}{|a-1|}\right)2^{m/s_0}\|f\|_{L^p}\|g\|_{L^q}|E|^{1-1/p-1/q}. \quad\square
\end{aligned}$$

5. Estimates for the terms I and II

The terms I and II in (3.1) are very similar. For instance, one can write the second term as follows:

$$\begin{aligned}\mathrm{II}(f,g)(x) &= \sum_{k_2} \int_{\mathbb{R}^2} \mathbf{m}(\xi + a\eta)\Big(\sum_{k_1 \le k_2 - |i| - 5} \widehat{f}(\xi)\widehat{\psi}_{k_1}(\xi)\Big) \Big(\widehat{g}(\eta)\widehat{\psi}_{k_2}(\eta)\Big)\, \mathrm{e}^{2\pi \mathrm{i} x(\xi+\eta)} d\xi d\eta \\ &= \sum_{k_2} \int_{\mathbb{R}^2} \mathbf{m}(\xi + a\eta)\Big(\widehat{f}(\xi)\widehat{\varphi}_{k_2-|i|-5}(\xi)\Big) \Big(\widehat{g}(\eta)\widehat{\psi}_{k_2}(\eta)\Big)\, \mathrm{e}^{2\pi \mathrm{i} x(\xi+\eta)} d\xi d\eta\end{aligned} \tag{5.1}$$

where $\widehat{\varphi_k}(\xi) = \widehat{\varphi}(2^{-k}\xi)$. Let Ψ and Φ be Schwartz functions with $\mathrm{supp}(\widehat{\Psi}) \subseteq \{\xi : 1/2^{10} \le |\xi| \le 2^{10}\}$. Note that

$$\widehat{\Psi}_{k+i}(\xi + a\eta)\widehat{\Psi}_k(\xi+\eta) = 1$$

if (ξ,η) lies in the support of $\widehat{\varphi}_{k-|i|-5}(\xi)\,\widehat{\psi}_k(\eta)$. Thus by inserting the function $\widehat{\Psi}_{k+i}(\xi+a\eta)\widehat{\Psi}_k(\xi+\eta)$ inside of the integral in (5.1) we have

$$\begin{aligned}&\mathrm{II}(f,g)(x)\\ &= \sum_k \int_{\mathbb{R}^2} \Big(\mathbf{m}(\xi + a\eta)\widehat{\Psi}_{k+i}(\xi+a\eta)\Big)\Big(\widehat{f}(\xi)\widehat{\varphi}_{k-|i|-5}(\xi)\Big)\Big(\widehat{g}(\eta)\widehat{\psi}_k(\eta)\Big)\,\widehat{\Psi}_k(\xi+\eta)\, \mathrm{e}^{2\pi\mathrm{i}x(\xi+\eta)} d\xi d\eta \\ &= \sum_k \int_{\mathbb{R}} \Bigg(\int_{\mathbb{R}^2} \mathcal{F}\big[\mathbf{m}(\cdot)\widehat{\Psi}_{k+i}(\cdot)\big](y) \Big(\widehat{f * \varphi_{k-|i|-5}}(\xi)\Big)\Big(\widehat{g * \psi_k}(\eta)\Big)\widehat{\psi}_k(\eta)\Big)\widehat{\Psi}_k(\xi+\eta)\, \mathrm{e}^{2\pi\mathrm{i}[\xi(x+y)+\eta(x+ay)]} d\xi d\eta \Bigg) dy.\end{aligned}$$

Let

$$\begin{aligned}\mathrm{II}^i(f,g)(x) := \sum_k 2^{-k\gamma} \int \mathcal{F}\big[\mathbf{m}(\cdot)\widehat{\Psi}_{k+i}(\cdot)\big](y) \\ \Bigg(\int_{\mathbb{R}^2} \Big(\widehat{f * \varphi_{k-|i|-5}}(\xi)\Big)\Big(\widehat{g * \psi_k}(\eta)\Big)\,\widehat{\Psi}_k(\xi+\eta) \mathrm{e}^{2\pi\mathrm{i}[\xi(x+y)+\eta(x+ay)]} d\xi d\eta\Bigg) dy,\end{aligned}$$

then it suffices to show that

$$\|\mathrm{II}^i(f,g)\|_{L^{r,\infty}} \le C_\gamma 2^{|i|(3+\gamma)} \left(\frac{(1+|a|)^2}{|a|}\right) \|f\|_{L^p}\|g\|_{L^q} \tag{5.2}$$

where $1/r = 1/p + 1/q$, $1 < p, q < \infty$. In proving (5.2) we may assume that $\|f\|_{L^p} \ne 0 \ne \|g\|_{L^q}$. Define an "exceptional set" Ω by

$$\Omega := \Big\{x : \mathrm{M}_p f(x) > t\|f\|_{L^p}|E|^{-1/p}\Big\} \bigcup \Big\{x : \mathrm{M}_q g(x) > t\|g\|_{L^q}|E|^{-1/q}\Big\} \tag{5.3}$$

where $\mathrm{M}_p f(x) := \left(\sup_{x\in I} \frac{1}{|I|}\int_I |f(y)|^p dy\right)^{1/p}$. Since

$$\Big|\Big\{x : \mathrm{M}_p f(x) > t\Big\}\Big| \le Ct^{-p}\int |f(x)|^p dx,$$

if t is large enough, we have $|\Omega| \le C\,(t^{-p} + t^{-q})\,|E| \lesssim 1/100|E|$. Let $E' = E \setminus \Omega$, then $|E'| \simeq |E|$. Let $h = \chi_{E'}$, then it suffices to show that

$$\langle \mathrm{II}^i(f,g),h\rangle \le C_\gamma 2^{|i|(3+\gamma)}\left(\frac{(1+|a|)^2}{|a|}\right)\|f\|_{L^p}\|g\|_{L^q}|E|^{1-1/p-1/q},$$

where

$$\begin{aligned}&\langle \mathrm{II}^i(f,g),h\rangle\\ &=\sum_k 2^{-k\gamma}\int\int \mathcal{F}\big[m(\cdot)\widehat{\Psi}_{k+i}(\cdot)\big](y)\Big(f*\varphi_{k-|i|-5}(x+y)\Big)\Big(g*\psi_k(x+ay)\Big)\Big(h*\Psi_k(x)\Big)\,dxdy.\end{aligned}$$

By using Lemma 3.1 we obtain

$$|\langle \mathrm{II}^i(f,g),h\rangle| \lesssim \sum_{m\ge 0}\sum_k\sum_{n\in\mathbb{Z}}\frac{2^{i\gamma}}{(1+2^{m+i})^{\mathcal{N}-1}}2^{-k+m}\,\mathrm{F}_n^k(m)\,\mathrm{G}_n^k(m)\,\mathrm{H}_n^k(m)$$

where

$$\begin{aligned}\mathrm{F}_n^k(m) &:= \sup_{|y|\sim 1,\,\beta\in[0,1]}\big|f*\varphi_{k-|i|-5}(2^{-k+m}(n+\beta+y))\big|,\\ \mathrm{G}_n^k(m) &:= \sup_{\beta\in[0,1]}\Bigg(\int\limits_{|y|\sim 1}\big|g*\psi_k\big(2^{-k+m}(n+\beta+ay)\big)\big|\,dy\Bigg),\\ \mathrm{H}_n^k(m) &:= \int\limits_0^1\big|h*\Psi_k\big(2^{-k+m}(n+\beta)\big)\big|\,d\beta.\end{aligned}\tag{5.4}$$

Thus if we set $I_n^{k-m}=[2^{-k+m}n,2^{-k+m}(n+1)]$, then we have

$$\begin{aligned}&|\langle \mathrm{II}^i(f,g),h\rangle|\\ &\lesssim \sum_{m\ge 0}\frac{2^{i\gamma}}{(1+2^{m+i})^{\mathcal{N}-1}}\sum_{k,n}\int \mathrm{F}_n^k(m)\,\mathrm{G}_n^k(m)\,\mathrm{H}_n^k(m)\,\chi_{I_n^{k-m}}(x)dx\\ &\lesssim \sum_{m\ge 0}\frac{2^{i\gamma}}{(1+2^{m+i})^{\mathcal{N}-1}}\int\Big[\sup_{k,n}|\mathrm{F}_n^k(m)|\chi_{I_n^{k-m}}(x)\Big]\Big[\sum_{k,n}|\mathrm{G}_n^k(m)|^2\chi_{I_n^{k-m}}(x)\Big]^{\frac12}\Big[\sum_{k,n}|\mathrm{H}_n^k|^2\,\chi_{I_n^{k-m}}(x)\Big]^{1/2}dx.\end{aligned}$$

Thus it suffices to show that

$$\begin{aligned}\mathrm{II}_{\mathcal{J}}^{i,m}(f,g,h) &:= \sum_{I_n^{k-m}\in\mathcal{J}}\int \mathrm{F}_n^k(m)\,\mathrm{G}_n^k(m)\,\mathrm{H}_n^k(m)\,\chi_{I_n^{k-m}}(x)\,dx\\ &\lesssim 2^{m+|i|}\left(\frac{(1+|a|)^2}{|a|}\right)\|f\|_{L^p}\|g\|_{L^q}|E|^{1-1/p-1/q}\end{aligned}\tag{5.5}$$

for any finite collection $\mathcal{J}$ of dyadic intervals.

Lemma 5.1. *Let $\mathrm{F}_n^k(m)$, $\mathrm{G}_n^k(m)$ and $\mathrm{H}_n^k(m)$ be as in (5.4). If $x\in I_n^{k-m}=[2^{-k+m}n,2^{-k+m}(n+1)]$, then*

(1) $\mathrm{F}_n^k(m)\lesssim(1+2^{m-|i|})\mathrm{M}(\mathrm{M}f)(x)$,
(2) $\mathrm{G}_n^k(m)\lesssim\left(\frac{1+|a|}{|a|}\right)\mathrm{M}(g*\psi_k)(x)$,
(3) $\mathrm{H}_n^k(m)\lesssim\mathrm{M}(h*\Psi_k)(x)$.

Proof. The proof is similar to Lemma 4.4. □

Lemma 5.2. *Let* $\mathrm{F}_n^k(m)$, $\mathrm{G}_n^k(m)$ *and* $\mathrm{H}_n^k(m)$ *be as in* (5.4). *Let* $1 < p, q, s < \infty$ *then*

(1) $\left\| \sup_{k,n} \left(|\mathrm{F}_n^k(m)|\chi_{I_n^{k-m}}\right) \right\|_{L^p} \lesssim (1 + 2^{m-|i|})\|f\|_{L^p}$,

(2) $\left\| \left(\sum_{k,n} |\mathrm{G}_n^k(m)|^2 \chi_{I_n^{k-m}} \right)^{1/2} \right\|_{L^q} \lesssim \left(\frac{1+|a|}{|a|}\right) \|g\|_{L^q}$,

(3) $\left\| \left(\sum_{k,n} |\mathrm{H}_n^k(m)|^2 \ \chi_{I_n^{k-m}} \right)^{1/2} \right\|_{L^s} \lesssim \|h\|_{L^s}$.

Proof. The proof is similar to Lemma 4.5. □

Let $\mathcal{J}$ be a family of dyadic intervals, then for $1 < p, q, s < \infty$ we define

$$\begin{aligned}
\mathrm{size}^{(1)}_{\mathcal{J},\,p}\Big(\left(\mathrm{F}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}} \Big) &:= S\text{-}Size_{\mathcal{J},\,p}\Big(\left(\mathrm{F}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}} \Big),\\
\mathrm{size}^{(2)}_{\mathcal{J},\,q}\Big(\left(\mathrm{G}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}} \Big) &:= Size_{\mathcal{J},\,q}\Big(\left(\mathrm{G}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}} \Big),\\
\mathrm{size}^{(3)}_{\mathcal{J},\,s}\Big(\left(\mathrm{H}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}} \Big) &:= Size_{\mathcal{J},\,s}\Big(\left(\mathrm{H}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}} \Big),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{energy}^{(1)}_{\mathcal{J},\,p}\Big(\left(\mathrm{F}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}} \Big) &:= S\text{-}Energy_{\mathcal{J},\,p}\Big(\left(\mathrm{F}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}} \Big),\\
\mathrm{energy}^{(2)}_{\mathcal{J},\,q}\Big(\left(\mathrm{G}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}} \Big) &:= Energy_{\mathcal{J},\,q}\Big(\left(\mathrm{G}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}} \Big),\\
\mathrm{energy}^{(3)}_{\mathcal{J},\,s}\Big(\left(\mathrm{H}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}} \Big) &:= Energy_{\mathcal{J},\,s}\Big(\left(\mathrm{H}_n^k(m)\right)_{I_n^{k-m}\in\mathcal{J}} \Big).
\end{aligned}$$

Proposition 5.1 *(cf. Proposition 2.12 in [17]). Let us denote*

$$S_1 := \mathrm{size}^{(1)}_{\mathcal{J},\,p},\ E_1 := \mathrm{energy}^{(1)}_{\mathcal{J},\,p},\ S_2 := \mathrm{size}^{(2)}_{\mathcal{J},\,q},\ E_2 := \mathrm{energy}^{(2)}_{\mathcal{J},\,q},\ S_3 := \mathrm{size}^{(3)}_{\mathcal{J},\,s},\ E_3 := \mathrm{energy}^{(3)}_{\mathcal{J},\,s}.$$

Then

$$\mathrm{II}^{i,m}_{\mathcal{J}}(f,g,h) \lesssim S_1^{1-p\theta_1} E_1^{p\theta_1} S_2^{1-q\theta_2} E_2^{q\theta_2} S_3^{1-s\theta_3} E_3^{s\theta_3}, \tag{5.6}$$

for any $0 \le \theta_1,\ \theta_2,\ \theta_3 < 1$ *such that*

$$\theta_1 + \theta_2 + \theta_3 = 1 \text{ and } 1 - p\theta_1 > 0,\ 1 - q\theta_2 > 1,\ 1 - s\theta_3 > 0. \tag{5.7}$$

In particular for any $1 < p, q < \infty$, *there exists* $1 < s < \infty$ *such that* $1/p + 1/q + 1/s > 1$. *For such* $1 < p, q, s < \infty$, *there are* $0 \le \theta_1,\ \theta_2,\ \theta_3 < 1$ *satisfying* (5.7).

Proof. The proof is similar to Proposition 4.1. □

Lemma 5.3. *Let* $\mathrm{F}_n^k(m)$, $\mathrm{G}_n^k(m)$ *and* $\mathrm{H}_n^k(m)$ *be as in* (5.4). *Let* $\widetilde{I} := 2(1+|a|)I$. *For* $1 < p, q, s < \infty$, *we have*

(1) $\left\| \sup_{I_n^{k-m}\subseteq I} \left(|\mathrm{F}_n^k(m)|\chi_{I_n^{k-m}}\right) \right\|_{L^p}^p \le C_N(1 + 2^{m-|i|}) \int |f(x)|^p \left(1 + 2^{m-|i|}\frac{\mathrm{dist}(I,x)}{|I|}\right)^{-N} dx$,

(2) $\left\| \left(\sum_{I_n^{k-m}\subseteq I} |\mathrm{G}_n^k(m)|^2 \chi_{I_n^{k-m}} \right)^{1/2} \right\|_{L^q}^q \le C_N \left(\frac{1+|a|}{|a|}\right)^q \int |g(x)|^q \left(1 + 2^m \frac{\mathrm{dist}(\widetilde{I},x)}{|\widetilde{I}|}\right)^{-N} dx$,

(3) $\left\| \left(\sum_{I_n^{k-m}\subseteq I} |\mathrm{H}_n^k(m)|^2 \chi_{I_n^{k-m}} \right)^{1/2} \right\|_{L^s}^s \le C_N \int |h(x)|^s \left(1 + 2^m \frac{\mathrm{dist}(I,x)}{|I|}\right)^{-N} dx.$

Proof. The proof is similar to Lemma 4.7. $\square$

Corollary 5.1. *Let* $\mathrm{F}_n^k(m)$, $\mathrm{G}_n^k(m)$ *and* $\mathrm{H}_n^k(m)$ *be as in* (5.4). *Let* $\widetilde{I} := 2(1+|a|)I$. *For* $1 < p, q, s < \infty$, *we have*

(1) $\mathrm{size}^{(1)}_{\mathcal{J},\,p}\Big(\big(\mathrm{F}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}} \Big) \lesssim (1+2^{m-|i|}) \sup_{I\in\mathcal{J}} \left(\frac{1}{|I|} \int |f(x)|^p \left(1+2^{m-|i|}\frac{\mathrm{dist}(I,x)}{|I|}\right)^{-N} dx \right)^{1/p}$,

(2) $\mathrm{size}^{(2)}_{\mathcal{J},\,q}\Big(\big(\mathrm{G}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}} \Big) \lesssim \left(\frac{1+|a|}{|a|}\right) \sup_{I\in\mathcal{J}} \left(\frac{1}{|I|} \int |g(x)|^q \left(1+2^{m}\frac{\mathrm{dist}(\widetilde{I},x)}{|\widetilde{I}|}\right)^{-N} dx \right)^{1/q}$,

(3) $\mathrm{size}^{(3)}_{\mathcal{J},\,s}\Big(\big(\mathrm{H}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}} \Big) \lesssim \sup_{I\in\mathcal{J}} \left(\frac{1}{|I|} \int |h(x)|^s \left(1+2^{m}\frac{\mathrm{dist}(I,x)}{|I|}\right)^{-N} dx \right)^{1/s}$.

Lemma 5.4. *Let* $\mathrm{F}_n^k(m)$, $\mathrm{G}_n^k(m)$ *and* $\mathrm{H}_n^k(m)$ *be as in* (5.4). *Let* $\widetilde{I} := 2(1+|a|)I$. *For* $1 < p, q, s < \infty$, *we have*

(1) $\mathrm{energy}^{(1)}_{\mathcal{J},\,p}\Big(\big(\mathrm{F}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}} \Big) \lesssim (1+2^{m-|i|})(1+2^{-m+|i|})\|f\|_{L^p} \lesssim 2^{m+|i|}\|f\|_{L^p}$,

(2) $\mathrm{energy}^{(2)}_{\mathcal{J},\,q}\Big(\big(\mathrm{G}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}} \Big) \lesssim \left(\frac{(1+|a|)^{1+1/q}}{|a|}\right) \|g\|_{L^q}$,

(3) $\mathrm{energy}^{(3)}_{\mathcal{J},\,s}\Big(\big(\mathrm{H}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}} \Big) \lesssim \|h\|_{L^s}$.

Proof. For any $R > 0$, note that

$$\frac{1}{|I|}\int |f(x)|^p \left(1+R\,\frac{\mathrm{dist}(I,x)}{|I|}\right)^{-N} dx \lesssim (1+R^{-1})\,\mathrm{M}(|f|^p)(x)\,\chi_I(x).$$

By using this estimate, the proof is similar to Lemma 4.8. $\square$

Let Ω be as in (5.3). Split the collection of dyadic intervals $\mathcal{J}$ as $\bigcup_d \mathcal{J}_d$ where

$$\mathcal{J}_d := \left\{ I : 1+\mathrm{dist}(I,\Omega^c)/|I| \simeq 2^d \right\}.$$

If $I\in\mathcal{J}_d$, then $2^d I\cap\Omega^c \neq \emptyset$, and so by Corollary 5.1 and the definition of the set Ω one has

$$S_1(d) := \mathrm{size}^{(1)}_{\mathcal{J}_d,\,p}\Big(\big(\mathrm{F}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}_d} \Big) \lesssim 2^d 2^{m+|i|}\|f\|_{L^p}|E|^{-1/p},$$

$$S_2(d) := \mathrm{size}^{(2)}_{\mathcal{J}_d,\,q}\Big(\big(\mathrm{G}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}_d} \Big) \lesssim 2^d \left(\frac{1+|a|}{|a|}\right) \|g\|_{L^q}|E|^{-1/q},$$

$$S_3(d) := \mathrm{size}^{(3)}_{\mathcal{J}_d,\,s}\Big(\big(\mathrm{H}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}_d} \Big) \lesssim 2^{-Nd},$$

and

$$E_1(d) := \mathrm{energy}^{(1)}_{\mathcal{J}_d,\,p}\Big(\big(\mathrm{F}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}_d} \Big) \lesssim 2^{m+|i|}\|f\|_{L^p},$$

$$E_2(d) := \mathrm{energy}^{(2)}_{\mathcal{J}_d,\,q}\Big(\big(\mathrm{G}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}_d} \Big) \lesssim \left(\frac{(1+|a|)^{1+1/q}}{|a|}\right) \|g\|_{L^q},$$

$$E_3(d) := \mathrm{energy}^{(3)}_{\mathcal{J}_d,\,s}\Big(\big(\mathrm{H}_n^k(m)\big)_{I_n^{k-m}\in\mathcal{J}_d} \Big) \lesssim |E|^{1/s}.$$

By Proposition 5.1, for any $1 < s < \infty$ with $1/p+1/q+1/s > 1$, there are $0 \le \theta_1, \theta_2, \theta_3 < 1$ satisfying (5.7) and

$$
\begin{aligned}
\mathrm{II}^{i,m}_{\mathcal{J}}(f,g,h) &\le \sum_d \mathrm{II}^{i,m}_{\mathcal{J}_d}(f,g,h)\\
&\lesssim \sum_d S_1(d)^{1-p\theta_1}E_1(d)^{p\theta_1}S_2(d)^{1-q\theta_2}E_2(d)^{q\theta_2}S_3(d)^{1-s\theta_3}E_3(d)^{s\theta_3}\\
&\lesssim 2^{m+|i|}\left(\frac{(1+|a|)^2}{|a|}\right)\|f\|_{L^p}\|g\|_{L^q}|E|^{1-1/p-1/q}. \qquad \square
\end{aligned}
$$

Appendix A

A.1. Proof of (1.3)

Let $0<\gamma<1$ and H^γ be as in (1.1). Since $\int_{|y|>\epsilon}\frac{1}{y|y|^\gamma}\,dy=0$, we have

$$
\begin{aligned}
H^\gamma(f,g)(x) &= \lim_{\epsilon\to 0}\int\limits_{|y|>\epsilon}\frac{f(x-y)g(x-ay)}{y|y|^\gamma}\,dy\\
&= \lim_{\epsilon\to 0}\int\limits_{|y|>\epsilon}\left(\int \widehat{f}(\xi)\mathrm{e}^{2\pi\mathrm{i}(x-y)\xi}\,d\xi\right)\left(\int \widehat{g}(\eta)\mathrm{e}^{2\pi\mathrm{i}(x-ay)\eta}\,d\eta\right)\frac{dy}{y|y|^\gamma}\\
&= \iint \lim_{\epsilon\to 0}\left(\int\limits_{|y|>\epsilon}\frac{\mathrm{e}^{-2\pi\mathrm{i}y(\xi+a\eta)}-1}{y|y|^\gamma}dy\right)\widehat{f}(\xi)\,\widehat{g}(\eta)\,\mathrm{e}^{2\pi\mathrm{i}x(\xi+\eta)}\,d\xi d\eta.
\end{aligned}
$$

Thus for $t:=\xi+a\eta\neq 0$, by change of variable $y\to y/t$

$$
\begin{aligned}
m^\gamma(t) := \lim_{\epsilon\to 0}\int\limits_{|y|>\epsilon}\frac{\mathrm{e}^{-2\pi\mathrm{i}yt}-1}{y|y|^\gamma}dy &= sgn(t)\,|t|^\gamma\left(\lim_{\epsilon\to 0}\int\limits_{|y|>\epsilon|t|}\frac{\mathrm{e}^{-2\pi\mathrm{i}y}-1}{y|y|^\gamma}dy\right)\\
&= sgn(t)\,|t|^\gamma\left(\int\frac{\mathrm{e}^{-2\pi\mathrm{i}y}-1}{y|y|^\gamma}dy\right).
\end{aligned}
$$

A.2. Proof of (1.8)

Let $|a|\sim 2^i$ for some $i\in\mathbb{Z}$ and $a\neq 0,1$, then

$$
\begin{aligned}
|\xi+a\eta|^\gamma &= \sum_{k_1,k_2\in\mathbb{Z}}|\xi+a\eta|^\gamma\,\widehat{\psi}(2^{-k_1}\xi)\widehat{\psi}(2^{-k_2}\eta)\\
&= \sum_{k_1\ge k_2+i+4} + \sum_{k_1\le k_2+i-4} + \sum_{k_2+i-4<k_1<k_2+i+4}\\
&:= m_1(\xi,\eta)+m_2(\xi,\eta)+m_3(\xi,\eta).
\end{aligned}
$$

Note that

$$
m_1(\xi,\eta)=\left(\sum_{k_1\ge k_2+i+4}\frac{|\xi+a\eta|^\gamma}{2^{k_1\gamma}}\left(\frac{2^{k_1}}{|\xi|}\right)^\gamma\widehat{\psi}(2^{-k_1}\xi)\widehat{\psi}(2^{-k_2}\eta)\right)|\xi|^\gamma := \widetilde{m_1}(\xi,\eta)|\xi|^\gamma,
$$

$$
m_2(\xi,\eta)=\left(\sum_{k_1\le k_2+i-4}\frac{|\xi+a\eta|^\gamma}{2^{k_2\gamma}}\widehat{\psi}(2^{-k_1}\xi)\left(\frac{2^{k_2}}{|\eta|}\right)^\gamma\widehat{\psi}(2^{-k_2}\eta)\right)|\eta|^\gamma := \widetilde{m_2}(\xi,\eta)|\eta|^\gamma.
$$

If $k_1 \geq k_2+i+4$ and $\widehat{\psi}(2^{-k_1}\xi)\widehat{\psi}(2^{-k_2}\eta) \neq 0$, then $|\xi+a\eta| \sim |\xi| \sim 2^{k_1} \sim |(\xi,\eta)|$, and so for any multi-indices β we have

$$\left|\partial^\beta[\widetilde{m_1}(\xi,\eta))]\right| \lesssim |(\xi,\eta)|^{-|\beta|}. \tag{A.1}$$

Similarly, if $k_1 \leq k_2+i-4$ and $\widehat{\psi}(2^{-k_1}\xi)\widehat{\psi}(2^{-k_2}\eta) \neq 0$, then $|\xi+a\eta| \sim |a\eta| \sim |a|2^{k_2} \sim |(\xi,\eta)|$, and so for any multi-indices β we have

$$\left|\partial^\beta[\widetilde{m_2}(\xi,\eta))]\right| \lesssim |(\xi,\eta)|^{-|\beta|}. \tag{A.2}$$

For the part m_3, we have

$$m_3(\xi,\eta) = \left(\sum_{k_2+i-4<k_1<k_2+i+4} \frac{|\xi+a\eta|^\gamma}{2^{k_1\gamma}}\left(\frac{2^{k_1}}{|\xi|}\right)^\gamma \widehat{\psi}(2^{-k_1}\xi)\widehat{\psi}(2^{-k_2}\eta)\right)|\xi|^\gamma := \widetilde{m_3}(\xi,\eta)|\xi|^\gamma.$$

If $k_2+i-4 < k_1 < k_2+i-4$ and $\widehat{\psi}(2^{-k_1}\xi)\widehat{\psi}(2^{-k_2}\eta) \neq 0$, then $|\xi+a\eta| \lesssim 2^{k_1}$, thus for any multi-indices β

$$\left|\partial^\beta[\widetilde{m_3}(\xi,\eta)]\right| \lesssim |\xi+a\eta|^{-|\beta|}. \tag{A.3}$$

We have

$$T_\Gamma^\gamma(f,g)(x) = T_{m_1}(f,g)(x) + T_{m_2}(f,g)(x) + T_{m_3}(f,g)(x),$$

where

$$T_{m_1}(f,g)(x) := \int_{\mathbb{R}^2} \widetilde{m}_1(\xi,\eta)\left(|\xi|^\gamma \widehat{f}(\xi)\right)\widehat{g}(\eta)\,\mathrm{e}^{2\pi\mathrm{i}x(\xi+\eta)}\,d\xi d\eta,$$

$$T_{m_2}(f,g)(x) := \int_{\mathbb{R}^2} \widetilde{m}_2(\xi,\eta)\,\widehat{f}(\xi)\left(|\eta|^\gamma \widehat{g}(\eta)\right)\mathrm{e}^{2\pi\mathrm{i}x(\xi+\eta)}\,d\xi d\eta,$$

$$T_{m_3}(f,g)(x) := \int_{\mathbb{R}^2} \widetilde{m}_3(\xi,\eta)\left(|\xi|^\gamma \widehat{f}(\xi)\right)\widehat{g}(\eta)\,\mathrm{e}^{2\pi\mathrm{i}x(\xi+\eta)}\,d\xi d\eta.$$

By (A.1) and (A.2), the multipliers $\widetilde{m_1}$ and $\widetilde{m_2}$ satisfy the classical Marcinkiewicz–Mikhlin–Hörmander condition, thus by applying the Coifman–Meyer theorem we obtain

$$\|T_{m_1}(f,g)\|_{L^r} \lesssim \|D^\gamma f\|_{L^p}\|g\|_{L^q},$$

$$\|T_{m_2}(f,g)\|_{L^r} \lesssim \|f\|_{L^p}\|D^\gamma g\|_{L^q},$$

provided that $1 < p,q < \infty$, $1/p+1/q = 1/r$ and $0 < r < \infty$. For the operator T_{m_3}, by (A.3) we can apply the results for bilinear operators with nonsmooth symbols (1.4) to obtain

$$\|T_{m_3}(f,g)\|_{L^r} \leq C\|D^\gamma f\|_{L^p}\|g\|_{L^q}$$

for $1 < p,q < \infty$ satisfying $1/r = 1/p+1/q$ and for $r > 2/3$.

A.3. Proof of Lemma 4.1

Let

$$S^N(f)(x) := \left(\sum_{k,n} \left|f * \psi_{k+j} * (2^{-k}(n+2^N))\right|^2 \chi_{I_n^k}(x)\right)^{1/2},$$

then we aim to prove that

$$\|S^N(f)\|_{L^p} \lesssim (1 + N + 2^{|j|})\|f\|_{L^p} \quad \text{for } 1 < p < \infty.$$

For $p = 2$,

$$\begin{aligned}
&\left\|\left(\sum_{k,n} \left|f * \psi_{k+j}(2^{-k}(n+2^N))\right|^2 \chi_{I_n^k}(x)\right)^{1/2}\right\|_{L^2}^2 \\
&= \sum_{k,n} \left|f * \psi_{k+j}(2^{-k}(n+2^N))\right|^2 2^{-k} \\
&= \sum_k 2^{-k} \sum_n f * \psi_{k+j}(2^{-k}(n+2^N))\, \overline{f * \psi_{k+j}(2^{-k}(n+2^N))}.
\end{aligned}$$

Since $\{\psi_{k+j}\}_{k\in\mathbb{Z}}$ is a lacunary family, it is enough to prove that

$$\left|\sum_n f * \psi_{k+j}(2^{-k}(n+2^N))\, \overline{f * \psi_{k+j}(2^{-k}(n+2^N))}\right| \lesssim 2^k \|f\|_{L^2}^2. \tag{A.4}$$

Let $a_{k,j,n} := \overline{f * \psi_{k+j}(2^{-k}(n+2^N))}$, then

$$\begin{aligned}
\sum_n |a_{k,j,n}|^2 &= \sum_n a_{k,j,n}\, f * \psi_{k+j}(2^{-k}(n+2^N)) \\
&= \int f(z) \sum_n a_{k,j,n}\psi_{k+j}(2^{-k}(n+2^N) - z)\, dz \\
&\le \|f\|_{L^2} \left\|\sum_n a_{k,j,n}\, \psi_{k+j}(2^{-k}(n+2^N) - z)\right\|_{L^2(dz)}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\left\|\sum_n a_{k,j,n}\psi_{k+j}(2^{-k}(n+2^N) - z)\right\|_{L^2(dz)}^2 \\
&= \sum_{n,n'} a_{k,j,n}\, \overline{a_{k,j,n'}} \int \psi_{k+j}(2^{-k}(n+2^N) - z)\, \overline{\psi_{k+j}(2^{-k}(n'+2^N) - z)}\, dz \\
&= \sum_{n,n'} a_{k,j,n}\, \overline{a_{k,j,n'}} \int \left|\widehat{\psi}\left(2^{-k-j}\xi\right)\right|^2 \mathrm{e}^{2\pi \mathrm{i} 2^{-k}(n-n')\xi}\, d\xi \\
&\lesssim \sum_{n,n'} |a_{k,j,n}|\, |a_{k,j,n'}|\, \frac{2^{k+j}}{(1 + 2^j|n-n'|)^{10}} \\
&\lesssim 2^k \sum_n |a_{k,j,n}|^2.
\end{aligned}$$

Thus we obtain

$$\sum_n |a_{k,j,n}|^2 \lesssim \|f\|_{L^2} 2^{k/2} \Big(\sum_n |a_{k,j,n}|^2 \Big)^{1/2},$$

and so (A.4) follows. If $p = 1$, then by a Calderón–Zygmund decomposition with height λ, there are disjoint dyadic intervals $\{J\}$ such that

$$\sum_J |J| \leq \frac{1}{\lambda} \sum_J \int_J |f(x)|\, dx \leq \frac{1}{\lambda} \|f\|_{L^1}.$$

Set $\Omega = \bigcup J$,

$$|f(x)| \leq \lambda \quad \text{for } x \in \Omega^c.$$

Write $f = g + b$ where

$$g = f\chi_{\Omega^c} + \sum_J \left(\frac{1}{|J|} \int_J f(x)\, dx \right) \chi_J$$

and $b = f - g = \sum_J b_J$ with

$$b_J = \left(f - \frac{1}{|J|} \int_J f(x)\, dx \right) \chi_J.$$

Then since

$$\frac{1}{|J|} \int_J |f(x)|\, dx \leq \frac{2}{|2J|} \int_{2J} |f(x)|\, dx \leq 2\lambda$$

we have $\|g\|_{L^\infty} \leq 2\lambda$. These properties imply that

$$|\{x : S^N f(x) > \lambda\}| \leq |\{x : S^N g(x) > \lambda/2\}| + |\{x : S^N b(x) > \lambda/2\}|. \tag{A.5}$$

To estimate the first term we use the L^2-boundedness of S^N and write

$$|\{x : S^N g(x) > \lambda/2\}| \leq \frac{4}{\lambda^2} \|S^N g\|_{L^2}^2 \lesssim \frac{1}{\lambda^2} \|g\|_{L^2}^2 \lesssim \frac{1}{\lambda^2} (\lambda \|g\|_{L^1}) \lesssim \frac{1}{\lambda} \|f\|_{L^1}. \tag{A.6}$$

For the second term, for any dyadic interval J, there are only $(N+1)$ disjoint dyadic intervals $J^1, J^2, \cdots, J^{N+1}$ having the same length as J and such that, if $|I| \leq |J|$ and $I_N \subset J$, then I be comes subinterval in one of $J^1, J^2, \cdots, J^{N+1}$ (see page 152 in [17]). Define

$$\Omega_J := 5J \cup 5J^1 \cup 5J^2 \cup \cdots \cup 5J^{N+1}.$$

We write

$$|\{x : S^N b(x) > \lambda/2\}| \leq |\{x \in \cup_J \Omega_J : S^N b(x) > \lambda/2\}| + |\{x \in (\cup_J \Omega_J)^c : S^N b(x) > \lambda/2\}|. \tag{A.7}$$

We estimate the first term by

$$|\{x \in \cup_J \Omega_J : S^N b(x) > \lambda/2\}| \le |\cup_J \Omega_J| \lesssim N \sum_J |J| \lesssim N \frac{1}{\lambda} \|f\|_{L^1}. \tag{A.8}$$

For the second term

$$\begin{aligned} |\{x \in (\cup_J \Omega_J)^c : S^N b(x) > \lambda/2\}| &\le \frac{2}{\lambda} \int\limits_{(\cup_J \Omega_J)^c} S^N(b)(x)\,dx \le \frac{2}{\lambda} \sum_J \int\limits_{(\cup_J \Omega_J)^c} S^N(b_J)(x)\,dx \\ &\le \frac{2}{\lambda} \sum_J \int\limits_{(\Omega_J)^c} S^N(b_J)(x)\,dx. \end{aligned} \tag{A.9}$$

We claim that

$$\int\limits_{(\Omega_J)^c} S^N b_J(x)\,dx \lesssim \lambda 2^{|j|} |J|. \tag{A.10}$$

Assuming (A.10), one gets

$$\sum_J \int\limits_{(\Omega_J)^c} S^N b_J(x)\,dx \lesssim \lambda 2^{|j|} \sum_J |J| \lesssim |\Omega| \lesssim \lambda 2^{|j|} \|f\|_{L^1}, \tag{A.11}$$

and so by (A.7), (A.8), (A.9) and (A.11)

$$|\{x : S^N b(x) > \lambda/2\}| \lesssim \frac{N + 2^{|j|}}{\lambda} \|f\|_{L^1}. \tag{A.12}$$

By (A.5), (A.6) and (A.12)

$$|\{x : S^N f(x) > \lambda/2\}| \lesssim \frac{N + 2^{|j|}}{\lambda} \|f\|_{L^1}.$$

The left-hand side of (A.10) is smaller than

$$\sum_{k,n} \int\limits_{(\Omega_J)^c} |b_J * \psi_{k+j}(2^{-k}(n + 2^N))|\, \chi_{I_n^k}(x)\,dx = \sum_{k,n\,:\,2^{-k} \le |J|} + \sum_{k,n\,:\,2^{-k} > |J|} := \mathrm{A} + \mathrm{B}.$$

Estimates for A. Since $|I_n^k| \le |J|$ and $I_n^k \cap (\Omega_J)^c \ne \emptyset$, one must have $\mathrm{dist}(2^{-k}(n + 2^N), J) \ge 2|J|$. Using this

$$\begin{aligned} \mathrm{A} &= \sum_{\substack{k,n\,:\,2^{-k} \le |J|, \\ \mathrm{dist}(2^{-k}(n+2^N),J) \ge |J|}} \int\limits_{(\Omega_J)^c} |b_J * \psi_{k+j}(2^{-k}(n + 2^N))|\, \chi_{I_n^k}(x)\,dx \\ &\lesssim \sum_{\substack{k,n\,:\,2^{-k} \le |J|, \\ \mathrm{dist}(2^{-k}(n+2^N),J) \ge |J|}} \int\limits_{(\Omega_J)^c} |b_J(y)|\, \frac{2^j}{(1 + 2^{k+j}|2^{-k}(n + 2^N) - y|)^{10}}\,dy \\ &\lesssim \sum_{k\,:\,2^{-k} \le |J|} \frac{1}{(1 + 2^{k+j}|J|)^5} \int\limits_{(\Omega_J)^c} |b_J(y)| \left(\sum_n \frac{2^j}{(1 + 2^{k+j}|2^{-k}(n + 2^N) - y|)^5} \right) dy \end{aligned}$$

$$\lesssim \sum_{k\,:\,2^{-k}\le|J|} \frac{1}{(1+2^{k+j}|J|)^5} \int\limits_{(\Omega_J)^c} |b_J(y)|\,dy$$
$$\lesssim 2^{-j}\|b_J\|_{L^1} \lesssim 2^{-j}\lambda|J|.$$

Estimates for B. Let y_J be the center of the interval J, then

$$\begin{aligned} b_J*\psi_{k+j}(2^{-k}(n+2^N)) &= \int b_J(y)\Big[\psi_{k+j}(2^{-k}(n+2^N)-y)-\psi_{k+j}(2^{-k}(n+2^N)-y_J)\Big]dy \\ &= \int b_J(y)\left(\int\limits_{y_J}^{y}\frac{d}{dt}\left[\psi_{k+j}(2^{-k}(n+2^N)-t)\right]dt\right)dy \\ &= \int b_J(y)\left(\int\limits_{y_J}^{y}2^{2(k+j)}\psi'(2^{k+j}(2^{-k}(n+2^N)-t))\,dt\right)dy. \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{B} &= \sum_{k,\,n\,:\,2^{-k}>|J|}\int\limits_{(\Omega_J)^c}|b_J*\psi_{k+j}(2^{-k}(n+2^N))|\,\chi_{I_n^k}(x)\,dx \\ &\lesssim \sum_{k\,:\,2^{-k}>|J|}\int|b_J(y)|\left(\int\limits_{y_J}^{y}\sum_n\frac{2^{k+2j}}{(1+2^{k+j}|2^{-k}(n+2^N)-t|)^{10}}\,dt\right)dy \\ &\lesssim \sum_{k\,:\,2^{-k}>|J|}2^{k+j}|J|\,\|b_J\|_{L^1}\lesssim 2^j\|b_J\|_{L^1}\lesssim 2^j\lambda|J|. \end{aligned}$$

For $1<p\le 2$, the result follows by interpolating weak-L^1 and L^2 estimates. Let $\{\epsilon_{k,n}\}$ be independent and identically distributed random variables with $P(\epsilon_{k,n}=\pm1)=1/2$. We consider the operators

$$S_\epsilon^N f(x) := \sum_{k,n}\epsilon_{k,n}\, f*\psi_{k+j}*(2^{-k}(n+2^N))\,\chi_{I_n^k}(x).$$

As for S^N we obtain

$$\|S_\epsilon^N f\|_{L^2} = \|S^N f\|_{L^2}\lesssim N\|f\|_{L^2},\quad \text{and}\quad \|S_\epsilon^N f\|_{L^{1,\infty}}\lesssim (N+2^{|j|})\|f\|_{L^1},$$

which holds uniformly in $\epsilon_{k,n}=\pm1$. Thus by interpolation for $1<p\le 2$ we have

$$\|S_\epsilon^N f\|_{L^p}\lesssim (N+2^{|j|})\|f\|_{L^p}. \tag{A.13}$$

And by duality we obtain (A.13) for $2<p<\infty$. Then by Khinchine's inequality, if we take average over all $\epsilon_{k,n}=\pm1$, we obtain

$$\|S^N f\|_{L^p}^p = C_p\mathbb{E}\left(\|S_\epsilon^N\|_{L^p}^p\right)\lesssim (N+2^{|j|})^p\|f\|_{L^p}^p.$$

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