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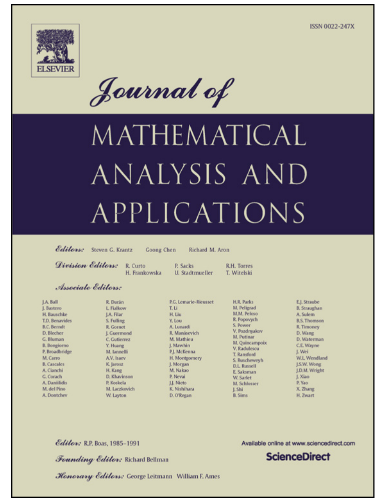

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Quasi-Variational Inequality Problems with Non-Compact Valued Constraint Maps

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Abstract

Quasi-variational inequality problems correspond to variational inequality problems in which the constraint set depends on the variable. They are playing nowadays an increasing role in the modelization of real life problem, in particular, because they provide a perfect framework for the reformulation of generalized Nash equilibrium problems. Our aim in this work is to establish the existence of solutions for quasi-variational inequalities defined by a non monotone map and a constraint map which possibly admits unbounded values. The key tools are the use of coercivity conditions and Himmelberg fixed point theorem. Applications to existence of generalized Nash equilibrium is also considered.

Keywords: Quasi-variational inequality, Pseudomonotone maps, Quasimonotone maps, Coercivity conditions, Nash equilibrium

2000 MSC: 49J40, 90C26, 90B10

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1. Introduction

Variational inequality problem was first introduced by Hartman and Stampacchia in 1966. Variational inequality theory acts an important role in applied mathematics, in particular for the problems of optimal control, optimization theory, equilibrium problems, partial differential equations and its related fields. Let X be a Banach space and X^* its topological dual endowed with weak* topology. Given a non-empty subset K of X and a set-valued map $T : K \rightrightarrows X^*$, the Stampacchia variational inequality problem $S(T,K)$ consists in finding a point $x \in K$ such that

$$\text{there exists } x^* \in T(x) \text{ with } \langle x^*, y - x\rangle \geq 0,\ \forall y \in K.$$

Classical existence results for this problem require a compactness assumption on the set K (see, [14]). Recently, Aussel and Hadjisavvas [7] proved an existence result for Stampacchia variational inequality problem over unbounded set by using the following coercivity condition for the map T:

$$(\bar{C}) \qquad \begin{array}{l} \exists\, \rho > 0,\ \forall\, x \in K \setminus \bar{B}(0,\rho),\ \exists\, y \in K \text{ with } ||y|| < ||x|| \\ \text{such that } \forall\, x^* \in T(x),\ \langle x^*, x - y\rangle \geq 0. \end{array}$$

A quasi-variational inequality corresponds to a variational inequality in which the constraint set is depending on the variable. More precisely, given two set-valued maps $T : X \rightrightarrows X^*$ and $K : D \rightrightarrows D$ (where D is a non-empty subset of X), a solution of the quasi-variational inequality problem QVI(T,K) is a point $x \in D$ such that

$$x \in K(x) \text{ and } \exists\, x^* \in T(x) \text{ with } \langle x^*, y - x\rangle \geq 0,\ \forall\, y \in K(x).$$

It is worth to mention that the quasi-variational inequality problem plays an important role in the study of quasi-optimization problem and generalized Nash equilibrium problem (see, e.g., [6, 13]). For some important results on the existence of solutions for the quasi-variational inequality problem, one can refer to [6, 18–20].

Again, similarly as for variational inequalities, most of the existence results for quasi-variational inequalities require some compactness of the set D and/or of the values of the constraint map K (see, e.g., [6, 20]). But very few results with possibly unbounded constraint maps have been proved in the literature and they are stated with a monotonicity assumption on T; see [2, 21]. Our aim in this work is to establish such existence result under weak monotonicity and regularity assumptions on the considered set-valued maps. The present work can be seen as an extension to quasi-variational inequalities of Bianchi et al. [10] where existence results for variational inequalities were proved by considering different coercivity conditions. Note that in [9], we have used similar set of weak monotonicity and regularity assumptions to prove the existence of the so-called "projected solution" for non-self quasi-variational inequalities. Nevertheless, both works differ since in [9], the aim is to face quasi-variational inequalities with non-self but compact-valued constraint map while here we focus on quasi-variational inequalities defined by self constraint map with (possibly) non-compact values.

The paper is organized as follows. After introducing the main concepts and notations in Section 2, the concept of coercivity condition in the context of quasi-variational inequality is discussed in Subsection 3.1. Then, in Subsection 3.2, the existence of solutions for pseudomonotone and quasimonotone quasi-variational inequalities with non-compact valued constraint map is proved under coercivity assumptions in a finite dimensional space. The case of a product of finite dimension and infinite dimension sets is considered in Subsection 3.3 for a particular case. Finally, an existence result for quasiconvex generalized Nash equilibrium problem with unbounded strategy sets is deduced.

2. Preliminaries

In this section we first recall some definitions and notations which will be used in the sequel. As it will be observed in Remark 1 (c), one technical assumption of the main existence results of this work forces them to be restricted to the finite dimensional setting. Nevertheless, in this section and subsection 3.1, the main concepts are given in the context of Banach spaces since actually the forthcoming Proposition 1 and Proposition 2 hold true in such general spaces.

Let X be a Banach space, X^* its topological dual and $\langle \cdot, \cdot \rangle$ be the duality pairing. The topological dual X^* is endowed with the weak* topology. Let S be a non-empty subset of X. The topological interior, the closure and the convex hull of S will be denoted by $\text{int}S$, $\text{cl}S$ and $\text{conv}S$, respectively. We denote the open ball and the closed ball in X with center x and radius $\varepsilon > 0$ by $B(x, \varepsilon)$ and $\bar{B}(x, \varepsilon)$, respectively. The symbol $T : X \rightrightarrows X^*$ stands for a set-valued map T from X to X^*. The domain and the graph of a set-valued map T will be denoted by $\text{dom}T$ and $\text{Gr}T$, respectively. For any $x, y \in X$, the notations $[x, y]$ and $]x, y[$ will be used for the sets $\{tx+(1-t)y : t \in [0, 1]\}$ and $\{tx + (1 - t)y : t \in]0, 1[\}$, respectively.

Let us now recall some classical definitions of generalized monotonicity of a set-valued map. A set-valued map $T : X \rightrightarrows X^*$ is called

− *monotone* on a subset K of X if,

$$\langle y^* - x^*, y - x \rangle \geq 0, \quad \text{for all } x, y \in K,\ x^* \in T(x) \text{ and } y^* \in T(y),$$

− *pseudomonotone* on a subset K of X if, for any $x, y \in K$,

$$\exists\, x^* \in T(x) \text{ with } \langle x^*, y - x \rangle \geq 0 \quad \Rightarrow \quad \langle y^*, y - x \rangle \geq 0,\ \forall y^* \in T(y),$$

− *quasimonotone* on K if, for any $x, y \in K$,

$$\exists\, x^* \in T(x) \text{ with } \langle x^*, y - x \rangle > 0 \quad \Rightarrow \quad \langle y^*, y - x \rangle \geq 0,\ \forall y^* \in T(y).$$

Clearly any monotone map is also pseudomonotone and pseudomonotonicity implies quasimonotonicity. Immediate examples of pseudomonotone maps that are not monotone, respecticvely of quasimonotone maps that are not pseudomonotone, are the gradient of the differentiable functions $x \mapsto x^3$, respectively the gradient of the function

$$x \mapsto \begin{cases} x^3 & \text{if } x < 0 \\ 0 & \text{if } x \in [0,1] \\ (x-1)^3 & \text{otherwise.} \end{cases}$$

More generally the subdifferential of any lower semicontinuous pseudoconvex (resp. quasiconvex) function is pseudomonotone (respectively quasimonotone), thus providing a wide family of such maps. The interested reader can refer to [4, 16] for definitions and associated results.

The classical definitions of upper semi-continuity, lower semi-continuity and closeness of a set-valued map can be found in [3].

Recently, Hadjisavvas [15] introduced the following two refinements of semicontinuity. A set-valued map $T : X \rightrightarrows X^*$ is called *lower sign-continuous* on a convex set $K \subseteq X$ if for all $x, y \in K$,

$$\forall t \in]0,1[, \quad \inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x \rangle \geq 0 \quad \Rightarrow \quad \inf_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0,$$

where $x_t = tx + (1-t)y$. And the map T is called *upper sign-continuous* on K if for all $x, y \in K$,

$$\forall t \in]0,1[, \quad \inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x \rangle \geq 0 \quad \Rightarrow \quad \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0.$$

It is easily follows from the definitions that every lower sign-continuous map is also upper sign-continuous. It is worth to note that each upper (respectively lower) semi-continuous map is upper (respectively lower) sign-continuous. Also, if T is upper (respectively lower) hemicontinuous, then the map T is upper (respectively lower) sign-continuous on K (see [8]). We say that the set-valued map T is *locally upper sign-continuous* [5] at $x \in X$ if there exist a convex neighbourhood V_x of x and an upper sign-continuous submap $\Phi_x : V_x \rightrightarrows X^*$ where for each $v \in V_x$, $\Phi_x(v)$ is a non-empty, convex and w^*-compact set satisfying $\Phi_x(v) \subseteq T(v) \setminus \{0\}$. The concept of locally upper sign-continuity acts an important role for the study of normal operator of quasiconvex functions due to the fact that a cone-valued map need not be upper semi-continuous (see [8]). More information and results on (locally) upper sign-continuous map can be found in [5].

Finally, the map T is said to be *dually lower semi-continuous* [5] on a set $K \subseteq X$ if, for each $x \in K$ and for each sequence $(y_k)_k \subseteq K$ with $y_k \to y$,

$$\liminf_k \sup_{y_k^* \in T(y_k)} \langle y_k^*, x - y_k \rangle \leq 0 \quad \Rightarrow \quad \sup_{y^* \in T(y)} \langle y^*, x - y \rangle \leq 0.$$

Given a set $K \subseteq X$ and a map $T : X \rightrightarrows X^*$, the notation $S^*(T, K)$ is used for the set of star solutions, also called "nontrivial solutions", of the Stampacchia variational inequality, i.e.,

$$S^*(T, K) := \{x \in K : \exists\, x^* \in T(x) \setminus \{0\} \text{ with } \langle x^*, y - x\rangle \geq 0,\ \forall y \in K\},$$

while $M(T, K)$ stands for the set of solutions of the Minty variational inequality, i.e.,

$$M(T, K) := \{x \in K : \langle y^*, y - x\rangle \geq 0,\ \forall y \in K,\ \forall y^* \in T(y)\}.$$

It is easy to note that the set $M(T, K)$ is closed and convex whenever K is closed and convex.

Let D be a non-empty subset of X. For two set-valued maps $T : X \rightrightarrows X^*$ and $K : D \rightrightarrows D$, we denote by $QVI^*(T, K)$, the set of star solutions of the Stampacchia quasi-variational inequality, i.e.,

$$QVI^*(T, K) := \left\{ x \in K(x) : \exists\, x^* \in T(x) \backslash \{0\} \text{ with } \langle x^*, y - x\rangle \geq 0,\ \forall y \in K(x) \right\}.$$

3. Main Results

3.1. Coercivity Conditions for Quasi-Variational Inequalities

In [10], Bianchi et al. defined and discussed different coercivity conditions for pseudomonotone and quasimonotone Stampacchia and Minty variational inequalities. In this subsection, let us adapt those coercivity conditions to the context of quasi-variational inequalities.

The main coercivity condition, denoted by $(\bar{C}_\mu)$ that we will consider is defined as follows: let D be a non-empty convex subset of a Banach space

X. Let $T : X \rightrightarrows X^*$ and $K : D \rightrightarrows D$ be two set-valued maps. For every $\mu \in D$, the coercivity condition $(\bar{C}_\mu)$ holds at μ if:

$$(\bar{C}_\mu) \qquad \exists\, \rho_\mu > 0,\ \forall x \in K(\mu) \setminus \bar{B}(0, \rho_\mu), \exists\, y \in K(\mu) \text{ with } ||y|| < ||x|| \\ \text{such that } \forall x^* \in T(x),\ \langle x^*, x - y\rangle \geq 0.$$

Whenever the constraint map K is constant (that is, $K(\mu) = D\ \forall \mu$), the coercivity condition $(\bar{C}_\mu)$ simply coincides with the condition $(\bar{C})$. Further, in the case of constant constraint map, it is easy to see that $(\bar{C}_\mu)$ is equivalent to the coercivity condition (C'), considered in [10].

Example 1. *In order to show that the coercivity condition $(\bar{C}_\mu)$ is not too restrictive, let us consider the quasi-variational inequality defined by $D = (\mathbb{R}^+)^2$, $K(\mu_1, \mu_2) = \{(x_1, x_2) \in D : x_2 - \mu_2 \leq 0\} = \mathbb{R}^+ \times [0, \mu_2]$ and the map T being the Clarke subdifferential of the Liptchitz function f defined on $\mathbb{R}^2$ by*

$$f(x_1, x_2) = \begin{cases} x_1^2 + x_2^2 & \text{if } \|(x_1, x_2)\|_2 \leq 1 \\ 1 & \text{otherwise.} \end{cases}$$

One has therefore

$$T(x_1, x_2) = \partial^C f(x_1, x_2) = \begin{cases} \{(2x_1, 2x_2)\} & \text{if } \|(x_1, x_2)\|_2 < 1 \\ [0, +\infty[\ \{(x_1, x_2)\} & \text{if } \|(x_1, x_2)\|_2 = 1 \\ \{(0, 0)\} & \text{otherwise.} \end{cases}$$

For any $\mu = (\mu_1, \mu_2) \in D$ set $\rho_\mu = 1$. Then for any $x \in K(\mu) \setminus \bar{B}((0, 0), 1)$, one has $\langle x^*, x - y\rangle = 0$, for any $y = \alpha x + (1 - \alpha)x/\|x\|$ with $\alpha \in]0, 1]$.

Another important coercivity condition, called (C), has been used and compared with (C') in [10]. In the context of QVI, this condition (C) would turn out to express as follows: for every $\mu \in D$, the coercivity condition (C_μ) holds at μ if:

$$(C_\mu) \qquad \exists\, n_\mu \in \mathbb{N},\ \forall x \in K(\mu) \setminus K_{n_\mu} \text{ (where } K_{n_\mu} = \{z \in K(\mu) : ||z|| \leq n_\mu\}\text{)},$$
$$\exists\, y \in K_{n_\mu} \text{ with such that } \forall x^* \in T(x),\ \langle x^*, x - y\rangle \geq 0.$$

It is important to note that conditions $(\bar{C}_\mu)$ and (C_μ) are trivially satisfied for any $\mu \in D$ for which $K(\mu)$ is bounded. Indeed, in this case one can simply consider ρ_μ (or n_μ) greater than $\mathrm{diam}(0, K(\mu)) := \max\{||y|| : y \in K(\mu)\}$.

In the case of variational inequalities, the interrelations between $(\bar{C})$, (C) and the non-emptiness of the solution set of $S(T, K)$ have been described in [10, Theorem 3.2]. A partial counterpart for quasi-variational inequalities is given in the following result.

Proposition 1. *Let D be a non-empty convex subset of a Banach space X. Let $T : X \rightrightarrows X^*$ and $K : D \rightrightarrows D$ be two set-valued maps. Consider the following statements:*

- *(i) $QVI(T, K) \neq \emptyset$;*
- *(ii) there exists $\mu \in D$ such that the condition (C_μ) holds;*
- *(iii) there exists $\mu \in D$ such that the condition $(\bar{C}_\mu)$ holds.*

Then, one always has $(ii) \Rightarrow (iii)$. Moreover, $(i) \Rightarrow (ii)$ provided the map T is pseudomonotone on D.

Proof. The implication $(ii) \Rightarrow (iii)$ clearly comes from the fact that, for any $x \in K(\mu) \setminus K_{n_\mu}$, $[K(\mu) \cap B(0, n_\mu)] \subset [K(\mu) \cap B(0, ||x||)]$. Let us assume that T is pseudomonotone. In order to prove $(i) \Rightarrow (ii)$, let us pick $\mu \in QVI(T, K)$

and choose $n_\mu \in \mathbb{N}$ with $n_\mu > ||\mu||$. By the pseudomonotonicity of T, one has $\langle x^*, x - \mu \rangle \geq 0$, for any $x \in K(\mu)$ and $x^* \in T(x)$. Therefore the coercivity condition (C_μ) is satisfied for $y = \mu$. □

As it can be observed from the above proof, the items (ii) and (iii) of Proposition 1 can actually be specified as follows: for any $\mu \in QVI(T, K)$, condition $(\bar{C}_\mu)$ (respectively, (C_μ)) holds. We should note that, in the above Proposition 1, the implication $(i) \Rightarrow (ii)$ or $(i) \Rightarrow (iii)$ do not hold in general if the map T is not pseudomonotone. The following example illustrates this situation.

Example 2. *Let us consider the subset $D = \mathbb{R} \times [0, \infty[$ of $\mathbb{R}^2$ and let $T : D \subseteq \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined by for all $(x_1, x_2) \in D$,*

$$T(x_1, x_2) = \begin{cases} \{(0,1)\} & \text{if } x_2 \neq 1 \\ \{(t,1) : t \in [0,1]\} & \text{if } x_2 = 1. \end{cases}$$

and the constraint map $K : D \rightrightarrows D$ by

$$K(x_1, x_2) =]-\infty, x_1] \times [0, \infty[.$$

Then T is not pseudomonotone (take $x = (0,0)$ and $y = (-2,1)$) and $(0,0) \in QVI(T, K)$ (which is thus a nonempty set). But the coercivity condition $(\bar{C}_\mu)$ (respectively, (C_μ)) does not hold for any $\mu \in D$ (consider for example $\mu = (-2, 0)$ and for any $\rho_\mu > 0$, $x = (-2\rho_\mu, 0)$).

Contrary to [10, Theorem 3.2], the above Proposition 1 does not establish a full equivalence of the three considered items. Indeed, as it will be seen in the forthcoming Theorems 1 and 3, the following uniform coercivity condition

$(\bar{C}^u)$ is required to deduce the existence of solutions for quasi-variational inequalities.

Definition 1. Let D be a non-empty subset of a Banach space X. Let $T : X \rightrightarrows X^*$ and $K : D \rightrightarrows D$ be two set-valued maps. Then the quasi-variational inequality $QVI(T,K)$ is said to satisfy the *uniform coercivity condition* $(\bar{C}^u)$ if:

(a) the coercivity condition $(\bar{C}_\mu)$ holds for all $\mu \in D$;

(b) there exists $\rho > \sup\{\rho_\mu : \mu \in D\}$ such that $K(\mu) \cap \bar{B}(0,\rho) \neq \emptyset$ for all $\mu \in D$.

Simple examples for which the coercivity conditions (C_μ), $(\bar{C}_\mu)$ or $(\bar{C}^u)$ are satisfied will be given in the forthcoming Example 3 and Remark 1 *b*).

3.2. General Existence Results

The following theorem is one of our main results. It establishes the existence of solution to the quasi-variational inequality problem for non-compact valued constraint maps.

Theorem 1. *Let D a non-empty closed convex subset of $\mathbb{R}^n$. Let $T : D \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $K : D \rightrightarrows D$ be two set-valued maps. Then, QVI(T,K) admits at least one solution if the following properties hold:*

(i) T is pseudomonotone and locally upper sign-continuous on D;

(ii) the map K is closed, lower semi-continuous and convex valued map with int$K(\mu) \neq \emptyset$ for all $\mu \in D$;

(iii) the quasi-variational inequality $QVI(T,K)$ satisfies the uniform coercivity condition $(\bar{C}^u)$.

In order to prove the above theorem, the following lemma will be used.

Lemma 1. *Let D be a non-empty convex subset of $\mathbb{R}^n$. Suppose that $K : D \rightrightarrows D$ is a lower semi-continuous map with closed convex values. Then, for any $\rho > 0$, the map $K_\rho : D \rightrightarrows D$ defined by $K_\rho(\mu) = K(\mu) \cap \bar{B}(0, \rho)$ is lower semi-continuous provided int$(K_\rho(\mu)) \neq \emptyset$ for all $\mu \in D$.*

Proof. Let $\mu \in D$ and $y \in K(\mu) \cap B(0, \rho)$ be arbitrary. Let $(\mu_n)_n$ be an arbitrary sequence such that $\mu_n \to \mu$ as $n \to \infty$. As $y \in K(\mu)$ and K is lower semi-continuous, there exists a sequence $y_n \in K(\mu_n)$ such that $y_n \to y$. Thus there exits a sequence $(z_n)_n$ such that $z_n \in K(\mu_n) \cap \bar{B}(0, \rho)$ $\forall\, n \in \mathbb{N}$ and $(z_n)_n$ converges to y. As $y \in K(\mu) \cap B(0, \rho)$ is arbitrary, we have $[K(\mu) \cap B(0, \rho)] \subseteq \mathrm{Liminf}_{\mu_n \to \mu} K_\rho(\mu_n)$ and thus

$$\mathrm{cl}(K(\mu) \cap B(0, \rho)) \subseteq \mathrm{cl}(\mathrm{Liminf}_{\mu_n \to \mu} K_\rho(\mu_n)) = \mathrm{Liminf}_{\mu_n \to \mu} K_\rho(\mu_n). \tag{1}$$

Now, we claim that

$$\mathrm{cl}(K(\mu) \cap B(0, \rho)) = K(\mu) \cap \bar{B}(0, \rho). \tag{2}$$

Since $[K(\mu) \cap B(0, \rho)] \subseteq K(\mu) \cap \bar{B}(0, \rho)$ and $K(\mu)$ is closed, $\mathrm{cl}(K(\mu) \cap B(0, \rho)) \subseteq K(\mu) \cap \bar{B}(0, \rho)$. Let $z \in K(\mu) \cap \bar{B}(0, \rho)$. As $K(\mu) \cap \bar{B}(0, \rho)$ is convex and $\mathrm{int}(K(\mu) \cap \bar{B}(0, \rho)) \neq \emptyset$, there exists a sequence $(u_n)_n \subset \mathrm{int}(K(\mu) \cap \bar{B}(0, \rho)) = \mathrm{int}(K(\mu)) \cap B(0, \rho))$ converging to z. Thus, it follows that $z \in \mathrm{cl}(K(\mu) \cap B(0, \rho))$. Therefore our claim (2) is true. By (1) and (2), we have

$$K_\rho(\mu) = K(\mu) \cap \bar{B}(0, \rho) \subseteq \mathrm{Liminf}_{\mu_n \to \mu} K_\rho(\mu_n).$$

Therefore K_ρ is lower semi-continuous at μ. □

Proof of Theorem 1. Define a map $G : D \rightrightarrows D$ by $G(x) := S(T, K_\rho(x))$ for any $x \in D$ where $K_\rho : D \rightrightarrows D$ is defined by $K_\rho(\mu) := K(\mu) \cap \bar{B}(0, \rho)$ for all $\mu \in D$. Let $\mu \in D$ be arbitrary. As $K(\mu) \cap \bar{B}(0, \rho)$ is non-empty, we have $K(\mu) \cap B(0, \rho) \neq \emptyset$. Indeed, let $z \in K(\mu) \cap \bar{B}(0, \rho)$ with $||z|| = \rho$. Then, according to $(\bar{C}_\mu)$ and since $\rho_\mu < \rho$, there exists $z' \in K(\mu)$ such that $||z'|| < ||z||$, that is, $||z'|| < \rho$.

Let $p \in K(\mu) \cap B(0, \rho)$ be arbitrary. Since $K(\mu)$ is convex and $\mathrm{int}K(\mu) \neq \emptyset$, there exists a sequence $(p_n)_{n\in\mathbb{N}}$ with $p_n \in \mathrm{int}K(\mu)$ and $(p_n)_n$ converges to p. Thus $p_n \in \mathrm{int}K(\mu) \cap B(0, \rho)$ for sufficiently large n and hence $\mathrm{int}K(\mu) \cap B(0, \rho) \neq \emptyset$. Therefore it follows that

$$\mathrm{int}(K_\rho(\mu)) = \mathrm{int}(K(\mu) \cap \bar{B}(0, \rho)) = \mathrm{int}K(\mu) \cap B(0, \rho) \neq \emptyset. \tag{3}$$

Since $\mu \in D$ is arbitrary in (3) and according to Lemma 1, the map K_ρ is lower semi-continuous. Also, the closedness of K_ρ is clear from the closedness of K and $\bar{B}(0, \rho)$. Finally, it is easy to see that K_ρ is convex valued. Therefore K_ρ is closed, lower semi-continuous and convex valued map with $\mathrm{int}K_\rho(\mu) \neq \emptyset$ for all $\mu \in D$. Hence it follows from [5, Proposition 4.2] that the map G is closed. Now, $G(D)$ is a compact subset of $\mathbb{R}^n$ since $G(D) \subseteq K(D) \cap \bar{B}(0, \rho)$ and K is closed. Thus, the map G is upper semi-continuous on D.

It follows from [5, Lemma 3.1, (ii)] and the pseudomonotonicity and locally upper sign-continuity of T that

$$G(x) = S(T, K_\rho(x)) = M(T, K_\rho(x)), \quad \forall\, x \in D. \tag{4}$$

Since K_ρ is compact-valued and combining (4) with [7, Lemma 2.1] and [7, Proposition 2.1], the map G has non-empty and convex values.

Hence, G is an upper semi-continuous map with non-empty, convex and closed values. Also, $G(D)$ is contained in a compact set $K(D) \cap \bar{B}(0, \rho)$ of D. According to the fixed point theorem due to Himmelberg [17, Theorem 2], the map G has a fixed point. This means that there exists a point $x_0 \in K_\rho(x_0) = K(x_0) \cap \bar{B}(0, \rho)$ such that

$$\exists\, x_0^* \in T(x_0) \text{ with } \langle x_0^*, y - x_0 \rangle \geq 0, \ \forall\, y \in K(x_0) \cap \bar{B}(0, \rho). \tag{5}$$

We now claim that there exists $y_0 \in K(x_0) \cap B(0, \rho)$ such that

$$\forall\, x^* \in T(x_0), \ \langle x^*, x_0 - y_0 \rangle \geq 0. \tag{6}$$

Indeed, if $||x_0|| = \rho$, then by $(\bar{C}_{x_0})$ and since $\rho > \rho_{x_0}$, (6) comes by considering $x = x_0$ in $(\bar{C}_{x_0})$. If $||x_0|| < \rho$, then one can simply take $y_0 = x_0$. In both cases, by combining (6) with (5), one has $\langle x_0^*, x_0 - y_0 \rangle = 0$.

Now, let y be an arbitrary element of $K(x_0)$. Then there exists $t \in (0, 1)$ such that $(1 - t)y + ty_0 \in K(x_0) \cap \bar{B}(0, \rho)$. Therefore, from (5), we have $\langle x_0^*, (1 - t)y + ty_0 - x_0 \rangle \geq 0$ and thus $\langle x_0^*, y - x_0 \rangle \geq 0$. The point y being an arbitrary element of $K(x_0)$, x_0 is a solution of $QVI(T, K)$. $\square$

Let us now give a simple example of quasi-variational inequality for which the existence of a solution can be proved using Theorem 1 while the classical existence theorems fail.

Example 3. *Let us consider the subset* $D = (\mathbb{R}^+)^2$ *and let* $T : D \subseteq \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ *be defined by for all* $(x_1, x_2) \in D$,

$$T(x_1, x_2) = \begin{cases} (1, 1) & \text{if } x_2 \neq 1 \\ \{(t, t) : t \in [1, 2]\} & \text{if } x_2 = 1. \end{cases}$$

Then, T is upper semi-continuous on D with non-empty, convex and compact values. Hence T is locally upper sign-continuous on D. Also, it is easy to check that T is pseudomonotone on D.

Let us define the constraint map $K : D \rightrightarrows D$ as follows:

$$K(x_1, x_2) = \begin{cases} [0,1] \times [0,\infty[& \text{if } 0 \leq x_1 < 1, \\ [0,x] \times [0,\infty[& \text{if } x_1 \geq 1. \end{cases}$$

One can easily verify that the map K is closed, lower semi-continuous and convex-valued with int$K(\mu) \neq \emptyset$ for all $\mu \in D$. Consider $\rho_\mu = 1$ for all $\mu \in D$. Then the coercivity condition $(\bar{C}_\mu)$ holds for all $\mu \in D$ for the set-valued map T and the constraint map K. Moreover, for any $\rho > \sup\{\rho_\mu : \mu \in D\} = 1$, $K(\mu) \cap \bar{B}(0,\rho) \neq \emptyset$ for all $\mu \in D$. Thus the uniform coercivity condition $(\bar{C}^u)$ is also satisfied. Hence, according to Theorem 1, the quasi-variational inequality $QVI(T,K)$ admits at least one solution.

Nevertheless classical existence results (see e.g. [2, 21]) cannot apply for this very simple example since there are based on a monotonicity assumption on T and here this map is clearly not monotone on D (consider the elements $(x,x^*) = ((0,1),(2,2))$ and $(y,y^*) = ((1,1),(1,1))$ of GrT). Note that actually $\bar{x} = (0,0)$ is a trivial solution of this quasi-variational inequality. Finally, the existence results proved in [6] (see for example Proposition 3.2 therein) cannot be applied to this example since D is not bounded (and thus not compact).

Remark 1. *a) As a corollary to the above theorem, we can deduce [6, Proposition 3.2] whenever the set D is compact.*

b) Let us observe that the classical Kakutani fixed point theorem cannot be used in the above proof because the set D is not assumed to be bounded.

c) Due to assumption (ii) and the uniform coercivity condition, Theorem 1 cannot be extended to infinite dimensional setting. Indeed, as observed in the proof of the theorem, for any $\mu \in D$, $K(\mu) \cap \bar{B}(0, \rho)$ is a compact set with non-empty interior, thus forcing K to be defined on a finite dimensional space.

d) Let us observe that under the assumptions of Theorem 1, the solution set $QVI(T, K)$ is closed. Indeed, let $\{x_n\}$ be a sequence in D such that $x_n \in QVI(T, K)$ for all n and $x_n \to x$ as $n \to \infty$. Since D is closed, $x \in D$. Define the map $S : D \rightrightarrows D$ where for any $\mu \in D$, $S(\mu) = S(T, K(\mu))$. According to [5, Proposition 4.2], the map S is closed under the assumptions of Theorem 1. As $x_n \in S(x_n)$ for all n, we have that $x \in S(x)$ or in other words $x \in QVI(T, K)$.

e) Note nevertheless that even under the set of assumptions of Theorem 1 the solution set $QVI(T, K)$ of the quasi-variational inequality may not be compact. Indeed if one consider the set D of $\mathbb{R}^2$ and the constraint map K defined in Example 3 and slightly modifies the map T by setting $T(x) = \{(0, 1)\}$, for any x in D, then one can easily verify that the unbounded set $D^* = [0, +\infty[\times \{0\}$ is included in the solution set $QVI(T, K)$.

In [21], Tian *et al.* established the following existence result for quasi-variational inequality on a non-compact set of an infinite dimensional space (under some coercivity condition) without assuming the non-emptiness of $\text{int}K(.)$. But on the other hand, the map T is assumed to be monotone and upper hemi-continuous, both being stronger assumptions than those of Theorem 1. Moreover in their existence result, the values of T are additionally required to be convex and compact.

Theorem 2. *[21] Let D be a non-empty convex subset of a locally convex Hausdorff topological vector space E. Let $T : D \rightrightarrows E^*$ be a monotone and upper hemi-continuous map with compact convex values and $K : D \rightrightarrows D$ be a closed lower semi-continuous map with non-empty closed convex values. Suppose that there exist a non-empty compact convex set $Z \subseteq D$ and a non-empty subset $M \subseteq Z$ such that*

(i) $K(M) \subseteq Z$ *and* $K(x) \cap Z \neq \emptyset$ *for all* $x \in Z$;

(ii) for each $x \in Z \setminus M$, *there exists* $y \in K(x) \cap Z$ *with* $\inf_{u \in T(y)} \langle u, x - y \rangle > 0$.

Then QVI(T, K) admits at least one solution.

In our next existence result, the pseudomonotonicity assumption made on map T in Theorem 1 is reduced to an hypothesis of quasimonotonicity but the semi-continuity of this map is reinforced.

Theorem 3. *Let D be a non-empty closed convex subset of $\mathbb{R}^n$. Let $T : D \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $K : D \rightrightarrows D$ be two set-valued maps. Then, $QVI^*(T, K)$ admits at least a solution if the following properties hold:*

(i) T is quasimonotone, locally upper sign-continuous and dually lower semi-continuous on D;

(ii) the map K is closed, lower semi-continuous and convex valued map with int$K(\mu) \neq \emptyset$ for all $\mu \in D$;

(iii) the quasi-variational inequality $QVI(T, K)$ satisfies the uniform coercivity condition $(\bar{C}^u)$.

To prove Theorem 3, the following lemma is useful. It is a direct consequence of [7, Lemma 2.1] and [7, Proposition 2.1].

Lemma 2. *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map and K be a non-empty convex and compact subset of $\mathbb{R}^n$. If T is quasimonotone and locally upper sign-continuous on K with convex values, then $S^*(T, K)$ is non-empty.*

Proof of Theorem 3. Define a map $G^* : D \rightrightarrows D$ by $G^*(x) := S^*(T, K_\rho(x))$ for any $x \in D$ where $K_\rho : D \rightrightarrows D$ is given by $K_\rho(\mu) := K(\mu) \cap \bar{B}(0, \rho)$ for all $\mu \in D$. Following the same lines as in the proof of Theorem 1, one can deduce that K_ρ is closed, lower semi-continuous and convex valued map with $\mathrm{int}K_\rho(\mu) \neq \emptyset$ for all $\mu \in D$. Hence, it follows from [5, Proposition 4.3] that the map G^* is closed. Thus, the set-valued map G^* is upper semi-continuous on D as $G^*(D) \subseteq K(D) \cap \bar{B}(0, \rho)$ and K is closed.

Let x be an arbitrary element of D. Combining hypothesis $i)$ and the non-emptiness of the interior of the values of K, it follows from [5, Lemma 3.1 $i)$ and $iv)$] that

$$G^*(x) = S^*(T, K_\rho(x)) = M(T, K_\rho(x)). \tag{7}$$

Since $K_\rho(x)$ is compact, Lemma 2 implies that $G^*(x) \neq \emptyset$. Moreover, by (7), we clearly have that G^* is closed-valued and convex-valued.

Hence G^* is an upper semi-continuous map with non-empty convex closed values. Also, $G^*(D)$ is contained in a compact set $K(D) \cap \bar{B}(0, \rho)$ of D. According to the fixed point theorem due to Himmelberg [17, Theorem 2], the map G^* has a fixed point. This means that there exists a point $x_0 \in K_\rho(x_0) = K(x_0) \cap \bar{B}(0, \rho)$ such that

$$\exists\, x_0^* \in T(x_0) \setminus \{0\} \text{ with } \langle x_0^*, y - x_0 \rangle \geq 0, \ \forall\, y \in K(x_0) \cap \bar{B}(0, \rho).$$

Similarly to the proof of Theorem 1, it can be proved that $\langle x_0^*, y - x_0 \rangle \geq 0$ for all $y \in K(x_0)$. $\square$

In the case where D is a compact set, one can recover [6, Proposition 3.3] from the above theorem.

Remark 2. *By the same arguments as in Remark 1 d) but using [5, Proposition 4.3], one can prove that the solution set $QVI^*(T, K)$ is closed, provided that the assumptions of Theorem 3 are satisfied.*

Moreover the example of Remarks 1 e) also shows that the solution set $QVI^*(T, K)$ may be non compact since $D^* = [0, +\infty[\times\{0\} \subset QVI^*(T, K)$.

3.3. Quasi-variational Inequalities with Product Constraint Maps

Motivated by the approach used in [11, 12] for generalized Nash equilibrium problems, we now consider a quasi-variational inequality, where the constraint map K is defined by a product $K(x) = \prod_{\nu=1}^{N} K_\nu(x)$, where for any ν, the component-maps K_ν are either single-valued or convex-valued map with $intK_\nu(\mu) \neq \emptyset$ for all $\mu \in D$. As it will be shown in the forthcoming Section 4, this specific case is particularly adapted to the quasi-variational inequalities coming from the reformulation of generalized Nash equilibrium problems. Moreover, it gives an example in which the non-emptiness of the interior of the values of the constraint map K does hold true.

Let us first define the following concept of semi-continuity. Let X be a Banach space with topological dual X^*. A set-valued map $T : X \rightrightarrows X^*$ is said to be *strongly dual lower semi-continuous* on a set $M \subseteq X$ if, for each $x, y \in M$ and each sequences $(x_k)_k$ and $(y_k)_k$ of M converging to x and y respectively,

$$\liminf_k \sup_{y_k^* \in T(y_k)} \langle y_k^*, x_k - y_k \rangle \leq 0 \quad \Rightarrow \quad \sup_{y^* \in T(y)} \langle y^*, x - y \rangle \leq 0. \tag{8}$$

It is worth to note that the concept of strongly dual lower semi-continuous map is slightly different from the concept of dually lower semi-continuity introduced in Subsection 2. Indeed, the above implication is assumed to hold for any sequence $(x_k)_k$ converging to x. Hence, any strongly dual lower semi-continuous map is dually lower semi-continuous. Moreover, any lower semi-continuous map is also strongly dual semi-continuous. Indeed, for any $y \in T(y)$ there exists a sequence $(y_k^*)_k$ converging to y^* with $y_k^* \in T(y_k)$ for any k. Thus if the left hand side of (8) holds true then $\liminf_k \langle y_k^*, x_k - y_k \rangle \leq 0$ and the conclusion follows by the continuity of the scalar product. Note that a concept similar to strongly dual lower semi-continuity has been already used in [1, Th. 4.2] to establish the lower semi-continuity of the solution map of a perturbed Stampacchia variational inequality.

Theorem 4. *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map and D be a non-empty closed convex subset of $\mathbb{R}^n$. Suppose that $K : D \rightrightarrows D$ is defined by $K(x) := \prod_{\nu=1}^N K_\nu(x)$, where $K_\nu : D \rightrightarrows \mathbb{R}^{n_\nu}$ for $\nu = 1, \ldots, N$. Then, $QVI^*(T, K)$ admits at least a solution if the following properties hold:*

(i) T is quasimonotone, locally upper sign-continuous and strongly dual lower semi-continuous on D;

(ii) for each ν, $K_\nu : D \rightrightarrows \mathbb{R}^{n_\nu}$ are either single-valued continuous map or closed lower semi-continuous convex-valued map with int$K_\nu(\mu) \neq \emptyset$ for all $\mu \in D$;

(iii) the quasi-variational inequality $QVI(T, K)$ satisfies the uniform coercivity condition $(\bar{C}^u)$.

To prove the above theorem, we need the following lemma from [5].

Lemma 3. *[5, Lem. 4.1] Let* $(Q_k)_{k\geq 1}$ *be a sequence of convex subsets of* $\mathbb{R}^n$ *with* $int Q_k \neq \emptyset$ $\forall k$. *Assume that there exists a convex set* $Q \subseteq \mathbb{R}^n$ *such that* $Q \subseteq Liminf_{k\to\infty} int Q_k$. *If* $int Q \neq \emptyset$, *then for each* $y \in int Q$, *there is* $k_0 \in \mathbb{N}$ *such that* $y \in int Q_k$ *for all* $k \geq k_0$.

Proof of Theorem 4. Define a map $G^* : D \rightrightarrows D$ by $G^*(\mu) := S^*(T, K(\mu) \cap \bar{B}(0,\rho))$ for any $\mu \in D$. As in the beginning of the proof of Theorem 3, by combining hypothesis (i) and the non-emptiness of the values of K, one can deduce that for any $\mu \in D$, $G^*(\mu) = S^*(T, K(\mu) \cap \bar{B}(0,\rho)) = M(T, K(\mu) \cap \bar{B}(0,\rho))$ and thus that the values of G^* are non-empty, closed and convex.

Let us show now that the map G^* is closed. So, let $\{(\mu_n, x_n)\}_n \subseteq GrG$ be an arbitrary sequence such that $((\mu_n, x_n))_n$ converging to (μ, x). Thus $x_n \in K(\mu_n) \cap \bar{B}(0,\rho)$. Since K_ν is closed for each $\nu = 1, 2, \ldots, N$, we have $x \in K(\mu)$ and hence $x \in K(\mu) \cap \bar{B}(0,\rho)$. It remains to show that $\langle x^*, y - x\rangle \geq 0$ for any $y \in K(\mu) \cap \bar{B}(0,\rho)$ and some $x^* \in T(x) \setminus \{0\}$. It will be done in three steps.

Since $K(\mu) \cap \bar{B}(0,\rho) \neq \emptyset$, we have from the uniform coercivity condition that $K(\mu) \cap B(0,\rho) \neq \emptyset$. Let I_1 be the set of those ν for which K_ν is single-valued and $I_2 = \{1, \ldots, N\} \setminus I_1$.

So, as a first step, let us consider an arbitrary element $p = (p^1, \ldots, p^N)$ of $K(\mu) \cap B(0,\rho)$ such that $p^\nu \in \mathrm{int} K_\nu(\mu)$ for $\nu \in I_2$. Clearly, $p^\nu = x^\nu$ for $\nu \in I_1$. For each $\nu \in I_2$, since $K_\nu(\mu_n)$ (for any n) and $K_\nu(\mu)$ are convex with non-empty interior and the map K_ν is lower semi-continuous, it follows from Lemma 3 that there exists $j_\nu \in \mathbb{N}$ such that $p^\nu \in \mathrm{int} K_\nu(\mu_n)$, $\forall n \geq j_\nu$. Let us define a sequence $(p_n = (p_n^1, \ldots, p_n^N))_n$ where for each $n \in \mathbb{N}$, $p_n^\nu = x_n^\nu$ for $\nu \in I_1$ and $p_n^\nu = p^\nu$ otherwise. Thus $p_n \in K(\mu_n) \cap B(0,\rho)$, for any $n >$

$j = \max_{\nu \in I_2} j_\nu$. Clearly the convergence of $(x_n)_n$ to x immediately implies that $(p_n)_n$ converges to p. Since $x_n \in S^*(T, K(\mu) \cap \bar{B}(0,\rho)) = M(T, K(\mu) \cap \bar{B}(0,\rho))$ for all $n \in \mathbb{N}$, we have

$$\langle p_n^*, p_n - x_n \rangle \geq 0, \quad \forall p_n^* \in T(p_n) \text{ and } \forall n \geq j.$$

By the strong dual lower semi-continuity of T at p, we have $\langle p^*, p - x \rangle \geq 0$ for all $p^* \in T(p)$.

Second, let $z = (z^1, \ldots, z^N)$ be an arbitrary element of $K(\mu) \cap B(0,\rho)$. Then there exists a sequence $(z_n = (z_n^1, \ldots, z_n^N))_n$ in $K(\mu) \cap B(0,\rho)$ such that $(z_n)_n$ converges to z and $(z_n^\nu)_n \subseteq \mathrm{int} K_\nu(\mu)$ for each $\nu \in I_2$. Thus it follows from the above paragraph that for each $n \in \mathbb{N}$,

$$\langle z_n^*, z_n - x \rangle \geq 0, \quad \forall z_n^* \in T(z_n).$$

By the dual lower semi-continuity of T at z, we have $\langle z^*, z - y \rangle \geq 0$ for all $z^* \in T(z)$.

Finally, let y be an arbitrary element of $K(\mu) \cap \bar{B}(0,\rho)$. Since $K(\mu) \cap B(0,\rho) \neq \emptyset$ and $K(\mu) \cap B(0,\rho)$ is convex, there exists a sequence $(y_n = (y_n^1, \ldots, y_n^N))_n \subseteq K(\mu) \cap B(0,\rho)$ such that $(y_n)_n$ converges to y. Hence, it follows from the above paragraph and continuity arguments that $\langle y^*, y - x \rangle \geq 0$ for all $y^* \in T(y)$. Therefore $x \in M(T, K(\mu) \cap \bar{B}(0,\rho)) = S^*(T, K(\mu) \cap \bar{B}(0,\rho))$, showing that the map G^* is closed.

Now following the same arguments as in the end of the proof of Theorem 1, there exist $x_0 \in K(x_0) \cap \bar{B}(0,\rho)$ and $x_0^* \in T(x_0) \setminus \{0\}$ such that $\langle x_0^*, y - x_0 \rangle \geq 0, \ \forall y \in K(x_0)$. □

Let us end this section by observing that, by slightly adapting the above proof, one can easily extend Theorem 4 to the following infinite dimensional setting:

Proposition 2. *Let $I = I_1 \cup I_2$ be a finite subset of natural numbers and for each $i \in I_1$, X_i be a Banach space with topological dual X_i^* equipped with the weak* topology. Let $T : \prod_{i \in I_1} X_i \times \prod_{j \in I_2} \mathbb{R}^{n_j} \rightrightarrows \prod_{i \in I_1} X_i^* \times \prod_{j \in I_2} \mathbb{R}^{n_j}$ be a set-valued map and D be a non-empty convex subset of $\prod_{i \in I_1} X_i \times \prod_{j \in I_2} \mathbb{R}^{n_j}$. Suppose that $K : D \rightrightarrows D$ is defined by $K(\mu) = \prod_{i \in I_1} K_i(\mu) \times \prod_{j \in I_2} K_j(\mu)$, where $K_i : D \rightrightarrows X_i$ for $i \in I_1$ and $K_j : D \rightrightarrows \mathbb{R}^{n_j}$ for $j \in I_2$. Then, $QVI^*(T, K)$ admits at least a solution if the following conditions hold:*

(i) T is quasimonotone, locally upper sign-continuous and strongly dual lower semi-continuous on D;

(ii) for each $i \in I_1$, K_i are closed lower semi-continuous convex-valued map with int$K_j(\mu) \neq \emptyset$ for all $\mu \in D$ and for each $j \in I_2$, K_j are single-valued continuous map;

(iii) the quasi-variational inequality $QVI(T, K)$ satisfies the uniform coercivity condition $(\bar{C}^u)$ and the set $D \cap \bar{B}(0, \rho)$ is compact.

4. Application to GNEP

The Nash equilibrium problems are non-cooperative games in which the payoff/cost function that any player tends to maximize/minimize depends on the strategy vector of the concurrent players. The generalized Nash equilibrium problems (GNEP) is a Nash equilibrium problem in which the strategy set of each player is also depending of the strategies of the other players. This situation occurs, typically, as soon as the the amount of available exchange commodies are limited.

To be more precise, suppose that there are N players in the non-cooperative game, each player ν controlling the strategy variable $x^\nu \in \mathbb{R}^{n_\nu}$. According to classical notation, x denotes the vector formed by all strategy variables:

$$x = (x^1, \ldots, x^N) \quad \text{and} \quad n = n_1 + n_2 + \cdots + n_N,$$

while $x^{-\nu}$ stands for the vector formed by all players' decision variables except the ones of player ν. Hence the notation $x = (x^\nu, x^{-\nu})$ is commonly used in the literature (see, e.g., [14]). On the other hand, for each player ν, the strategy x^ν belongs to the set $K_\nu(x^{-\nu}) \subseteq \mathbb{R}^{n_\nu}$ that depends on the strategy variables of the other players. For the given strategy vector $x^{-\nu}$, the aim of the player ν is to choose a strategy x^ν which solves the following optimization problem

$$(P_\nu) \qquad \min_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}), \quad \text{subject to} \quad x^\nu \in K_\nu(x^{-\nu}),$$

where $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ is the *cost function* of the player ν that depends on both his own strategy x^ν as well as the strategy vector $x^{-\nu}$ of the rival players. For any given strategy vector $x^{-\nu}$ of the rival players, we denote the solution set of the problem (P_ν) by $Sol_\nu(x^{-\nu})$. The Generalized Nash equilibrium problem (GNEP) consists in finding a vector $\bar{x}$ such that $\bar{x}^\nu \in Sol_\nu(\bar{x}^{-\nu})$, for any ν. Then such a vector $\bar{x}$ is called a *Generalized Nash equilibrium* of the GNEP. The Generalized Nash equilibrium problem reduces to the classical Nash equilibrium problem if for each $\nu = 1, \ldots, N$, $K_\nu(x^{-\nu}) = C_\nu$ for some $C_\nu \subseteq \mathbb{R}^{n_\nu}$, that is, the sets $K_\nu(x^{-\nu})$ do not depend on the rival players' strategies.

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Let us now recall the following reformulation of a generalized Nash equilibrium problem in terms of quasi-variational inequality.

Lemma 4. *[13] For any $\nu = 1, \ldots, N$, let C_ν be a non-empty subset of $\mathbb{R}^{n_\nu}$, $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ and $K_\nu : C_{-\nu} = \Pi_{i=1, i \neq \nu}^N C_i \rightrightarrows \mathbb{R}^{n_\nu}$ be such that θ_ν is continuously differentiable in both variables and convex with respect to the x^ν variable and the constraint maps K_ν are closed convex valued maps. Then a point $\bar{x}$ is a generalized Nash equilibrium of $GNEP((\theta_\nu)_\nu, (K_\nu)_\nu)$ if and only if it is a solution of the quasi-variational inequality $QVI(F, K)$ where $K : C = \prod_\nu C_\nu \rightrightarrows \mathbb{R}^n$ is defined by $K(x) = \prod_{\nu=1}^p K_\nu(x^{-\nu})$ and the map $F : \mathbb{R}^n \to \mathbb{R}^n$ is such that $F(y) = \prod_{\nu=1}^p \nabla_\nu \theta_\nu(\cdot, y^{-\nu})(y^\nu)$.*

Now, we derive the following result on the existence of generalized Nash equilibrium by combining the Theorem 4 and Lemma 4.

Theorem 5. *For any $\nu = 1, \ldots, N$, let C_ν be a non-empty closed convex subset of $\mathbb{R}^{n_\nu}$, $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ and $K_\nu : C_{-\nu} = \Pi_{i=1, i \neq \nu}^N C_i \rightrightarrows C_\nu$, where θ_ν is continuously differentiable in both variables and convex with respect to the x^ν variable and the constraint maps K_ν are either single-valued continuous map or closed lower semi-continuous convex-valued map with $intK_\nu(\mu^{-\nu}) \neq \emptyset$ for all $\mu^{-\nu} \in C_{-\nu}$. Assume that for every for every $\mu = (\mu^\nu, \mu^{-\nu}) \in C = \prod_\nu C_\nu$,*

$$\exists\, \rho_\mu > 0,\ \forall\, x \in \prod_\nu K_\nu(\mu^{-\nu}) \setminus \bar{B}(0, \rho_\mu), \exists\, y \in \prod_\nu K_\nu(\mu^{-\nu}) \text{ with}$$
$$||y|| < ||x|| \text{ such that } \langle \nabla_\nu \theta_\nu(x), x^\nu - y^\nu \rangle \geq 0 \text{ for each } \nu,$$

and there exists $\rho > \max\{\rho_\mu : \mu \in C\}$ so that $\prod_\nu K_\nu(\mu^{-\nu}) \cap \bar{B}(0, \rho)$ is non-empty for all $\mu \in C$. Then the $GNEP((\theta_\nu)_\nu, (K_\nu)_\nu)$ defined by the function $(\theta_\nu)_\nu$ and the maps $(K_\nu)_\nu$ admits at least one Generalized Nash equilibrium.

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