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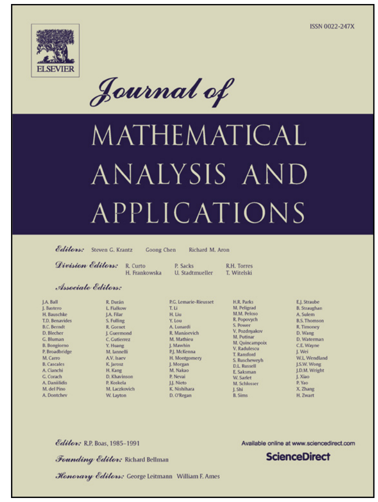

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A Remark on Denjoy's inequality for PL circle homeomorphisms with two break points [1]

Akhtam Dzhalilov[2] Alisher Jalilov[3], Dieter Mayer[4]

Abstract

It is well known that for a P-homeomorphism f of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ with irrational rotation number ρ_f the Denjoy's inequality $|\log Df^{q_n}| \leq V$ holds, where V is the total variation of $\log Df$ and q_n, $n \geq 1$, are the first return times of f. Let h be a piecewise-linear (PL) circle homeomorphism with two break points a_0, c_0, irrational rotation number ρ_h and total jump ratio $\sigma_h = 1$. Denote by $\mathbf{B}_n(h)$ the partition determined by the break points of h^{q_n} and by μ_h the unique h-invariant probability measure. It is shown that the derivative Dh^{q_n} is constant on every element of $\mathbf{B}_n(h)$ and takes either two or three values. Furthermore we prove, that $\log Dh^{q_n}$ can be expressed in terms of μ_h- measures of some intervals of the partition $\mathbf{B}_n(h)$ multiplied by the logarithm of the jump ratio $\sigma_h(a_0)$ of h at the break point a_0.

1 Introduction

Let f be an orientation preserving homeomorphism of the circle $S^1 \equiv \mathbb{R}/\mathbb{Z}$ with lift $F : \mathbb{R} \to \mathbb{R}$, which is continuous, increasing and fulfills $F(\hat{x}+1) = F(\hat{x}) + 1$ for $\hat{x} \in \mathbb{R}$ a lift of x. The circle homeomorphism f is then defined by $f(x) = F(\hat{x})$ $(mod\, 1)$, $x \in S^1$. The **rotation number** ρ_f is defined by $\rho_f := \lim\limits_{n\to\infty} \frac{F^n(\hat{x})-\hat{x}}{n}$ $(mod\, 1)$. Here and below, F^i denotes the i-th iteration of the lift F. It is well known, that the rotation number ρ_f does not depend on the point $\hat{x} \in \mathbb{R}$ and is irrational if and only if f has no periodic points (see [5]). The rotation number ρ_f is invariant under topological conjugations. We shall assume the rotation number ρ_f to be irrational throughout this paper. We use the continued fraction representation $\rho_f = 1/(k_1 + 1/(k_2 + ...)) := [k_1, k_2, ..., k_n, ...)$ of the rotation number ρ_f. Denote by $p_n/q_n = [k_1, k_2, ..., k_n]$, $n \geq 1$, its $n-$th convergent. The numbers $q_n, n \geq 1$ are the **first return times** of f and satisfy the recursive relations $q_{n+1} = k_{n+1}q_n + q_{n-1}$ for $n \geq 1$, where $q_0 = 1$, and $q_1 = k_1$.

A natural extension of circle diffeomorphisms are piecewise smooth homeomorphisms with break points or shortly, the class of P-homeomorphisms.

The class of **P-homeomorphisms** consists of orientation preserving circle homeomorphisms f which are differentiable except at a finite or countable number of break points x_b, at which the one-sided positive derivatives Df_- and Df_+ exist, which do not coincide and for which there exist constants $0 < c_1 < c_2 < \infty$, such that

- $c_1 < Df_-(x_b) < c_2$ and $c_1 < Df_+(x_b) < c_2$ for all $x_b \in BP(f)$, the set of break points of f in S^1;
- $c_1 < Df(x) < c_2$ for all $x \in S^1 \backslash BP(f)$;
- $\log Df$ has finite total variation in S^1.

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The ratio $\sigma_f(x_b) = \frac{Df_-(x_b)}{Df_+(x_b)}$ is called the **jump ratio** of f at x_b and $\sigma_f = \prod_{x_b} \sigma_f(x_b)$ its **total jump ratio**.

Piecewise linear PL circle homeomorphisms are the simplest examples of class P-homeomorphisms. They occur in many other areas of mathematics such as group theory, homotopy theory and logic via the Thompson groups. PL circle homeomorphisms with two break points were considered by Herman in [13] to obtain homeomorphisms of arbitrary irrational rotation number which admit no invariant probability measure equivalent to Lebesgue measure. We recall Herman's theorem.

Theorem 1.1. *(See [13].) Let h be a PL circle homeomorphism with two break points and irrational rotation number ρ_h. Then the h- invariant probability measure μ_h is absolutely continuous with respect to Lebesgue measure l if and only if its break points belong to the same orbit.*

In the following we denote PL circle homeomorphisms always by h. The invariant measures of piecewise smooth P- homeomophisms with a finite number of break points were studied by many authors (see for instance [13], [4], [22], [20], [10],[1], [12]). Dzhalilov, Mayer and Safarov proved in [12], that the invariant measures of piecewise $C^{2+\varepsilon}$ ($\epsilon > 0$) smooth P- homeomorphisms f with irrational rotation number ρ_f and non trivial total jump ratio σ_f are singular w.r.t. Lebesgue measure. In this case the conjugacy φ between f and the linear rotation f_ρ is a singular function.

The behaviour of Df^{q_n} is a classical problem in the theory of circle homeomorphisms and plays a key role for their dynamics (see for instance [14], [15], [18], [19], [6]). Let us recall two of Denjoy's classical results generalized to P-homeomorphisms (see [7],[13]). The first one is a theorem on the bounds of Df^{q_n}:

Theorem 1.2. *Let f be a P-homeomorphism with irrational rotation number ρ_f. Then for any x with $f^s(x) \notin BP(f)$, $0 \le s < q_n$ the following inequality holds:*

$$|\log Df^{q_n}(x)| \le V, \tag{1}$$

where V is the total variation of $\log Df$ in S^1.

The other one is an important fact concerning the expectation of $\log Df^{q_n}$

Theorem 1.3. *Let f be a P-homeomorphism with irrational rotation number ρ_f. Then for every $n \ge 1$*

$$\int_{S^1} \log Df^{q_n}(x) d\mu_f = 0 \tag{2}$$

Inequality (1) is called **Denjoy's inequality**. If Denjoy's inequality holds for the map f with irrational rotation number $\rho := \rho_f$, then this implies the existence of a map $\varphi : S^1 \to S^1$ conjugating f and the linear rotation f_ρ with lift $F_\rho(\hat{x}) = \hat{x} + \rho$ (see [5]). In this case, the conjugation φ, satisfying $f = \varphi^{-1} \circ f_\rho \circ \varphi$, is an essentially unique homeomorphism of the circle. Another natural question arising is to ask for the smoothness of the conjugation φ and its dependence on the smoothness of the homeomorphism f. Since the conjugating map φ and the unique f-invariant measure μ_f are related by $\varphi(x) = \mu_f([0, x])$ (see [5]), regularity properties of the conjugating map φ imply corresponding properties of the density of the absolutely continuous invariant measure μ_f. Fundamental results on smoothness of h were obtained by V.I. Arnold [3], M. Herman [13], J.C. Yoccoz [23], Y. Katznelson and D. Ornstein [14], K. Khanin and Y. Sinai [15], respectively K.Khanin and A.Teplinskii [18]. All the results on smoothness properties of φ show its close relations to sharp estimates in Denjoy's inequality and arithmetic properties of the rotation number ρ_f. For example, it was shown in [14], [15] and [19] for a $C^{2+\varepsilon}(S^1)$ diffeomorphism f with irrational rotation number ρ_f, that the conjugation φ is C^1-smooth if the sequence $\mathbf{K}_n := \sup_{S^1} |\log Df^{q_n}(x)|$ tends to zero exponentially fast . Y. Katznelson and D. Ornstein [14] have shown, that convergence of the sum of the sequence $\mathbf{K}_n^2$ is enough for absolute continuity of the conjugating map φ. Indeed, they proved for a wider class of circle diffeomorphisms the following theorem.

Theorem 1.4. *(See[14]). If $\log Df$ is absolutely continuous on the circle, $D\log Df \in L^p$ for some $p > 1$, and the rotation number ρ_f is irrational, then*

$$\sum_{n=1}^{\infty} \boldsymbol{K}_n^2 < \infty. \tag{3}$$

In particular, if the rotation number is of bounded type i.e. the entries k_i's in the continued fraction expansion of ρ_f are bounded, then the conjugacy φ and its inverse φ^{-1} are absolutely continuous with square-summable derivatives.

Our main goal in the present paper is to express for a piecewise-linear (PL) homeomorphism h with two break points a_0 and c_0 and total jump ratio $\sigma_h = 1$ the derivative Dh^{q_n} of h^{q_n} by the jump ratio $\sigma_h(a_0)$ and the μ_h-measures of intervals of the partition $B_n(h)$ of S^1 determined by the break points of h^{q_n}. Thereby μ_h denotes the unique invariant probability measure of h. To start, take some $x_0 \in S^1$. Using its orbit $\{x_i = f^i(x_0),\ i \in \mathbb{Z}\}$ one defines a sequence of natural partitions of the circle. Namely, let $I_0^{(n)}(x_0)$ be the closed interval in S^1 with endpoints x_0 and $x_{q_n} = f^{q_n}(x_0)$. Notice, in the clockwise orientation of the circle the point x_{q_n} is for n odd to the left of x_0, and to its right for n even. If we denote by $I_i^{(n)}(x_0) = f^i(I_0^{(n)}(x_0)), i \geq 1$, the iterates of the interval $I_0^{(n)}(x_0)$ under f, then it is well known, that the set $\xi_n(x_0)$ of intervals with mutually disjoint interior, defined as

$$\xi_n(x_0) = \{I_i^{(n-1)}(x_0),\ 0 \leq i < q_n\} \cup \{I_j^{(n)}(x_0),\ 0 \leq j < q_{n-1}\},$$

determines for any n a partition of the circle. The partition $\xi_n(x_0)$ is called the n-th **dynamical partition** of the point x_0.

Consider now an arbitrary P-homeomorphism f with irrational rotation number ρ_f and two break points a_0 and c_0, which are not on the same orbit. Denote by $\frac{p_n}{q_n}$ the partial convergents of ρ_f. We will next determine the location of the break points of f^{q_n} and the derivative Df^{q_n} on S^1. Obviously the map f^{q_n} has $2\,q_n$ break points denoted by $BP_f^n := BP_f^n(a_0) \cup BP_f^n(c_0)$ with $BP_f^n(a_0) := \{a_0^*, a_{-1}^*, ..., a_{-q_n+1}^*\}$, respectively $BP_f^n(c_0) := \{c_0^*, c_{-1}^*, ..., c_{-q_n+1}^*\}$, where $a_{-i}^* = f^{-i}(a_0)$, respectively $c_{-i}^* = f^{-i}(c_0)$, $0 \leq i \leq q_n - 1$. It is clear, that these break points of the map f^{q_n} define a partition $B_n(f)$ of the circle S^1 into $2\,q_n$ intervals with pairwise non-intersecting interior.

Let $\xi_n(a_0^*)$ be the n-th dynamical partition determined by the break point $a_0^* = a_0$ with respect to the map f. Then one has for the second break point c_0^* either $c_0^* \in I_{i_0}^{(n)}(a_0)$ for some $0 \leq i_0 < q_{n-1}$, or $c_0^* \in I_{j_0}^{(n-1)}(a_0) = f^{j_0}((a_0, a_{-q_n}]) \cup f^{j_0}((a_{-q_n}, a_{q_{n-1}}))$ for some $0 \leq j_0 < q_n$, i.e. $c_0^* \in f^{j_0}((a_0, a_{-q_n}])$ or $c_0^* \in f^{j_0}((a_{-q_n}, a_{q_{n-1}}))$. The two last cases we have to treat separately. The following three lemmas describe the location of the break points of f^{q_n} in intervals of certain n-th dynamical partitions .

Lemma 1.5. *Assume $c_0^* \in I_{i_0}^{(n)}(a_0^*)$ for some i_0 with $0 \le i_0 < q_{n-1}$. Then the break points a_{-i}^*, c_{-i}^*, $0 \le i \le q_n - 1\}$ of f^{q_n} belong to the following elements of the dynamical partition $\xi_n(a_0^*)$ (see also Fig 3.1):*

- $a_0^* \in I_0^{(n)}(a_0^*)$;
- $c_{-i_0+s}^* = f^s(c_{-i_0}^*) \in I_s^{(n)}(a_0^*),\ 0 \le s \le i_0$;
- $a_{-q_n+s}^* = f^s(a_{-q_n}) \in f^s(a_0^*, a_{-q_n}]) \subset I_s^{(n-1)}(a_0^*),\quad 1 \le s \le i_0$;
- $a_{-q_n+s}^*$, $c_{-q_n-i_0+s}^* = f^s(c_{-q_n-i_0}) \in f^s((a_0^*, a_{-q_n}]) \subset I_s^{(n-1)}(a_0^*),\ i_0 + 1 \le s \le q_n - 1$.

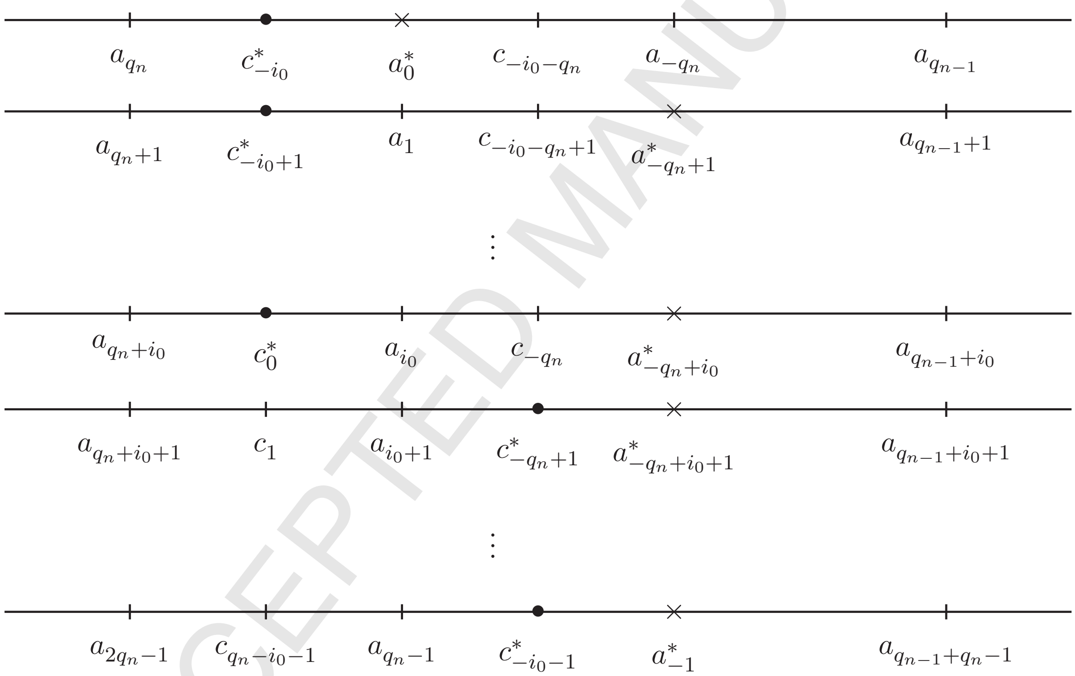


Fig. 3.1

Lemma 1.6. *Assume $c_0^* \in f^{i_0}((a_0^*, a_{-q_n}])$ for some $0 \le i_0 < q_n$. Then the break points of f^{q_n} belong to the following elements of the dynamical partition $\xi_n(c_{-i_0}^*)$ of the break point $c_{-i_0}^*$ (see Fig 3.2):*

- $c_{-i_0}^*, a_0^* \in I_0^{(n)}(c_{-i_0}^*)$
- $c_{-i_0+s}^* = f^s(c_{-i_0}^*),\quad a_{-q_n+s}^* = f^s(a_{-q_n}) \in f^s([c_{-i_0}^*, a_{-q_n}]) \subset I_s^{(n-1)}(c_{-i_0}^*),\quad 1 \le s \le i_0$;

- $c^*_{-q_n-i_0+s} = f^s(c^*_{-q_n-i_0})$, $a^*_{-q_n+s} = f^s(a_{-q_n}) \in f^s([c_{-i_0}, c_{-q_n}]) \subset I_s^{(n-1)}(c^*_{-i_0})$, $i_0 + 1 \leq s \leq q_n - 1$.

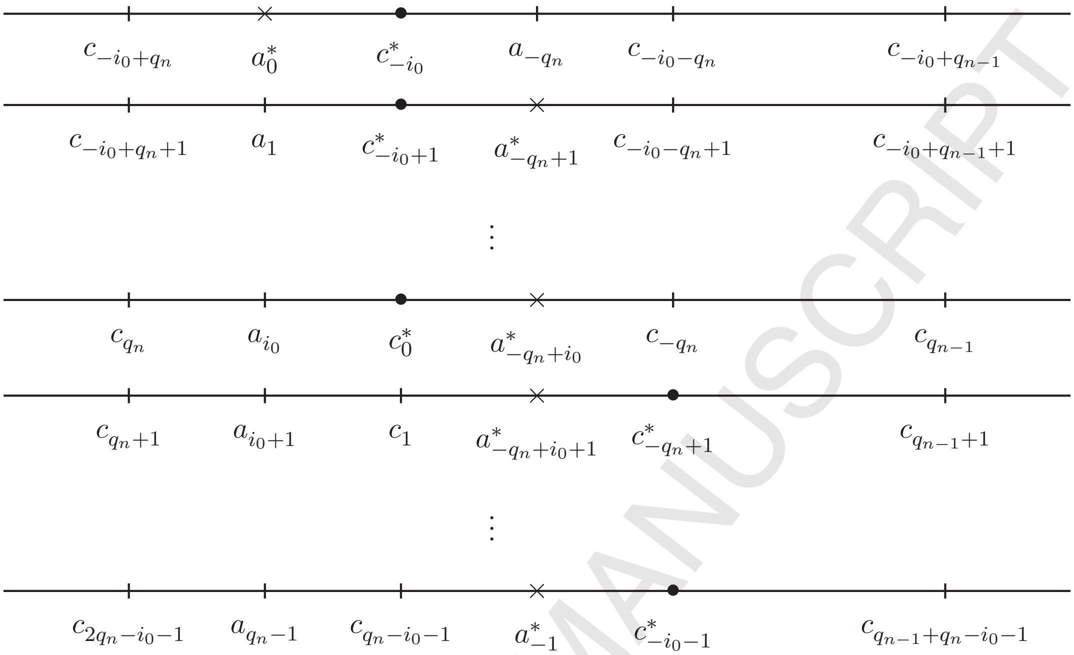
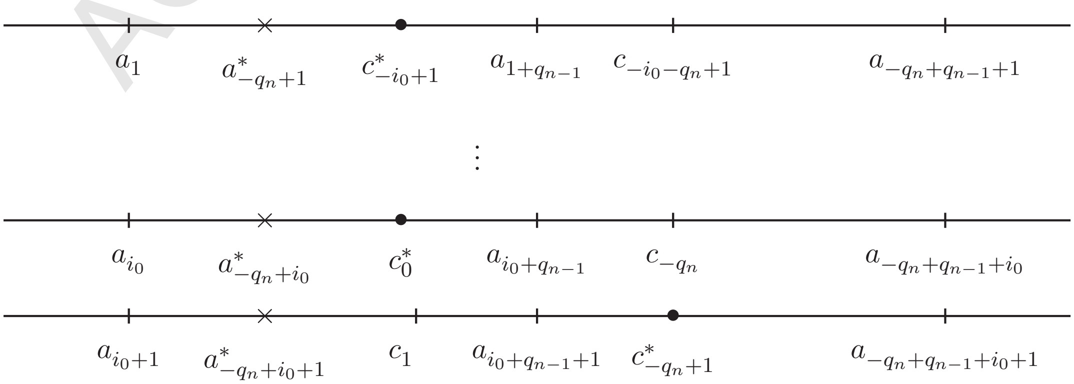


Fig. 3.2

Lemma 1.7. *If $c^*_0 \in f^{i_0}((a_{-q_n}, a_{q_{n-1}}])$ for some i_0 with $0 \leq i_0 < q_n$, the break points of f^{q_n} are located in the following elements of the dynamical partition $\xi_n(a^*_{-q_n+1})$ of the break point $a^*_{-q_n+1}$ (see also Fig 3.3):*

- $a^*_{-q_n+1+s} = f^s(a^*_{-q_n+1})$, $c^*_{-i_0+1+s} = f^s(c^*_{-i_0+1}) \in I_s^{(n-1)}(a^*_{-q_n+1})$, $0 \leq s \leq i_0 - 1$;
- $a^*_{-q_n+i_0+1+s} = f^s(a^*_{-q_n+i_0+1})$, $c^*_{-q_n+1+s} = f^s(c^*_{-q_n+1}) \in I_{i_0+s}^{(n-1)}(a^*_{-q_n+1})$, $0 \leq s \leq q_n - i_0 - 1$.



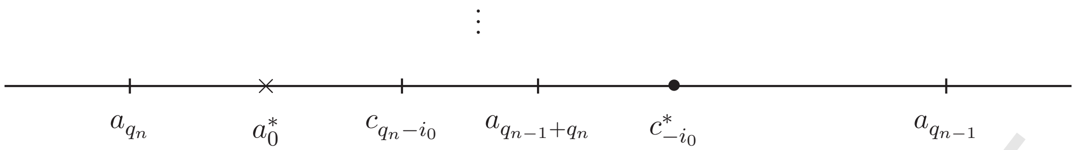


Fig. 3.3

Next we consider a P-homeomorphism f with irrational rotation number ρ_f and two break points $a_0^* := a_0$, $a_{i_0}^* := f^{i_0}(a_0)$, $i_0 > 0$, on the same orbit. Put $n_{i_0} := min\{n : q_n \geq i_0\}$. Assume that $n > n_{i_0}$. If the total jump ratio $\sigma_f = 1$, the map f^{q_n} has $2i_0$ break points

$$a_{-q_n+1}^* := a_{-q_n+1},\ a_{-q_n+2}^* := a_{-q_n+2},\ \dots,\ a_{-q_n+i_0}^* := a_{-q_n+i_0}$$

and

$$a_1^* := a_1, a_2^* := a_2, ..., a_{i_0}^* := a_{i_0}.$$

If $\sigma_f \neq 1$ the map f^{q_n} has $q_n + i_0$ break points

$$a_{-q_n+1}^* := a_{-q_n+1},\ a_{-q_n+2}^* := a_{-q_n+2},\ \dots,\ a_0^* := a_0,\ \dots,\ a_{i_0}^* := a_{i_0}.$$

One has the following

Lemma 1.8. *Assume f is a P-homeomorphism with irrational rotation number ρ_f and two break points $a_0^* := a_0$, $a_{i_0}^* := f^{i_0}(a_0)$, $i_0 > 0$, on the same orbit. Choose $n > n_{i_0}$.*

1) If $\sigma_f = 1$, then one finds for the break points $a_{-q_n+s+1}^*$, a_{s+1}^* of f^{q_n}

- $a_{-q_n+s+1}^*,\ a_{s+1}^* \in f^s([a_1^*, a_{-q_n+1}^*]) \subset I_{s+1}^{(n-1)}(a_0^*) \in \xi_n(a_0^*),\ 0 \leq s \leq i_0 - 1$ *;*

(see Fig 3.4)

2) if $\sigma_f \neq 1$, we have

- $a_0^* \subset I_0^{(n-1)}(a_0^*)$;
- $a_{-q_n+1+s}^*,\ a_{1+s}^* \in f^s([a_{i_0+1}, a_{-q_n+i_0+1}^*]) \subset I_{i_0+1+s}^{(n-1)}(a_0^*), \quad 0 \leq s \leq q_n - i_0 - 2.$
- $a_{s+1}^* \in f^s([a_1^*, a_{-q_n+1}^*]) \subset I_{1+s}^{(n-1)}(a_0^*), \quad i_0 \leq s \leq q_n - i_0 - 1.$

Fig. 3.4

Lemmas 1.5 to 1.8 show the location of the break points of f^{q_n} on elements of different n-th dynamical partitions determined by the map f, respectively their order along the circle. Indeed these lemmas hold true also for any pure rotation f_ρ with ρ_f irrational and any two points $a_0, c_0 \in S^1$, whose preimages under $f_\rho^{q_n}$ correspond to the break points of the P-homeomorphism f^{q_n}.

Next we apply these Lemmas to a PL circle homeomorphism h with irrational rotation number ρ_h, when the two break points $a_0^* = 0, c_0^* = c_0$ are not on the same orbit and the total jump ratio $\sigma_h = 1$.

In case of Lemma 1.5 and Lemma 1.7 the break points $PB_n(a_0^*)$ originating from $a_0^* = 0$ and the break points $PB_n(c_0^*)$ originating from $c_0^* = c_0$ alternate in their order along the circle S^1 . Let n be odd. Obviously these break points define a system of disjoint subintervals of the circle, given in case of the assumption in Lemma 1.5 by (see Fig 3.1)

(4) $$[c^*_{-i_0+s}, a^*_{-q_n+s}],\ 1 \le s \le i_0,$$

respectively

(5) $$[c^*_{-i_0-q_n+s}, a^*_{-q_n+s}],\ i_0+1 \le s \le q_n.$$

We combine these subintervals to the subsets

$$A_n := \bigcup_{s=1}^{i_0} [c^*_{-i_0+s}, a^*_{-q_n+s}],\ B_n := \bigcup_{s=i_0+1}^{q_n} [c^*_{-i_0-q_n+s}, a^*_{-q_n+s}].$$

In case of the assumption in Lemma 1.7 the subintervals are given by (see Fig 3.3)

(6) $$[a^*_{-q_n+s}, c^*_{-i_0+s}],\ 1 \le s \le i_0,$$

respectively

(7) $$[a^*_{-q_n+s}, c^*_{-i_0-q_n+s}],\ i_0+1 \le s \le q_n,$$

which we combine to the subsets

$$A_n := \bigcup_{s=1}^{i_0} [a^*_{-q_n+s}, c^*_{-i_0+s}],\ B_n := \bigcup_{s=i_0+1}^{q_n} [a^*_{-q_n+s}, c^*_{-i_0-q_n+s}],$$

For n even, the orientation of the above intervals has to be reversed. Therefore in case of Lemma 1.5 we have the following system of disjoint intervals

(8) $$[a^*_{-q_n+s}, c^*_{-i_0+s}],\ 1 \le s \le i_0,$$

respectively

(9) $$[a^*_{-q_n+s}, c^*_{-i_0-q_n+s}],\ i_0+1 \le s \le q_n.$$

In case of Lemma 1.7 one finds

(10) $$[c^*_{-i_0+s}, a^*_{-q_n+s}],\ 1 \le s \le i_0,$$

respectively

(11) $$[c^*_{-i_0-q_n+s}, a^*_{-q_n+s}],\ i_0+1 \le s \le q_n.$$

In case of Lemma 1.5 and n even, respectively in case of Lemma 1.7 and n odd, the subsets A_n and B_n can be defined as before. The above constructions show, that the boundaries of every interval in the subsets A_n and B_n consist of break points from $PB_n(a_0^*)$ respectively $PB_n(c_0^*)$. In the following we abbreviate the jump ratio of h at the break point a_0^* by

$$\sigma := \sigma_h(a_0^*) = \frac{Dh_-(0)}{Dh_+(0)}.$$

We can then formulate our first main result.

Theorem 1.9. *Let h be a PL circle homeomorphism with irrational rotation number ρ_h and two break points $a_0^* = 0$ and $c_0^* := c_0$, whose total jump ratio $\sigma_h = 1$, and which lie on different orbits. Assume c_0^* fulfills the assumptions of Lemma 1.5 repectively Lemma 1.7 for some i_0 with $0 \leq i_0 < q_{n-1}$. Then in case of Lemma 1.5*

$$(Dh^{q_n}(x))^{(-1)^n} = \begin{cases} \sigma^{\mu_h(A_n \cup B_n)-1}, & \text{if } x \in A_n \cup B_n \\ \sigma^{\mu_h(A_n \cup B_n)}, & \text{if } x \in S^1 \setminus (A_n \cup B_n); \end{cases} \tag{12}$$

respectively in case of Lemma 1.7 ,

$$(Dh^{q_n}(x))^{(-1)^{n+1}} = \begin{cases} \sigma^{\mu_h(A_n \cup B_n)-1}, & \text{if } x \in A_n \cup B_n \\ \sigma^{\mu_h(A_n \cup B_n)}, & \text{if } x \in S^1 \setminus (A_n \cup B_n). \end{cases} \tag{13}$$

Theorem 1.9 shows that Dh^{q_n} is constant on every element of $\mathbf{B}_n(h)$ and takes only two values under the assumptions of Lemmas 1.5 and 1.7. Moreover, the values of Dh^{q_n} are determined by the jump ratio $\sigma = \sigma_h(a_0^*)$ and the μ_h-measure of $A_n \cup B_n$.
In case of the assumption on c_0^* in Lemma 1.6 we can define again a system of disjoint subintervals determined by the elements in $\mathbf{B}_n(h)$. Let n be odd. Then these subintervals are as follows (see Fig 3.2):

$$[c^*_{-i_0+s}, a^*_{-q_n+s}],\ 1 \leq s \leq i_0, \tag{14}$$

respectively

$$[a^*_{-q_n+s}, c^*_{-i_0-q_n+s}],\ i_0 + 1 \leq s \leq q_n. \tag{15}$$

For n even, the orientation of the above intervals has to be reversed. To determine in the case of Lemma 1.6 the values of Df^{q_n} we define

$$A_n := \bigcup_{s=1}^{i_0} [c^*_{-i_0+s}, a^*_{-q_n+s}],\ \ B_n := \bigcup_{s=i_0+1}^{q_n} [a^*_{-q_n+s}, c^*_{-i_0-q_n+s}]. \tag{16}$$

Then the following theorem holds.

Theorem 1.10. *Let h be a PL circle homeomorphism with two break points $a_0^* = a_0$ and $c_0^* = c_0$ with $\sigma_h = 1$, which lie on different orbits. Assume c_0^* fulfills the assumption of Lemma 1.6 for some i_0 with $0 \leq i_0 < q_n$. Then for all $n \geq 1$*

$$(Dh^{q_n}(x))^{(-1)^n} = \begin{cases} \sigma^{\mu_h(A_n)-\mu_h(B_n)-1}, & \text{if} x \in A_n, \\ \sigma^{\mu_h(A_n)-\mu_h(B_n)+1}, & \text{if } x \in B_n, \\ \sigma^{\mu_h(A_n)-\mu_h(B_n)}, & \text{if } x \notin A_n \cup B_n. \end{cases} \tag{17}$$

It remains to discuss the case of a PL-homeomorphism h with irrational rotation number ρ_h and two break points $a_0^* = 0$ and $a_{i_0}^* = h^{i_0}(a_0^*)$, $i_0 > 0$, on the same orbit. In this case the break points of h^{q_n} alternate in their order along the circle S^1. Denote by $U_n(a_s^*)$, $1 \leq s \leq i_0$, the closed intervals with endpoints a_s^* and $a^*_{-q_n+s}$. Obviously these subintervals are disjoint. Lemma 1.8 implies, that $U_n(a_s^*) \subset I_s^{(n-1)}(a_0^*)$, $1 \leq s \leq i_0$. Next we define for every $n \geq 1$

$$U_n = \bigcup_{s=1}^{i_0} U_n(a_s^*). \tag{18}$$

Then one has

Theorem 1.11. *Let h be a PL circle homeomorphism with two break points $a_0^* = 0$ and $a_{i_0}^* = h^{i_0}(a_0)$, $i_0 > 0$, with $\sigma_h = 1$, which lie on the same orbit. Put $n_{i_0} := min\{n : q_n \geq i_0\}$. For $n > n_{i_0}$ one finds*

$$(Dh^{q_n}(x))^{(-1)^{n+1}} = \begin{cases} \sigma^{\mu_h(U_n)}, & \textit{if } x \in U_n \\ \sigma^{\mu_h(U_n)-1}, & \textit{if } x \in S^1 \setminus U_n, \end{cases} \tag{19}$$

2 Proof of the Lemmas 1.5 - 1.8

We start with the proofs of Lemmas 1.5 - 1.7.

Proof of Lemma 1.5. Remember, that for arbitrary $x \in S^1$ the points $x_{q_n} = f^{q_n}(x)$ and $x_{-q_n} = f^{-q_n}(x)$ lie on opposite sides of x. Assume $c_0^* = c_0 \in I_{i_0}^n(a_0^*)$ for some $0 \leq i_0 < q_{n-1}$ and hence $c_{-i_0}^* \in I_0^n(a_0^*)$ (see Fig 3.1). Suppose n to be odd. Then we have in the clockwise order on S^1:

$$a_{q_n} \prec c_{-i_0}^* \prec a_0^* \prec c_{-i_0-q_n} \prec a_{-q_n} \prec a_{q_{n-1}}.$$

Since f is orientation preserving we get also

$$f^s(a_{q_n}) \prec f^s(c_{-i_0}^*) \prec f^s(a_0^*) \prec f^s(c_{-i_0-q_n}) \prec f^s(a_{-q_n}) \prec f^s(a_{q_{n-1}})$$

for all $0 \leq s \leq i_0$, which proves the first three assertions of Lemma 1.5.
It is also obvious, that

$$f^s(a_0^*) \prec f^s(c_{-i_0-q_n}) \prec f^s(a_{-q_n}) \prec f^s(a_{q_{n-1}})$$

for all $i_0 + 1 \leq s \leq q_n - i_0$, which proves the last assertion of Lemma 1.5.

Proof of Lemma 1.6. The interval $J_0^n(c_{-i_0}^*) = [c_{-i_0+q_n}, c_{-i_0+q_{n-1}}]$ contains only the two break points $a_0^*, c_{-i_0}^*$ of f^{q_n}. More precisely, we have (see Fig 3.2)

$$c_{-i_0+q_n} \prec a_0^* \prec c_{-i_0}^* \prec a_{-q_n} \prec c_{-i_0-q_n} \prec c_{-i_0+q_{n-1}}$$

which implies the first assertion of Lemma 1.6.

Next, the renormalization interval $J_1^n(c_0^*) = [c_{-i_0+q_n+1}, c_{-i_0+q_{n-1}+1}]$ contains also two break points of f^{q_n}, namely $a_{-q_n+1}^*$ and $c_{-i_0+1}^*$. The last two break points belong to the interval $I_1^{n-1}(c_{-i_0}^*)$. We have (see Fig 3.2)

$$c_{-i_0+1}^* \prec a_{-q_n+1}^* \prec c_{-i_0-q_n+1} \prec c_{-i_0+q_{n-1}+1}.$$

Applying the map f^s for $0 \leq s \leq i_0 - 1$ leads to

$$f^s(c_{-i_0+1}^*) \prec f^s(a_{-q_n+1}^*) \prec f^s(c_{-i_0-q_n+1}) \prec f^s(c_{-i_0+q_{n-1}+1})$$

which implies the second assertion of Lemma 1.6.

It is also clear, that

$$c_1 \prec a_{-q_n+i_0+1}^* \prec c_{-q_n+1}^* \prec c_{q_{n-1}+1}$$

Hence for $1 \leq s \leq q_n - i_0 - 2$

$$f^s(c_1) \prec f^s(a_{-q_n+i_0+1}^*) \prec f^s(c_{-q_n+1}^*) \prec f^s(c_{q_{n-1}+1}).$$

which implies the third assertion of Lemma 1.6.

Proof of Lemma 1.7. Consider the n-th dynamical partition $\xi_n(a^*_{-q_n+1})$ of the break point $a^*_{-q_n+1}$. To determine the location of the break points of f^{q_n} in the intervals of $\xi_n(a^*_{-q_n+1})$ under the assumption of Lemma 1.7, we use the structure of this dynamical partition and the monotonicity of f to arrive for $0 \leq s \leq q_n - 1$ at

$$f^s(a_1) \prec f^s(a^*_{-q_n+1}) \prec f^s(c^*_{-i_0+1}) \prec f^s(a_{q_{n-1}+1}) \prec f^s(c_{-i_0-q_n+1}) \prec f^s(a_{-q_n+q_{n-1}+1}).$$

It is easy to see that the first i_0 of these relations imply the first i_0 claims of Lemma 1.7, and the last $q_n - i_0$ the remaining ones.

Proof of Lemma 1.8. We will prove the first assertion only. The second one can be proved similarly. It is clear that (see Fig 3.4)

$$f^s(a_{q_n+1}) \prec f^s(a^*_1) \prec f^s(a^*_{-q_n+1}) \prec f^s(a_{q_{n-1}}+1)$$

for all $0 \leq s \leq i_0 - 1$. Consequently, $f^s(a^*_1), f^s(a^*_{-q_n+1}) \in I_s^{(n-1)}((a^*_1))$, $0 \leq s \leq i_0 - 1$, which proves the assertion of Lemma 1.8.

3 Proof of Theorems 1.9., 1.10. and 1.11.

Proof of Theorem 1.9.

We prove only the case of Lemma 1.5. The case of Lemma 1.7 can be proved analogously. We furthermore restrict ourselves to the case when n is odd. The even case can be handled similarly. In case c^*_0 fulfills the assumption of Lemma 1.5, we have $a_{q_n} \prec c^*_{-i_0} \prec a^*_0 \prec c_{-i_0-q_n} \prec a_{-q_n} \prec a_{q_{n-1}}$ (see Fig. 3.1). Obviously the function Dh^{q_n} on the circle S^1 is constant on every interval of the partition $\mathbf{B}_n(h)$ determined by all break points of h^{q_n}. It makes jumps determined by the jump ratio $\sigma = \sigma_h(a^*_0)$ at the break points $BP_n(a^*_0)$ and by the jump ratio σ^{-1} at the break points $BP_n(c^*_0)$. Taking into account Lemma 1.5 and the structure of the dynamical partitions it follows that the points of $BP_n(a^*_0)$ and $BP_n(c^*_0)$ alternate in their order around S^1 (see Fig. 3.1). We "renumerate" all break points $BP_n(a^*_0)$ and $BP_n(c^*_0)$ of h^{q_n} in the counter-clockwise direction as $a^{(1)} := a^*_0$, $a^{(2)} := a^*_{-1}$, $a^{(3)} := a^*_{-2}, \cdots, a^{(q_n)} := a^*_{-q_n+1}$, respectively $c^{(1)} := c^*_{-i_0}$, $c^{(2)} := c^*_{-i_0-1}$, $c^{(3)} := c^*_{-i_0-2}, \cdots, c^{(q_n)} := c^*_{-i_0+1}$. Then we have

$$a^{(1)} \prec c^{(q_n)} \prec a^{(q_n)} \prec ... \prec c^{(2)} \prec a^{(2)} \prec c^{(1)} \prec a^{(1)}$$

It is clear that

$A_n \bigcup B_n = \bigcup\limits_{s=1}^{q_n} [c^{(s)}, a^{(s)}]$, and

$S^1 \setminus (A_n \bigcup B_n) = \bigcup\limits_{s=1}^{q_n} (a^{(s+1)}, c^{(s)}) \bigcup (a^{(1)}, c^{(q_n)})$.

Next we determine the values of Dh^{q_n}. For $s > 1$ we have

$$Dh^{q_n}([c^{(s)}, a^{(s)}]) = Dh^{q_n}_-(a^{(s)}) = \sigma Dh^{q_n}_+(a^{(s)}) = \sigma Dh^{q_n}([a^{(s)}, c^{(s-1)}])$$
$$= \sigma Dh^{q_n}_-(c^{(s-1)}) = \sigma\sigma^{-1} Dh^{q_n}_+(c^{(s-1)}) = Dh^{q_n}_+(c^{(s-1)}) = Dh^{q_n}([c^{(s-1)}, a^{(s-1)}])$$

So we get

$$Dh^{q_n}([c^{(s)}, a^{(s)}]) = Dh^{q_n}([c^{(s-1)}, a^{(s-1)}])$$

Iterating the last relation leads to

$$Dh^{q_n}([c^{(s)}, a^{(s)}]) = Dh^{q_n}([c^{(1)}, a^{(1)}]) \equiv Dh_-^{q_n}(a^{(1)}) = \sigma Dh_+^{q_n}(a^{(1)}) = \sigma Dh_+^{q_n}(a_0^*)$$

Hence Dh^{q_n} takes the constant value $\sigma Dh_+^{q_n}(a_0^*)$ on $A_n \bigcup B_n$.

Next we show, that Dh^{q_n} takes the constant value $Dh_+^{q_n}(a_0^*)$ on $S^1 \setminus (A_n \bigcup B_n)$. For this we determine Dh^{q_n} first on the interval $(a^{(1)}, c^{(q_n)})$. Obviously

$$Dh^{q_n}((a^{(1)}, c^{(q_n)})) = Dh_+^{q_n}(a^{(1)}).$$

On the other hand one has for $s \geq 1$

$$Dh^{q_n}(a^{(s+1)}, c^{(s)}) = Dh_-^{q_n}(c^{(s)}) = \sigma^{-1} Dh_+^{q_n}(c^{(s)}) = \sigma^{-1} Dh^{q_n}([c^{(s)}, a^{(s)}]).$$

The last relation together with

$$Dh^{q_n}([c^{(s)}, a^{(s)}]) = \sigma Dh_+^{q_n}(a_0^*)$$

implies, that for every $s \geq 1$

$$Dh^{q_n}(a^{(s+1)}, c^{(s)}) = Dh_+^{q_n}(a_0^*).$$

For the proof of (12) it is enough to prove under the assumption of n being odd and therefore $a_{q_n} \prec c^*_{-i_0} \prec a_0^* \prec c_{-i_0-q_n} \prec a_{-q_n} \prec a_{q_{n-1}}$, that

$$Dh_+^{q_n}(a_0^*) = \sigma^{(-1)^{n+1}\mu_h(U_n) - \delta_{1,(-1)^{n+1}}}, \tag{20}$$

where $\delta_{1,(-1)^{n+1}} = 1$ for n odd, respectively $\delta_{1,(-1)^{n+1}} = 0$ for n even. Notice that the last equation is true also for n even. Since h^{q_n} is an orientation preserving homeomorphism with irrational rotation number and the same invariant measure μ_h as the map h, we get from Theorem 1.3.

$$\int_{S^1} \log Dh^{q_n}(x) d\mu_h(x) = 0. \tag{21}$$

As mentioned above, the function Dh^{q_n} is constant on the subsets $U_n := A_n \cup B_n$ and $\bar{U}_n = S^1 \setminus U_n$. Therefore

$$\int_{S^1} \log Dh^{q_n}(x) d\mu_h(x) = \int_{U_n} \log Dh^{q_n}(x) d\mu_h(x) + \int_{\bar{U}_n} \log Dh^{q_n}(x) d\mu_h(x) = 0$$

Inserting the constant values of Dh^{q_n} on the sets U_n respectively $\bar{U}_n$ one finds

$$\int_{U_n} \log Dh^{q_n}(x) d\mu_h = \mu_h(U_n) \log(\sigma Dh_+^{q_n}(a_0^*)),$$

$$\int_{U_n} \log Dh^{q_n}(x) d\mu_h = \mu_h(\bar{U}_n) \log Dh_+^{q_n}(a_0^*) = [1 - \mu_f(U_n)] \log Dh_+^{q_n}(a_0^*),$$

and therefore

$$\mu_h(U_n)\log(\sigma Dh_+^{q_n}(a_0^*)) + [1-\mu_h(U_n)]\log Dh_+^{q_n}(a_0^*) = 0.$$

This shows that $\mu_h(U_n)\log\sigma = -\log Dh_+^{q_n}(a_0^*)$ respectively $Dh_+^{q_n}(a_0^*) = \sigma^{-\mu_h(U_n)}$, and hence formula (20) holds for n odd. For n even, the proof of formula (20) proceeds similarly. Theorem 1.9 is therefore completely proved.

Proof of Theorem 1.10.

We will prove only the following equation

$$Dh_+^{q_n}(a_0^*) = \sigma^{(-1)^n(\mu_h(A_n(i_0))-\mu_h(B_n(i_0))-\delta_{1,(-1)^{n-1}}} \tag{22}$$

for n odd. Since the rotation number ρ_h of h is irrational and its break points a_0^* and c_0^* are on different orbits, all the intervals in A_n and B_n are pairwise disjoint. For all $x \in B_n$ one has obviously $Dh^{q_n}(x) = Dh_+^{q_n}(a_0^*)$. But at the break point $c_{i_0}^*$ the function $Dh^{q_n}(x)$ makes the jump $Dh_+^{q_n}(c_{i_0}^*)/Dh_-^{q_n}(c_{i_0}^*) = Dh_+(c_{i_0}^*)/Dh_-(c_{i_0}^*) = \sigma$, and therefore it takes the constant value $Dh^{q_n}(x) = \sigma\ Dh_+^{q_n}(a_0^*)$ in this interval containing no break point of h^{q_n}. Indeed, this holds true for all intervals without break points, i.e. for $x \notin A_n \cup B_n$. The left boundary point of any interval in A_n belongs to the set $BP(c_0^*)$ and hence the function $Dh^{q_n}(x)$ makes at these break points the jump $Dh_+(c_{i_0}^*)/Dh_-(c_{i_0}^*) = \sigma$ and therefore takes the constant value $Dh^{q_n}(x) = \sigma^2\ Dh_+^{q_n}(a_0^*)$ for any $x \in A_n$. This proves assertion (17).

To prove assertion (22) we use again

$$\int_{S^1} \log Dh^{q_n}(x)d\mu_h(x) = 0,$$

and the possible values of the function Dh^{q_n} discussed above. Then

$$\log(\sigma^2\ Dh_+^{q_n}(a_0^*))\mu_f h(A_n) + \log(Dh_+^{q_n}(a_0^*))\mu(B_n) + \log(\sigma\ Dh_+^{q_n}(a_0^*))\mu(\bar{U}_n^*) = 0,$$

where $\bar{U}_n^* = S^1 \setminus (A_n \cup B_n)$. Hence

$$(\log\sigma)\{\mu_h(A_n) - \mu_h(B_n)\} + \log Dh_+^{q_n}(a_0^*)) + \log\sigma = 0$$

This proves equation (22) for n odd. The proof of the theorem for n even is similar. Theorem 1.10 is therefore completely proved.

Proof of Theorem 1.11.

Let h be a PL circle homeomorphism with two break points a_0^* and $a_{i_0}^* = f^{i_0}(a_0),\ i_0 > 0$, and irrational rotation number ρ_h. Assume $n > n_0$. Then h has $2i_0$ break points. Put $BP_h^n := BP_h^n(a_1^*) \cup BP_h^n(a_{-q_n+1}^*)$ with

$$BP_h^n(a_1^*) = \{a_1^*, a_2^*, ..., a_{i_0}^*\},$$

respectively

$$BP_h^n(a_{-q_n+1}^*) = \{a_{-q_n+1}^*, a_{-q_n+2}^*, ..., a_{-q_n+i_0}^*\},$$

where $a_s^* = f^s(a_0),\ a_{-q_n+s}^* = f^s(a_{-q_n}),\ 1 \le s \le i_0$. For the proof of Theorem 1.11 it is sufficient to prove the following formula

$$Dh_+^{q_n}(a_0^*) = \sigma^{(-1)^{n+1}\mu_h(U_n)-\delta_{1,(-1)^{n+1}}} \tag{23}$$

The partition $\mathbf{B}_n(h)$ determined by all break points of h^{q_n} has $2i_0$ closed intervals with disjoint interior. The map Dh^{q_n} is piecewise constant with constant values on the element of $\mathbf{B}_n(h)$. The first assertion of Lemma 1.8 implies that the intervals in $U_n = \{[a_s^*, a_{-q_n+s}^*], 1 \leq s \leq i_0\}$ are pairwise disjoint. Hence the intervals of $\bar{U}_n = S^1 \setminus U_n$ are also pairwise disjoint. Next we conclude that • the break points of $BP_h^n(a_1^*)$ and $BP_h^n(a_{-q_n+1}^*)$ alternate in their order on S^1;

• the intervals in U_n and $\bar{U}_n$ alternate in their order on S^1; Denote by $\bar{U}_n(a_s^*)$ the closed interval in $\bar{U}_n$ with right endpoint a_s^*, $1 \leq s \leq i_0$. It is easy to see that at each point a_s^* of $BP_h^n(a_1^*)$

$$Dh_+^{q_n}(a_s^*) = Dh_+^{q_n}(a_0^*),\ Dh_-^{q_n}(a_s^*) = Dh_-^{q_n}(a_0^*),\ 1 \leq s \leq i_0. \tag{24}$$

It is clear that the intervals $U_n(a_s^*)$ and $\bar{U}_n(a_s^*)$ are neighbours with common endpoint a_s^*. It is obvious that

$$\frac{Dh_-^{q_n}(a_s^*)}{Dh_+^{q_n}(a_s^*)} = \sigma_h(a_0^\star) = \sigma,\ 1 \leq s \leq i_0.$$

The last relation together with (24) implies $Dh^{q_n}(x) = \sigma Dh_+^{q_n}(a_0^*)$ if $x \in U_n$ respectively $Dh^{q_n}(x) = Dh_+^{q_n}(a_0^*)$ if $x \in S^1 \setminus U_n$. Remains to determine the value of $Dh_+^{q_n}(a_0^*)$. From Theorem 1.3 we obtain

$$\int_{S^1} \log Dh^{q_n}(x) d\mu_h = \int_{U_n} \log Dh^{q_n}(x) d\mu_h + \int_{\bar{U}_n} \log Dh^{q_n}(x) d\mu_h = 0.$$

Hence $\mu_h(U_n) \log Dh_+^{q_n}(a_0^*) + \mu_h(\bar{U}_n) \log Dh_-^{q_n}(a_0^*) = 0$.
Inserting $\mu_h(\bar{U}_n) = 1 - \mu_h(U_n)$ respectively $Dh_-^{q_n}(a_0^*) = \sigma_h Dh_+^{q_n}(a_0^*)$ we get relation (23). Theorem 1.11 hence is proved.

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Figure captions

Figure 3.1:
Location of the $2\,q_n$ break points of f^{q_n} in the dynamical partition $\xi_n(a_0^*)$ if $c_0^* \in I_{i_0}^{(n)}(a_0^*)$ for some $0 \leq i_0 < q_{n-1}$.

Figure 3.2:
Location of the $2\,q_n$ break points of f^{q_n} in the dynamical partition $\xi_n(c_{-i_0}^*)$ if $c_0^* \in f^{i_0}((a_0^*, a_{-q_n}])$ for some $0 \leq i_0 < q_{n-1}$.

Figure 3.3:
Location of the $2\,q_n$ break points of f^{q_n} in the dynamical partition $\xi_n(a_{-q_n+1}^*)$ if $c_0^* \in f^{i_0}((a_{-q_n}, a_{q_{n-1}}])$ for some $0 \leq i_0 < q_{n-1}$.

Figure 3.4:
Location of the $2\,i_0$ break points of f^{q_n} in the dynamical partition $\xi_n(a_0^*)$ if $c_0^* = f^{i_0}(a_0) = a_{i_0}^*$ for some $i_0 > 0$ and $\sigma_f = 1$.