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A note on Riemann–Liouville processes

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A B S T R A C T

In this note, it is proved that under certain conditions, Riemann–Liouville processes can arise from the temporal structures of the functional fluctuation limits of the occupation times of a type of spatial inhomogeneous branching particle system with infinite variances.

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1. Introduction

Let $\mathcal{M}(\mathrm{d}s)$ be a β-stable random measure on $(\mathbb{R}, \mathcal{B})$ with control measure λ and it is totally skewed to the right, where $\beta \in (0, 2]$, $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}$ and λ is the Lebesgue measure. For every $t \geq 0$, let $f_t(s) \equiv \mathbf{1}_{\{0 \leq s < t\}}(t-s)^\alpha$ with $\alpha > -1/\beta$. From Samorodnitsky and Taqqu [12, Chapter 3], we know that the stochastic integral

$$W_{\alpha,\beta}(t) := \int_{\mathbb{R}} f_t(s)\mathcal{M}(\mathrm{d}s) = \int_0^t (t-s)^\alpha \mathcal{M}(\mathrm{d}s) \tag{1.1}$$

defines a β-stable random process. When $\beta \neq 1$, its finite-dimensional distribution is determined by the characteristic function

$$\mathbb{E}\Big(\mathrm{e}^{i\sum_{k=1}^n \theta_k W_{\alpha,\beta}(t_k)}\Big) = \exp\Big\{-\int_{\mathbb{R}} |\Phi(s)|^\beta (1 - i\mathrm{sign}(\Phi(s))\tan(\frac{\beta\pi}{2}))\mathrm{d}s\Big\}, \tag{1.2}$$

where $0 \leq t_1 < t_2 < \cdots < t_n$, $\theta_k \in \mathbb{R}$ for all $1 \leq k \leq n$ and $\Phi(s) = \sum_{k=1}^n \theta_k f_{t_k}(s)$. Furthermore, for non-negative $\theta_1, \theta_2, \cdots, \theta_n$,

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$$\mathbb{E}\Big(\mathrm{e}^{-\sum_{k=1}^{n}\theta_k W_{\alpha,\beta}(t_k)}\Big) = \exp\Big\{-\int_{\mathbb{R}} \Phi^{\beta}(s)\mathrm{d}s/\cos(\beta\pi/2)\Big\}. \tag{1.3}$$

In this study, we refer to this as the (β,α)-type Riemann–Liouville process (abbreviated as (β,α)-RLP).

The (β,α)-RLP is self-similar with the index $H = \alpha + 1/\beta$, i.e. for any $c \geq 0$,

$$\{W_{\alpha,\beta}(ct), t \geq 0\} \stackrel{d}{=} \{c^{\alpha+1/\beta}W_{\alpha,\beta}(t), t \geq 0\}.$$

This can be viewed as the fractional integration of a β-stable Lévy process that is totally skewed to the right and that is not symmetric when $\beta < 2$. Moreover, $\{W_{\alpha,\beta}(t)\}$ is a fractional martingale (Hu et al. [6]) when $1 < \beta \leq 2$. In addition, we note that the (β,α)-RLP defined in this study differs slightly from the "RLP" defined in Lifshits and Simon [11] where a RLP is supposed to be symmetric.

A branching particle system is a type of stochastic model where particles move independently and evolve according to a given law, where this system is applicable in various fields including physics and mathematical biology. It is interesting to study how branching particle systems can be used to reconstruct or explain stochastic phenomena or models. For example, branching particle systems have been used to reconstruct measure-valued processes (see Le Gall [7] and the references therein), Gaussian fields (see Li and Xiao [9,10]), Gaussian processes, and stable processes (see Bojdecki et al. [1,3] and the references therein). The aim of this note is to find the (β,α)-RLPs for some branching particle systems.

The branching particle systems considered in this note are assumed to be defined as follows. At the beginning, the particles are distributed according to a Poisson random measure with intensity measure λ in the space $\mathbb{R}^d$. The particles then move and evolve independently, where they move according to a symmetric α-stable Lévy process. The lifetime of each particle is an exponential random variable with parameter 1. If a particles dies at site x, it immediately splits into several particles according to a branching mechanism with the generating function

$$f(s) = s + \frac{\sigma(x)(1-s)^{1+\beta}}{1+\beta}, \qquad \sigma(x) \in [0,1],\ 0 < \beta \leq 1.$$

New particles start their movements and evolution from their "birthplaces" according to the aforementioned law. When $\sigma(x) \equiv 1$, the branching particle systems are simply the classical (d,α,β)-branching particle systems (e.g., see Gorostiza and Wakolbinger [5]). In this study, we use the function $\sigma(x)$ to focus on the situation where the particles may have different branching mechanisms at different sites (e.g., see Dawson and Fleischmann [4]) and, for convenience, we refer to the models as the σ-mixed (α,β)-systems. In particular, we focus on the case where $d = 1 < \alpha < 1+\beta \leq 2$ and $0 < \int_{\mathbb{R}} \sigma(x)\mathrm{d}x < \infty$ in the sequel.

Let $N(s)$ denote the empirical measure of the σ-mixed (α,β)-system at time s, i.e., for each measurable set $A \subset \mathbb{R}$, $N(s)(A)$ is the number of particles at time s in set A. Let

$$Y(t) = \int_0^t (N(s) - \mathbb{E}(N(s)))\mathrm{d}s, \quad t \geq 0,$$

which is called the occupation time fluctuation process, where $\mathbb{E}(N(s))$ is the expectation functional understood as $\langle \mathbb{E}(N(s)), \phi\rangle = \mathbb{E}(\langle N(s), \phi\rangle)$ for all $\phi \in \mathcal{S}(\mathbb{R})$, the space of smooth rapidly decreasing functions, and $\langle \mu, f\rangle$ denotes the integral $\int f\mathrm{d}\mu$ for a measure μ and a measurable function f. We find that as $T \to \infty$, the functional limit of $Y(T\cdot)/F_T$ with $F_T = T^{(2+\beta)/(1+\beta)-1/\alpha}$ exists, where its temporal structure comprises a $(1+\beta, 1-1/\alpha)$-RLP.

The remainder of this paper is organized as follows. In Section 2, we state our result and present some preliminary facts. In Section 3, we give the proof of the main result. Unless stated otherwise, in the sequel, let $K := \int_{\mathbb{R}} \sigma(x)\mathrm{d}x/(1+\beta)$, and M, M_1, and M_2 are unspecified positive finite constants that are not necessarily the same in each occurrence.

2. Preliminaries

Let $N(s)$ be the random counting measure of the σ-mixed (α, β)-system. Let ξ be the symmetric α-stable Lévy process. Denote its semigroup by $\{L_t\}_{t\geq 0}$ and the transition density by p_t, i.e.,

$$L_t f(x) := \mathbb{E}(f(\xi(t+s))|\xi(s) = x) = \int_{\mathbb{R}} p_t(y-x)f(y)\mathrm{d}y$$

for all $s, t \geq 0$, $x \in \mathbb{R}$ and bounded measurable functions f (to avoid ambiguity, we sometimes write $L_t f(x)$ as $L_t(f(\cdot))(x)$). It is well known that

$$p_{tu}(x) = t^{-1/\alpha} p_u(xt^{-1/\alpha}), \quad x \in \mathbb{R},\ t, u > 0, \tag{2.1}$$

and that

$$0 \leq p_1(x) \leq p_1(0) = \frac{\Gamma(1/\alpha)}{\alpha\pi}, \qquad x \in \mathbb{R}. \tag{2.2}$$

For every bounded and integrable function f, define

$$G_t f(x) = \int_0^t L_s f(x)\mathrm{d}s. \tag{2.3}$$

By (2.1) and (2.2), it is easy to see that a constant M exists such that for $t > 1$, $\alpha \in (1, 2)$

$$G_t f(x) \leq M t^{1-1/\alpha}. \tag{2.4}$$

Standard analysis of branching particle systems (e.g., see Bojdecki et al. [1]) shows that

$$\mathbb{E}(\langle N(t), \phi\rangle) = \int_{\mathbb{R}} L_t\phi(x)\mathrm{d}x = \int_{\mathbb{R}} \phi(x)\mathrm{d}x = \langle \lambda, \phi\rangle.$$

The occupation time fluctuation process $Y = \{Y(t), t \geq 0\}$ can be rewritten as follows:

$$\langle Y(t), \phi\rangle = \int_0^t \langle N(s) - \lambda, \phi\rangle \mathrm{d}s$$

for every $\phi \in \mathcal{S}(\mathbb{R})$. Let $F_T = T^{(2+\beta)/(1+\beta)-1/\alpha}$ and $Y_T(\cdot) = Y(T\cdot)/F_T$. Thus, we have the following main result presented in this note.

Theorem 2.1. *Suppose that $d = 1 < \alpha < 1 + \beta \leq 2$ and $\int_{\mathbb{R}} \sigma(x)\mathrm{d}x < \infty$. As $T \to \infty$, the process Y_T converges weakly to $K_1\chi\lambda$ in $C([0,1], \mathcal{S}'(\mathbb{R}))$, where $\chi(\cdot)$ is the $(1+\beta, 1-1/\alpha)$-RLP and the constant $K_1 = \frac{\Gamma(1/\alpha)}{(\alpha-1)\pi}\big[-K\cos\big(\frac{(1+\beta)\pi}{2}\big)\big]^{1/(1+\beta)}$.*

Remark 2.1. Li [8] studied the special case where $\beta = 1$ and obtained a Gaussian Riemann–Liouville process that is symmetric. In the present study, Theorem 2.1 generalizes the main result reported by Li [8].

Remark 2.2. Bojdecki et al. [1,2] studied the classical (d, α, β)-systems under different initial values and obtained some new but relatively complicated stable processes. Compared with their stable processes, the $(1+\beta, 1-1/\alpha)$-RLP is easier to understand and it is closely related to the fractional integrals.

For any $n > 0$, let $0 \le t_1 < t_2 < \cdots < t_n \le 1$. $0 \le \phi_k \in \mathcal{S}(\mathbb{R})$ for every $k = 1, 2, \cdots, n$. Let

$$\varphi(x,t) = \sum_{k=1}^{n} \phi_k(x)\mathbf{1}_{(0,t_k]}(t) \quad \text{and} \quad \psi_T(x,t) = \frac{1}{F_T}\varphi(x, \frac{t}{T}). \tag{2.5}$$

Define

$$V_{\psi_T}(x,t,r) := 1 - \mathbb{E}_x\Big(\exp\Big\{ -\int_0^t \langle N(s), \psi_T(\cdot, r+s)\rangle \mathrm{d}s \Big\}\Big) \tag{2.6}$$

and

$$J_{\psi_T}(x,t,r) := \int_0^t L_s\psi_T(\cdot, r+s)(x)\mathrm{d}s. \tag{2.7}$$

Then,

$$V_{\psi_T}(x,t,r) \le J_{\psi_T}(x,t,r). \tag{2.8}$$

Furthermore, by some standard arguments (e.g., see Li [8]), we find that

$$\begin{aligned} V_{\psi_T}(x,t,r) = &\int_0^t L_s\big[\psi_T(\cdot, r+s)\big(1 - V_{\psi_T}(\cdot, t-s, r+s)\big)\big](x)\mathrm{d}s \\ &- \frac{1}{1+\beta}\int_0^t L_s[\sigma(\cdot)V_{\psi_T}^{1+\beta}(\cdot, t-s, r+s)](x)\mathrm{d}s, \end{aligned} \tag{2.9}$$

and that

$$\mathbb{E}\Big(\exp\Big\{ -\sum_{k=1}^{n} \langle Y_T(t_k), \phi_k\rangle \Big\}\Big) = \exp\Big(I_1(T,\psi_T) - I_2(T,\psi_T) + I_3(T,\psi_T)\Big), \tag{2.10}$$

where

$$I_1(T,\psi_T) = \frac{1}{1+\beta}\int_{\mathbb{R}} \sigma(x)\mathrm{d}x \int_0^T J_{\psi_T}^{1+\beta}(x, T-s, s)\mathrm{d}s, \tag{2.11}$$

$$I_2(T,\psi_T) = \frac{1}{1+\beta}\int_{\mathbb{R}} \sigma(x)\mathrm{d}x \int_0^T \Big[J_{\psi_T}^{1+\beta}(x, T-s, s) - V_{\psi_T}^{1+\beta}(x, T-s, s)\Big]\mathrm{d}s, \tag{2.12}$$

and

$$I_3(T,\psi_T) = \int_{\mathbb{R}} \mathrm{d}x \int_0^T \psi_T(x,s)V_{\psi_T}(x, T-s, s)\mathrm{d}s. \tag{2.13}$$

The inequality $a^{1+\beta} - b^{1+\beta} \le (a^\beta + b^\beta)(a-b)$ for all $a \ge b > 0$ implies that

$$I_2(T,\psi_T) \le \frac{2}{1+\beta}\Big[I_{21}(T,\psi_T) + \frac{1}{1+\beta} I_{22}(T,\psi_T)\Big], \tag{2.14}$$

where

$$I_{21}(T,\psi_T) = \int_{\mathbb{R}} \sigma(x)\mathrm{d}x \int_0^T J^\beta_{\psi_T}(x,T-s,s)\mathrm{d}s \int_s^T L_{u-s}\big(\psi_T(\cdot,u)J_{\psi_T}(\cdot,T-u,u)\big)(x)\mathrm{d}u, \tag{2.15}$$

$$I_{22}(T,\psi_T) = \int_{\mathbb{R}} \sigma(x)\mathrm{d}x \int_0^T J^\beta_{\psi_T}(x,T-s,s)\mathrm{d}s \int_s^T L_{u-s}(\sigma(\cdot)J^{1+\beta}_{\psi_T}(\cdot,T-u,u))(x)\mathrm{d}u. \tag{2.16}$$

3. Proof of the main result

To prove Theorem 2.1, it is sufficient to prove the finite-dimensional convergence of $\{Y_T\}_{T\ge1}$ plus their tightness. We prove this in two lemmas. For convenience, in this section, $1/\alpha$ is denoted by $\bar{\alpha}$.

Lemma 3.1. *Under the assumptions of Theorem 2.1, $Y_T \to K_1\chi\lambda$ in finite-dimensional distributions.*

Proof. By (1.3) the process $\chi = \{\chi(t)\}$ has finite-dimensional distributions determined by the Laplace functions

$$\mathbb{E}(\mathrm{e}^{-\sum_{k=1}^n \theta_k\chi(t_k)}) = \exp\Big\{-\int_{\mathbb{R}}\Big[\sum_{k=1}^n \theta_k(t_k-s)^{1-\bar{\alpha}}\mathbf{1}_{(0,t_k]}(s)\Big]^{1+\beta}\mathrm{d}s/\cos((1+\beta)\pi/2)\Big\},$$

where $\theta_k > 0$, $k = 1,2,\cdots,n$ and $0 = t_0 \le t_1 < \cdots < t_n \le 1$. Lemma 3.4 given by Bojdecki et al. [1] and the same argument used by Bojdecki et al. [1, Lemma 3.5] ensure the sufficiency of proving that

$$\begin{aligned}\lim_{T\to\infty}\mathbb{E}\Big(\exp\Big\{-\sum_{k=1}^n\langle Y_T(t_k),\phi_k\rangle\Big\}\Big) &= \mathbb{E}\Big(\exp\Big\{-K_1\sum_{k=1}^n\chi(t_k)\langle\lambda,\phi_k\rangle\Big\}\Big)\\ &= \exp\Big\{K\int_{\mathbb{R}}\Big(\frac{\Gamma(\bar{\alpha})}{(\alpha-1)\pi}\sum_{k=1}^n\int_{\mathbb{R}}\phi_k(y)\mathrm{d}y(t_k-s)^{1-\bar{\alpha}}\mathbf{1}_{(0,t_k]}(s)\Big)^{1+\beta}\mathrm{d}s\Big\}\end{aligned} \tag{3.1}$$

for any non-negative $\phi_1,\cdots,\phi_n \in \mathcal{S}(\mathbb{R})$. According to (2.10), we divide the proof into three steps.

Step 1. We consider the limit of $I_1(T,\psi_T)$. By applying (2.1), (2.5), and (2.7) to (2.11), we find that

$$\begin{aligned}I_1(T,\psi_T) &= \frac{1}{1+\beta}\int_{\mathbb{R}}\sigma(x)\mathrm{d}x\int_0^T\Big(\int_0^{T-s}L_u\psi_T(x,s+u)\mathrm{d}u\Big)^{1+\beta}\mathrm{d}s\\ &= \frac{T^{2+\beta}}{(1+\beta)F_T^{1+\beta}}\int_{\mathbb{R}}\sigma(x)\mathrm{d}x\int_0^1\Big(\int_0^{1-s}\sum_{k=1}^n L_{Tu}\phi_k(x)\mathbf{1}_{(0,t_k]}(s+u)\mathrm{d}u\Big)^{1+\beta}\mathrm{d}s\\ &= \frac{T^{2+\beta-(1+\beta)\bar{\alpha}}}{(1+\beta)F_T^{1+\beta}}\int_{\mathbb{R}}\sigma(x)\mathrm{d}x\int_0^1\Big(\sum_{k=1}^n\Phi_k(s,T)\Big)^{1+\beta}\mathrm{d}s,\end{aligned}$$

where

$$\Phi_k(s,T) = \int_0^{1-s} u^{-\bar{\alpha}} \mathbf{1}_{(0,t_k]}(s+u) \int_{\mathbb{R}} p_1((x-y)(Tu)^{-\bar{\alpha}})\phi_k(y)\mathrm{d}y\mathrm{d}u.$$

By the dominated convergence theorem, it is easy to see that

$$\begin{aligned}\lim_{T\to\infty} \Phi_k(s,T) &= \int_0^{1-s} u^{-\bar{\alpha}} \mathbf{1}_{(0,t_k]}(s+u)\mathrm{d}u \int_{\mathbb{R}} p_1(0)\phi_k(y)\mathrm{d}y \\ &= \frac{\alpha}{\alpha-1}(t_k-s)^{1-\bar{\alpha}} \mathbf{1}_{(0,t_k]}(s) \int_{\mathbb{R}} p_1(0)\phi_k(y)\mathrm{d}y. \end{aligned} \tag{3.2}$$

By substituting $F_T^{1+\beta} = T^{2+\beta-(1+\beta)\bar{\alpha}}$ and the right-hand side of (2.2) into (3.2), we find that as $T \to \infty$,

$$\lim_{T\to\infty} I_1(T,\psi_T) = K \int_0^1 \left(\frac{\Gamma(\bar{\alpha})}{(\alpha-1)\pi} \sum_{k=1}^n (t_k-s)^{1-\bar{\alpha}} \mathbf{1}_{(0,t_k]}(s) \int_{\mathbb{R}} \phi_k(y)\mathrm{d}y \right)^{1+\beta} \mathrm{d}s. \tag{3.3}$$

Step 2. We consider the limit of $I_2(T,\psi_T)$. By applying (2.3) and (2.5) to (2.7), we find that

$$J_{\psi_T}(x, T-u, u) \le \frac{M}{F_T} G_T \sum_{k=1}^n \phi_k(x) \tag{3.4}$$

for any $u \le T$. By applying (2.4) and (3.4) to (2.15), we obtain a constant $M > 0$ such that

$$\begin{aligned} I_{21}(T,\psi_T) &\le \frac{MT}{F_T^{2+\beta}} \int_{\mathbb{R}} \sigma(x) \Big[\Big(\int_0^T L_v \sum_{k=1}^n \phi_k(x)\mathrm{d}v \Big)^{\beta} \int_0^T L_u \Big(\sum_{k=1}^n \phi_k \int_0^T L_w \sum_{k=1}^n \phi_k \mathrm{d}w \Big)(x)\mathrm{d}u \Big] \mathrm{d}x \\ &\le \frac{MT^{1+(1-\bar{\alpha})(2+\beta)}}{F_T^{2+\beta}} \int_{\mathbb{R}} \sigma(x)\mathrm{d}x. \end{aligned} \tag{3.5}$$

Therefore, by substituting $F_T = T^{(2+\beta)/(1+\beta)-\bar{\alpha}}$ into (3.5), we can readily obtain

$$I_{21}(T,\psi_T) \le \frac{M}{T^{1/(1+\beta)}} \int_{\mathbb{R}} \sigma(x)\mathrm{d}x \to 0. \tag{3.6}$$

Furthermore, applying (3.4) to (2.16) leads to

$$I_{22}(T,\psi_T) \le \frac{MT}{F_T^{1+2\beta}} \int_{\mathbb{R}} \sigma(x) \Big(G_T \sum_{k=1}^n \phi_k(x) \Big)^{\beta} G_T \Big(\sigma (G_T \sum_{k=1}^n \phi_k)^{1+\beta} \Big)(x)\mathrm{d}x \tag{3.7}$$

for some $M > 0$. From (2.4), (3.7), and the fact that $\int_{\mathbb{R}} \sigma(x)\mathrm{d}x < \infty$, it follows that

$$I_{22}(T,\psi_T) \le \frac{M}{T^{\bar{\alpha}-1/(1+\beta)}} \to 0, \tag{3.8}$$

where the convergence is due to the assumption that $\beta > \alpha - 1$. Note that $I_2(T,\psi_T) \ge 0$ because of (2.8). By combining (2.12) with (3.5) and (3.8), we find that as $T \to \infty$,

$$I_2(T,\psi_T) \to 0. \tag{3.9}$$

Step 3. We consider the limit of $I_3(T,\psi_T)$. Let

$$I_{31}(T,\psi_T) := \int_{\mathbb{R}} \mathrm{d}x \int_0^T \psi_T(x,s) J_{\psi_T}(x,T-s,s)\mathrm{d}s. \tag{3.10}$$

From (2.13), (2.8), (3.4), and (2.4), it follows that

$$0 \le I_3(T,\psi_T) \le I_{31}(T,\psi_T) \le \frac{T^{2-\bar{\alpha}}M}{F_T^2} \int_{\mathbb{R}} \sum_{k=1}^{n} \phi_k(x)\mathrm{d}x \tag{3.11}$$

for some $M > 0$. By substituting $F_T = T^{(2+\beta)/(1+\beta)-\bar{\alpha}}$ into (3.11), the fact that $\beta \le 1 < 2\alpha - 1$ for all $\alpha \in (1,2)$ indicates that

$$I_3(T,\psi_T) \to 0. \tag{3.12}$$

After combining (2.10) with (3.3), (3.9), and (3.12), we arrive at (3.1). □

Lemma 3.2. *Under the assumptions of Theorem 2.1, $\{Y_T\}_{T\ge 1}$ is tight in $C([0,1],\mathcal{S}'(\mathbb{R}))$.*

Proof. The proof is analogous to that given by Bojdecki et al. [1, Proposition 3.3]. For simplicity, we omit some common details.

For any given $v,u \in [0,1]$ with $v < u$, and $n > \frac{2}{u-v}$, let $h_n \in \mathcal{S}(\mathbb{R})$ satisfy $\mathrm{supp}(h_n) \subset [v, v+\frac{1}{n}] \cup [u-\frac{1}{n}, u]$, and $h_n \le 0$ on $[v, v+\frac{1}{n}]$ with $\int_v^{v+\frac{1}{n}} h_n(s)\mathrm{d}s = -1$ and $h_n \ge 0$ on $[u, u-\frac{1}{n}]$ with $\int_{u-\frac{1}{n}}^{u} h_n(s)\mathrm{d}s = 1$.

According to the discussion given by Bojdecki et al. [1, Proposition 3.3], it is sufficient to prove that for any given nonnegative $\phi \in \mathcal{S}(\mathbb{R}^d)$, the constants $a \ge 1$, $b > 0$, and $M > 0$ exist such that for all $T \ge 1$, $0 \le v < u \le 1$, $n \ge 2/(u-v)$, and $1 > \delta > 0$,

$$\int_0^{1/\delta} \Big(1 - \mathrm{Re}\Big(\mathbb{E}\Big[\exp\{-i\omega\langle \tilde{Y}_T, \phi h_n\rangle\}\Big]\Big)\Big)\mathrm{d}\omega \le \frac{M}{\delta^a}(u-v)^{1+b}, \tag{3.13}$$

where $\tilde{Y}$ is defined as $\tilde{X}$ by Bojdecki et al. [1] (see (3.6) therein). Note that

$$\mathbb{E}\Big[\exp\Big\{-i\omega\langle\tilde{Y}_T, \phi h_n\rangle\Big\}\Big] = \exp\Big\{I_1(T, i\omega\psi_{T,n}) - I_2(T, i\omega\psi_{T,n}) + I_3(T, i\omega\psi_{T,n})\Big\},$$

and that

$$|V_{i\omega\psi_{T,n}}| \le J_{\omega\psi_{T,n}},$$

where

$$\psi_{T,n}(x,s) = \frac{1}{F_T}\phi(x)\tilde{h}_n\Big(\frac{s}{T}\Big) \ \text{ and } \ \tilde{h}_n(s) = \int_s^1 h_n(t)\mathrm{d}t.$$

It is easy to check that

$$\begin{cases} |I_1(T, i\omega\psi_{T,n}) - I_2(T, i\omega\psi_{T,n})| \le \omega^{1+\beta} I_1(T,\psi_{T,n}); \\ |I_3(T, i\omega\psi_{T,n})| \le \omega^2 I_{31}(T,\psi_{T,n}). \end{cases} \tag{3.14}$$

In the following, we estimate $I_1(T,\psi_{T,n})$ and $I_{31}(T,\psi_{T,n})$ based on the above.

First, we estimate the upper bound of $I_1(T,\psi_{T,n})$. By the same arguments leading to (3.3), we can readily find that a constant $M>0$ exists that depends on ϕ such that

$$I_1(T,\psi_{T,n}) \le M\int_0^1 \mathrm{d}s\Big(\int_s^1 h_n(t)(t-s)^{1-\bar\alpha}\mathrm{d}t\Big)^{1+\beta}. \tag{3.15}$$

Let $H_n(s)=\int_s^1 h_n(t)(t-s)^{1-\bar\alpha}\mathrm{d}t$. Then, by the assumptions on h_n, we have

$$0\le H_n(s)\le \begin{cases} 0, & s\in(u,1],\\ (u-s)^{1-\bar\alpha}, & s\in[v,u],\\ (u-s)^{1-\bar\alpha}-(v-s)^{1-\bar\alpha}, & s\in[0,v).\end{cases} \tag{3.16}$$

Therefore, (3.15) indicates that

$$\begin{aligned} I_1(T,\psi_{T,n}) &\le M\int_0^v ((u-s)^{1-\bar\alpha}-(v-s)^{1-\bar\alpha})^{1+\beta}\mathrm{d}s + M\int_v^u (u-s)^{(1-\bar\alpha)(1+\beta)}\mathrm{d}s\\ &\le M\int_0^1((u-v+s)^{1-\bar\alpha}-s^{1-\bar\alpha})^{1+\beta}\mathrm{d}s+\frac{M(u-v)^{1+(1-\bar\alpha)(1+\beta)}}{1+(1-\bar\alpha)(1+\beta)}. \end{aligned} \tag{3.17}$$

We observe that for each $x\in(0,1]$,

$$\begin{aligned} \int_0^1((x+s)^{1-\bar\alpha}-s^{1-\bar\alpha})^{1+\beta}\mathrm{d}s &= x^{1+(1-\bar\alpha)(1+\beta)}\int_0^{1/x}[(1+u)^{1-\bar\alpha}-u^{1-\bar\alpha}]^{1+\beta}\mathrm{d}u\\ &\le x^{1+(1-\bar\alpha)(1+\beta)}\Big(1+\int_1^\infty[(1-\bar\alpha)u^{-\bar\alpha}]^{1+\beta}\mathrm{d}u\Big), \end{aligned}$$

where we use the facts that for every $u\ge 0$, $0\le(1+u)^{1-\bar\alpha}-u^{1-\bar\alpha}\le 1$ and

$$(1+u)^{1-\bar\alpha}-u^{1-\bar\alpha}=\int_u^{1+u}(1-\bar\alpha)x^{-\bar\alpha}\mathrm{d}x\le(1-\bar\alpha)u^{-\bar\alpha}.$$

Since $\bar\alpha(1+\beta)>1$,

$$\int_1^\infty[(1-\bar\alpha)u^{-\bar\alpha}]^{1+\beta}\mathrm{d}u<\infty.$$

We can obtain a positive constant M such that

$$\int_0^1((x+s)^{1-\bar\alpha}-s^{1-\bar\alpha})^{1+\beta}\mathrm{d}s\le Mx^{1+(1-\bar\alpha)(1+\beta)}$$

for all $x\in(0,1]$. Therefore, a constant $M>0$ exists that is independent of u,v such that

$$\int_0^1 ((u-v+s)^{1-\bar{\alpha}} - s^{1-\bar{\alpha}})^{1+\beta} \mathrm{d}s \le M(u-v)^{1+(1-\bar{\alpha})(1+\beta)}. \tag{3.18}$$

(3.17) and (3.18) show that a constant $M > 0$ exists that only depends on ϕ such that

$$I_1(T, \psi_{T,n}) \le M(u-v)^{1+(1-\bar{\alpha})(1+\beta)} \tag{3.19}$$

for all $T > 1$ and $n > 2/(u-v)$.

Next, we estimate $I_{31}(T, \psi_{T,n})$. From (3.10), it follows that

$$I_{31}(T, \psi_{T,n}) \le \frac{1}{F_T^2} \int_0^T \tilde{h}_n\Big(\frac{s}{T}\Big) \mathrm{d}s \int_0^{T-s} \tilde{h}_n\Big(\frac{s+r}{T}\Big) \mathrm{d}r \int_{\mathbb{R}} \phi(x) L_r \phi(x) \mathrm{d}x.$$

Note that from the assumptions on h_n,

$$0 \le \tilde{h}_n(s) = \int_s^1 h_n(t) \mathrm{d}t \le \mathbf{1}_{[u,v]}(s)$$

for all $s \in [0, 1]$. Therefore, by using (2.4)

$$I_{31}(T, \psi_{T,n}) \le \frac{T^2}{F_T^2} \int_v^u \mathrm{d}s \int_s^u \mathrm{d}r \int_{\mathbb{R}} \phi(x) L_{T(r-s)} \phi(x) \mathrm{d}x \le M \frac{T^{2-\bar{\alpha}}}{F_T^2} \int_v^u \mathrm{d}s \int_s^u (r-s)^{-\bar{\alpha}} \mathrm{d}r. \tag{3.20}$$

After substituting $F_T = T^{(2+\beta)/(1+\beta)-\bar{\alpha}}$ into (3.20), we find that

$$I_{31}(T, \psi_{T,n}) \le \frac{M}{T^{2/(1+\beta)-\bar{\alpha}}} (u-v)^{2-\bar{\alpha}} \le M(u-v)^{2-\bar{\alpha}} \tag{3.21}$$

since $\beta \le 1 < 2\alpha - 1$. From (3.19) and (3.21), a positive constant M exists that only depends on ϕ such that for all $T > 1$ and $n \ge 2/(u-v)$,

$$|I_{31}(T, \omega\psi_{T,n})| + |I_1(T, \omega\psi_{T,n})| \le M(\omega^2 + \omega^{1+\beta})|u-v|^{2-\bar{\alpha}}. \tag{3.22}$$

Since $|1 - \mathrm{e}^z| \le |z|$ when $|\mathrm{e}^z| \le 1$,

$$\Big|1 - \mathrm{Re}\Big(\mathbb{E}\Big[\exp\big\{-i\omega\langle \tilde{Y}_T, \phi h_n\rangle\big\}\Big]\Big)\Big| \le |I_1(T, i\omega\psi_{T,n}) - I_2(T, i\omega\psi_{T,n})| + |I_3(T, i\omega\psi_{T,n})|.$$

Therefore, (3.14) and (3.22) together imply that for $\delta \in (0, 1)$

$$\int_0^{1/\delta} \Big(1 - \mathrm{Re}\Big(\mathbb{E}\Big[\exp\big\{-i\omega\langle \tilde{Y}_T, \phi h_n\rangle\big\}\Big]\Big)\Big) \mathrm{d}\omega \le \frac{M}{3\delta^3} |u-v|^{2-\bar{\alpha}},$$

which completes the proof of (3.13) and the proof of Lemma 3.2. □

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References

[1] T. Bojdecki, L. Gorostiza, A. Talarczyk, A long range dependence stable process and an infinite variance branching system, Ann. Probab. 35 (2007) 500–527.

[2] T. Bojdecki, L. Gorostiza, A. Talarczyk, Occupation times of branching systems with initial inhomogeneous Poisson states and related superprocesses, Electron. J. Probab. 14 (2009) 1328–1371.

[3] T. Bojdecki, A. Talarczyk, Particle picture interpretation of some Gaussian processes related to fractional Brownian motion, Stochastic Process. Appl. 122 (2012) 2134–2154.

[4] D.A. Dawson, K. Fleischmann, Critical dimension for a model of branching in a random medium, Probab. Theory Related Fields 70 (1985) 315–334.

[5] L.G. Gorostiza, A. Wakolbinger, Persistence criteria for a class of critical branching particle systems in continuous time, Ann. Probab. 19 (1991) 266–288.

[6] Y. Hu, D. Nualart, J. Song, Fractional martingales and characterization of the fractional Brownian motion, Ann. Probab. 37 (2009) 2404–2430.

[7] J.F. Le Gall, Spatial Branching Processes, Random Snakes and Partial Differential Equations, Birkhäuser Verlag, Basel, 1999.

[8] Y. Li, Riemann–Liouville processes arising from branching particle systems, Stoch. Dyn. 13 (3) (2013) 1250022.

[9] Y. Li, Y. Xiao, Occupation time fluctuations of weakly degenerate branching systems, J. Theoret. Probab. 25 (2012) 1119–1152.

[10] Y. Li, Y. Xiao, A class of fractional Brownian fields from branching systems and their regularity properties, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 16 (3) (2013) 1350023.

[11] M.A. Lifshits, T. Simon, Small deviations for fractional stable processes, Ann. Inst. Henri Poincaré 41 (2005) 725–752.

[12] G. Samorodnitsky, M.S. Taqqu, Stable Non-Gaussian Random Processes, Chapman and Hall, New York, 1994.