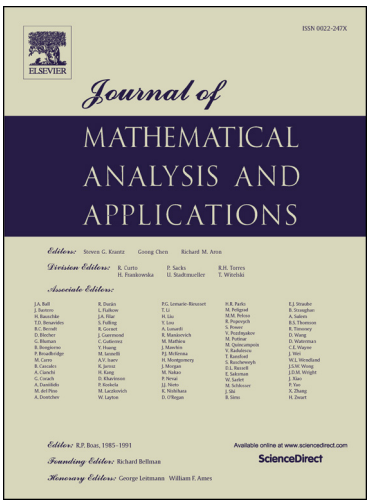
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Approximate fixed points of α-nonexpansive mappings

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Abstract

In this paper, we define the class of (α, β)-nonexpansive mappings which is properly larger than the class of α-nonexpansive mappings and prove that every (α, β)-nonexpansive mapping $T : C \to C$ has an approximate fixed point sequence, where C is a nonempty bounded subset of a Banach space X, $\alpha > 0$ and $\beta \geq 0$. This, in particular, gives an affirmative answer to the open question posed by Ariza-Ruiz and et al. concerning the existence of an approximate fixed point sequence for α-nonexpansive mappings [D. Ariza-Ruiz, C. Hernández Linares, E. Llorens-Fuster, E. Moreno-Gálvez, On α-nonexpansive mappings in Banach spaces, Carpathian J. Math., 32(1) (2016), 13-28].

Keywords: fixed point, approximate fixed point sequence, (α, β)-nonexpansive mapping, normal structure.
2000 MSC: 47H10.

1. Introduction and Preliminares

In the following we assume that C is a nonempty subset of a Banach space $(X, \|.\|)$ and $T : C \to X$ is a mapping. T is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$. A point $p \in C$ is called a fixed point for T when $Tp = p$ and the set of the fixed points of T will be denoted by $\mathrm{Fix}(T)$. A sequence (x_n) in C is called an approximate fixed point sequence (a.f.p.s., for short) for T if $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. T is said to be a quasi-nonexpansive, if $\mathrm{Fix}(T) \neq \emptyset$ and for every $p \in \mathrm{Fix}(T)$ and $x \in C$, $\|Tx - Tp\| \leq \|x - p\|$.
In the recent years, several generalizations of nonexpansive mappings have received attention and their fixed point theory have been studied by many authors, see [1, 3, 6, 7, 10, 12, 13, 15] and the references therein.

In 2011, Aoyama and Kohsaka [3] introduced the α-nonexpansive mappings and proved that every α-nonexpansive mapping $T : C \to C$ with $\alpha < 1$ has a fixed point, where C is a convex closed bounded subset of a uniformly convex Banach space X. The class of α-nonexpansive mappings is important because it contains the nonspreading

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mappings defined by Kohsaka and Takahashi in 2008 [11], the hybrid mappings introduced by Takahashi in 2010 [16] and, in the setting of Hilbert spaces, the λ-hybrid mappings due to Aoyama et al. introduced in 2010 [2].
Recently Ariza-Ruiz et al. [4] presented a further study about α-nonexpansive mappings and proved that every $\frac{1}{2}$-nonexpansive mapping $T : C \to C$ has an a.f.p.s., provided that C is a convex closed bounded subset of a Banach space X. They also posed the question of the existence of an a.f.p.s. for an α-nonexpansive mapping $T : C \to C$, where $0 < \alpha < 1$.
In this paper we define the class of (α, β)-nonexpansive mappings which is properly larger than the class of α-nonexpansive mappings and prove that if C be a nonempty bounded subset of a Banach space X then every (α, β)-nonexpansive mapping $T : C \to C$ with $\alpha > 0$ and $\beta \geq 0$, has an a.f.p.s., which gives an affirmative answer to the above mentioned question posed in [4].

Throughout the paper, let $(X, \|.\|)$ be a real Banach space. For any subset D of X and each $x \in X$, set $r_x(D) = \sup_{y \in D} \|x - y\|$. A point $x \in D$ is said to be *nondiametral* if $r_x(D) < \operatorname{diam}(D)$. A convex subset C of X is said to have *normal structure* if each bounded, convex subset K of C with $\operatorname{diam}(K) > 0$ contains a nondiametral point. A Banach space X has normal structure if each convex subset of X has normal structure. Note that every uniformly convex Banach space enjoys normal structure, but the converse does not hold. We also need the following definition in the sequal.

Definition 1.1. *[12] Let C be a nonempty closed bounded subset of a Banach space X. We say that the mapping $T : C \to C$ satisfies condition (L) (or it is an (L)-type mapping) on C provided that it fulfills the following two conditions.*

(i) If a set $D \subseteq C$ is nonempty closed convex and T invariant, (i.e., $T(D) \subseteq D$), then there exists an a.f.p.s. for T in D.

(ii) For any a.f.p.s. (x_n) of T in C and each $x \in C$,

$$\limsup_{n\to\infty} \|x_n - Tx\| \leq \limsup_{n\to\infty} \|x_n - x\|.$$

The class of mappings that satisfy condition (L), properly includes the nonexpansive mappings and many of its generalizations.

2. Main result

We begin with the following definition.

Definition 2.1. *[3] For a given real number $\alpha < 1$, a mapping $T : C \to X$ is said to be α-nonexpansive if for all $x, y \in C$,*

$$||Tx - Ty||^2 \leq \alpha||Tx - y||^2 + \alpha||Ty - x||^2 + (1 - 2\alpha)||x - y||^2.$$

Now, we introduce a two-parametric class of nonlinear mappings.

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Definition 2.2. *Let C be a nonempty subset of a Banach space X and let $\alpha, \beta \in \mathbb{R}$. We say that a mapping $T : C \to X$ is (α, β)-nonexpansive if for each $x, y \in C$,*

$$\begin{aligned}\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 \\ + \beta\|x - Tx\|^2 + \beta\|y - Ty\|^2 + (1 - 2\alpha - 2\beta)\|x - y\|^2.\end{aligned}$$

Remark 2.1. Let C be a nonempty subset of Banach space X and let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta < 0$. Then the unique (α, β)-nonexpansive mapping $T : C \to X$ is the identity mapping $I : C \to C$. Indeed, taking $x = y$ in Definition 2.2, we get $2(\alpha + \beta)\|x - Tx\|^2 \geq 0$, which yields, $Tx = x$ for each $x \in C$.

Note that every generalized nonexpansive mappings in the sense of Goebel, Kirk and Shimi, $T : C \to X$ [5, 8, 9, 14] is (α, β)-nonexpansive. To see this, let $T : C \to X$ be a generalized nonexpansive mapping in the sense of Goebel, Kirk and Shimi, so

$$\|Tx - Ty\| \leq a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|) + c(\|x - Ty\| + \|y - Tx\|),$$

for all $x, y \in C$, where a, b, c are non-negative constants with $a + 2b + 2c \leq 1$. Thus,

$$\|Tx - Ty\| \leq (1 - 2b - 2c)\|x - y\| + b(\|x - Tx\| + \|y - Ty\|) + c(\|x - Ty\| + \|y - Tx\|),$$

and then by the convexity of the function $x \to x^2$, we have

$$\|Tx - Ty\|^2 \leq (1 - 2b - 2c)\|x - y\|^2 + b\|x - Tx\|^2 + b\|y - Ty\|^2 + c\|x - Ty\|^2 + c\|y - Tx\|^2.$$

The following example shows that the class of (α, β)-nonexpansive mappings is properly larger than both classes of α-nonexpansive mappings and generalized nonexpansive mappings in the sense of Goebel, Kirk and Shimi.

Example 2.1. *Let $T : [0, 1] \to [0, 1]$ be defined as follows:*

$$T(x) = \begin{cases} \frac{7}{8}, & x \neq 1 \\ \frac{1}{4}, & x = 1. \end{cases}$$

For each $x \in [0, 1)$, let

$$\begin{aligned}f(x) := \frac{1}{1000}|\frac{7}{8} - 1|^2 + \frac{1}{1000}|\frac{1}{4} - x|^2 \\ + \frac{8}{9}|\frac{7}{8} - x|^2 + \frac{8}{9}|\frac{1}{4} - 1|^2 + (1 - \frac{2}{1000} - \frac{16}{9})|x - 1|^2.\end{aligned}$$

Since f is non-decreasing on $[0, 1)$ and $|\frac{7}{8} - \frac{1}{4}|^2 < f(0)$, so for each $x \in [0, 1)$ we have $|T(x) - T(1)|^2 = |\frac{7}{8} - \frac{1}{4}|^2 \leq f(x)$, that is, T is a $(\frac{1}{1000}, \frac{8}{9})$-nonexpansive mapping.
Now suppose that T is $\overline{\alpha}$-nonexpansive for some $\overline{\alpha} \in [0, 1)$. Letting $(x, y) = (0, 1)$ and $(x, y) = (1, \frac{7}{8})$ in Definition 2.1, we respectively get that $\overline{\alpha} \leq \frac{39}{123}$ and $\overline{\alpha} \geq 1$, a contradiction.
Notice that $\frac{7}{8} \in \mathrm{Fix}(T)$ and $|T(\frac{7}{8}) - T(1)| = \frac{5}{8} > \frac{1}{8} = |\frac{7}{8} - 1|$. Then, T is not a quasi-nonexpansive mapping and so T is not a generalized nonexpansive mapping in the sense of Goebel, Kirk and Shimi.

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Let l_∞ denote the Banach space of bounded real sequences with the supremum norm. It is well known that there exists a bounded linear functional μ on l_∞, called Banach limit, that satisfies the following conditions.

(i) If $(t_n) \in l_\infty$ with $t_n \geq 0$ for every $n \in \mathbb{N}$, then $\mu(t_n) \geq 0$.

(ii) If $t_n = 1$ for every $n \in \mathbb{N}$, then $\mu(t_n) = 1$.

(iii) For every $(t_n) \in l_\infty$, $\mu(t_n) = \mu(t_{n+1})$.

The following theorem is our main result.

Theorem 2.1. *Let X be a Banach space, C be a nonempty bounded subset of X and let $T : C \to C$ be an (α, β)-nonexpansive mapping with $\alpha > 0$ and $\beta \geq 0$. Then T has an a.f.p.s.*

Proof. Let μ be a Banach limit and let $x \in C$. We first show that for each $j \in \mathbb{N} \cup \{0\}$,

$$\mu\|T^{n+1}x - T^{n-j}x\|^2_{n\geq j} \leq C_j\mu\|T^{n+1}x - T^{n-j-1}x\|^2_{n\geq j+1}, \tag{2.1}$$

where $C_j = \frac{j+1}{j+2}$. Replacing x by $T^{n-1}x$ and y by T^nx in Definition 2.2, for each $n \in \mathbb{N}$. We obtain

$$\begin{aligned}\|T^{n+1}x - T^nx\|^2 \leq \alpha\|T^{n+1}x - T^{n-1}x\|^2 + \beta\|T^nx - T^{n-1}x\|^2 \\ + \beta\|T^{n+1}x - T^nx\|^2 + (1 - 2\alpha - 2\beta)\|T^nx - T^{n-1}x\|^2,\end{aligned}$$

and so

$$\begin{aligned}\mu\|T^{n+1}x - T^nx\|^2 \leq \alpha\mu\|T^{n+1}x - T^{n-1}x\|^2 + \beta\mu\|T^{n+1}x - T^nx\|^2 \\ + \beta\mu\|T^{n+1}x - T^nx\|^2 + (1 - 2\alpha - 2\beta)\mu\|T^{n+1}x - T^nx\|^2.\end{aligned}$$

Then,

$$\mu\|T^{n+1}x - T^nx\|^2 \leq \frac{1}{2}\mu\|T^{n+1}x - T^{n-1}x\|^2,$$

which is just our claim for $j = 0$. Now, suppose that the claim holds for some $0 \leq j$. Replacing x by T^nx and y by $T^{n-j-2}x$ in Definition 2.2, for each $n \geq j + 2$. We obtain

$$\begin{aligned}&\|T^{n+1}x - T^{n-j-1}x\|^2 \leq \alpha\|T^{n+1}x - T^{n-j-2}x\|^2 + \alpha\|T^nx - T^{n-j-1}x\|^2 \\ &+ \beta\|T^{n+1}x - T^nx\|^2 + \beta\|T^{n-j-1}x - T^{n-j-2}x\|^2 + (1 - 2\alpha - 2\beta)\|T^nx - T^{n-j-2}x\|^2,\end{aligned}$$

and so

$$\begin{aligned}&\mu\|T^{n+1}x - T^{n-j-1}x\|^2 \leq \alpha\mu\|T^{n+1}x - T^{n-j-2}x\|^2 + \alpha\mu\|T^{n+1}x - T^{n-j}x\|^2 \\ &+ \beta\mu\|T^{n+1}x - T^nx\|^2 + \beta\mu\|T^{n+1}x - T^nx\|^2 + (1 - 2\alpha - 2\beta)\mu\|T^{n+1}x - T^{n-j-1}x\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}(2\alpha + 2\beta)\mu\|T^{n+1}x - T^{n-j-1}x\|^2 \leq \alpha\mu\|T^{n+1}x - T^{n-j-2}x\|^2 \\ + \alpha C_j\mu\|T^{n+1}x - T^{n-j-1}x\|^2 + (2\beta C_0C_1...C_j)\mu\|T^{n+1}x - T^{n-j-1}x\|^2.\end{aligned}$$

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Then,

$$\mu\|T^{n+1}x - T^{n-j-1}x\|^2 \leq \frac{\alpha}{\alpha(2-C_j)+2\beta(1-C_0C_1...C_j)}\mu\|T^{n+1}x - T^{n-j-2}x\|^2$$
$$\leq \frac{1}{2-C_j}\mu\|T^{n+1}x - T^{n-j-2}x\|^2 = C_{j+1}\mu\|T^{n+1}x - T^{n-j-2}x\|^2,$$

and so (2.1) holds for each $j \in \mathbb{N} \cup \{0\}$.
From (2.1), we deduce that for each $x \in C$, $j \in \mathbb{N}$ and each sufficiently large $n \in \mathbb{N}$,

$$\mu\|T^{n+1}x - T^{n}x\|^2 \leq (C_0C_1...C_{j-1})\mu\|T^{n+1}x - T^{n-j}x\|^2$$
$$= \frac{1}{j+1}\mu\|T^{n+1}x - T^{n-j}x\|^2 \leq \frac{(\mathrm{diam}(D))^2}{j+1},$$

and so by letting $j \to \infty$, we obtain

$$\mu\|T^{n+1}x - T^{n}x\|^2 = 0.$$

On the other hand we know that

$$0 \leq \liminf_{n\to\infty} \|T^{n+1}x - T^{n}x\|^2 \leq \mu\|T^{n+1}x - T^{n}x\|^2 = 0.$$

Hence, there exists an increasing sequence $(n_k) \subseteq \mathbb{N}$ such that

$$\lim_{k\to\infty} \|T^{n_k+1}x - T^{n_k}x\| = 0.$$

Therefore, $(T^{n_k}x)_k$ is an a.f.p.s. for T. □

The following example shows there exists an (α, β)-nonexpansive mapping without a.f.p.s.

Example 2.2. *Let $T : [0,1] \to [0,1]$ be defined as follows:*

$$T(x) = \begin{cases} \frac{3}{4}, & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{4}, & \frac{1}{2} < x \leq 1. \end{cases}$$

We show that T is an $(-1,3)$-nonexpansive mapping without a.f.p.s. It is straightforward to show that

$$2|x-\frac{3}{4}|^2 + 2|y-\frac{3}{4}|^2 - 3|x-y|^2 \geq 0 = |Tx-Ty|^2, \quad \forall x, y \in [0, \frac{1}{2}],$$

$$2|x-\frac{1}{4}|^2 + 2|y-\frac{1}{4}|^2 - 3|x-y|^2 \geq 0 = |Tx-Ty|^2, \quad \forall x, y \in (\frac{1}{2}, 1],$$

$$-|x-\frac{1}{4}|^2 - |y-\frac{3}{4}|^2 + 3|x-\frac{3}{4}|^2 + 3|y-\frac{1}{4}|^2 - 3|x-y|^2 \geq \frac{1}{4} = |Tx-Ty|^2, \quad \forall x \in [0, \frac{1}{2}], y \in (\frac{1}{2}, 1].$$

Then T is an $(-1,3)$-nonexpansive mapping. Also for each $x \in [0,1]$, $|x - Tx| \geq \frac{1}{4}$ and so T has no a.f.p.s.

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Now, the following problem naturally arises.

Problem 2.1. *Determine all* $(\alpha, \beta) \in \mathbb{R}^2$ *such that every* (α, β)*-nonexpansive mapping* $T : C \to C$ *has a.f.p.s., where* C *is an arbitrary bounded subset of the Banach space* X*.*

The following corollary gives an affirmative answer to the open question raised by Ariza-Ruiz et al. [4] concerning the existence of an a.f.p.s. for α-nonexpansive mappings.

Corollary 2.1. *Let* $T : C \to C$ *be an* α*-nonexpansive mapping for some* $0 < \alpha < 1$*, where* C *is a nonempty bounded subset of the Banach space* X*. Then* T *has an a.f.p.s.*

From Corollary 2.1 and Theorem 6.2 in [4], we immediately get the following.

Corollary 2.2. *Let* $T : C \to C$ *be an* α*-nonexpansive mapping for some* $0 < \alpha < 1$*, where* C *is a nonempty closed bounded subset of* X*. Then* T *satisfies condition* (L)*.*

Remark 2.2. Let $T : [0, 1] \to [0, 1]$ be the mapping which is defined in Example 2.2. Then T is an (α, β)-nonexpansive mapping which does not satisfy condition (L) (note that T is not quasi-nonexpansive).

The following corollary, which is an extension of Corollary 6.5 in [4], follows immediately from Corollary 2.1 and Corollary 6.4 of [4].

Corollary 2.3. *Let* C *be a nonempty weakly compact convex subset of the Banach space* X *with normal structure. Assume that* $T : C \to C$ *is an* α*-nonexpansive mapping for some* $0 \leq \alpha < 1$*. Then,* T *has a fixed point.*

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