J. Math. Anal. Appl. 482 (2020) 123531

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Remark on maximal inequalities for Bessel processes

Cloud Makasu

Department of Maths and Applied Maths, University of the Western Cape, Private Bag X17, Bellville 7535, Cape Town, South Africa

ARTICLE INFO

Article history:
Received 8 February 2019
Available online 24 September 2019
Submitted by V. Pozdnyakov

Keywords:
Bessel processes
Maximal inequalities
Optimal stopping problem
Reverse Young inequality

ABSTRACT

For a Bessel process $X = (X_t)_{t \geq 0}$ with dimension $\alpha > 0$ starting at zero, a result of Dubins, Shepp and Shiryaev (1993) states that there exists a constant $\gamma(\alpha)$ depending only on α such that

$$\mathbf{E}\left(\max_{0 \leq t \leq \tau} X_t\right) \leq \gamma(\alpha)\sqrt{\mathbf{E}(\tau)}$$

for any stopping time τ of X. In this paper, we give an explicit form of the constant $\gamma(\alpha)$ in the case $0 < \alpha \leq 1$. The present result complements the known case when $\alpha > 1$ treated in Graversen and Peskir (1998).

1. Introduction

Let $X = (X_t)_{t \geq 0}$ be a Bessel process with dimension $\alpha > 0$ starting at $x \geq 0$, given by

$$X_t = x + \int_0^t \frac{\alpha - 1}{2X_r}\, dr + B_t, \tag{1.1}$$

where $B = (B_t)_{t \geq 0}$ is a standard one-dimensional Brownian motion. It is well known that the process X is a non-negative continuous Markov process, a submartingale for $\alpha \geq 1$, a supermartingale when $\alpha \leq 0$ and is not a semimartingale in the case $0 < \alpha < 1$. For a detailed account on various properties of Bessel processes, see for instance [4] and [7]. Throughout, we shall use the notation $\mathbf{P}_x$ for the probability measure, and $\mathbf{E}_x$ for the expectation of X starting at x. For the case when X is strictly starting at zero, we shall drop the subscripts.

The motivation of this paper is a question raised and left open in the following result proved by Dubins, Shepp and Shiryaev [2].

E-mail address: cmakasu@uwc.ac.za.

https://doi.org/10.1016/j.jmaa.2019.123531
0022-247X/

Theorem 1.1. *Let $X = (X_t)_{t\geq 0}$ be a Bessel process (1.1) with dimension $\alpha > 0$, starting at zero under $\mathbf{P}$. Then*

$$\mathbf{E}\left(\max_{0\leq t\leq \tau} X_t\right) \leq \gamma(\alpha)\sqrt{\mathbf{E}(\tau)} \tag{1.2}$$

for all stopping times τ of X, where $\gamma(\alpha) = \sqrt{4s_1(\alpha)}$ with $s_1(\alpha)$ being a (unique) root of the equation $g_*(s) = 0$ for a function $s \mapsto g_*(s)$ which is a (unique) nonnegative solution of the nonlinear equation

$$g'(s) = \frac{2-\alpha}{2c\left(s^{2-\alpha}g^{\alpha-1}(s) - g(s)\right)} \tag{1.3}$$

such that $g_*(s) \leq s$ for all $s \geq 0$, and with growth rate $\frac{g_*(s)}{s} \to 1$ as $s \to \infty$, and provided that $c = 1$.

The question raised in [2] is that of finding an explicit form for the constant $\gamma(\alpha)$ in (1.2). For the case $\alpha > 1$, this question was addressed by Graversen and Peskir [3]. In this paper, we shall settle this question in the remaining case $0 < \alpha \leq 1$. Our starting point is the optimal stopping problem (1.5) used in the proof of the result (1.2) proved in [2].

Let $X = (X_t)_{t\geq 0}$ be a Bessel process given by (1.1) with dimension $\alpha > 0$ starting at $x \geq 0$, and let

$$S_t = \left(\max_{0\leq u\leq t} X_u\right) \vee s \tag{1.4}$$

for $0 \leq x \leq s$.

Consider the following optimal stopping problem:

$$\Phi_\alpha(x, s) := \sup_\tau \mathbf{E}_{x,s}\left(S_\tau - c\tau\right), \tag{1.5}$$

where the supremum is taken over all stopping times τ for X such that $\mathbf{E}_{x,s}(\tau) < \infty$, $\mathbf{E}_{x,s}$ denotes the expectation with respect to the probability law $\mathbf{P}_{x,s}$ of the process (X, S) starting at (x, s) such that $0 \leq x \leq s$, and $c > 0$ is a positive constant here and in the sequel.

It is proved in [2] that $\Phi_\alpha(x, s)$ admits the form

$$\Phi_\alpha(x, s) = s + \frac{c}{2-\alpha}g_*^2(s) - \frac{2c}{\alpha(2-\alpha)}x^{2-\alpha}g_*^\alpha(s) + \frac{c}{\alpha}x^2 \tag{1.6}$$

for $g_*(s) \leq x \leq s$, where $s \mapsto g_*(s)$ is a (unique) nonnegative solution of the nonlinear equation (1.3) such that $g_*(s) \leq s$ for all $s \geq 0$, and $g_*(s)/s \to 1$ as $s \to \infty$.

We note that the optimal stopping problem (1.5) and $\Phi_\alpha(x, s)$ given in (1.6) will play an important role in this paper. On a different note, it is essential to point out that the main difficulty in the question raised in [2] lies in establishing explicit bounds for a nonnegative solution of the nonlinear equation (1.3). This issue will be dealt with in the next section.

2. Maximal inequalities for Bessel processes

In this section, we shall now state and prove the main result of this paper. The use of a comparison principle to establish a one-sided explicit estimate for a nonnegative solution of Eq. (1.3) is the key argument in our proof.

Theorem 2.1. *If $X = (X_t)_{t\geq 0}$ is a Bessel process given by (1.1) with dimension $0 < \alpha \leq 1$, starting at zero under* $\mathbf{P}$*, then*

$$\mathbf{E}\left(\max_{0\leq t\leq \tau} X_t\right) \leq \frac{1}{\sqrt{2-\alpha}}\sqrt{\mathbf{E}(\tau)} \tag{2.1}$$

for any stopping time τ of X. Inequality (2.1) is sharp.

Proof. Fix $c > 0$, $0 < \alpha \leq 1$, and let

$$D_g = \{(s, g) \mid g(s) < s \text{ for all } s \geq 0\}$$

be given.

Define a real-valued continuous function $\Psi(s, g(s))$ on D_g by

$$\Psi(s, g(s)) = \frac{2-\alpha}{2c\left(s^{2-\alpha}g^{\alpha-1}(s) - g(s)\right)}.$$

We now proceed to establish the following assertion. Given a real-valued continuous function $\Psi(s, g(s))$ on D_g, there exists a continuous nonnegative function $\phi(s, g(s))$ on D_g such that

$$\Psi(s, g(s)) \leq \phi(s, g(s)). \tag{2.2}$$

Applying the reverse Young inequality (see [5], [6]), we have

$$s^{2-\alpha}g^{\alpha-1}(s) - g(s) \geq (2-\alpha)(s - g(s))$$

on D_g. This immediately implies that

$$\Psi(s, g(s)) \leq \frac{1}{2c(s - g(s))}$$

which proves the assertion (2.2).

Now consider (1.3) on D_g and a related nonlinear equation

$$h'(s) = \frac{1}{2c(s - h(s))} \tag{2.3}$$

on D_h. Clearly,

$$h_*(s) = s - \frac{1}{2c}$$

is the maximal solution of (2.3) and $h_*(s) < s$ for all $s \geq 0$.

Consequently, using a comparison principle [8], any nonnegative solution $s \mapsto g_*(s)$ of (1.3) satisfies the estimate

$$g_*(s) \leq h_*^+(s) \tag{2.4}$$

on D_g, where h_*^+ denotes the positive part of h_*.

It follows from (1.5), (1.6) and using the estimate (2.4) that

$$\mathbf{E}\left(\max_{0\leq t\leq\tau} X_t\right) \leq c\mathbf{E}(\tau)+\Phi_\alpha(0,0)$$
$$= c\mathbf{E}(\tau)+\frac{c}{2-\alpha}g_*^2(0)$$
$$\leq c\mathbf{E}(\tau)+\frac{1}{4c(2-\alpha)}. \tag{2.5}$$

Hence,

$$\mathbf{E}\left(\max_{0\leq t\leq\tau} X_t\right) \leq \inf_{c>0}\left(c\mathbf{E}(\tau)+\frac{1}{4c(2-\alpha)}\right)$$
$$= \frac{1}{\sqrt{2-\alpha}}\sqrt{\mathbf{E}(\tau)} \tag{2.6}$$

with the minimum attained at $c=\dfrac{1}{2\sqrt{(2-\alpha)\mathbf{E}(\tau)}}$. This establishes the inequality (2.1).

The proof will be complete once it is shown that equality in (2.1) is attained. We consider the case $\alpha = 1$ in (1.1). Let $X = (B_t)_{t\geq 0}$ be a standard Brownian motion starting at zero, and let

$$\tau_a = \inf\{t > 0 \mid |B_t| = a\} \quad (a > 0)$$

be a stopping time for B.

Using Dynkin's theorem (see [9], Theorem 1), then $\mathbf{E}(\tau_a) = a^2$. Consequently, $\sqrt{E(\tau_a)} = a$, which is the right-hand side of the inequality (2.1). On the other hand, we have $\mathbf{E}\left(\max_{0\leq t\leq\tau_a} B_t\right) = a$ on the left-hand side of (2.1). This proves the sharpness of the inequality (2.1). The proof of the theorem is now complete. □

Remark 2.1. In the special case $\alpha = 1$, $X = (B_t)_{t\geq 0}$ is a standard Brownian motion, then it follows from (2.1) that

$$\mathbf{E}\left(\max_{0\leq t\leq\tau} B_t\right) \leq \sqrt{\mathbf{E}(\tau)} \tag{2.7}$$

which is one of the main results proved in Dubins and Schwarz [1]. The maximal inequality (2.7) is sharp.

Acknowledgment

This work was completed while the author was visiting the Department of Mathematics, University of Bari (Italy), and while on leave from the University of the Western Cape. It is a pleasure to thank an anonymous referee for the useful remarks.

References

[1] L.E. Dubins, G. Schwarz, A sharp inequality for submartingales and stopping times, Astérisque 157/158 (1988) 129–145.
[2] L.E. Dubins, L.A. Shepp, A.N. Shiryaev, Optimal stopping rules and maximal inequalities for Bessel processes, Theory Probab. Appl. 38 (1993) 226–261.
[3] S.E. Graversen, G. Peskir, Maximal inequalities for Bessel processes, J. Inequal. Appl. 2 (1998) 99–119.
[4] N. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North Holland, 1981.
[5] Y. Manasrah, F. Kittaneh, A generalization of two refined Young inequalities, Positivity 19 (2015) 757–768.

[6] D.S. Mitrinović, P.M. Vasić, Analytic Inequalities, Springer-Verlag, Berlin, 1970.
[7] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, Springer, Berlin, 1991.
[8] J. Szarski, Differential Inequalities, Polish Scientific Publishers, Warszawa, 1965.
[9] J. Takeuchi, First hitting time for Bessel processes, Proc. Japan Acad. Ser. A Math. Sci. 61 (1985) 246–248.