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A Generalization of the Hyers–Ulam–Rassias Stability of the Pexider Equation

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In this paper we prove a generalization of the stability of the Pexider equation $f(x+y) = g(x) + h(y)$ in the spirit of Hyers, Ulam, Rassias, and Găvruta.

Key Words: Hyers–Ulam–Rassias stability; Jensen functional equation; Pexider equation.

1. INTRODUCTION

In 1940, Ulam [16] had raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

In 1941, Hyers [4] proved that if $f\colon V \to X$ is a mapping satisfying

$$\| f(x+y) - f(x) - f(y) \| \le \delta$$

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for all $x, y \in V$, where V and X are Banach spaces and δ is a given positive number, then there exists a unique additive mapping $T\colon V \to X$ such that

$$\| f(x) - T(x) \| \leq \delta$$

for all $x \in V$. Throughout the paper, we denote by X a Banach space.

In 1978, Rassias [11] showed a generalization of the result of Hyers in the following theorem:

THEOREM 1.1. *Let V be a normed space and let $f\colon V \to X$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exist $\theta \geq 0$ and $p < 1$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in V \setminus \{0\}$. Then there exists a unique linear mapping $T\colon V \to X$ such that

$$\|T(x) - f(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p$$

for all $x \in V \setminus \{0\}$.

Gajda [1], following the same approach as in Rassias [11], got the result for the case $p > 1$. However, it was verified that the result for the case $p = 1$ does not hold (see [1, 15]). Recently, Găyruta [2] also obtained a further generalization of the Hyers–Rassias theorem (see also [5, 8, 10, 12–14]).

The authors [9] obtained the Hyers–Ulam–Rassias stability of the Jensen equation in the following theorem.

THEOREM 1.2. *Let V be a normed space and let $f\colon V \to X$ be a mapping. Assume that there exist $\theta \geq 0$ and $p \neq 1$ such that*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in V \setminus \{0\}$. Then there exists a unique additive mapping $T\colon V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \leq \frac{\theta(3 + 3^p)}{|3 - 3^p|}\|x\|^p$$

for all $x \in V \setminus \{0\}$.

In this paper, using the ideas from the papers of Hyers [4], Rassias [11], and Găvruta [2], we prove a generalization of the stability of the Pexider equation.

THEOREM 1.3. *Let V be a normed space and let $f, g, h\colon V \to X$ be mappings. Assume that there exist $\theta \geq 0$ and $p \neq 1$ such that*

$$\|f(x+y) - g(x) - h(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in V \setminus \{0\}$. Then there exists a unique additive mapping $T\colon V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \leq \frac{4\theta(3+3^p)}{2^p|3-3^p|}\|x\|^p$$

for all $x \in V \setminus \{0\}$.

2. STABILITY IN THE CASE $p < 1$

We denote by G an abelian group such that there does not exist $x \neq 0$ with $2x = 0$ or $3x = 0$. Let E be a subset of G such that $nx \in$ for any integer n and for all $x \in E$. We also denote by $\varphi\colon E \setminus \{0\} \times E \setminus \{0\} \to [0, \infty)$ a mapping such that

$$\tilde{\varphi}(x, y) := \sum_{k=0}^{\infty} 3^{-k}\varphi(3^k x, 3^k y) < \infty \tag{1}$$

for all $x, y \in E \setminus \{0\}$. Tue authors [9] obtained the following lemma.

LEMMA 2.1. *Let $f\colon E \to X$ be a mapping such that*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \varphi(x, y)$$

for all $x, y \in E \setminus \{0\}$ with $(x+y)/2 \in E$. Then there exists a unique mapping $T\colon E \to X$ such that

$$T(x+y) = T(x) + T(y) \qquad \text{for all } x, y \in E \text{ with } \quad x + y \in E,$$

$$\|f(x) - T(x) - f(0)\| \leq 3^{-1}(\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x)) \qquad \text{for all } x \in E \setminus \{0\},$$

and

$$\lim_{n\to\infty} \frac{f(3^n x)}{3^n} = T(x) \qquad \text{for } x \in E.$$

From Lemma 2.1, we can modify the results of [3] and [7] in the following theorem.

THEOREM 2.2. *Let $f, g, h\colon E \to X$ be mappings such that*

$$\|f(x+y) - g(x) - h(y)\| \le \varphi(x,y) \tag{2}$$

for all $x, y \in E \setminus \{0\}$ with $x + y \in E$. Then there exists a unique mapping $T\colon 2E \to X$ such that

$$T(x+y) = T(x) + T(y) \qquad \text{for all } x, y \in 2E \quad \text{with } x + y \in 2E, \tag{3}$$

$$\begin{aligned} &\|f(x) - T(x) - f(0)\| \\ &\quad \le \frac{1}{3}\left[\tilde{\varphi}\left(\frac{x}{2}, \frac{-x}{2}\right) + \tilde{\varphi}\left(\frac{-x}{2}, \frac{x}{2}\right) + \tilde{\varphi}\left(\frac{x}{2}, \frac{x}{2}\right) + 2\tilde{\varphi}\left(\frac{-x}{2}, \frac{-x}{2}\right)\right. \\ &\qquad \left. + \tilde{\varphi}\left(\frac{-x}{2}, \frac{3x}{2}\right) + \tilde{\varphi}\left(\frac{3x}{2}, \frac{-x}{2}\right) + \tilde{\varphi}\left(\frac{3x}{2}, \frac{3x}{2}\right)\right], \end{aligned} \tag{4}$$

for $x \in 2E \setminus \{0\}$ and

$$\lim_{n\to\infty} \frac{f(3^n x)}{3^n} = \lim_{n\to\infty} \frac{g(3^n x)}{3^n} = \lim_{n\to\infty} \frac{h(3^n x)}{3^n} = T(x) \qquad \text{for } x \in 2E, \tag{5}$$

where $2E = \{2x \mid x \in E\}$.

Proof. From (2), we get

$$\begin{aligned} &\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \\ &\quad \le \left\|f\left(\frac{x+y}{2}\right) - g\left(\frac{x}{2}\right) - h\left(\frac{y}{2}\right)\right\| + \left\|f\left(\frac{x+y}{2}\right) - g\left(\frac{y}{2}\right) - h\left(\frac{x}{2}\right)\right\| \\ &\qquad + \left\|f(x) - g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| + \left\|f(y) - g\left(\frac{y}{2}\right) - h\left(\frac{y}{2}\right)\right\| \\ &\quad \le \varphi\left(\frac{x}{2}, \frac{y}{2}\right) + \varphi\left(\frac{y}{2}, \frac{x}{2}\right) + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(\frac{y}{2}, \frac{y}{2}\right) \end{aligned}$$

for all $x, y \in 2E \setminus \{0\}$ with $(x+y)/2 \in 2E$. Let

$$\varphi_1(x,y) = \varphi\left(\frac{x}{2}, \frac{y}{2}\right) + \varphi\left(\frac{y}{2}, \frac{x}{2}\right) + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(\frac{y}{2}, \frac{y}{2}\right) \qquad \text{for all } x, y \in 2E \setminus \{0\}.$$

From Lemma 2.1, there exists a unique mapping $T\colon 2E \to X$ satisfying (3), (4), and

$$\lim_{n\to\infty} \frac{f(3^n x)}{3^n} = T(x) \qquad \text{for } x \in 2E. \tag{6}$$

Let $x \in 2E \setminus \{0\}$. Replacing x by $3^n x$ and y by $3^n x$ in (2), we get

$$\left\| \frac{f(3^n x + 3^n x)}{3^n} - \frac{g(3^n x) + h(3^n x)}{3^n} \right\| \le \frac{1}{3^n} \varphi(3^n x, 3^n x). \tag{7}$$

Taking the limit in (7), we obtain

$$\lim_{n \to \infty} \frac{g(3^n x) + h(3^n x)}{3^n} = T(2x) = 2T(x) \tag{8}$$

from (1), (3), and (6). Replacing x by $3^{n+1}x$ and y by $3^n x$ and dividing 3^n on both sides, the inequality (2) implies

$$\left\| \frac{f(3^{n+1} x + 3^n x)}{3^n} - \frac{g(3^{n+1} x) + h(3^n x)}{3^n} \right\| \le \frac{1}{3^n} \varphi(3^{n+1} x, 3^n x). \tag{9}$$

Replacing x by $3^n x$ and y by $3^{n+1}x$ and dividing 3 on both sides, the inequality (2) implies

$$\left\| \frac{f(3^n x + 3^{n+1} x)}{3^n} - \frac{g(3^n x) + h(3^{n+1} x)}{3^n} \right\| \le \frac{1}{3^n} \varphi(3^n x, 3^{n+1} x). \tag{10}$$

From (9) and (10),

$$\begin{aligned} &\left\| \frac{g(3^{n+1} x) - h(3^{n+1} x)}{3^n} - \frac{g(3^n x) - h(3^n x)}{3^n} \right\| \\ &\qquad \le \frac{1}{3^n} \varphi(3^{n+1} x, 3^n x) + \frac{1}{3^n} \varphi(3^n x, 3^{n+1} x). \end{aligned}$$

Let $\varepsilon > 0$ be given. Since

$$\sum_{n=1}^{\infty} \frac{1}{3^n} \big(\varphi(3^{n+1} x, 3^n x) + \varphi(3^n x, 3^{n+1} x) \big) = \tilde{\varphi}(3x, x) + \tilde{\varphi}(x, 3x) < \infty,$$

there exists $M > 0$ such that

$$\frac{1}{3^j} \big(\varphi(3^{j+1} x, 3^j x) + \varphi(3^j x, 3^{j+1} x) \big) < \varepsilon \qquad \text{for all } j \ge M. \tag{11}$$

On the other hand, there exists $M' \ge M$ such that

$$\frac{g(3^M x) - h(3^M x)}{3^{M'}} < \varepsilon. \tag{12}$$

From (11) and (12), we obtain

$$\begin{aligned}
&\left\| \frac{g(3^j x) - h(3^j x)}{3^j} \right\| \\
&\quad \leq \left\| \frac{g(3^M x) - h(3^M x)}{3^j} \right\| \\
&\qquad + \sum_{k=0}^{j-M-1} \left\| \frac{g(3^{M+k+1} x) - h(3^{M+k+1} x)}{3^{j-k-M} 3^{M+k}} - \frac{g(3^{M+k} x) h(3^{M+k} x)}{3^{j-k-M} 3^{M+k}} \right\| \\
&\quad \leq \frac{1}{3^{j-M'}} \left\| \frac{g(3^M x) - h(3^M x)}{3^{M'}} \right\| + \sum_{k=0}^{j-M-1} \frac{1}{3^{j-k-M}} \varepsilon \\
&\quad \leq \frac{1}{3} \varepsilon + \frac{1}{2} \varepsilon < \varepsilon
\end{aligned}$$

for all $j > M'$. Hence we have

$$\lim_{n \to \infty} \frac{g(3^n x) - h(3^n x)}{3^n} = 0. \tag{13}$$

By (8) and (13), we conclude that

$$\lim_{n \to \infty} \frac{g(3^n x)}{3^n} = \lim_{n \to \infty} \frac{h(3^n x)}{3^n} = T(x).$$

▮

THEOREM 2.3. *Let V be a vector space and let E be a subset of V satisfying the following conditions*:

(i) *$rx \in E$ for all $x \in E$ and $|r| \geq 1$,*

(ii) *if x is a nonzero element of V, then there exists $n_x \in N$ such that $n_x x \in E$,*

(iii) *$0 \in E$.*

Let f, g, h: $E \to X$ be mappings such that

$$\|f(x + y) - g(x) - h(y)\| \leq \varphi(x, y)$$

for all $x, y \in E \setminus \{0\}$ with $x + y \in E$. Then there exists a unique additive mapping T: $V \to X$ satisfying (4) *and* (5).

Proof. By the conditions (i) and (iii), we get a unique mapping T': $2E \to X$ satisfying (3), (4), and (5) in Theorem 2.2. If $x \in V$, there exists an $n_x \in N$ such that $n_x x \in 2E$ by the condition (ii). We can define a mapping $T: V \to X$ by

$$T(x) = \begin{cases} T'(x) & \text{for } x \in 2E, \\ n_x^{-1} T'(n_x x) & \text{for } x \notin 2E. \end{cases}$$

If $x, y \in V$, we can choose an $n \in N$ such that nx, ny, and $n(x+y) \in 2E$ by the conditions (i) and (ii). From this, we get

$$\|T(x+y) - T(x) - T(y)\|$$
$$= n^{-1}\|T(n(x+y)) - T(nx) - T(ny)\| = 0.$$

This completes the proof. ∎

Applying Theorem 2.3, we obtain Corollary 2.4.

COROLLARY 2.4. *Let V be a normed space and let $E = \{x \in V: \|x\| > a\} \cup \{0\}$ for a fixed $a \geq 0$. Let f, g, h: $E \to X$ be mapping such that*

$$\|f(x+y) - g(x) - h(y)\| \leq \varphi(x, y)$$

for all $x, y \in E \setminus \{0\}$ with $x + y \in E$. Then there exists a unique additive mapping T: $V \to X$ satisfying (4) *and* (5).

The following corollary is a generalization of [6, Theorem 1].

COROLLARY 2.5. *Let V be a normed space. For a fixed a with $0 \leq a < 3$, let ψ: $(a, \infty) \to R^+$ be a function such that*

(i) $\psi(ts) \leq \psi(t)\psi(s)$ *for all $t, s > a$, and*

(ii) $\psi(3)/3 < 1$.

Let f, g, h: $V \to X$ be mappings such that

$$\|f(x+y) - g(x) - h(y)\| \leq \psi(\|x\|) + \psi(\|y\|)$$

for all x, y with $\|x\|, \|y\| > a$.

Then there exists a unique additive mapping T: $V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \leq \frac{12\psi(\|x/2\|) + 4\psi(\|3x/2\|)}{3 - \psi(3)}$$

for all $x \in V$ with $\|x\| > 2a$.

Proof. Let $E = \{x \in V: \|x\| > a\} \cup \{0\}$ and let $\varphi(x, y) = \psi(\|x\|) + \psi(\|y\|)$ for all $x, y \in E \setminus \{0\}$. Then we get

$$\tilde{\varphi}(x, y) = \sum_{n=0}^{\infty} 3^{-n} \varphi(3^n x, 3^n y)$$

$$\leq \sum_{n=0}^{\infty} (\psi(3)/3)^n (\psi(\|x\|) + \psi(\|y\|))$$

$$= \frac{\psi(\|x\|) + \psi(\|y\|)}{1 - \psi(3)/3} < \infty$$

from (i) and (ii). Applying Corollary 2.4, there exists a unique additive mapping $T\colon V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \leq \frac{12\psi(\|x/2\|) + 4\psi(\|3x/2\|)}{3 - \psi(3)}$$

for all $x \in V$ with $\|x\| > 2a$. ∎

COROLLARY 2.6. *Let V be a normed space. Let $p < 1$ and $0 \leq a < 3$. Let $f, g, h\colon V \to X$ be mappings such that*

$$\|f(x+y) - g(x) - h(y)\| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in V$ with $\|x\|, \|y\| > a$.

Then there exists a unique additive mapping $T\colon V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \leq \frac{4(3 + 3^p)}{2^p(3 - 3^p)}\|x\|^p \quad \text{for all } x \in V \text{ with } \|x\| > 2a.$$

Proof. Define $\psi\colon (a, \infty) \to R^+$ by $\psi(t) = t^p$ and apply Corollary 2.5. ∎

3. STABILITY IN THE CASE $p > 1$

Let $\phi\colon E \setminus \{0\} \times E \setminus \{0\} \to [0, \infty)$ be a mapping such that

$$\tilde{\phi}(x, y) := \sum_{k=0}^{\infty} 3^k \phi(3^{-k}x, 3^{-k}y) < \infty$$

for all $x, y \in E \setminus \{0\}$. The authors [9] proved the following lemma.

LEMMA 3.1. *Let V be a vector space and let E be a subset of V satisfying the following conditions*:

(i) *$rx \in E$ for all $x \in E$ and $|r| \leq 1$,*

(ii) *if x is a nonzero element of V, there exists $n \in N$ such that $n^{-1}x \in E$.*

Let $f\colon E \to X$ be a mapping such that

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \le \phi(x,y)$$

for any $x, y \in E \setminus \{0\}$ with $(x+y)/2 \in E$. Then there exists a unique additive mapping $T\colon V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \le \tilde{\phi}(3^{-1}x, 3^{-1}(-x)) + \tilde{\phi}(3^{-1}(-x), x)$$

for all $x \in E \setminus \{0\}$ and

$$T(x) = \lim_{n\to\infty} 3^n(f(3^{-n}x) - f(0)).$$

THEOREM 3.2. *Let V be a vector space and let E be a subset of V satisfying the following conditions*:

(i) *$rx \in E$ for all $x \in E$ and $|r| \le 1$,*

(ii) *if x is a nonzero element of V, there exists $n_x \in N$ such that $n_x^{-1}x \in E$.*

Let $f, g, h\colon E \to X$ be mappings such that

$$\|f(x+y) - g(x) - h(y)\| \le \phi(x,y) \tag{14}$$

for all $x, y \in E \setminus \{0\}$ with $x + y \in E$. Then there exists a unique additive mapping $T\colon V \to X$ such that

$$\begin{aligned}\|f(x) - T(x) - f(0)\| &\le \tilde{\phi}\left(\frac{x}{6}, \frac{-x}{6}\right) + \tilde{\phi}\left(\frac{-x}{6}, \frac{x}{6}\right) + \tilde{\phi}\left(\frac{x}{6}, \frac{x}{6}\right) + 2\tilde{\phi}\left(\frac{-x}{6}, \frac{-x}{6}\right)\\ &\quad + \tilde{\phi}\left(\frac{-x}{6}, \frac{x}{2}\right) + \tilde{\phi}\left(\frac{x}{2}, \frac{-x}{6}\right) + \tilde{\phi}\left(\frac{x}{2}, \frac{x}{2}\right)\end{aligned} \tag{15}$$

for all $x \in E \setminus \{0\}$ and

$$\lim_{n\to\infty} 3^n(f(3^{-n}x) - f(0)) = T(x) \qquad \text{for } x \in E. \tag{16}$$

Proof. From (14), we get

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\|\\ &\quad\le \left\|f\left(\frac{x+y}{2}\right) - g\left(\frac{x}{2}\right) - h\left(\frac{y}{2}\right)\right\| + \left\|f\left(\frac{x+y}{2}\right) - g\left(\frac{y}{2}\right) - h\left(\frac{x}{2}\right)\right\|\\ &\quad\quad + \left\|f(x) - g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| + \left\|f(y) - g\left(\frac{y}{2}\right) - h\left(\frac{y}{2}\right)\right\|\\ &\quad\le \phi\left(\frac{x}{2}, \frac{y}{2}\right) + \phi\left(\frac{y}{2}, \frac{x}{2}\right) + \phi\left(\frac{x}{2}, \frac{x}{2}\right) + \phi\left(\frac{y}{2}, \frac{y}{2}\right)\end{aligned}$$

for all $x, y \in E \setminus \{0\}$ with $(x+y)/2 \in E$. From Lemma 3.1, there exists a unique mapping $T\colon V \to X$ such that f satisfies (15) and (16).

Remark. If ϕ is defined on $E \times E$, then we can replace x by 0 and y by 0 and the inequality (14) implies

$$\|3^n(f(0) - g(0) - h(0))\| \le 3^n\phi(0,0).$$

Taking the limit in the above, we obtain

$$\lim_{n\to\infty} 3^n\|f(0) - g(0) - h(0)\| = 0.$$

Hence,

$$f(0) = g(0) + h(0).$$

Replacing x by $3^{-n}x$ and y by 0, the inequality (14) implies

$$\begin{aligned}&\|3^n(f(3^{-n}x) - f(0)) - 3^n(g(3^{-n}x) + h(0) - f(0))\|\\ &\quad \le 3^n\phi(3^{-n}x, 0) \end{aligned} \tag{17}$$

for $x \in E$. Taking the limit in (17), we obtain

$$\lim_{n\to\infty} 3^n(g(3^{-n}x) - g(0)) = T(x) \qquad \text{for } x \in E.$$

Similarly, we obtain

$$\lim_{n\to\infty} 3^n(h(3^{-n}x) - h(0)) = T(x) \qquad \text{for } x \in E.$$

COROLLARY 3.3. *Let V be a normed space and let $E = \{x \in V\colon \|x\| < a\}$ for a fixed $a > 0$. Let $f, g, h\colon E \to X$ be mappings such that*

$$\|f(x+y) - g(x) - h(y)\| \le \phi(x,y)$$

for all $x, y \in E \setminus \{0\}$ with $x + y \in E$. Then there exists a unique additive mapping $T\colon V \to X$ such that

$$\begin{aligned}&\|f(x) - T(x) - f(0)\|\\ &\quad\le \tilde{\phi}\left(\frac{x}{6}, \frac{-x}{6}\right) + \tilde{\phi}\left(\frac{-x}{6}, \frac{x}{6}\right) + \tilde{\phi}\left(\frac{x}{6}, \frac{x}{6}\right) + 2\tilde{\phi}\left(\frac{-x}{6}, \frac{-x}{6}\right)\\ &\qquad + \tilde{\phi}\left(\frac{-x}{6}, \frac{x}{2}\right) + \tilde{\phi}\left(\frac{x}{2}, \frac{-x}{6}\right) + \tilde{\phi}\left(\frac{x}{2}, \frac{x}{2}\right)\end{aligned}$$

for all $x \in E \setminus \{0\}$.

Proof. Apply Theorem 3.2.

COROLLARY 3.4. *Let V be a normed space and let $E = \{x \in V: \|x\| < a\}$ for a fixed $a > 3$. Let a function $\psi: (0, a) \to R^+$ satisfy*

(i) $\psi(ts) \geq \psi(t)\psi(s)$ *for all* $0 < t, s < a$ *and*

(ii) $\psi(3)/3 > 1$.

Let $f, g, h: V \to X$ be mappings such that

$$\|f(x+y) - g(x) - h(y)\| \leq \psi(\|x\|) + \psi(\|y\|)$$

for all $x, y \in E \setminus \{0\}$. Then there exists a unique additive mapping $T: V \to X$ such that

$$\|f(x) - T(x) - f(0)\| \leq \frac{12\psi(\|x/6\|) + 4\psi(\|x/2\|)}{1 - 3/\psi(3)} \quad \text{for all } x \in E \setminus \{0\}.$$

Proof. Let $\phi(x, y) = \psi(\|x\|) + \psi(\|y\|)$ for all $x, y \in E \setminus \{0\}$. We get

$$\begin{aligned} \tilde{\phi}(x, y) &= \sum_{n=0}^{\infty} 3^n \phi(3^{-n}x, 3^{-n}y) \\ &= \sum_{n=0}^{\infty} 3^n (\psi(\|3^{-n}x\|) + \psi(\|3^{-n}y\|)) \\ &\leq \sum_{n=0}^{\infty} (3/\psi(3))^n (\psi(\|x\|) + \psi(\|y\|)) \\ &= \frac{\psi(\|x\|) + \psi(\|y\|)}{1 - 3/\psi(3)} < \infty \end{aligned}$$

from (i) and (ii). Applying Corollary 3.3, the proof is completed. ∎

COROLLARY 3.5. *Let V be a normed space. Let $p > 1$ and $a > 3$. Let $f, g, h: V \to X$ be mappings such that*

$$\|f(x+y) - g(x) - h(y)\| \leq \|x\|^p + \|y\|^p \quad \text{for all } x, y \text{ with } 0 < \|x\|, \|y\| < a.$$

Then there exists a unique additive mapping $T: V \to X$ *such that*

$$\|f(x) - T(x) - f(0)\| \leq \frac{4(3^p + 3)}{2^p(3^p - 3)}\|x\|^p$$

$$\text{for all } x \quad \text{with} \quad 0 \leq \|x\| < a.$$

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