Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE DIRECT

J. Math. Anal. Appl. 278 (2003) 243–249

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Note

A note on polyharmonic functions

Stevo Stević

Matematički Fakultet, Studentski Trg 16, 11000 Beograd, Serbia, Yugoslavia

Received 2 July 2001

Submitted by L. Chen

Abstract

In this note we prove the following theorem:
Suppose $0 < p < \infty$ and $\alpha > -1$. Then there is a constant $C = C(p, m, n, \alpha)$ such that

$$\int\limits_B \left|u(x)\right|^p \left(1 - |x|\right)^\alpha \, dV(x) \leqslant C\left(\left|u(0)\right|^p + \int\limits_B \left|\nabla u(x)\right|^p \left(1 - |x|\right)^{p+\alpha} \, dV(x)\right),$$

for all polyharmonic functions u of order m, on the unit ball $B \subset \mathbf{R}^n$.

Keywords: Polyharmonic functions; Weight function; Bergman space; Distortion

1. Introduction

Throughout this note n is an integer greater than or equal to 2, D is a domain in the Euclidean space $\mathbf{R}^n$, $B(a, r) = \{x \in \mathbf{R}^n \mid |x - a| < r\}$ denotes the open ball centered at a of radius r, where $|x|$ denotes the norm of $x \in \mathbf{R}^n$ and B is the open unit ball in the n-dimensional Euclidean space $\mathbf{R}^n$. $S = \partial B = \{x \in \mathbf{R}^n \mid |x| = 1\}$ is the Euclidean boundary of B. By $dV(x)$ we denote the Lebesgue volume measure on B, $d\sigma$ the surface measure on S and $d\sigma_N$ the normalized surface measure on S. Let $\omega(r)$, $0 \leqslant r < 1$, be a positive weight function which is integrable on $(0, 1)$. We extend ω on B by setting $\omega(x) = \omega(|x|)$.

For $0 < p < \infty$ the weighted Bergman space $b^p_\omega(B)$ is the space of all real harmonic functions u on B such that

E-mail address: sstevo@matf.bg.ac.yu.

0022-247X/03/$ – see front matter
doi:10.1016/S0022-247X(02)00326-8

$$\|u\|_{\omega,p} = \left(\int_B \left|u(x)\right|^p \omega(x)\,dV(x)\right)^{1/p} < +\infty.$$

If $\omega(x) = (1-|x|)^\alpha$, we use the notation $\|u\|_{\alpha,p}$.

For weighted Bergman spaces of analytic functions see, for example, in [1,5] and the references therein. Basic facts about unweighted harmonic Bergman spaces can be found in [2].

For $u \in C(B)$ we usually write $M_p^p(u,r) = \int_S |u(r\zeta)|^p\,d\sigma_N(\zeta)$, $0 \leqslant r < 1$, for the integral means of u. The norm in $b_\omega^p(B)$ can then be written

$$\|u\|_{\omega,p} = \left(\sigma_n \int_0^1 M_p^p(u,r)\omega(r)r^{n-1}\,dr\right)^{1/p},$$

where $\sigma_n = \sigma(S)$.

For a given weight ω we define the function

$$\psi(r) = \psi_\omega(r) \stackrel{\text{def}}{=} \frac{1}{\omega(r)} \int_r^1 \omega(u)\,du, \quad 0 \leqslant r < 1,$$

and we call it the distortion function of ω. We put $\psi(x) = \psi(|x|)$ for $x \in B$.

Definition 1 [5]. We say that a weight ω is admissible if it satisfies the following conditions:

(a) There is a positive constant $A = A(\omega)$ such that

$$\omega(r) \geqslant \frac{A}{1-r} \int_r^1 \omega(u)\,du, \quad \text{for } 0 \leqslant r < 1;$$

(b) There is a positive constant $B = B(\omega)$ such that

$$\omega'(r) \leqslant \frac{B}{1-r}\omega(r), \quad \text{for } 0 \leqslant r < 1;$$

(c) For each sufficiently small positive δ there is a positive constant $C = C(\delta,\omega)$ such that

$$\sup_{0 \leqslant r < 1} \frac{\omega(r)}{\omega(r + \delta\psi(r))} \leqslant C.$$

Observe that (a) implies $A\psi(r) \leqslant 1 - r$ thus for sufficiently small positive δ we have $r + \delta\psi(r) < 1$ and the quantity in the denominator of the fraction in (c) is well defined.

The following theorem was established in [5]:

Theorem A. *Suppose $1 \leqslant p < \infty$ and ω is an admissible weight with distortion function ψ. Let*

$$L(f) = \int_U \left|f(z)\right|^p \omega(z)\,dm(z) \quad \textit{and}$$

$$R(f) = \left|f(0)\right|^p + \int_U \left|f'(z)\right|^p \psi(z)^p \omega(z)\, dm(z),$$

then there are finite positive constants C and C' independent of f such that

$$CR(f) \leqslant L(f) \leqslant C'R(f) \tag{1}$$

for all analytic functions f on the unit disc U, where $dm(z) = r\,dr\,d\theta/\pi$ denotes the normalized Lebesgue area measure on U.

In [6] we showed that the first inequality in (1) holds also when $p \in (0,1)$. In [7] we generalized Theorem A in the case of harmonic functions on the unit ball $B \subset \mathbf{R}^n$. It is an open problem whether the second inequality in (1) holds when $p \in (0,1)$. In order to solve the open problem we proved the following theorem, which is a special case of Theorem 2 in [4].

Theorem B. *Suppose $0 < p < \infty$ and $\alpha > -1$. Then there is a constant $C = C(p,\alpha)$ such that*

$$\int_U \left|f(z)\right|^p \left(1-|z|\right)^\alpha dm(z) \leqslant C\left(\left|f(0)\right|^p + \int_U \left|f'(z)\right|^p \left(1-|z|\right)^{p+\alpha} dm(z)\right),$$

for all $f \in H(U)$.

A real valued function u is called polyharmonic of order m on a domain D if $u \in C^\infty(D)$ and $\Delta^m u \equiv 0$, where m is a positive integer, Δ denotes the Laplacian and $\Delta^m u = \Delta^{m-1}(\Delta u)$. We denote by $\mathcal{H}_m(B)$ the space of polyharmonic functions of order m on B. In particular, $\mathcal{H}_1(B)$ is the class of all harmonic functions on B.

The purpose of the note is to prove an analogous theorem to Theorem B in the case of polyharmonic functions on the unit ball. We were motivated by [3]. We prove the following theorem.

Theorem 1. *Suppose $0 < p < \infty$, $\alpha > -1$ and $u \in \mathcal{H}_m(B)$. Then there is a constant $C = C(p,\alpha)$ such that*

$$\int_B \left|u(x)\right|^p \left(1-|x|\right)^\alpha dV(x) \leqslant C\left(\left|u(0)\right|^p + \int_B \left|\nabla u(x)\right|^p \left(1-|x|\right)^{p+\alpha} dV(x)\right).$$

2. Auxiliary results

In order to prove the main result we need several auxiliary results. Throughout the following proofs C denotes a positive constant which may change from line to line.

Lemma 1 was essentially proved in Lemma 5 in [3], because $u \in \mathcal{H}_m(B)$ implies $D^\beta u \in \mathcal{H}_m(B)$ for every multi-index β.

Lemma 1. *Let $0 < p < \infty$. Then for every multi-index β,*

$$\left|D^{\beta}u(a)\right|^{p} \leqslant \frac{C}{r^{n}} \int\limits_{B(a,r)} \left|D^{\beta}u\right|^{p}\,dV \quad \textit{whenever } B(a,r) \subset B,$$

for all $u \in \mathcal{H}_m(B)$ and some constant C depending only on β, p, m and n.

Lemma 2 [3, Lemma 7]. *Let $g(r)$ be a nonnegative continuous function on the interval $[0,1)$, $\lambda > 0$ and let $\alpha > -1$. Then there is a constant $C = C(\alpha, \lambda)$ such that*

$$\int\limits_{0}^{1} g^{\lambda}(r)(1-r)^{\alpha}\,dr \leqslant C\left(\max_{r\in[0,1/2]} g^{\lambda}(r) + \int\limits_{0}^{1}\left|g\left(\frac{1+r}{2}\right) - g(r)\right|^{\lambda}(1-r)^{\alpha}\,dr\right).$$

One can easily prove the following lemma.

Lemma 3. *Let $p > 0$, $u \in C^{1}(B)$, $0 \leqslant r < 1$. Then*

$$\left|M_p^p(\rho,u) - M_p^p(r,u)\right| \leqslant (\rho - r)^{p} \int\limits_{S} \sup_{r<t<\rho}\left|\nabla u(t\zeta)\right|^{p}\,d\sigma(\zeta) \quad \textit{for } p \in (0,1]$$

and

$$\left|M_p(\rho,u) - M_p(r,u)\right| \leqslant (\rho - r)\left(\int\limits_{S} \sup_{r<t<\rho}\left|\nabla u(t\zeta)\right|^{p}\,d\sigma(\zeta)\right)^{1/p} \quad \textit{for } p \geqslant 1,$$

for every ρ and r, such that $0 \leqslant r < \rho < 1$.

Lemma 4. *Suppose $0 < p < \infty$ and $\alpha > -1$. Then there is a constant $C = C(p, \alpha, m, n)$ such that*

$$\begin{aligned} M_{\infty}^{p}(u,1/2) &= \max_{|x|\leqslant 1/2}\left|u(x)\right|^{p} \\ &\leqslant C\left(\left|u(0)\right|^{p} + \int\limits_{B}\left|\nabla u(x)\right|^{p}\left(1-|x|\right)^{p+\alpha}\,dV(x)\right), \end{aligned}$$

for all $u \in \mathcal{H}_m(B)$.

Proof. Since

$$u(x_0) - u(0) = \int\limits_{0}^{1} u'(tx_0)\,dt = \int\limits_{0}^{1}\left\langle \nabla u(tx_0), x_0\right\rangle dt,$$

by elementary inequalities we obtain

$$\left|u(x_0)\right|^{p} \leqslant c_p\left(\left|u(0)\right|^{p} + |x_0|^{p} \max_{|x|\leqslant 1/2}\left|\nabla u(x)\right|^{p}\right), \tag{2}$$

for each $x_0 \in \overline{B(0,1/2)}$, where $c_p = 1$ for $0 < p < 1$ and $c_p = 2^{p-1}$ for $p \geqslant 1$.

On the other hand by Lemma 1 we obtain

$$\left|\nabla u(x)\right|^p \leqslant C \int\limits_{B(x,1/4)} \left|\nabla u(y)\right|^p \, dV(y)$$

for each $x \in \overline{B(0,1/2)}$ and consequently

$$\max_{|x|\leqslant 1/2} \left|\nabla u(x)\right|^p \leqslant \max\left\{C4^{p+\alpha}, C\right\} \int\limits_{B(0,3/4)} \left|\nabla u(y)\right|^p \left(1-|y|\right)^{p+\alpha} dV(y). \tag{3}$$

From (2) and (3) the result follows. □

Similarly we can prove the following lemma.

Lemma 5. *Suppose* $0 < p < \infty$ *and* $\alpha > -1$. *Then there is a constant* $C = C(p, \alpha, m, n)$ *such that*

$$M_\infty^p(\nabla u, 3/4) \leqslant C \int\limits_B \left|\nabla u(x)\right|^p \left(1-|x|\right)^{p+\alpha} dV(x),$$

for all $u \in \mathcal{H}_m(B)$.

The proof of the following lemma is analogous to the proof of Lemma 6 in [3].

Lemma 6. *Let* $u \in \mathcal{H}_m(B)$, $0 < p < \infty$, $\alpha > -1$ *and* $f^+(r\zeta) = \sup\{|f(t\zeta)| \mid r < t < (1+r)/2\}$, $0 \leqslant r < 1$, $\zeta \in S$. *Then there is a constant* $C = C(p, m, n, \alpha)$ *such that*

$$\int_0^1 M_p^p\left(|\nabla u|^+, r\right)(1-r)^{p+\alpha} r^{n-1} \, dr \leqslant C \int_0^1 M_p^p(\nabla u, r)(1-r)^{p+\alpha} r^{n-1} \, dr.$$

3. Proof of the theorem

In this section we prove the main result in this paper.

Proof of Theorem 1. Let $p \in (0, 1]$. Then by Lemma 2 ($\lambda = 1$), Lemmas 3, 4, 5 and 6 we obtain

$$\begin{aligned} \|u\|_{\alpha,p}^p &= \sigma_n \int_0^1 M_p^p(u,r)(1-r)^\alpha r^{n-1} \, dr \leqslant \sigma_n \int_0^1 M_p^p(u,r)(1-r)^\alpha \, dr \\ &\leqslant C\left(M_p^p(u, 1/2) + \int_0^1 \left|M_p^p\left(u, (1+r)/2\right) - M_p^p(u,r)\right|(1-r)^\alpha \, dr\right) \end{aligned}$$

$$
\begin{aligned}
&\leqslant C\left(\max_{|x|\leqslant 1/2}\big|u(x)\big|^p+\int_0^1 M_p^p\big(|\nabla u|^+,r\big)(1-r)^{p+\alpha}\,dr\right)\\
&= C\left(\max_{|x|\leqslant 1/2}\big|u(x)\big|^p+\int_0^{1/2}+\int_{1/2}^1 M_p^p\big(|\nabla u|^+,r\big)(1-r)^{p+\alpha}\,dr\right)\\
&\leqslant C\left(\max_{|x|\leqslant 1/2}\big|u(x)\big|^p+C\max_{|x|\leqslant 3/4}\big|\nabla u(x)\big|^p\right.\\
&\qquad\left.+\,2^{n-1}\int_{1/2}^1 M_p^p\big(|\nabla u|^+,r\big)(1-r)^{p+\alpha}r^{n-1}\,dr\right)\\
&\leqslant C\left(\max_{|x|\leqslant 1/2}\big|u(x)\big|^p+\int_0^1 M_p^p(\nabla u,\rho)(1-\rho)^{p+\alpha}\rho^{n-1}\,d\rho\right)\\
&\leqslant C\left(\big|u(0)\big|^p+\int_B\big|\nabla u(x)\big|^p\big(1-|x|\big)^{p+\alpha}\,dV(x)\right).
\end{aligned}
$$

In the case $p > 1$ we apply Lemma 2 for $\lambda = p$ and Lemma 3 for $p > 1$. □

Let $|\nabla^k u|$ denote the norm of the kth gradient of u which is given by

$$\big|\nabla^2 u\big| = \sqrt{u_{x_1x_1}^2+\cdots+u_{x_nx_n}^2+2u_{x_1x_2}^2+\cdots+2u_{x_{n-1}x_n}^2}$$

and $|\nabla^k u|$ in a similar way.

By Theorem 1 and some simple calculations we obtain.

Corollary 1. *Suppose* $0 < p < \infty$, $\alpha > -1$, $k \in \mathbf{N}$ *and* $u \in \mathcal{H}_m(B)$. *Then there is a constant* $C = C(p, m, n, k, \alpha)$ *such that*

$$
\begin{aligned}
&\int_B\big|u(x)\big|^p\big(1-|x|\big)^\alpha\,dV(x)\\
&\qquad\leqslant C\left(\sum_{i=1}^{k-1}\big|\nabla^k u(0)\big|^p+\int_B\big|\nabla^k u(x)\big|^p\big(1-|x|\big)^{pk+\alpha}\,dV(x)\right).
\end{aligned}
$$

References

[1] A. Aleman, A. Siskakis, Integration operators on Bergman spaces, Indiana Univ. Math. J. 46 (1997) 337–356.
[2] S. Axler, P. Bourdon, W. Ramey, Harmonic Function Theory, Springer, New York, 1992.
[3] M. Pavlović, Decompositions of L^p and Hardy spaces of polyharmonic functions, J. Math. Anal. Appl. 216 (1997) 499–509.

[4] Shi Ji-Huai, Inequalities for the integral means of holomorphic functions and their derivatives in the unit ball of $\mathbf{C}^n$, Trans. Amer. Math. Soc. 328 (2) (1991) 619–637.
[5] A. Siskakis, Weighted integrals of analytic functions, Acta Sci. Math. 66 (2000) 651–664.
[6] S. Stević, A note on weighted integrals of analytic functions, Bull. Greek Math. Soc. 46 (2002) 3–9.
[7] S. Stević, Weighted integrals of harmonic functions, Studia Sci. Math. Hung. 39 (1–2) (2002) 87–96.