



# Uniqueness theorem for quasilinear $2n$ th-order equations

Jiří Benedikt<sup>1</sup>

University of West Bohemia, Centre of Applied Mathematics, Univerzitní 22, 306 14 Plzeň, Czech Republic

Received 26 August 2003

Submitted by R. Manásevich

---

## Abstract

We are concerned with existence and uniqueness of the solution of initial value problems for quasilinear  $2n$ th-order equations of the type

$$(-1)^n (|u^{(n)}|^{p-2} u^{(n)})^{(n)} = \lambda |u|^{q-2} u,$$

where  $n \in \mathbb{N}$ ,  $\lambda \in \mathbb{R}$  and  $p, q > 1$ . We show that there exists a global solution for  $p \geq q$ , while the solution can “blow-up” for  $p < q$ . On the other hand, there is at most one solution for  $p \leq q$ , and for  $p > q$  we give an example of nonuniqueness. We prove the uniqueness theorem for a general equation, involving nonconstant coefficients and jumping nonlinearity.

© 2004 Elsevier Inc. All rights reserved.

**Keywords:** Existence and uniqueness of solution; Continuous dependence on initial conditions; Jumping nonlinearity

---

## 1. Main results

In the whole paper we use the notation

$$\psi_m(s) = \begin{cases} |s|^{m-2}s & \text{for } s \neq 0, \\ 0 & \text{for } s = 0, \end{cases}$$

and  $m^* = m/(m-1)$ , where  $m > 1$  ( $\psi_m$  and  $\psi_{m^*}$  are then inverse functions).

---

*E-mail address:* [benedikt@kma.zcu.cz](mailto:benedikt@kma.zcu.cz).

*URL:* <http://www.cam.zcu.cz/members/benedikt>.

<sup>1</sup> This research has been supported by the Grant Agency of the Czech Republic, No. 201/03/0671.

Recently, the study of ordinary quasilinear problems forced some authors to prove existence and uniqueness of the solution of the corresponding initial value problem (uniqueness theorem). Namely, for the second-order initial value problem (for the  $p$ -Laplacian)

$$\begin{cases} -(\psi_p(u'(t)))' = \lambda \psi_q(u(t)), & t \geq t_0, \\ u(t_0) = \alpha, & u'(t_0) = \beta, \end{cases} \quad (1)$$

with  $\lambda \geq 0$  and  $p = q > 1$  (homogeneous case), local existence and uniqueness of the solution was proved by del Pino in [5]. This result was later generalized by Drábek and Manásevich in [7] for general  $p, q > 1$  (nonhomogeneous case). Both assuming  $\lambda \geq 0$  since the studied eigenvalue problem has clearly no negative eigenvalues.

In [1], counterexamples to global existence and uniqueness were introduced. The first one (see [1, Remark 3.3]) shows that for  $\lambda < 0$ ,  $p < q$  and suitable initial conditions the solution of (1) diverges for  $t$  tending to some finite value (so-called “blow-up”). The second one (see [1, Remark 4.6]) shows that for  $\lambda < 0$ ,  $p > q$  and  $\alpha = \beta = 0$  there is more than one solution (local, i.e., even global). But the result of this paper implies that for  $p \geq q$  there exists a global solution of (1), and if  $p \leq q$  or  $|\alpha| + |\beta| > 0$ , then there is at most one solution. This makes the question of existence and uniqueness of the solution of the nonhomogeneous second-order initial value problem (1) closed. Unfortunately, this is not the case with the higher-order problems.

Local existence and uniqueness of the solution of the fourth-order initial value problem (for the  $p$ -biharmonic equation)

$$\begin{cases} (\psi_p(u''(t)))'' = \lambda \psi_q(u(t)), & t \geq t_0, \\ u(t_0) = \alpha, & u'(t_0) = \beta, & \psi_p(u''(t_0)) = \gamma, & (\psi_p(u''(t)))'|_{t=t_0} = \delta, \end{cases} \quad (2)$$

where  $\lambda > 0$  and  $p = q > 1$ , was proved by Drábek and Ôtani in [8]. This result was then generalized in [1] for  $p, q > 1$  arbitrary, nonconstant coefficients and jumping nonlinearity.

The situation for (2) is similar to that for (1) (see [1]). Only  $\lambda > 0$  alternates with  $\lambda < 0$ . If  $p \geq q$ , then there exists a global solution of (2), and for  $p < q$  and  $\lambda > 0$  there is a counterexample to global existence. On the other hand, if  $p \leq q$  or at least one of the constants  $\alpha, \beta, \gamma, \delta$  is not zero, then there exists at most one solution, and if  $p > q$ ,  $\alpha = \beta = \gamma = \delta = 0$  and  $\lambda > 0$ , then there are at least two solutions. For  $\lambda = 0$  the proof of existence and uniqueness is trivial.

As we see, for (2) the question of global existence for  $p < q$  and  $\lambda < 0$ , and the question of uniqueness for  $p > q$ ,  $\alpha = \beta = \gamma = \delta = 0$  and  $\lambda < 0$ , remain open. But for the second-order problem (1) in the corresponding cases ( $p < q$  and  $\lambda > 0$  for global existence and  $p > q$ ,  $\alpha = \beta = 0$  and  $\lambda > 0$  for uniqueness) the uniqueness theorem was proved by Drábek and Manásevich in [7]. The reason why we cannot use the same technique for the higher-order problems is that they used the first integral of (1),

$$\frac{|u'(t)|^p}{p^*} + \lambda \frac{|u(t)|^q}{q} = \frac{|\beta|^p}{p^*} + \lambda \frac{|\alpha|^q}{q}. \quad (3)$$

If  $\lambda > 0$ , then for  $\alpha = \beta = 0$ , (3) immediately implies  $u(t) \equiv 0$ . Moreover, there can be obviously no “blow-up” when  $\lambda > 0$ . In spite of that (3) can be generalized for  $2n$ th-order equation (see [2, (19)]), for  $n > 1$  it does not have these consequences. Nevertheless, if we suppose that the solution of (2) does not change its sign on some right neighborhood of  $t_0$ ,

then it is not difficult to prove the uniqueness for  $\lambda < 0$  and  $\alpha = \beta = \gamma = \delta = 0$ . But it is not known whether the solution can change its sign on arbitrarily small right neighborhood of  $t_0$  or not. In [2] a fourth-order analogue of (3) is used to prove that the solution of (2) with the zero initial conditions is monotone, but only for  $\lambda > 0$ , unfortunately. Note that an analogous problem occurs also for existence.

To mention an article that deals with a second-order initial value problem with nonconstant coefficients, we can name the paper of Walter [17] that considers the homogeneous equation

$$-(t^\alpha \psi_p(u'(t)))' = t^\alpha q(t) \psi_p(u(t)),$$

where  $q$  is a continuous function. The term  $t^\alpha$  can be obtained, e.g., by transformation of the radial  $p$ -Laplacian to one dimension (then  $\alpha = N - 1$ , where  $N$  is the dimension). We can also apply the generalized Prüfer transformation, developed by Elbert [9] to

$$-(a(t) \psi_p(u'(t)))' = b(t) \psi_p(u(t))$$

with  $a \in C^1$ ,  $b \in C$ . For  $p \geq 2$  we obtain a Lipschitzian right-hand side, so the global existence and uniqueness follow then from the classical theory. Note that there is no analogous transformation for the higher-order equations.

There is also another group of mathematicians studying oscillation properties of equations, similar to that in (1). They obtained a nonhomogeneous equation, as a generalization of the Emden–Fowler equation

$$u''(t) + a(t)|u(t)|^r \operatorname{sgn} u(t) = 0, \quad (4)$$

where  $r > 0$ . Note that  $|s|^r \operatorname{sgn} s \equiv \psi_q(s)$  for  $q = r + 1$ . If we rewrite (4) as a system of two first-order equations, it can be generalized to

$$u_i'(t) = (-1)^{i-1} a_i(t) |u_{3-i}(t)|^{r_i} \operatorname{sgn} u_{3-i}(t), \quad i = 1, 2, \quad (5)$$

with  $r_1, r_2 > 0$  (see [16]). Putting  $a_1 \equiv 1$  and  $a_2 \equiv \lambda$ , (5) is equivalent to the equation in (1) for  $p = r_1^{-1} + 1$  and  $q = r_2 + 1$ . It is proved in [16, Theorem 9.1, p. 63] that neither  $u_1$  nor  $u_2$  can “blow-up” if  $r_1 r_2 \leq 1$ . But this condition corresponds to  $p \geq q$  for (1). On the other hand, [16, Theorem 9.2, p. 64] states that for  $r_1 r_2 \geq 1$ , (5) with zero initial conditions has only the trivial solution. But this is a consequence of uniqueness for (1) since  $r_1 r_2 \geq 1$  corresponds to  $p \leq q$ . The proofs in [16] are based on [14, Lemma 4.4, p. 48].

Our aim is to cover and generalize all the mentioned results. In order to provide a uniqueness theorem, usable for most quasilinear problems, we take a general equation with nonconstant coefficients and jumping nonlinearity. We consider the initial value problem

$$\begin{cases} (-1)^n (a(t) \psi_p(u^{(n)}(t)))^{(n)} = b_1(t) \psi_{q_1}(u^+(t)) - b_2(t) \psi_{q_2}(u^-(t)), & t \in \mathcal{I}, \\ u^{(i)}(t_0) = \alpha_i, & (a(t) \psi_p(u^{(n)}(t)))^{(j)}|_{t=t_0} = \beta_j, \quad i, j \in \{0, 1, \dots, n-1\}, \end{cases} \quad (6)$$

where  $\mathcal{I} = [t_0, t_1]$ ,  $t_0 < t_1$ , or  $\mathcal{I} = [t_0, \infty)$ ,  $n \in \mathbb{N}$ ,  $a, b_1, b_2 \in C(\mathcal{I})$ ,  $a > 0$ ,  $p, q_1, q_2 > 1$ ,  $\alpha_0, \dots, \alpha_{n-1}, \beta_0, \dots, \beta_{n-1} \in \mathbb{R}$ ,  $u^+ = \max\{u, 0\}$  (positive part of  $u$ ) and  $u^- = \max\{-u, 0\}$  (negative part of  $u$ ). Note that  $u = u^+ - u^-$ .

Our main results are the following

**Proposition 1** (Local existence). *There exists  $\varepsilon > 0$  such that (6) has a solution on  $\mathcal{I} = [t_0, t_0 + \varepsilon]$ .*

**Theorem 2** (Global existence). *Let  $p \geq \max\{q_1, q_2\}$ . Then (6) has a solution on  $\mathcal{I} = [t_0, \infty)$ .*

**Proposition 3** (Local uniqueness). *Let at least one of the following two conditions hold true:*

- $\sum_{i=0}^{n-1} |\alpha_i| + \sum_{j=0}^{n-1} |\beta_j| > 0$ ,
- $p \leq \min\{q_1, q_2\}$ .

*Moreover, let at least one of the following four conditions hold true:*

- $\alpha_0 = \alpha_1 = \dots = \alpha_{n-1} = 0$ ,
- $\sum_{j=0}^{n-1} |\beta_j| > 0$ ,
- $p \leq 2$ ,
- *neither  $b_1$  nor  $b_2$  changes its sign on  $[t_0, t_1]$  for some  $t_1 > t_0$ .*

*Then there exists  $\varepsilon > 0$  such that (6) has at most one solution on  $\mathcal{I} = [t_0, t_0 + \varepsilon]$ .*

**Corollary 4** (Global uniqueness). *Let  $p \leq \min\{q_1, q_2\}$ . Moreover, let  $p \leq 2$  or for any  $\tilde{t} \in \mathcal{I}$ ,  $\tilde{t} < \sup \mathcal{I}$ , there exist  $\varepsilon > 0$  such that neither  $b_1$  nor  $b_2$  changes its sign on  $[\tilde{t}, \tilde{t} + \varepsilon]$ . Then (6) has at most one solution on  $\mathcal{I}$ .*

**Remark 5.** An example of a continuous function that changes its sign on arbitrarily small right neighborhood of a  $\tilde{t} \in \mathbb{R}$  is the function

$$t \mapsto (t - \tilde{t}) \sin \frac{1}{t - \tilde{t}}. \quad (7)$$

**Remark 6.** One can also deal with nonconstant  $p, q_1, q_2 \in C(\mathcal{I})$ ,  $p, q_1, q_2 > 1$  (confer, e.g., [12] and references therein). The reader is invited to follow our proofs substituting

$$\inf\{p(t): t \geq t_0\} \geq \sup\{q_k(t): t \geq t_0, k \in \{1, 2\}\} \quad (8)$$

for  $p \geq \max\{q_1, q_2\}$  in Theorem 2 and

$$\sup\{p(t): t \in \mathcal{I}\} \leq \inf\{q_k(t): t \in \mathcal{I}, k \in \{1, 2\}\} \quad (9)$$

for  $p \leq \min\{q_1, q_2\}$  in Proposition 3 (and Corollary 4).

For the special case of (6),

$$\begin{cases} (-1)^n (\psi_p(u^{(n)}(t)))^{(n)} = \lambda \psi_q(u(t)), & t \in \mathcal{I}, \\ u^{(i)}(t_0) = \alpha_i, & (\psi_p(u^{(n)}(t)))^{(j)}|_{t=t_0} = \beta_j, \quad i, j \in \{0, 1, \dots, n-1\}, \end{cases} \quad (10)$$

where  $\lambda \in \mathbb{R}$  is a constant, the results are summarized in Tables 1 and 2. Note that for (10) the last condition of the latter four in Proposition 3 is satisfied trivially because  $b_1 = b_2 = \lambda$  is a constant here.

Table 1  
Existence of a solution of (10) on  $\mathcal{I} = [t_0, \infty)$

$p \geq q$	YES (Theorem 2)	
$p < q$	$(-1)^n \lambda > 0$	NO (Example [1, Remark 3.3]); blow-up for suitable initial conditions
	$\lambda = 0$	YES (trivial)
	$(-1)^n \lambda < 0$	? (YES for $n = 1$ ; see [7])

Table 2  
Uniqueness of a solution of (10) on  $\mathcal{I} = [t_0, t_0 + \varepsilon]$  for some  $\varepsilon > 0$

$\sum_{i=0}^{n-1}  \alpha_i  + \sum_{j=0}^{n-1}  \beta_j  > 0$	YES (Proposition 3)	
$\alpha_0 = \dots = \alpha_{n-1}$ $= \beta_0 = \dots = \beta_{n-1} = 0$	$p \leq q$	YES (Proposition 3)
	$p > q$	$(-1)^n \lambda > 0$ NO (Example [1, Remark 4.6])
		$\lambda = 0$ YES (trivial)
		$(-1)^n \lambda < 0$ ? (YES for $n = 1$ ; see [7])

Let us rewrite the problem (6) as an equivalent initial value problem for a system of  $2n$  differential equations of the first order. Denoting  $u_0 := u$  and  $u_n := a\psi_p(u^{(n)})$  we obtain

$$\begin{cases} u'_i(t) = u_{i+1}(t), & i \in \{0, \dots, n-2\}, \\ u'_{n-1}(t) = c(t)\psi_{p^*}(u_n(t)), \\ u'_{n+j}(t) = u_{n+j+1}(t), & j \in \{0, \dots, n-2\}, \\ u'_{2n-1}(t) = (-1)^n(b_1(t)\psi_{q_1}(u_0^+(t)) - b_2(t)\psi_{q_2}(u_0^-(t))), & t \in \mathcal{I}, \\ u_i(t_0) = \alpha_i, & i \in \{0, \dots, n-1\}, \\ u_{n+j}(t_0) = \beta_j, & j \in \{0, \dots, n-1\}, \end{cases} \quad (11)$$

where  $c(t) = \psi_{p^*}(1/a(t))$  (thus  $c \in C(\mathcal{I})$ ,  $c > 0$ ). Here one can see that the existence and uniqueness problem for (11) is not trivial and it cannot be inferred from the classical theory (see, e.g., [13, pp. 31–35]). Continuous dependence of the solution of (11) as an element of  $(C(\mathcal{I}))^{2n}$  on the initial conditions and parameters is a standard consequence of uniqueness of the solution. The reader is invited to follow the proof of [4, Theorem 4.1, p. 59] (cf. [1, Corollary 1.9]).

There are also another possibilities how to generalize (1) or (5). For example, del Pino et al. [6, Appendix] gave sufficient conditions for existence and uniqueness of the solution of

$$\begin{cases} -(\psi_p(u'(t)))' = f(t, u(t)), & t \geq t_0, \\ u(t_0) = \alpha, & u'(t_0) = \beta, \end{cases} \quad (12)$$

where  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous and  $f(t, 0) = 0$  for all  $t \in \mathbb{R}$ . As for  $f$  Carathéodory and the solution of (12) in the Carathéodory sense, García-Huidobro et al. [11, Lemma 2.1] discussed boundedness of the solution, and Manásevich et al. [15, Lemma 2.1] proved (under some assumptions on  $f$ ) that (12) with  $\alpha = \beta = 0$  has only the trivial solution.

Our aim is to provide the uniqueness theorem for the study of symmetric boundary value problems (see [2,3,5–9,11,17]). Elias and Pinkus [10, Appendix A] generalized (5) to a cyclic system of  $m$  first-order equations without any symmetry

$$\begin{aligned} u'_{i-1}(t) &= b_i(t) |u_i(t)|^{r_i} \operatorname{sgn} u_i(t), \quad i \in \{1, \dots, m-1\}, \\ u'_{m-1}(t) &= b_0(t) |u_0(t)|^{r_0} \operatorname{sgn} u_0(t), \end{aligned} \quad (13)$$

where  $b_i \in C$ ,  $b_i > 0$  and  $r_i > 0$ ,  $i \in \{0, \dots, m-1\}$ . For  $r_0 r_1 \dots r_{m-1} \leq 1$  the solution of the corresponding initial value problem is globally extendable (see [10, Theorem A.1]), and for nonzero initial conditions, the solution is locally unique (see [10, Theorem A.2]). Let us choose  $m = 2n$ ,  $b_i \equiv 1$  and  $r_i = 1$  for  $i \in \{1, \dots, n-1, n+1, \dots, 2n-1\}$  in (13), and  $b_1 \equiv b_2 > 0$  and  $q_1 = q_2$  in (6). Then our results (Theorem 2 and Proposition 3) meet with these in [10]. Indeed, for  $b_1, b_2 > 0$  and nonzero initial conditions the first of the former two conditions and the last of the latter four conditions in Proposition 3 are satisfied.

This paper is organized as follows. In Section 2 we define a solution of (6) and (11). Section 3 contains proofs of Proposition 1 and Theorem 2, and Section 4 is devoted to a proof of Proposition 3. Finally, in Section 5 we formulate some open problems, close related to our results.

## 2. Preliminaries

To prove our main results we adopt the techniques used in [1].

We define a solution of (6) via a solution of (11). Hence we transfer the problem of existence and uniqueness of the solution of (6) to the equivalent problem for (11).

**Definition 7.** By a *solution of (11)* we understand a vector function  $\mathbf{u} = [u_0, \dots, u_{2n-1}]^T$  of the class  $(C^1(\mathcal{I}))^{2n}$  which satisfies the equations in (11) at every point of  $\mathcal{I}$ , and fulfills the initial conditions in (11).

By a *solution of the problem (6)* we understand a function  $u$  of the class  $C^n(\mathcal{I})$ , such that  $[u, \dots, u^{(n-1)}, a\psi_p(u^{(n)}), \dots, (a\psi_p(u^{(n)}))^{(n-1)}]^T$  is a solution of the corresponding problem (11).

## 3. Existence

**Proof of Proposition 1.** Since the right-hand side of (11) is continuous, the local existence is a consequence of the Cauchy–Peano theorem. It is also possible to use the fact that  $\mathbf{u}$  is a solution of (11) if and only if the couple  $[u_0, u_n]^T$  is a fixed point of the operator  $T : (C(\mathcal{I}))^2 \rightarrow (C(\mathcal{I}))^2$  defined by

$$\begin{aligned} T(u, v) = & \left[ \sum_{i=0}^{n-1} \alpha_i t^i + \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} c(\tau) \psi_{p^*}(v(\tau)) d\tau, \right. \\ & \sum_{j=0}^{n-1} \beta_j t^j + \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} (-1)^n \\ & \left. \times (b_1(\tau) \psi_{q_1}(u^+(\tau)) - b_2(\tau) \psi_{q_2}(u^-(\tau))) d\tau \right]^T. \end{aligned}$$

The reader is invited to prove that there exists  $\varepsilon > 0$  for which the Schauder fixed point theorem guarantees existence of at least one fixed point of  $T$ .  $\square$

**Proof of Theorem 2.** To prove that the local solution can be extended to  $\infty$ , we have to show that any solution of (11) is bounded on any finite interval. We assume  $p \geq \max\{q_1, q_2\}$ . It obviously suffices to prove existence of a solution on  $\mathcal{I} = [t_0, t_1]$  for arbitrary  $t_1 > t_0$ .

We consider the auxiliary problem

$$\begin{cases} \hat{u}'_i(t) = \hat{u}_{i+1}(t), & i \in \{0, \dots, n-2\}, \\ \hat{u}'_{n-1}(t) = C\psi_{p^*}(\hat{u}_n(t)), \\ \hat{u}'_{n+j}(t) = \hat{u}_{n+j+1}(t), & j \in \{0, \dots, n-2\}, \\ \hat{u}'_{2n-1}(t) = B\psi_p(\hat{u}_0(t)), & t \in \mathcal{I}, \\ \hat{u}_i(t_0) = \hat{\alpha}_i, & i \in \{0, \dots, n-1\}, \\ \hat{u}_{n+j}(t_0) = \hat{\beta}_j, & j \in \{0, \dots, n-1\}, \end{cases} \quad (14)$$

where  $B, C \in \mathbb{R}$  are constants satisfying  $|b_1(t)| \leq B$ ,  $|b_2(t)| \leq B$  and  $|c(t)| \leq C$  for all  $t \in [t_0, t_1]$ . One can easily check that the vector function  $\hat{\mathbf{u}} = [\hat{u}_0, \dots, \hat{u}_{2n-1}]^T$  with

$$\begin{aligned} \hat{u}_i &= Kr^i e^{r(t-t_0)}, \quad i \in \{0, \dots, n-1\}, \\ \hat{u}_{n+j} &= \left(\frac{Kr^n}{C}\right)^{p-1} (r(p-1))^j e^{r(p-1)(t-t_0)}, \quad j \in \{0, \dots, n-1\}, \end{aligned}$$

$K > 0$  arbitrary and

$$r = \left(\frac{BC^{p-1}}{(p-1)^n}\right)^{1/(np)},$$

is a solution of (14) for

$$\begin{aligned} \hat{\alpha}_i &= Kr^i, \quad i \in \{0, \dots, n-1\}, \\ \hat{\beta}_j &= \left(\frac{Kr^n}{C}\right)^{p-1} (r(p-1))^j, \quad j \in \{0, \dots, n-1\}. \end{aligned}$$

We now choose  $K$  big enough to be  $|\alpha_i| < \hat{\alpha}_i$  and  $|\beta_j| < \hat{\beta}_j$  for all  $i, j \in \{0, \dots, n-1\}$ . We prove that any solution  $\mathbf{u} = [u_0, \dots, u_{2n-1}]^T$  of (11) on  $\mathcal{I} = [t_0, t_1]$  satisfies

$$|u_i(t)| \leq \hat{u}_i(t) \quad \text{for all } i \in \{0, \dots, 2n-1\} \text{ and } t \in [t_0, t_1]. \quad (15)$$

Since  $|u_0(t_0)| < \hat{u}_0(t_0)$ , the set

$$T = \{t \in [t_0, t_1]: |u_0(t)| \leq \hat{u}_0(t)\}$$

is nonempty and closed. Thus there exists

$$t_m = \max\{t \in [t_0, t_1]: [t_0, t] \subseteq T\}.$$

If we assume  $K \geq 1$ , we have  $\hat{u}_0(t) \geq 1$  for all  $t \in [t_0, t_m]$ . Consequently, for any  $t \in [t_0, t_m]$ ,

$$|u'_{2n-1}(t)| \leq B|u_0(t)|^{q-1} \leq B(\hat{u}_0(t))^{q-1} \leq B(\hat{u}_0(t))^{p-1} = \hat{u}'_{2n-1}(t),$$

where  $q = q_1$  if  $u_0(t) \geq 0$ , and  $q = q_2$  if  $u_0(t) < 0$ . It yields that for every  $t \in [t_0, t_m]$ ,

$$|u_{2n-1}(t)| \leq |\beta_{n-1}| + \int_{t_0}^t |u'_{2n-1}(\tau)| d\tau < \hat{\beta}_{n-1} + \int_{t_0}^t \hat{u}'_{2n-1}(\tau) d\tau = \hat{u}_{2n-1}(t).$$

By the same way we then show that for all  $t \in [t_0, t_m]$  we have  $|u_{n+j}(t)| < \hat{u}_{n+j}(t)$  for  $j = n-2, n-3, \dots, 0$ . Hence

$$|u'_{n-1}(t)| \leq C|u_n(t)|^{p^*-1} < C(\hat{u}_n(t))^{p^*-1} = \hat{u}'_{n-1}(t)$$

for any  $t \in [t_0, t_m]$ , which again implies  $|u_i(t)| < \hat{u}_i(t)$  for  $i = n-1, n-2, \dots, 0$ . If it was  $t_m < t_1$ , then the inequality  $|u_0(t_m)| < \hat{u}_0(t_m)$  would contradict the maximality of  $t_m$ . Thus  $t_m = t_1$ , which proves (15). Now the proof of Theorem 2 can be completed using the standard continuation arguments. One can also avoid these arguments. If we compute an upper bound for  $\varepsilon$  (see the proof of Proposition 1) that guarantees the local existence, it is readily seen that if the solution is bounded on an interval  $[t_0, t_1]$ , then there exists a minimal upper bound  $\varepsilon_0 > 0$ . The local existence is then guaranteed on  $[t, t + \varepsilon_0]$  for any  $t \in [t_0, t_1]$ , and so the solution can be continued over  $[t_0, t_1]$  in a finite number of steps.  $\square$

#### 4. Uniqueness

Let us now prove the local uniqueness (Proposition 3), i.e., that (11) has at most one solution on  $\mathcal{I} = [t_0, t_0 + \varepsilon]$  for  $\varepsilon > 0$  small enough. In the proof we distinguish whether

$$\alpha_i \neq 0 \quad \text{for some } i \in \{0, \dots, n-1\} \quad \text{and} \quad \alpha_k = 0 \quad \text{for } k < i, \quad (16)$$

or

$$\alpha_0 = \alpha_1 = \dots = \alpha_{n-1} = 0, \quad (17)$$

and whether

$$\beta_j \neq 0 \quad \text{for some } j \in \{0, \dots, n-1\} \quad \text{and} \quad \beta_k = 0 \quad \text{for } k < j, \quad (18)$$

or

$$\beta_0 = \beta_1 = \dots = \beta_{n-1} = 0. \quad (19)$$

Thus we divide the proof into four lemmata: Lemma 8 for (16) and (18), Lemma 9 for (17) and (18), Lemma 10 for (16) and (19), and Lemma 11 for (17) and (19). In the first three cases  $u_0$  does not change its sign on  $\mathcal{I}$  for sufficiently small  $\varepsilon > 0$ , and so the equation in the fourth row in (11) has the form

$$u'_{2n-1}(t) = b(t)\psi_q(u_0(t)), \quad (20)$$

where  $b = (-1)^n b_1$  or  $b = (-1)^n b_2$ , and  $q = q_1$  or  $q = q_2$ , if  $u_0 \geq 0$  or  $u_0 \leq 0$ , respectively.

In the following proofs we denote by  $A, B, C \geq 1$  constants such that

$$|a(t)| \leq A, \quad |b_1(t)| \leq B, \quad |b_2(t)| \leq B, \quad |c(t)| \leq C$$



for all  $t \in \mathcal{I}$ . Consequently, for any  $t \in \mathcal{I}$ ,

$$|a(t)| \geq C^{1-p} > 0, \quad |c(t)| \geq A^{1-p^*} > 0.$$

In all proofs we can assume  $t_0 = 0$  without loss of generality.

**Lemma 8.** *Let (16) and (18) hold true. Then there exists  $\varepsilon > 0$  such that (11) has at most one solution on  $\mathcal{I} = [t_0, t_0 + \varepsilon]$ .*

**Proof.** Let both  $u$  and  $v$  be solutions of (11) where the fourth row takes the form (20). The former  $n$  equations yield

$$u_n(t) - v_n(t) = a(t)(\psi_p(u_0^{(n)}(t)) - \psi_p(v_0^{(n)}(t))),$$

and from the latter  $n$  equations we conclude

$$u_n^{(n)}(t) - v_n^{(n)}(t) = b(t)(\psi_q(u_0(t)) - \psi_q(v_0(t)))$$

for all  $t \in \mathcal{I}$ . Hence

$$\begin{aligned} & a(t)(\psi_p(u_0^{(n)}(t)) - \psi_p(v_0^{(n)}(t))) \\ &= \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} b(\tau)(\psi_q(u_0(\tau)) - \psi_q(v_0(\tau))) d\tau. \end{aligned} \quad (21)$$

The assumption (18) implies for  $t \rightarrow 0_+$ ,

$$\frac{u_0^{(n)}(t)}{t^{j(p^*-1)}} = c(t)\psi_{p^*}\left(\frac{u_n(t)}{t^j}\right) \rightarrow c(0)\psi_{p^*}\left(\frac{\beta_j}{j!}\right) \neq 0, \quad (22)$$

and the same holds true for  $v_0^{(n)}$ . Hence there exist constants  $K_1, K_2 > 0$  such that for all  $t \in (0, \varepsilon]$  with  $\varepsilon > 0$  small enough we have

$$K_1 \leq \left| \frac{u_0^{(n)}(t)}{t^{j(p^*-1)}} \right| \leq K_2 \quad \text{and} \quad K_1 \leq \left| \frac{v_0^{(n)}(t)}{t^{j(p^*-1)}} \right| \leq K_2.$$

Since the function  $\psi_p'(s) = (p-1)|s|^{p-2}$  is continuous and positive on  $[K_1, K_2]$ , there exists a constant  $K_3 > 0$  satisfying  $\psi_p'(s) \geq K_3$  for all  $|s| \in [K_1, K_2]$ . Let us assume  $\varepsilon \leq 1$ . Then for all  $t \in (0, \varepsilon]$ ,

$$\begin{aligned} & |a(t)(\psi_p(u_0^{(n)}(t)) - \psi_p(v_0^{(n)}(t)))| \\ &= |a(t)|t^j \left| \psi_p\left(\frac{u_0^{(n)}(t)}{t^{j(p^*-1)}}\right) - \psi_p\left(\frac{v_0^{(n)}(t)}{t^{j(p^*-1)}}\right) \right| \\ &\geq C^{1-p}t^j \left| \int_{v_0^{(n)}(t)/t^{j(p^*-1)}}^{u_0^{(n)}(t)/t^{j(p^*-1)}} \psi_p'(\sigma) d\sigma \right| \geq C^{1-p}K_3t^{j(2-p^*)}|u_0^{(n)}(t) - v_0^{(n)}(t)|. \end{aligned} \quad (23)$$

The assumption (16) yields for  $\tau \rightarrow 0_+$ ,

$$\frac{u_0(\tau)}{\tau^i} \rightarrow \frac{\alpha_i}{i!} \neq 0,$$

and the same for  $v_0$ , too. Consequently, there exist constants  $K_4, K_5 > 0$  satisfying for all  $\tau \in (0, \varepsilon]$  with  $\varepsilon > 0$  small enough,

$$K_4 \leq \left| \frac{u_0(\tau)}{\tau^i} \right| \leq K_5 \quad \text{and} \quad K_4 \leq \left| \frac{v_0(\tau)}{\tau^i} \right| \leq K_5. \quad (24)$$

Again, the continuity of  $\psi'_q(s)$  on  $[K_4, K_5]$  imply existence of a constant  $K_6 > 0$  such that  $\psi'_q(s) \leq K_6$  for all  $|s| \in [K_4, K_5]$ . Hence for all  $\tau \in (0, \varepsilon]$ ,

$$\left| \psi_q \left( \frac{u_0(\tau)}{\tau^i} \right) - \psi_q \left( \frac{v_0(\tau)}{\tau^i} \right) \right| = \left| \int_{v_0(\tau)/\tau^i}^{u_0(\tau)/\tau^i} \psi'_q(\sigma) d\sigma \right| \leq K_6 \tau^{-i} |u_0(\tau) - v_0(\tau)|. \quad (25)$$

Clearly, for any  $\tau \in \mathcal{I}$ ,

$$\begin{aligned} |u_0(\tau) - v_0(\tau)| &= \left| \int_0^\tau \frac{(\tau - \sigma)^{n-1}}{(n-1)!} (u_0^{(n)}(\sigma) - v_0^{(n)}(\sigma)) d\sigma \right| \\ &\leq \frac{\tau^n}{(n-1)!} \|u_0^{(n)} - v_0^{(n)}\|_{C(\mathcal{I})}, \end{aligned} \quad (26)$$

where  $\|\cdot\|_{C(\mathcal{I})}$  denotes the classical sup norm. We suppose that  $\varepsilon \leq 1$ . Then combining (25) and (26) we obtain for adjusted right-hand side of (21),

$$\begin{aligned} &\left| \int_0^t \frac{(t - \tau)^{n-1}}{(n-1)!} b(\tau) \tau^{i(q-1)} \left( \psi_q \left( \frac{u_0(\tau)}{\tau^i} \right) - \psi_q \left( \frac{v_0(\tau)}{\tau^i} \right) \right) d\tau \right| \\ &\leq \frac{K_6 B}{((n-1)!)^2} t^{2n+i(q-2)} \|u_0^{(n)} - v_0^{(n)}\|_{C(\mathcal{I})} \end{aligned} \quad (27)$$

for all  $t \in (0, \varepsilon]$ . Finally, we put (21), (23) and (27) together and pass to the maximum over  $t \in \mathcal{I}$  to get

$$\|u_0^{(n)} - v_0^{(n)}\|_{C(\mathcal{I})} \leq \frac{K_6 B C^{p-1}}{K_3 ((n-1)!)^2} \varepsilon^{2n+i(q-2)+j(p^*-2)} \|u_0^{(n)} - v_0^{(n)}\|_{C(\mathcal{I})}.$$

Since

$$2n + i(q-2) + j(p^*-2) > 2n - 2(n-1) = 2,$$

it must be  $\|u_0^{(n)} - v_0^{(n)}\|_{C(\mathcal{I})} = 0$  for sufficiently small  $\varepsilon > 0$ . It immediately yields  $\mathbf{u} = \mathbf{v}$  because  $\mathbf{u}$  and  $\mathbf{v}$  fulfill the same initial conditions, and so the lemma is proved.  $\square$

**Lemma 9.** *Let (17) and (18) hold true. Then there exists  $\varepsilon > 0$  such that (11) has at most one solution on  $\mathcal{I} = [t_0, t_0 + \varepsilon]$ .*

**Proof.** As in the proof of Lemma 8, let  $\mathbf{u}$  and  $\mathbf{v}$  be solutions of (11) where the fourth row has the form (20). The former  $n$  equations imply

$$u_0^{(n)}(t) - v_0^{(n)}(t) = c(t) (\psi_{p^*}(u_n(t)) - \psi_{p^*}(v_n(t)))$$

for all  $t \in \mathcal{I}$ . Consequently, for every  $t \in \mathcal{I}$ ,

$$b(t)(u_0(t) - v_0(t)) = b(t) \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} c(\tau) (\psi_{p^*}(u_n(\tau)) - \psi_{p^*}(v_n(\tau))) d\tau. \quad (28)$$

As in the proof of Lemma 8, due to (18) we have (22) as  $t \rightarrow 0_+$ . Thus there exist constants  $K_1, K_2 > 0$  satisfying

$$K_1 \leq \left| \frac{u_0^{(n)}(\tau)}{\tau^{j(p^*-1)}} \right| \leq K_2$$

for all  $\tau \in (0, \varepsilon]$  with  $\varepsilon > 0$  small enough. Hence for any  $t \in \mathcal{I}$ ,

$$K_1 g(t) \leq |u_0(t)| = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} |u_0^{(n)}(\tau)| d\tau \leq K_2 g(t),$$

where

$$g(t) = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} \tau^{j(p^*-1)} d\tau.$$

The continuity and positivity of  $\psi'_{q^*}(s)$  on  $[K_1, K_2]$  yield existence of a constant  $K_3 > 0$  such that  $\psi'_{q^*}(s) \geq K_3$  for all  $|s| \in [K_1, K_2]$ . For all  $t \in (0, \varepsilon]$  we have  $g(t) > 0$ , and so

$$\begin{aligned} |b(t)(u_0(t) - v_0(t))| &= |b(t)| g(t) \left| \psi_{q^*} \left( \psi_q \left( \frac{u_0(t)}{g(t)} \right) \right) - \psi_{q^*} \left( \psi_q \left( \frac{v_0(t)}{g(t)} \right) \right) \right| \\ &= |b(t)| g(t) \left| \int_{\psi_q(v_0(t)/g(t))}^{\psi_q(u_0(t)/g(t))} \psi'_{q^*}(\sigma) d\sigma \right| \\ &\geq K_3 (g(t))^{2-q} |u_n^{(n)}(t) - v_n^{(n)}(t)|. \end{aligned} \quad (29)$$

Due to (18),

$$\frac{u_n(\tau)}{\tau^j} \rightarrow \frac{\beta_j}{j!} \neq 0$$

as  $\tau \rightarrow 0_+$  (the same for  $v_n$ ), and so there exist constants  $K_4, K_5 > 0$  such that for all  $\tau \in (0, \varepsilon]$  with  $\varepsilon > 0$  sufficiently small,

$$K_4 \leq \left| \frac{u_n(\tau)}{\tau^j} \right| \leq K_5 \quad \text{and} \quad K_4 \leq \left| \frac{v_n(\tau)}{\tau^j} \right| \leq K_5.$$

The continuity of  $\psi'_{p^*}(s)$  on  $[K_4, K_5]$  imply here existence of  $K_6 > 0$  such that  $\psi'_{p^*}(s) \leq K_6$  for any  $|s| \in [K_4, K_5]$ . Hence

$$\begin{aligned} \left| \psi_{p^*} \left( \frac{u_n(\tau)}{\tau^j} \right) - \psi_{p^*} \left( \frac{v_n(\tau)}{\tau^j} \right) \right| &= \left| \int_{v_n(\tau)/\tau^j}^{u_n(\tau)/\tau^j} \psi'_{p^*}(\sigma) d\sigma \right| \\ &\leq K_6 \tau^{-j} |u_n(\tau) - v_n(\tau)|. \end{aligned} \quad (30)$$

Analogously as (26) we can derive

$$|u_n(\tau) - v_n(\tau)| \leq \frac{\tau^n}{(n-1)!} \|u_n^{(n)} - v_n^{(n)}\|_{C(\mathcal{I})} \quad (31)$$

for all  $\tau \in \mathcal{I}$ . Thus we can estimate the right-hand side of (28) as follows:

$$\begin{aligned} & \left| b(t) \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} c(\tau) \tau^{j(p^*-1)} \left( \psi_{p^*} \left( \frac{u_n(\tau)}{\tau^j} \right) - \psi_{p^*} \left( \frac{v_n(\tau)}{\tau^j} \right) \right) d\tau \right| \\ & \leq \frac{K_6 B C}{(n-1)!} t^{n-j} g(t) \|u_n^{(n)} - v_n^{(n)}\|_{C(\mathcal{I})}. \end{aligned} \quad (32)$$

If we assume  $\varepsilon \leq 1$ , then clearly  $g(t) \leq 1$  and even  $(g(t))^{q-1} \leq 1$  for all  $t \in \mathcal{I}$ . Putting (28), (29) and (32) together and passing to the maximum over  $t \in \mathcal{I}$  we arrive at

$$\|u_n^{(n)} - v_n^{(n)}\|_{C(\mathcal{I})} \leq \frac{K_6 B C}{K_3(n-1)!} \varepsilon^{n-j} \|u_n^{(n)} - v_n^{(n)}\|_{C(\mathcal{I})}.$$

Since  $n-j \geq 1$ , this guarantees again  $u = v$  on  $\mathcal{I} = [0, \varepsilon]$  with  $\varepsilon > 0$  small enough.  $\square$

In the next lemma we assume (16) and (19), which means that the first two conditions of the latter four in Proposition 3 are not satisfied. Consequently, we suppose here that  $p \leq 2$  or that neither  $b_1$  nor  $b_2$  changes its sign on some right closed neighborhood of  $t_0$ .

**Lemma 10.** *Let (16) and (19) hold true. Moreover, let  $p \leq 2$  or there exist  $t_1 > t_0$  such that neither  $b_1$  nor  $b_2$  changes its sign on  $[t_0, t_1]$ . Then there exists  $\varepsilon > 0$  such that (11) has at most one solution on  $\mathcal{I} = [t_0, t_0 + \varepsilon]$ .*

**Proof.** As in the previous proofs, we suppose that  $u$  and  $v$  are solutions of (11) with the fourth row in the form (20). As in the proof of Lemma 8, (27) holds true due to the assumption (16).

We divide the proof into two parts. In the first one (i) we assume  $p \leq 2$ , and in the second one (ii) we suppose that  $b$  does not change its sign on  $\mathcal{I}$ .

(i) The continuity of  $u_0^{(n)}$  and  $v_0^{(n)}$  implies existence of  $K_1 \geq 1$  such that  $|u_0^{(n)}(t)| \leq K_1$  and  $|v_0^{(n)}(t)| \leq K_1$ . Though  $\psi_p'(s)$  is not continuous on  $[-K_1, K_1]$ , due to  $p \leq 2$  it is clear that  $\psi_p'(s) \geq K_2$  for all  $s \in [-K_1, K_1] \setminus \{0\}$ , where  $K_2 = (p-1)K_1^{p-2} > 0$ . Hence

$$\begin{aligned} & |a(t)(\psi_p(u_0^{(n)}(t)) - \psi_p(v_0^{(n)}(t)))| \\ & \geq C^{1-p} \left| \int_{v_0^{(n)}(t)}^{u_0^{(n)}(t)} \psi_p'(\sigma) d\sigma \right| \geq C^{1-p} K_2 |u_0^{(n)}(t) - v_0^{(n)}(t)|. \end{aligned} \quad (33)$$

From (21), (27) and (33) we now conclude (passing to the maximum over  $t \in \mathcal{I}$ )

$$\|u_0^{(n)} - v_0^{(n)}\|_{C(\mathcal{I})} \leq \frac{K_6 B C^{p-1}}{K_2((n-1)!)^2} \varepsilon^{2n+i(q-2)} \|u_0^{(n)} - v_0^{(n)}\|_{C(\mathcal{I})}.$$

This again proves  $u = v$  on sufficiently small  $\mathcal{I}$ .

(ii) We can suppose that

$$f(t) := \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} \tau^{i(q-1)} |b(\tau)| d\tau > 0 \quad \forall t \in (0, \varepsilon].$$

Indeed, otherwise  $b(\tau) = 0$  for all  $\tau \in [0, t_0]$  for some  $t_0 > 0$ , and the uniqueness is then trivial. As in the proof of Lemma 8, (16) guarantees (24) for some  $K_4, K_5 > 0$  and  $\varepsilon > 0$  sufficiently small. Hence (20) yields

$$(K_4 \tau^i)^{q-1} |b(\tau)| \leq |u_n^{(n)}(\tau)| \leq (K_5 \tau^i)^{q-1} |b(\tau)|$$

for all  $\tau \in [0, \varepsilon]$ . Due to the assumptions (16) and that  $b$  does not change its sign on  $\mathcal{I}$ , neither  $u_n^{(n)}$  changes its sign. Thus for any  $t \in \mathcal{I}$ ,

$$|u_n(t)| = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} |u_n^{(n)}(\tau)| d\tau \geq K_4^{q-1} f(t)$$

(we assume (19)), and, similarly,  $|u_n(t)| \leq K_5^{q-1} f(t)$ . From the  $n$ th equation in (11) we now conclude

$$K_1 (f(t))^{p^*-1} \leq |u_0^{(n)}(t)| \leq K_2 (f(t))^{p^*-1},$$

where (assuming  $K_4 \leq 1 \leq K_5$ )

$$K_1 = A^{1-p^*} K_4^{(q-1)(p^*-1)} > 0 \quad \text{and} \quad K_2 = C K_5^{(q-1)(p^*-1)} > 0.$$

The same estimate one can derive also for  $v_0^{(n)}$ . The continuity and positivity of  $\psi_p'(s)$  on  $[K_1, K_2]$  guarantee existence of  $K_3 > 0$  satisfying  $\psi_p'(s) \geq K_3$  for all  $|s| \in [K_1, K_2]$ . Hence for any  $t \in (0, \varepsilon]$ ,

$$\begin{aligned} & |a(t)(\psi_p(u_0^{(n)}(t)) - \psi_p(v_0^{(n)}(t)))| \\ & \geq C^{1-p} f(t) \left| \psi_p\left(\frac{u_0^{(n)}(t)}{f^{p^*-1}(t)}\right) - \psi_p\left(\frac{v_0^{(n)}(t)}{f^{p^*-1}(t)}\right) \right| \\ & = C^{1-p} f(t) \left| \int_{v_0^{(n)}(t)/f^{p^*-1}(t)}^{u_0^{(n)}(t)/f^{p^*-1}(t)} \psi_p'(\sigma) d\sigma \right| \geq K_3 (f(t))^{2-p^*} |u_0^{(n)}(t) - v_0^{(n)}(t)|. \end{aligned} \quad (34)$$

As in the proof of Lemma 8, (16) yields (25) for some  $K_6 > 0$ . Using this together with (26) we can estimate the right-hand side in (21) as follows:

$$\begin{aligned} & \left| \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} b(\tau) \tau^{i(q-1)} \left( \psi_q\left(\frac{u_0(\tau)}{\tau^i}\right) - \psi_q\left(\frac{v_0(\tau)}{\tau^i}\right) \right) d\tau \right| \\ & \leq \frac{K_6}{(n-1)!} t^{n-i} f(t) \|u_0^{(n)} - v_0^{(n)}\|_{C(\mathcal{I})}. \end{aligned} \quad (35)$$

If we assume  $\varepsilon \leq 1$ , then clearly

$$f(t) \leq \frac{t^{n+i(q-1)}}{(n-1)!} B,$$

and so (assuming  $B \geq 1$ )

$$(f(t))^{p^*-1} \leq \left( \frac{t^{n+i(q-1)}}{(n-1)!} \right)^{p^*-1} B^{p^*-1} \quad (36)$$

for every  $t \in \mathcal{I}$ . Putting (21) and (34)–(36) together and passing to the maximum for  $t \in \mathcal{I}$  we arrive at

$$\|u_0^{(n)} - v_0^{(n)}\|_{C(\mathcal{I})} \leq \frac{K_6 B^{p^*-1}}{K_3((n-1)!)^{p^*}} \varepsilon^{(n-i)p^* + iq(p^*-1)} \|u_0^{(n)} - v_0^{(n)}\|_{C(\mathcal{I})}.$$

Since  $(n-i)p^* + iq(p^*-1) > 1$ , this again means that  $u = v$  on  $\mathcal{I} = [0, \varepsilon]$  with  $\varepsilon > 0$  small enough.  $\square$

The last lemma deals with the case (17) and (19). Thus the first condition of the former two in Proposition 3 is not satisfied. Hence we suppose here that the second one is true.

**Lemma 11.** *Let (17) and (19) hold true. Moreover, let  $p \leq \min\{q_1, q_2\}$ . Then there exists  $\varepsilon > 0$  such that (11) has at most one solution on  $\mathcal{I} = [t_0, t_0 + \varepsilon]$ .*

**Proof.** Since for (17) and (19) the problem (11) has the trivial solution, our aim is to prove that any solution  $u$  of (11) satisfies  $u \equiv 0$  on  $\mathcal{I} = [0, \varepsilon]$  with some  $\varepsilon > 0$ .

We have

$$\begin{aligned} & |a(t)\psi_p(u_0^{(n)}(t))| \\ & \leq \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} (|b_1(\tau)|\psi_{q_1}(|u_0^+(\tau)|) + |b_2(\tau)|\psi_{q_2}(|u_0^-(\tau)|)) d\tau \end{aligned} \quad (37)$$

for all  $t \in \mathcal{I}$ . As (26) we can derive that for any  $\tau \in \mathcal{I}$ ,

$$|u_0(\tau)| \leq \frac{\tau^n}{(n-1)!} \|u_0^{(n)}\|_{C(\mathcal{I})}.$$

Since  $u_0(0) = 0$ , we can choose  $\varepsilon > 0$  so small that  $\varepsilon \leq 1$  and  $|u_0(\tau)| \leq 1$  for all  $\tau \in \mathcal{I}$ . Then due to the assumption  $p \leq \min\{q_1, q_2\}$ ,

$$|b_1(\tau)|\psi_{q_1}(|u_0^+(\tau)|) + |b_2(\tau)|\psi_{q_2}(|u_0^-(\tau)|) \leq B \left( \frac{\tau^n}{(n-1)!} \right)^{p-1} \|u_0^{(n)}\|_{C(\mathcal{I})}^{p-1}. \quad (38)$$

As  $u_0^{(n)}(0) = 0$ , we choose  $\varepsilon > 0$  sufficiently small to have also  $|u_0^{(n)}(t)| \leq 1$  for all  $t \in \mathcal{I}$ . Then combining (37), (38) and

$$|a(t)\psi_p(u_0^{(n)}(t))| \geq C^{1-p} |u_0^{(n)}(t)|^{p-1},$$

and taking the maximum for all  $t \in \mathcal{I}$ , we conclude

$$\|u_0^{(n)}\|_{C(\mathcal{I})}^{p-1} \leq \frac{BC^{p-1}}{((n-1)!)^p} \varepsilon^{np} \|u_0^{(n)}\|_{C(\mathcal{I})}^{p-1}.$$

This implies (similarly as in the previous proofs)  $\|u_0^{(n)}\|_{C(\mathcal{I})} = 0$ , i.e.,  $u \equiv 0$  on  $\mathcal{I} = [0, \varepsilon]$  for sufficiently small  $\varepsilon > 0$ .  $\square$

This completes the proof of Proposition 3. Corollary 4 is now a standard consequence of the local uniqueness.

## 5. Open problems

We leave the following problems open (cf. [1]).

(1) The first two open questions were already mentioned in Section 1 (see Tables 1 and 2). For  $p < q$ ,  $(-1)^n \lambda < 0$  and  $n > 1$ , it is an open question whether there exists a global solution of (10). On the other hand, for  $p > q$ ,  $(-1)^n \lambda < 0$  and  $n > 1$ , it is not known whether there can exist more than one (local) solution of (10).

(2) The counterexamples [1, Remark 3.3] and [1, Remark 4.6] show that the assumptions  $p \geq \max\{q_1, q_2\}$  and  $p \leq \min\{q_1, q_2\}$  in Theorem 2 and Corollary 4, respectively, cannot be left out. But it is an open question whether we can leave out the second condition in Corollary 4 which excludes the situation that  $p > 2$ , and  $b_1$  or  $b_2$  is a function of the type (7).

It was mentioned in Section 1 that the existence and uniqueness problem for (1) is closed. But if we consider nonconstant coefficients, it is not true even for the homogeneous problem, since it is not known whether there can exist more than one (local) solution of

$$\begin{cases} (-1)^n (\psi_p(u^{(n)}(t)))^{(n)} = b(t) \psi_p(u(t)), & t \in \mathcal{I}, \\ u^{(i)}(t_0) = \alpha_i, & (\psi_p(u^{(n)}(t)))^{(j)}|_{t=t_0} = 0, \quad i, j \in \{0, 1, \dots, n-1\}, \end{cases}$$

where  $p > 2$ , at least one of the constants  $\alpha_i$ ,  $i \in \{0, \dots, n-1\}$ , is not zero and  $b \in C(\mathcal{I})$  changes its sign on arbitrarily small neighborhood of  $t_0$  (like (7)).

(3) Is it possible to generalize our results to cover also the results from [6], [11], [15] or [10]? For example, the question of local uniqueness of the solution of (13) with zero initial conditions is not solved in [10]. However, from all the mentioned results one can deduce the conjecture that the solution of (13) with  $r_1 r_2 \dots r_n \geq 1$  and zero initial conditions must be trivial.

(4) As for the nonconstant  $p, q_1, q_2$  (see Remark 6), it is an open problem whether the conditions (8) and (9) can be weakened for example to  $p(t) \geq \max\{q_1(t), q_2(t)\}$  and  $p(t) \leq \min\{q_1(t), q_2(t)\}$ , respectively, for all  $t \in \mathcal{I}$ .

## Acknowledgments

The author is grateful to the referee for drawing the author's attention to the paper [10], and to Professor R. Manásevich for his valuable comments.

## References

- [1] J. Benedikt, Uniqueness theorem for  $p$ -biharmonic equations, *Electron. J. Differential Equations* 53 (2002) 1–17, <http://ejde.math.swt.edu>.
- [2] J. Benedikt, On simplicity of spectra of  $p$ -biharmonic equations, *Nonlinear Anal.*, in press.
- [3] J. Benedikt, On the discreteness of the spectra of the Dirichlet and Neumann  $p$ -biharmonic problem, *Abstr. Appl. Anal.*, in press.
- [4] E.A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw–Hill, New York, 1955.
- [5] M.A. del Pino, On a quasilinear second-order problem, Master's thesis, Univ. de Chile, Santiago, 1987 (in Spanish, "Sobre un problema cuasilineal de segundo orden").
- [6] M.A. del Pino, R.F. Manásevich, A.E. Murúa, Existence and multiplicity of solutions with prescribed period for a second order quasilinear O.D.E., *Nonlinear Anal.* 18 (1992) 79–92.
- [7] P. Drábek, R.F. Manásevich, On the closed solution to some nonhomogeneous eigenvalue problems with  $p$ -Laplacian, *Differential Integral Equations* 12 (1999) 773–788.
- [8] P. Drábek, M. Ôtani, Global bifurcation result for the  $p$ -biharmonic operator, *Electron. J. Differential Equations* 48 (2001) 1–19, <http://ejde.math.swt.edu>.
- [9] Á. Elbert, A half-linear second order differential equation, in: *Qualitative Theory of Differential Equation*, vols. I and II, North-Holland, Amsterdam, 1981, pp. 153–180.
- [10] U. Elias, A. Pinkus, Nonlinear eigenvalue problems for a class of ordinary differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* 132 (2002) 1333–1359.
- [11] M. García-Huidobro, R.F. Manásevich, F. Zanolin, A Fredholm-like result for strongly nonlinear second order ODE's, *J. Differential Equations* 114 (1994) 132–167.
- [12] P. Harjulehto, P. Hästö, An overview of variable exponent Lebesgue and Sobolev spaces, in: *Proceedings of the Workshop on Future Trends in Geometric Function Theory (Jyväskylä 2003)*, in press.
- [13] P. Hartman, *Ordinary Differential Equations*, Wiley, Baltimore, 1973.
- [14] I.T. Kiguradze, *Some Singular Boundary Value Problems For Ordinary Differential Equations*, Tbilisi Univ. Press, Tbilisi, 1975 (in Russian).
- [15] R.F. Manásevich, F.I. Njoku, F. Zanolin, Positive solutions for the one-dimensional  $p$ -Laplacian, *Differential Integral Equations* 8 (1995) 213–222.
- [16] J.D. Mirzov, *Asymptotic Properties of Solutions of Systems of Nonlinear Nonautonomous Ordinary Differential Equations*, Adygeja, Maikop, 1993 (in Russian).
- [17] W. Walter, Sturm–Liouville theory for the radial  $\Delta_p$ -operator, *Math. Z.* 227 (1998) 175–185.