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# The coefficient body of Bell representations of finitely connected planar domains

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**Abstract**

In this paper, we determine the homotopy type of the coefficient body of Bell representations of non-degenerate $n$-connected planar domains with $n \geqslant 3$. Also, by considering the isomorphism classes of rational functions, we get the precise number of those corresponding to Bell representations with same set of critical values. Further, the case of those with the same set of critical points is discussed.


*Keywords:* Conformal representation; Rational function; Hurwitz space

## 1. Introduction

In this paper, a *non-degenerate $n$-connected planar domain* is a subdomain $\Omega$ of the Riemann sphere $\hat{\mathbb{C}}$ such that $\hat{\mathbb{C}} - \Omega$ consists of exactly $n$ connected components each of which contains more than one point. We also assume that $n \geqslant 2$.

Then we know that every such $\Omega$ has a canonical representation as in the following theorem, which is called a *Bell representation* of it.

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**Theorem 1.1** [6]. *Every non-degenerate $n$-connected planar domain with $n \geqslant 2$ is mapped biholomorphically onto a domain $W_{\mathbf{a},\mathbf{b}}$ defined by*

$$\left\{ z \in \mathbb{C} \;\middle|\; \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

*with suitable complex vectors $\mathbf{a} = (a_1, a_2, \dots, a_{n-1})$ and $\mathbf{b} = (b_1, b_2, \dots, b_{n-1})$.*

This theorem is considered as a natural generalization of the classical Riemann mapping theorem for simply connected planar domains. Importance of such representations consists in such a fact that every domain $W_{\mathbf{a},\mathbf{b}}$ has algebraic kernel functions. To be precise, the function

$$f_{\mathbf{a},\mathbf{b}}(z) = z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}$$

is a proper holomorphic mapping from $W_{\mathbf{a},\mathbf{b}}$ onto the unit disc $U$ which is algebraic. Hence Bell's result in [1,2] implies the following

**Proposition 1.2.** *Every non-degenerate $n$-connected planar domain is biholomorphic to a domain with algebraic Bergman kernel and algebraic Szegő kernel.*

Here it is important to know the locus of the complex vectors $(\mathbf{a}, \mathbf{b})$ which correspond to non-degenerate $n$-connected planar domains.

**Definition 1.3.** For every $n \geqslant 2$, let $\mathbf{B}_n$ in $\mathbb{C}^{2n-2}$ be the set of all complex vectors

$$(\mathbf{a}, \mathbf{b}) = (a_1, a_2, \dots, a_{n-1}, b_1, b_2, \dots, b_{n-1})$$

in $\mathbb{C}^{2n-2}$ such that the corresponding domains

$$W_{\mathbf{a},\mathbf{b}} = \left\{ z \in \mathbb{C} \;\middle|\; \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

are non-degenerate $n$-connected planar domains.

We call $\mathbf{B}_n$ the *coefficient body* for non-degenerate $n$-connected canonical domains.

In this paper, we will investigate the geometric structure of $\mathbf{B}_n$.

## 2. A modified representation

To clarify the structure of the coefficient body, it is more convenient to consider the following modification. In the sequel, we assume that $n > 2$, since $\mathbf{B}_2$ and $\mathbf{B}_2^*$ are explicitly known (cf. [7]).

**Definition 2.1.** We set

$$\mathbf{B}_n^* = \{(a_1, \ldots, a_{n-1}, \mathbf{b}) \mid (a_1^2, \ldots, a_{n-1}^2, \mathbf{b}) \in \mathbf{B}_n\},$$

and call it the *modified coefficient body*.

Clearly, $\mathbf{B}_n^*$ is contained in

$$(\mathbb{C}^*)^{n-1} \times F_{0,n-1}\mathbb{C},$$

where

$$F_{0,n-1}\mathbb{C} = \{(z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1} \mid z_j \neq z_k \text{ if } j \neq k\}.$$

Also it is invariant under the symmetry

$$S_k : (a_1, \ldots, a_k, \ldots, a_{n-1}, \mathbf{b}) \mapsto (a_1, \ldots, -a_k, \ldots, a_{n-1}, \mathbf{b})$$

of $\mathbb{C}^{2n-2}$ for every $k$. And $\mathbf{B}_n$ can be identified with the quotient space of $\mathbf{B}_n^*$ by the action of the group $G = \langle S_1, \ldots, S_{n-1}\rangle$ generated by these symmetries. Thus $\mathbf{B}_n^*$ is $2^{n-1}$ sheeted holomorphic covering of $\mathbf{B}_n$ with the covering transformation group $G$.

Next, note that $\mathbf{B}_n^*$ is circular in the following sense.

**Proposition 2.2.** *For every $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^*$ and every $\theta \in \mathbb{R}$, $e^{i\theta}(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^*$.*

**Proof.** If $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^*$, then letting

$$g_{\mathbf{a},\mathbf{b}}(z) = z + \sum_{k=1}^{n-1} \frac{a_k^2}{z - b_k},$$

$$W_{\mathbf{a},\mathbf{b}}^* = \{z \in \mathbb{C} \mid |g_{\mathbf{a},\mathbf{b}}(z)| < 1\}$$

is a non-degenerate $n$-connected domain. Hence

$$\begin{aligned} W_{e^{i\theta}\mathbf{a}, e^{i\theta}\mathbf{b}}^* &= \left\{z \in \mathbb{C} \;\middle|\; \left|z + \sum_{k=1}^{n-1} \frac{(e^{i\theta} a_k)^2}{z - e^{i\theta} b_k}\right| < 1\right\} = \{z \in \mathbb{C} \mid |e^{i\theta} g_{\mathbf{a},\mathbf{b}}(e^{-i\theta} z)| < 1\} \\ &= e^{i\theta} W_{\mathbf{a},\mathbf{b}}^*, \end{aligned}$$

which is biholomorphic to $W_{\mathbf{a},\mathbf{b}}^*$, is a non-degenerate $n$-connected planar domain. □

Another important property is "star-shapedness" of $\mathbf{B}_n^*$.

**Proposition 2.3.** *For every $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^*$ and every $0 < r \leqslant 1$, $r(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^*$.*

**Proof.** Let $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^*$, and $g_{\mathbf{a},\mathbf{b}}(z)$ be as in the previous proof. And for every $0 < r < 1$, set

$$W_{\mathbf{a},\mathbf{b}}^r = \{|g_{\mathbf{a},\mathbf{b}}(z)| < 1/r\}.$$

It contains $W_{\mathbf{a},\mathbf{b}}^*$. And for every connected component $F$ of the preimage $g_{\mathbf{a},\mathbf{b}}^{-1}(E_r)$ of $E_r = \{w \in \mathbb{C} \mid |w| \geqslant 1/r\}$, $g_{\mathbf{a},\mathbf{b}}(z)$ gives a homeomorphic map of $F$ onto $E_r$. Since $E_r$ contains more than a point, so does each $F$. Hence $W^r_{\mathbf{a},\mathbf{b}}$ is also a non-degenerate $n$-connected domain.

Since $rW^r_{\mathbf{a},\mathbf{b}}$ is biholomorphic to $W^r_{\mathbf{a},\mathbf{b}}$, $rW^r_{\mathbf{a},\mathbf{b}}$ is also non-degenerate $n$-connected domain. Furthermore, since

$$rg_{\mathbf{a},\mathbf{b}}(z/r) = z + \sum_{k=1}^{n-1} \frac{(ra_k)^2}{z - rb_k} = g_{r\mathbf{a},r\mathbf{b}}(z),$$

$W^*_{r\mathbf{a},r\mathbf{b}} = rW^r_{\mathbf{a},\mathbf{b}}$ is a non-degenerate $n$-connected domain and therefore $r(\mathbf{a},\mathbf{b}) \in \mathbf{B}^*_{\mathbf{n}}$. $\square$

We now have the following property of $\mathbf{B}^*_n$ and $\mathbf{B}_n$.

**Theorem 2.4.** $\mathbf{B}^*_n$ *and hence* $\mathbf{B}_n$ *are domains and have the same homotopy type as that of*

$$\left(S^1\right)^{n-1} \times F_{0,n-1}\mathbb{C}.$$

**Corollary 2.5.** *The modified coefficient body* $\mathbf{B}^*_n$ *is a circular domain homeomorphic to* $\mathbf{B}_n$.

**Remark 2.6.** The fundamental group of $F_{0,n-1}\mathbb{C}$ is called the *pure braid group*, and its structure is well-known. See, for instance, [3].

The above theorem follows from the following two lemmas.

**Lemma 2.7.** *The coefficient body* $\mathbf{B}_n$ *is the set of all* $(\mathbf{a},\mathbf{b})$ *such that*

$$f'_{\mathbf{a},\mathbf{b}}(z) = 0$$

*has* $2n-2$ *solutions* $c_1,\ldots,c_{2n-2}$ *counted with multiplicities such that*

$$\left|f_{\mathbf{a},\mathbf{b}}(c_j)\right| < 1$$

*for every* $j$. *The set* $\mathbf{B}^*_n$ *is characterized in the same way. In particular,* $\mathbf{B}_n$ *and* $\mathbf{B}^*_n$ *are open subsets of* $\mathbb{C}^{2n-2}$.

**Proof.** $f_{\mathbf{a},\mathbf{b}}(z)$ has exactly $2n-2$ finite critical points $c_1,\ldots,c_{2n-2}$, i.e., zeros of $f'_{\mathbf{a},\mathbf{b}}(z)$, counted with multiplicities, and $(\mathbf{a},\mathbf{b})$ belongs to $\mathbf{B}_n$, i.e.,

$$\left\{z \in \mathbb{C} \mid \left|f_{\mathbf{a},\mathbf{b}}(z)\right| < 1\right\}$$

is a non-degenerate $n$-connected domain if and only if, for every connected component $F$ of the preimage

$$f^{-1}_{\mathbf{a},\mathbf{b}}(E)$$

of $E = \{w \in \mathbb{C} \mid |w| \geqslant 1\}$, $f_{\mathbf{a},\mathbf{b}}(z)$ gives a homeomorphic map of $F$ onto $E$. Hence we have the first assertion. The case of $\mathbf{B}^*_n$ is similar.

Next, since $c_j$ varies continuously with respect to $(\mathbf{a},\mathbf{b})$, $\mathbf{B}_n$ and $\mathbf{B}^*_n$ are open subsets of $\mathbb{C}^{2n-2}$. $\square$

Next set

$$\rho(\mathbf{b}) = \min_{j \neq k} |b_j - b_k|.$$

And for a sufficiently small $\varepsilon > 0$ with $\varepsilon \leqslant 1/(6n)$, we set

$$\mathbf{B}_n^\varepsilon = \left\{(\mathbf{a}, \mathbf{b}) \in \mathbb{C}^{2n-2} \mid \rho(\mathbf{b}) > 0,\ |b_k| \leqslant 1/2,\ 0 < |a_k| \leqslant \varepsilon\sqrt{\rho(\mathbf{b})},\ 1 \leqslant k \leqslant n-1\right\}.$$

Note that $\rho(\mathbf{b}) \leqslant 1$.

**Lemma 2.8.** $\mathbf{B}_n^*$ *has the same homotopy type as that of* $\mathbf{B}_n^\varepsilon$*.*

**Proof.** First we show that

$$\mathbf{B}_n^\varepsilon \subset \mathbf{B}_n^*.$$

Suppose that $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^\varepsilon$. If we set

$$C_k = \left\{z \in \mathbb{C} \mid |b_k - z| = \varepsilon\rho(\mathbf{b})\right\}$$

then $z \in C_k$ implies that $|z| \leqslant 2/3$, and

$$|b_j - z| \geqslant (1-\varepsilon)\rho(\mathbf{b}) > \rho(\mathbf{b})/2$$

for every $j \neq k$, and hence

$$\begin{aligned}\left|g_{\mathbf{a},\mathbf{b}}(z)\right| &\leqslant |z| + \sum_{j=1}^{n-1}\left|\frac{a_j^2}{z-b_j}\right| \leqslant \frac{2}{3} + \frac{\varepsilon^2\rho(\mathbf{b})}{\varepsilon\rho(\mathbf{b})} + (n-2)\frac{\varepsilon^2\rho(\mathbf{b})}{\rho(\mathbf{b})/2}\\ &= \frac{2}{3} + \big(1 + (2n-4)\big)\varepsilon < 1.\end{aligned}$$

On the other hand, if we set

$$\tilde{C}_k = \left\{|b_k - z| = \left|a_k^2\right|/2\right\}$$

then $|a_k^2|/2 < \varepsilon^2\rho(\mathbf{b})$, and $z \in \tilde{C}_k$ implies that

$$\begin{aligned}\left|g_{\mathbf{a},\mathbf{b}}(z)\right| &\geqslant \frac{|a_k^2|}{|z-b_k|} - |z| - \sum_{j\neq k}\left|\frac{a_j^2}{z-b_j}\right| \geqslant 2 - \frac{2}{3} - (n-2)\frac{\varepsilon^2\rho(\mathbf{b})}{\rho(\mathbf{b})/2}\\ &= 2 - \frac{2}{3} - (2n-4)\varepsilon^2 > 1.\end{aligned}$$

Thus

$$\left\{z \in \mathbb{C} \mid \left|g_{\mathbf{a},\mathbf{b}}(z)\right| = 1\right\}$$

has a component in

$$\left\{z \in \mathbb{C} \mid \left|a_k^2\right|/2 < |z - b_k| < \varepsilon\rho(\mathbf{b})\right\},$$

and $W_{\mathbf{a},\mathbf{b}}^*$ is disjoint from $\{|z - b_k| \leqslant |a_k^2|/2\}$, for every $k$, which implies that $W_{\mathbf{a},\mathbf{b}}^*$ is non-degenerate and $n$-connected.

Next for every $(\mathbf{a}_0, \mathbf{b}_0) \in \mathbf{B}_n^*$ with $\mathbf{a}_0 = (a_{1,0}, \dots, a_{n,0})$ and $\mathbf{b}_0 = (b_{1,0}, \dots, b_{n,0})$, let $\ell_{\mathbf{a}_0,\mathbf{b}_0}$ be the ray

$$\{(r\mathbf{a}_0, r\mathbf{b}_0) \mid 0 < r \leqslant 1\}.$$

Then by Proposition 2.3, $\ell_{\mathbf{a}_0,\mathbf{b}_0} \subset \mathbf{B}_n^*$. Also since $\rho(r\mathbf{b}_0) = r\rho(\mathbf{b}_0)$, we conclude that

$$|ra_{k,0}| = r|a_{k,0}| = \varepsilon' \sqrt{\rho(r\mathbf{b}_0)},$$

where

$$\varepsilon' = \sqrt{r}\,|a_{k,0}|/\sqrt{\rho(\mathbf{b}_0)},$$

which in turn tends to 0 as $r$ does.

Now, fix an $\varepsilon > 0$ with $\varepsilon \leqslant 1/(6n)$. Then, $(r\mathbf{a}_0, r\mathbf{b}_0) \in \mathbf{B}_n^\varepsilon$ for every sufficiently small $r$. Hence we can construct a deformation retraction

$$r_\varepsilon : \mathbf{B}_n^* \to \mathbf{B}_n^\varepsilon,$$

by mapping the point $(\mathbf{a}_0, \mathbf{b}_0)$ to the nearest point in $\mathbf{B}_n^\varepsilon$ along $\ell_{\mathbf{a}_0,\mathbf{b}_0}$. This retraction is clearly the identity on $\mathbf{B}_n^\varepsilon$, and we conclude the assertion. □

Here we give typical examples of points in $\mathbf{B}_3$. Consider the case that

$$f(z) = f_{4a^2,4a^2,b,-b}(z) = z + \frac{4a^2}{z-b} + \frac{4a^2}{z+b}$$

with $a, b \in \mathbb{C} - \{0\}$. Then Lemma 2.7 implies the following theorem.

**Theorem 2.9.** *The complex vector* $(4a^2, 4a^2, b, -b)$ *belongs to* $\mathbf{B}_3$ *if and only if*

$$\left|b^2 + 4a^2 + 4a\left(a^2 + b^2\right)^{1/2}\right| \cdot \left|b^2 - 2a^2 + 2a\left(a^2 + b^2\right)^{1/2}\right|^2 < |b|^4$$

*where the same value of* $(a^2 + b^2)^{1/2}$ *is taken in each term.*

**Proof.** Since $f'(z)$ has 4 roots

$$\left(b^2 + 4a^2 + 4a\left(a^2 + b^2\right)^{1/2}\right)^{1/2},$$

Lemma 2.7 implies that $(4a^2, 4a^2, b, -b)$ belongs to $\mathbf{B}_3$ if and only if

$$\begin{aligned}
&\left|b^2 + 4a^2 + 4a\left(a^2 + b^2\right)^{1/2}\right| \left|\frac{12a^2 + 4a(a^2 + b^2)^{1/2}}{4a^2 + 4a(a^2 + b^2)^{1/2}}\right|^2 \\
&\quad = \left|b^2 + 4a^2 + 4a\left(a^2 + b^2\right)^{1/2}\right| \left|\frac{(3a + (a^2 + b^2)^{1/2})(-a + (a^2 + b^2)^{1/2})}{b^2}\right|^2 \\
&\quad = \left|b^2 + 4a^2 + 4a\left(a^2 + b^2\right)^{1/2}\right| \left|\frac{b^2 - 2a^2 + 2a(a^2 + b^2)^{1/2})}{b^2}\right|^2 < 1. \qquad \square
\end{aligned}$$

**Example 2.10.** Let $a = 3/40$ and $b = 1/10$. Since $a$ and $b$ satisfy the inequality in Theorem 2.9, $(4a^2, 4a^2, b, -b)$ belongs to $\mathbf{B}_3$. In fact,

$$\left\{\pm\frac{\sqrt{7}}{10},\ \pm\frac{\sqrt{2}}{20}i\right\}$$

is the set of critical points of $f_{4a^2,4a^2,b,-b}$ and $|f_{4a^2,4a^2,b,-b}| < 1$ at each critical point.

## 3. Parametrization as the Hurwitz space

Sometimes, holomorphic functions are parametrized by the set of critical points, or that of the critical values, i.e., the images of critical points. Here we consider the parametrization using the critical values. Such a parametrization is usually considered for those functions in general position. In the sequel, we assume that $n > 2$.

**Definition 3.1.** Let $\Gamma$ be the set of all points $(\mathbf{a}, \mathbf{b})$ of $\mathbf{B}_n$ such that the corresponding rational function $f_{\mathbf{a},\mathbf{b}}$ has a non-simple critical point or has a pair of critical points whose images are the same. We call $\Gamma$ the *collision locus*.

Then for every point $(\mathbf{a}, \mathbf{b})$ in $\mathbf{B}_n - \Gamma$, the rational function $f_{\mathbf{a},\mathbf{b}}$ has $2n - 2$ simple critical values. We denote the set of simple critical values of $f_{\mathbf{a},\mathbf{b}}$ by

$$S_{\mathbf{a},\mathbf{b}} = \{\alpha_1, \ldots, \alpha_{2n-2}\},$$

where, letting $\{c_j\}_{j=1}^{2n-2}$ be the set of the simple critical points of $f_{\mathbf{a},\mathbf{b}}$, $\alpha_j = f_{\mathbf{a},\mathbf{b}}(c_j)$ for every $j$. This set can be considered as a point in the unordered configuration space $B_{0,2n-2}\mathbb{C}$, i.e., the quotient space of $F_{0,2n-2}\mathbb{C}$ by the symmetric group $S_{2n-2}$. Moreover by Lemma 2.7, we see that $S_{\mathbf{a},\mathbf{b}}$ is actually a point of the unordered configuration space $B_{0,2n-2}U$ for the unit disc $U$ (cf. [3]).

Thus we can define the projection

$$\pi_S : \mathbf{B}_n - \Gamma \to B_{0,2n-2}U$$

by setting

$$\pi_S(\mathbf{a}, \mathbf{b}) = S_{\mathbf{a},\mathbf{b}}.$$

We have the following theorem about the projection $\pi_S$.

**Theorem 3.2.** *The projection $\pi_S$ is a $(2n-2)!n^{n-3}$-sheeted proper holomorphic covering of $B_{0,2n-2}U$ for every $n > 2$.*

**Remark 3.3.** The number

$$\frac{(2n-2)!n^{n-3}}{n!}$$

is called a Hurwitz number. See, for instance, [5].

First recall that, for every point $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n - \Gamma$, the critical points $c_1, \ldots, c_{2n-2}$ of $f_{\mathbf{a},\mathbf{b}}$ are the solutions of the algebraic equation

$$\prod_{j=1}^{n-1} (z - b_j)^2 \left(1 - \sum_{k=1}^{n-1} \frac{a_k}{(z - b_k)^2}\right) = 0.$$

Hence $c_j$ moves holomorphically with respect to $(\mathbf{a}, \mathbf{b})$. Since so does the image $\alpha_j$ of $c_j$ for each $j = 1, \ldots, 2n-2$, the map $\pi_S$ is holomorphic.

Next we show by the following two lemmas that, for every point S in $B_{0,2n-2}U$, $\pi_S^{-1}(\mathrm{S})$ consists of $(2n-2)!n^{n-3}$ points.

**Definition 3.4.** The *marked Hurwitz space* $MH_{0,n}(1, \ldots, 1)$ of genus 0 and degree $n$ with type $(1, \ldots, 1)$ and with the ordered poles is the set of all isomorphism classes of rational functions in general position (i.e., with simple critical values) of degree $n$ such that poles are simple and ordered. Here we say that two such rational functions $f, g$ are *isomorphic* if there is a Möbius transformation $A$ such that

$$f = g \circ A$$

and $A$ maps poles of $f$ to those of $g$ keeping the order. (Cf. [8].)

**Lemma 3.5.** $\mathbf{B}_n - \Gamma$ *can be identified with the subset* $MH_nU$ *of marked Hurwitz space* $MH_{0,n}(1, \ldots, 1)$*, consisting of all isomorphism classes of rational functions whose critical values are in* $U$*, by the mapping* $\iota$ *which maps* $(\mathbf{a}, \mathbf{b})$ *to the isomorphism class of* $f_{\mathbf{a},\mathbf{b}}$.

**Proof.** By Lemma 2.7, every $f = f_{\mathbf{a},\mathbf{b}}$ with $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n - \Gamma$ determines a point in $MH_nU$. Here we always assume that the order of poles is $b_1, \ldots, b_{n-1}, \infty$.

Next suppose that $(\mathbf{a}', \mathbf{b}')$ is also in $\mathbf{B}_n - \Gamma$. If $g = f_{\mathbf{a}',\mathbf{b}'}$ is in the isomorphism class of $f$, then there is a Möbius transformation $A$ such that

$$f = g \circ A$$

and since $A$ maps poles of $f$ to those of $g$ keeping the order, $A$ fixes $\infty$ and hence is affine, which we write as $A(z) = pz + q$. Then

$$z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} = pz + q + \sum_{k=1}^{n-1} \frac{a'_k}{A(z) - b'_k}.$$

Hence $A$ should be the identity map. This implies that

$$(\mathbf{a}, \mathbf{b}) = \left(\mathbf{a}', \mathbf{b}'\right),$$

and hence $\iota$ is injective.

Finally, for every point P in $MH_nU$, take a representative (a rational function) $f$ in this class. Then the poles of $f$ are simple and ordered. By applying precomposition of a suitable Möbius transformation which sends $\infty$ to a pole if necessary, we may assume that $f$ has the form

$$f(z) = az + b + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}.$$

Again by another precomposition of an affine transformation, we may assume that $a = 1$, $b = 0$, i.e., $f = f_{\mathbf{a},\mathbf{b}}$ with some $(\mathbf{a}, \mathbf{b})$ in $\mathbf{B}_n - \Gamma$. Thus $\iota : \mathbf{B}_n - \Gamma \to MH_nU$ is surjective. $\square$

Now, fix a point $\mathrm{S} = \{\alpha_j\}_{j=1}^{2n-2}$ in $B_{0,2n-2}U$. And fix a set of mutually disjoint cuts (simple smooth arcs) $\ell_j$ from $\alpha_j$ to a mutually distinct boundary point $\omega_j$ of $U$ for every $j$. Here we assume that $\omega_1, \ldots, \omega_{2n-2}$ are located with this order (with respect to the counter-clockwise direction) on the boundary $\partial U$ of $U$.

**Lemma 3.6.** *The number of points in* $\pi_S^{-1}(\mathrm{S})$ *of* S *by* $\pi_S$ *is always*

$$(2n-2)!n^{n-3}.$$

**Proof.** For every point $(\mathbf{a}, \mathbf{b})$ in $\pi_S^{-1}(\mathrm{S})$, $f_{\mathbf{a},\mathbf{b}}$ gives a representative of the point $\iota((\mathbf{a}, \mathbf{b}))$ in $MH_nU$ over S. In other words, $f_{\mathbf{a},\mathbf{b}}$ gives an $n$-sheeted branched holomorphic covering of $\hat{\mathbb{C}}$ by $\hat{\mathbb{C}}$ with critical values S and ordered simple poles $b_1, \ldots, b_{n-1}, \infty$.

Recall that $f = f_{\mathbf{a},\mathbf{b}}$ also gives the branched covering of $U$ by $W_{\mathbf{a},\mathbf{b}}$. This covering can be reconstructed as follows: Set $D = U - \bigcup_{j=1}^{2n-2} \ell_j$. Then the preimage $f^{-1}(D)$ consists of $n$ domains $D_k$, the order of which is naturally defined as follows: Let $\gamma_k$ be the component of $f_{\mathbf{a},\mathbf{b}}^{-1}(\partial U)$ surrounding the $k$th pole. Then $D_k$ is the component whose boundary contains the part of $\gamma_k$ which is projected by $f_{\mathbf{a},\mathbf{b}}$ onto the subarc of $\partial U$ from $\omega_{2n-2}$ to $\omega_1$ (which contains no $\omega_j$).

Let $\ell_j^k$ be the "slit" on $D_k$ over $\ell_j$ (i.e., the part of the boundary corresponding to the preimage $f^{-1}(\ell_j)$ on $D_k$) for every $k$ and $j$. Then each $\ell_j^k$ is divided by some critical point into two arcs, which can be considered as two sides of the "slit" $\ell_j^k$. And for every $j$, there is a pair, say $\{D_{k(j)}, D_{k'(j)}\}$ such that sides of these "slits" are glued "crosswise" along $\ell_j^{k(j)}$ and $\ell_j^{k'(j)}$. (Here two sides of every other "slit" $\ell_j^k$ is glued trivially.) Hence we have a transposition $\sigma_j = (k(j)k'(j))$ of ordered $n$ sheets at $\ell_j$ when we move counter-clockwise along $\partial U$ for each $j$. Since $W_{\mathbf{a},\mathbf{b}}$ has exactly $n$ boundary components,

$$\sigma_{2n-2} \circ \cdots \circ \sigma_1$$

should be the identical permutation. And apply all such gluings as above, we can reconstruct the branched covering $f : W_{\mathbf{a},\mathbf{b}} \to U$.

Thus for every $(\mathbf{a}, \mathbf{b})$ in $\pi_S^{-1}(\mathrm{S})$ and with fixed cuts $\{\ell_j\}$, we have an ordered factorization of the identical permutation into $2n-2$ transpositions. And since $W_{\mathbf{a},\mathbf{b}}$ is connected, such transpositions generate the full symmetric group $S_n$.

Conversely, for every such an ordered factorization of the identical permutation, we can construct an $n$-sheeted branched covering of $\hat{\mathbb{C}}$ by itself, and hence also of $U$ by an $n$-connected domain $W$, having the set S as simple critical values. Then $W$ has $n$ boundary components, and hence by the argument in the proof of the main theorem in [6], we can find a point $(\mathbf{a}, \mathbf{b})$ in $\mathbf{B}_n - \Gamma$ such that $W_{\mathbf{a},\mathbf{b}}$ is biholomorphic to $W$ and $f_{\mathbf{a},\mathbf{b}}$ belongs to the isomorphism class of the covering projection of the above covering. In other words, $(\mathbf{a}, \mathbf{b}) \in \pi_S^{-1}(\mathrm{S})$. Also it is clear that different such factorizations give different branched covering structures, and hence different $(\mathbf{a}, \mathbf{b})$ in $\pi_S^{-1}(\mathrm{S})$ by Lemma 3.5.

On the other hand, it is known (cf. [5]) that the number of such (transitive minimal) ordered factorizations of the identical permutation on $\{1, \ldots, n\}$ into transpositions is

$$(2n-2)!n^{n-3},$$

which shows the assertion. $\square$

Finally, we have

**Lemma 3.7.** *$\pi_S$ is locally biholomorphic, and evenly covered (i.e., for every point* $\mathrm{S} \in B_{0,2n-2}U$*, there is a neighborhood $V$ of* S *such that every component of $\pi_S^{-1}(V)$ is biholomorphic to $V$).*

**Proof.** Fix a point S in $B_{0,2n-2}U$ and a point $(\mathbf{a}, \mathbf{b})$ in $\pi_S^{-1}(\mathrm{S})$ arbitrarily. Then it is classically well-known (or can be shown by a standard arguments in the quasiconformal deformation theory) that we can find a neighborhood $V$ of S and a holomorphic function $\phi$ of $V$ into $\mathbf{B}_n$ such that

$$\phi(\mathrm{S}) = (\mathbf{a}, \mathbf{b})$$

and for every $(\mathbf{a}', \mathbf{b}')$ in $\phi(V)$, $f_{\mathbf{a}',\mathbf{b}'}$ gives the same factorization of the identical permutation as $f_{\mathbf{a},\mathbf{b}}$ does. Here if $V$ is sufficiently small, we can consider the natural bijection between S and the set S$'$ of critical values of $f_{\mathbf{a}',\mathbf{b}'}$ for every $(\mathbf{a}', \mathbf{b}') \in \phi(V)$. And we take as the "slits" $\ell'_j$ for S$'$ the image of $\ell_j$ by a self-diffeomorphism of $U \cup \partial U$ which is the identity outside mutually disjoint simply connected, relatively compact, neighborhoods of each $\alpha_j$ in $U$ and induces the above bijection between S and S$'$.

Then from the construction, $\pi_S \circ \phi$ is the identity. And since the number of points in the preimage $\pi_S^{-1}(\mathrm{S})$ is a finite constant by above lemma, we conclude that $\pi_S$ is locally biholomorphic, and also evenly covered. $\Box$

Thus $\pi_S$ gives an unbranched $(2n-2)!n^{n-3}$-sheeted, holomorphic covering of $B_{0,2n-2}U$ by $\mathbf{B}_n - \Gamma$. In particular, it is proper, which completes the proof of Theorem 3.2.

**Example 3.8.** In the case $n = 3$, such ordered factorizations are

$$\big\{(pq), (pq), (pr), (pr)\big\},$$
$$\big\{(pq), (pr), (pr), (pq)\big\},$$
$$\big\{(pq), (pr), (rq), (pr)\big\},$$
$$\big\{(pq), (pr), (qp), (qr)\big\},$$

where we can take any bijection of $\{p, q, r\}$ to $\{1, 2, 3\}$. Hence we have 4! different ordered (transitive minimal) factorizations of the identical permutations on $\{1, 2, 3\}$.

## 4. Parametrization by the critical points

We call the set of all points in $(\mathbb{C}^*)^{n-1} \times F_{0,n-1}\mathbb{C}$ such that the corresponding rational function $f_{\mathbf{a},\mathbf{b}}$ has a non-simple critical point the *non-simple locus*, and we denote it by $\Delta$. Then $\Delta \subset \Gamma$, and for every point $(\mathbf{a}, \mathbf{b})$ in $(\mathbb{C}^*)^{n-1} \times F_{0,n-1}\mathbb{C} - \Delta$, we denote the set of simple critical points of $f_{\mathbf{a},\mathbf{b}}$ by

$$C_{\mathbf{a},\mathbf{b}} = \{c_1, \ldots, c_{2n-2}\}.$$

This set can be considered again as a point in the unordered configuration space $B_{0,2n-2}\mathbb{C}$. And we can define a holomorphic map

$$\pi_C : \left(\mathbb{C}^*\right)^{n-1} \times F_{0,n-1}\mathbb{C} - \Delta \to B_{0,2n-2}\mathbb{C}$$

by setting

$$\pi_C(\mathbf{a}, \mathbf{b}) = C_{\mathbf{a},\mathbf{b}}.$$

Similarly as in the previous section, we can show the following

**Theorem 4.1.** *For every point $C$ in $B_{0,2n-2}\mathbb{C}$, there are at most*

$$\frac{(2n-2)!}{n!}$$

*preimages of $C$ by $\pi_C$.*

The number

$$\frac{(2n-2)!}{n!(n-1)!}$$

is called the *$n$th Catalan number*. And the theorem can be shown by noting the following result.

**Lemma 4.2** [4,9]. *For every fixed $C$ in $B_{0,2n-2}\mathbb{C}$, there are*

$$\frac{(2n-2)!}{n!(n-1)!}$$

*classes of rational functions of degree $n$ which have $C$ as the set of critical points.*

Here two rational functions $f$ and $g$ are in the same class if there is a Möbius transformation $A$ such that

$$f = A \circ g.$$

But in this case, a class as above contains a rational function of the form

$$f(z) = z + \sum_{j=1}^{n-1} \frac{a_j}{z - b_j}$$

only if the image of $\infty$ is different from all critical values. And the image $\pi_C(\mathbf{B}_n - \Delta)$ seems to be mysterious.

**Example 4.3.** If we take

$$\left\{\pm\left(\sqrt[4]{3}/\sqrt{2}\right)(1 \pm i)\right\}$$

as the set of critical points (with $n = 3$), we have $2 = 4!/(3!2!)$ different classes determined by

$$f_1 = f_{-1,-1,1,-1} \; (= f_{-1,-1,-1,1})$$

and

$$f_2 = f_{1,1,i,-i}\ (= f_{1,1,-i,i}).$$

But since $|f_1((\sqrt[4]{3}/\sqrt{2})(1+i))| > 1$ and $|f_2((\sqrt[4]{3}/\sqrt{2})(1+i))| > 1$, for instance, all of $(-1,-1,1,-1)$, $(-1,-1,-1,1)$, $(1,1,i,-i)$, and $(1,1,-i,i)$ do not belong to $\mathbf{B}_3$.

On the other hand, if we take

$$\left\{\pm\frac{\sqrt{7}}{10},\ \pm\frac{\sqrt{2}}{20}i\right\}$$

as the set of critical points (with $n = 3$), we have 2 different classes determined by

$$f_1 = f_{a,a,b,-b}\ (= f_{a,a,-b,b}) \quad \text{with } a = \frac{9}{400},\ b = \frac{1}{10}$$

and

$$f_2 = f_{a',a',b',-b'}\ (= f_{a',a',-b',b'}) \quad \text{with } a' = \frac{1}{48},\ b' = \frac{1}{10}\sqrt{\frac{7}{6}}.$$

We already mentioned that $|f_1| < 1$ at each critical point in Example 2.10. Also $|f_2| < 1$ at each critical point and hence all of $(a,a,b,-b)$, $(a,a,-b,b)$, $(a',a',b',-b')$, and $(a',a',-b',b')$ belong to $\mathbf{B}_3$.

Next, the rational function

$$f(z) = z + \frac{6z-4}{(z-1)^2},$$

has $\{1,-1,2+\sqrt{3},2-\sqrt{3}\}$ as critical points. Here since $f(\infty) = f(1)$, the class of $f$ can contain no rational function $f_{\mathbf{a},\mathbf{b}}$ with $(\mathbf{a},\mathbf{b})$ belonging even to $(C^*)^2 \times F_{0,2}\mathbb{C}$.

Note that there is a natural smooth map of $\mathbf{B}_n$ to the (reduced) moduli space $M_{0,0,n}$ of a non-degenerate $n$-connected planar domain, which is a real $3n-6$ dimensional variety if $n > 2$. (Recall that two points $(\mathbf{a},\mathbf{b})$ and $(\mathbf{a}',\mathbf{b}')$ are mapped to the same point of $M_{0,0,n}$ if there is a biholomorphic map $F$ of $W_{\mathbf{a},\mathbf{b}}$ onto $W_{\mathbf{a}',\mathbf{b}'}$.)

Hence the preimages of a generic point of $M_{0,0,n}$, which we have called a *leaf* of the coefficient body $\mathbf{B}_n$ in [7], has a positive real dimension. And Theorems 3.2 and 4.1 imply the following

**Corollary 4.4.** *There is a non-trivial real parameter family* $\mathbf{F}$ *of Bell representations in a single leaf such that* $\mathbf{F}$ *is disjoint from* $\Gamma$*, and that* $\pi_C$ *and* $\pi_S$ *are injective on* $\mathbf{F}$.

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