

Integral inequalities for retarded Volterra equations

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Abstract

Integral inequalities are very useful in the qualitative analysis of differential and integral equations. Starting with [O. Lipovan, A retarded Gronwall-like inequality and its applications, J. Math. Anal. Appl. 252 (2000) 389–401], several recent investigations, see [O. Lipovan, A retarded integral inequality and its applications, J. Math. Anal. Appl. 285 (2003) 436–443; B.G. Pachpatte, Explicit bounds on certain integral inequalities, J. Math. Anal. Appl. 267 (2002) 48–61; B.G. Pachpatte, On some retarded integral inequalities and applications, J. Inequal. Pure Appl. Math. 3 (2002), Article 18; B.G. Pachpatte, On a certain retarded integral inequality and its applications, J. Inequal. Pure Appl. Math. 5 (2004), Article 19; B.G. Pachpatte, On some new nonlinear retarded integral inequalities, J. Inequal. Pure Appl. Math. 5 (2004), Article 80], were devoted to retarded integral inequalities. In this paper we consider the case of retarded Volterra integral equations. We establish bounds on the solutions and, by means of examples, we show the usefulness of our results in investigating the asymptotic behaviour of the solutions.

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1. The linear case

Theorem 1.1. *Let $k \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $a \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ with $(t, s) \mapsto \partial_t a(t, s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$. Assume in addition that α is nondecreasing and $\alpha(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

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$$u(t) \leq k(t) + \int_0^{\alpha(t)} a(t, s)u(s) ds, \quad t \geq 0, \quad (1.1)$$

then

$$u(t) \leq k(t) + e^{\int_0^{\alpha(t)} a(t, s) ds} \int_0^t e^{-\int_0^{\alpha(r)} a(r, s) ds} \partial_r \left(\int_0^{\alpha(r)} a(r, s)k(s) ds \right) dr, \quad t \geq 0. \quad (1.2)$$

Proof. Denote $z(t) = \int_0^{\alpha(t)} a(t, s)u(s) ds$. Our assumptions on a and α imply that z is nondecreasing on \mathbb{R}_+ . Hence, for $t \geq 0$, we have

$$\begin{aligned} z'(t) &= a(t, \alpha(t))u(\alpha(t))\alpha'(t) + \int_0^{\alpha(t)} \partial_t a(t, s)u(s) ds \\ &\leq a(t, \alpha(t))[k(\alpha(t)) + z(\alpha(t))]\alpha'(t) + \int_0^{\alpha(t)} \partial_t a(t, s)[k(s) + z(s)] ds \\ &\leq a[t, \alpha(t)][k(\alpha(t)) + z(t)]\alpha'(t) + \int_0^{\alpha(t)} \partial_t a(t, s)k(s) ds + z(t) \int_0^{\alpha(t)} \partial_t a(t, s) ds, \end{aligned}$$

or, equivalently,

$$z'(t) - z(t) \frac{d}{dt} \left(\int_0^{\alpha(t)} a(t, s) ds \right) \leq \frac{d}{dt} \left(\int_0^{\alpha(t)} a(t, s)k(s) ds \right).$$

Multiplying the above inequality by $e^{-\int_0^{\alpha(t)} a(t, s) ds}$, we get

$$\frac{d}{dt} (z(t) e^{-\int_0^{\alpha(t)} a(t, s) ds}) \leq e^{-\int_0^{\alpha(t)} a(t, s) ds} \frac{d}{dt} \left(\int_0^{\alpha(t)} a(t, s)k(s) ds \right).$$

Consider now the integral on the interval $[0, t]$ to obtain

$$z(t) \leq e^{\int_0^{\alpha(t)} a(t, s) ds} \int_0^t e^{-\int_0^{\alpha(r)} a(r, s) ds} \partial_r \left(\int_0^{\alpha(r)} a(r, s)k(s) ds \right) dr, \quad t \geq 0.$$

Combine the above inequality with $u(t) \leq k(t) + z(t)$ to get (1.2) and, with this, the proof is complete. \square

Corollary 1.1. Assume a, α are as in Theorem 1.1 and $k(t) \equiv k > 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies (1.1), then

$$u(t) \leq k e^{\int_0^{\alpha(t)} a(t, s) ds}, \quad t \geq 0. \quad (1.3)$$

Proof. Apply Theorem 1.1 to obtain

$$\begin{aligned} u(t) &\leq k + k e^{\int_0^{\alpha(t)} a(t,s) ds} \int_0^t e^{-\int_0^{\alpha(r)} a(r,s) ds} \partial_r \left(\int_0^{\alpha(r)} a(r,s) ds \right) dr \\ &= k + k e^{\int_0^{\alpha(t)} a(t,s) ds} (1 - e^{-\int_0^{\alpha(t)} a(t,s) ds}) = k e^{\int_0^{\alpha(t)} a(t,s) ds}, \quad t \geq 0. \quad \square \end{aligned}$$

Remark 1.1. We note that for $\partial_t a(t, s) \equiv 0$ in Corollary 1.1 we get an inequality obtained in [3]. If, in addition, $\alpha(t) = t$, the inequality given by Corollary 1.1 reduces to Gronwall's inequality [2].

Corollary 1.2. Let a, α be as in Theorem 1.1 and $k(t) \equiv k > 0$. Suppose $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a solution to the Volterra integral equation

$$u(t) = k + \int_0^{\alpha(t)} a(t, s) u(s) ds, \quad t \geq 0. \quad (1.4)$$

If $\lim_{t \rightarrow \infty} \int_0^{\alpha(t)} a(t, s) ds < \infty$, then u is bounded on \mathbb{R}_+ .

Proof. The conclusion follows immediately from Corollary 1.1. Note that the limit $\lim_{t \rightarrow \infty} \int_0^{\alpha(t)} a(t, s) ds$ always exists since the function $t \mapsto \int_0^{\alpha(t)} a(t, s) ds$ is nondecreasing on \mathbb{R}_+ . \square

Example. The function $a(t, s) = t/(1 + 2t + (1 + t)s^2)$, $t, s \geq 0$, satisfies the hypotheses in Corollary 1.2 for any nondecreasing $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with $\alpha(t) \leq t$, $t \geq 0$. In this case all solutions $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ of (1.4) are bounded.

Theorem 1.2. Let $a, b, k \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and assume that α is nondecreasing with $\alpha(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$u(t) \leq k(t) + a(t) \int_0^{\alpha(t)} b(s) u(s) ds, \quad t \geq 0, \quad (1.5)$$

then

$$u(t) \leq k(t) + a(t) \int_0^{\alpha(t)} e^{\int_r^{\alpha(t)} a(s) b(s) ds} b(r) k(r) dr, \quad t \geq 0. \quad (1.6)$$

Proof. Denote $z(t) = \int_0^{\alpha(t)} b(s) u(s) ds$. Then

$$\begin{aligned} z'(t) &= b(\alpha(t)) u(\alpha(t)) \alpha'(t) \leq b(\alpha(t)) [k(\alpha(t)) + a(\alpha(t)) z(\alpha(t))] \alpha'(t) \\ &\leq b(\alpha(t)) [k(\alpha(t)) + a(\alpha(t)) z(t)] \alpha'(t), \quad t \geq 0. \end{aligned}$$

Hence

$$z'(t) - z(t) b(\alpha(t)) a(\alpha(t)) \alpha'(t) \leq b(\alpha(t)) k(\alpha(t)) \alpha'(t).$$

Multiplying the above inequality by $e^{-\int_0^{\alpha(t)} a(s)b(s)ds}$, we get

$$\frac{d}{dt}(z(t)e^{-\int_0^{\alpha(t)} a(s)b(s)ds}) \leq e^{-\int_0^{\alpha(t)} a(s)b(s)ds} b(\alpha(t))k(\alpha(t))\alpha'(t), \quad t \geq 0.$$

Integrating on the interval $[0, t]$, we now deduce

$$\begin{aligned} z(t) &\leq e^{\int_0^{\alpha(t)} a(s)b(s)ds} \int_0^t e^{-\int_0^{\alpha(r)} a(s)b(s)ds} b(\alpha(r))k(\alpha(r))\alpha'(r) dr \\ &= \int_0^t e^{\int_{\alpha(r)}^{\alpha(t)} a(s)b(s)ds} b(\alpha(r))k(\alpha(r))\alpha'(r) dr \\ &= \int_0^{\alpha(t)} e^{\int_r^{\alpha(t)} a(s)b(s)ds} b(r)k(r) dr, \quad t \geq 0, \end{aligned}$$

after a change of variables performed in the last integral above. Now (1.6) follows by the above inequality together with $u(t) \leq k(t) + a(t)z(t)$. \square

Remark 1.2. Considering $\alpha(t) = t$ in Theorem 1.2, we obtain Morro's inequality [4].

Corollary 1.3. Let a, b, k, α be as in Theorem 1.2. Suppose $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a solution to the integral equation

$$u(t) = k(t) + a(t) \int_0^{\alpha(t)} b(s)u(s) ds, \quad t \geq 0.$$

If a, k are bounded on \mathbb{R}_+ and $\int_0^{\alpha(\infty)} b(s) ds < \infty$, then u is bounded on \mathbb{R}_+ .

Corollary 1.4. Let a, b, k, α be as in Theorem 1.2 with $k(t) \rightarrow 0$ as $t \rightarrow \infty$. Suppose $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a solution to the integral equation

$$u(t) = k(t) + a(t) \int_0^{\alpha(t)} b(s)u(s) ds, \quad t \geq 0.$$

If

$$\int_0^{\alpha(\infty)} a(s)b(s) ds < \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} a(t) \int_0^{\alpha(t)} b(r)k(r) dr = 0, \quad (1.7)$$

then $u(t) \rightarrow 0$ as $t \rightarrow \infty$. In particular, if $a(t), k(t) \rightarrow 0$ as $t \rightarrow \infty$ and $\int_0^{\alpha(\infty)} b(s) ds < \infty$, then $u(t) \rightarrow 0$ as $t \rightarrow \infty$.

Remark 1.3. To discuss the conditions in (1.7), we particularize $\alpha(t) = t$. The integral equation

$$u(t) = k(t) + a(t) \int_0^t b(s)u(s) ds, \quad t \geq 0,$$

has the exact solution

$$u(t) = k(t) + a(t) \int_0^t e^{\int_r^t a(s)b(s)ds} b(r)k(r)dr, \quad t \geq 0.$$

So, in order to have $u(t) \rightarrow 0$ as $t \rightarrow \infty$, both

$$\lim_{t \rightarrow \infty} k(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} a(t) \int_0^t b(r)k(r)dr = 0$$

must hold. Concerning the condition $\int_0^\infty a(s)b(s)ds < \infty$, the case $a(t) = k(t) = t^{-2}$, $b(t) = t^2$, shows that

$$\lim_{t \rightarrow \infty} k(t) = 0, \quad \lim_{t \rightarrow \infty} a(t) \int_0^t b(r)k(r)dr = 0 \quad \text{and} \quad \int_0^\infty a(s)b(s)ds = \infty,$$

can all hold simultaneously. Notice that in this setting, the solution equals

$$u(t) = (t+1)^{-2} + (e^t - 1)(t+1)^{-2} \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

This shows that both conditions in (1.7) are relevant.

2. The nonlinear case

Theorem 2.1. Let a, α be as in Theorem 1.1. Assume $k, w \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions with $k(0) > 0$, $w(t) > 0$ for $t > 0$ and $\int_1^\infty \frac{dt}{w(t)} = \infty$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$u(t) \leq k(t) + \int_0^{\alpha(t)} a(t,s)w(u(s))ds, \quad t \geq 0,$$

then

$$u(t) \leq G^{-1} \left(G(k(t)) + \int_0^{\alpha(t)} a(t,s)ds \right), \quad t \geq 0, \quad (2.8)$$

where $G(t) = \int_1^t \frac{ds}{w(s)}$, $t \geq 0$.

Proof. Let $T \geq 0$ be fixed and denote $z(t) = \int_0^{\alpha(t)} a(t,s)w(u(s))ds$, $t \geq 0$. Our assumptions on a, α imply that z is nondecreasing on \mathbb{R}_+ . Hence for $t \in [0, T]$ we have

$$\begin{aligned} z'(t) &= a(t, \alpha(t))w(u(\alpha(t)))\alpha'(t) + \int_0^{\alpha(t)} \partial_t a(t,s)w(u(s))ds \\ &\leq a(t, \alpha(t))\alpha'(t)w[k(\alpha(t)) + z(\alpha(t))] + \int_0^{\alpha(t)} \partial_t a(t,s)w[k(s) + z(s)]ds \end{aligned}$$

$$\begin{aligned} &\leq a(t, \alpha(t))\alpha'(t)w[k(\alpha(T)) + z(t)] + w[k(\alpha(T)) + z(t)] \int_0^{\alpha(t)} \partial_t a(t, s) ds \\ &\leq \left(a(t, \alpha(t))\alpha'(t) + \int_0^{\alpha(t)} \partial_t a(t, s) ds \right) w[k(T) + z(t)], \end{aligned}$$

and then

$$\frac{z'(t)}{w[k(T) + z(t)]} \leq \frac{d}{dt} \left(\int_0^{\alpha(t)} a(t, s) ds \right), \quad t \in [0, T]. \quad (2.9)$$

Integrating both sides of (2.9) on $[0, t]$, we get

$$G(k(T) + z(t)) \leq G(k(T)) + \int_0^{\alpha(t)} a(t, s) ds, \quad t \in [0, T],$$

or, equivalently,

$$k(T) + z(t) \leq G^{-1} \left[G(k(T)) + \int_0^{\alpha(t)} a(t, s) ds \right], \quad t \in [0, T]. \quad (2.10)$$

Note that the right-hand side of (2.10) is well defined as $G(\infty) = \infty$. Letting $t = T$ in the above relation, we obtain

$$u(T) \leq k(T) + z(T) \leq G^{-1} \left[G(k(T)) + \int_0^{\alpha(T)} a(T, s) ds \right],$$

and since $T \geq 0$ was arbitrarily chosen, we get (2.8). \square

Corollary 2.1. *Let a, k, α, w be as in Theorem 2.1. Suppose $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a solution to the nonlinear Volterra integral equation*

$$u(t) = k(t) + \int_0^{\alpha(t)} a(t, s)w(u(s)) ds, \quad t \geq 0. \quad (2.11)$$

If k is bounded and $\lim_{t \rightarrow \infty} \int_0^{\alpha(t)} a(t, s) ds < \infty$, then u is bounded.

Example. The functions $w(t) = (t + 1) \ln(t + 1)$, $k(t) \equiv k > 0$, $a(t, s) = t/(1 + 2t + (1 + t)e^s)$, $t, s \geq 0$, satisfy the hypotheses in Corollary 2.1 for any nondecreasing $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with $\alpha(t) \leq t$, $t \geq 0$. In this case all solutions $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ of (2.11) are bounded.

Theorem 2.2. Let $a, b, k \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and assume that a, k, α are nondecreasing functions with $\alpha(t) \leq t$ for $t \geq 0$. Let also $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function such that $w(t) > 0$ for $t > 0$ and $\int_1^\infty \frac{dt}{w(t)} = \infty$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$u(t) \leq k(t) + a(t) \int_0^{\alpha(t)} b(s)w(u(s)) ds, \quad t \geq 0, \quad (2.12)$$

then

$$u(t) \leq G^{-1} \left(G(k(t)) + a(t) \int_0^{\alpha(t)} b(s) ds \right), \quad t \geq 0, \quad (2.13)$$

where $G(t) = \int_1^t \frac{ds}{w(s)}$, $t \geq 0$.

Proof. Let $T \geq 0$ be fixed. Then for $t \in [0, T]$, relation (2.12) together with our hypotheses on a, k imply

$$u(t) \leq k(T) + a(T) \int_0^{\alpha(t)} b(s)w(u(s)) ds. \quad (2.14)$$

By the retarded version of Bihari's inequality (see [3]), relation (2.14) implies

$$u(t) \leq G^{-1} \left[G(k(T)) + a(T) \int_0^{\alpha(t)} b(s) ds \right], \quad t \in [0, T].$$

Now let $t = T$ in the above relation to obtain

$$u(T) \leq G^{-1} \left[G(k(T)) + a(T) \int_0^{\alpha(T)} b(s) ds \right],$$

and since $T \geq 0$ was arbitrarily chosen, we get (2.13). \square

Corollary 2.2. Let a, b, k, w, α be as in Theorem 2.2. Suppose $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a solution to the integral equation

$$u(t) = k(t) + a(t) \int_0^{\alpha(t)} b(s)w(u(s)) ds, \quad t \geq 0.$$

If a, k are bounded on \mathbb{R}_+ and $\int_0^{\alpha(\infty)} b(s) ds < \infty$, then u is bounded on \mathbb{R}_+ .

3. Reversed inequalities

Using step by step similar arguments to those in the proofs of Theorems 1.2, 2.2, one obtains

Theorem 3.1. Let $a, b, k \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and assume that α is nondecreasing with $\alpha(t) \geq t$ for $t \geq 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$u(t) \geq k(t) + a(t) \int_0^{\alpha(t)} b(s)u(s) ds, \quad t \geq 0,$$

then

$$u(t) \geq k(t) + a(t) \int_0^{\alpha(t)} e^{\int_r^{\alpha(t)} a(s)b(s) ds} b(r)k(r) dr, \quad t \geq 0.$$

Corollary 3.1. Assume a, b, k, α are as in Theorem 3.1. Suppose $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a solution to the integral equation

$$u(t) = k(t) + a(t) \int_0^{\alpha(t)} b(s)u(s) ds, \quad t \geq 0.$$

Then each of the following conditions is sufficient for u to be unbounded:

- (i) a is unbounded and $b, k, \alpha \not\equiv 0$;
- (ii) $\limsup_{t \rightarrow \infty} a(t) > 0$ and $\int_0^\infty b(s)k(s) ds = \infty$.

Theorem 3.2. Let $a, b, k \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with a, k are nonincreasing on \mathbb{R}_+ . Suppose α is nondecreasing and $\alpha(t) \geq t$ for $t \geq 0$. Let also $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function such that $w(t) > 0$ for $t > 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$u(t) \geq k(t) + a(t) \int_0^{\alpha(t)} b(s)w(u(s)) ds, \quad t \geq 0,$$

then

$$u(t) \geq G^{-1} \left(G(k(\alpha(t))) + a(t) \int_0^{\alpha(t)} b(s) ds \right), \quad t_1 \geq t \geq 0,$$

where $G(t) = \int_1^t \frac{ds}{w(s)}$, $t \geq 0$, and t_1 is chosen so that $G(k(\alpha(t))) + a(t) \int_0^{\alpha(t)} b(s) ds \in \text{Dom}(G^{-1})$, for all $t \in [0, t_1]$.

Setting $a(t) \equiv 1$, $k(t) \equiv k > 0$ in Theorem 3.2, we obtain the following inequality, which may be regarded as a reverse version of Bihari's inequality [1].

Corollary 3.2. Consider $k > 0$, $b \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and suppose α is nondecreasing and $\alpha(t) \geq t$ for $t \geq 0$. Let also $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function such that $w(t) > 0$ for $t > 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$u(t) \geq k + \int_0^{\alpha(t)} b(s)w(u(s)) ds, \quad t \geq 0,$$

then

$$u(t) \geq G^{-1} \left(G(k) + \int_0^{\alpha(t)} b(s) ds \right), \quad t_1 \geq t \geq 0,$$

where $G(t) = \int_1^t \frac{ds}{w(s)}$, $t \geq 0$, and t_1 is chosen so that $G(k) + \int_0^{\alpha(t)} b(s) ds \in \text{Dom}(G^{-1})$, for all $t \in [0, t_1]$.

Corollary 3.3. Assume k, b, α, w, G are as in Corollary 3.2. Suppose in addition $G(\infty) = \int_1^\infty \frac{dt}{w(t)} = L < \infty$. Let $u \in C([0, t_0], \mathbb{R}_+)$ be a solution to the integral equation

$$u(t) = k + \int_0^{\alpha(t)} b(s)w(u(s)) ds, \quad t \geq 0.$$

Suppose also that $[0, t_0)$ is the maximal interval of existence for u . If $T = \inf\{t \geq 0: G(k) + \int_0^{\alpha(t)} b(s) ds \geq L \text{ exists and is finite, then } t_0 \leq T$.

Proof. Suppose T exists and is finite and the maximal existence time t_0 satisfies $t_0 > T$. Take now $t < T$. Then $0 \leq G(k) + \int_0^{\alpha(t)} b(s) ds < L$ and hence $G(k) + \int_0^{\alpha(t)} b(s) ds \in \text{Dom}(G^{-1})$. By Corollary 3.2, we get

$$u(t) \geq G^{-1} \left(G(k) + \int_0^{\alpha(t)} b(s) ds \right), \quad 0 \leq t < T.$$

Letting $t \rightarrow T$ in the above relation, we deduce $\lim_{t \rightarrow T} u(t) \geq G^{-1}(L) = \infty$, which contradicts our assumption $t_0 > T$. \square

Example. Put $k = 1$, $b(t) = 1/(t + 1)$, $\alpha(t) = 2t$ and $w(t) = t^2$ in Corollary 3.3. For these choices we obtain $G(\infty) = 1$, $G(k) = 0$ and hence $T = (e - 1)/2$. Thus $t_0 \leq (e - 1)/2$.

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