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Positive periodic solutions of periodic neutral Lotka–Volterra system with state dependent delays ☆

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Abstract

By using a fixed point theorem of strict-set-contraction, some new criteria are established for the existence of positive periodic solutions of the following periodic neutral Lotka–Volterra system with state dependent delays

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = x_i(t)\Bigg[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) - \sum_{j=1}^{n} b_{ij}(t)x_j\big(t-\tau_{ij}\big(t,x_1(t),\dots,x_n(t)\big)\big)\\ - \sum_{j=1}^{n} c_{ij}(t)x_j'\big(t-\sigma_{ij}\big(t,x_1(t),\dots,x_n(t)\big)\big)\Bigg], \quad i=1,2,\dots,n,\end{aligned}$$

where $r_i, a_{ij}, b_{ij}, c_{ij} \in C(\mathbb{R},\mathbb{R}^+)$ $(i,j=1,2,\dots,n)$ are ω-periodic functions and $\tau_{ij},\sigma_{ij} \in C(\mathbb{R}^{n+1},\mathbb{R})$ $(i=1,2,\dots,n)$ are ω-periodic functions with respect to their first arguments, respectively.

Keywords: Positive periodic solution; Neutral functional differential equation; Lotka–Volterra system; k-Set contraction

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1. Introduction

Recently, by using the continuation theorem developed by Gaines and Mawhin [1], Yang and Cao [2] studied the existence of positive periodic solutions for the following neutral Lotka–Volterra system with delays

$$\frac{\mathrm{d}N_i(t)}{\mathrm{d}t} = N_i(t)\Bigg[a_i(t) - \sum_{j=1}^{n}\beta_{ij}(t)N_j(t) - \sum_{j=1}^{n} b_{ij}(t)N_j\big(t-\tau_{ij}(t)\big) - \sum_{j=1}^{n} c_{ij}(t)N'_{ij}\big(t-\gamma_{ij}(t)\big)\Bigg], \quad i=1,2,\ldots,n, \tag{1.1}$$

where $a_i, \beta_{ij}, b_{ij}, c_{ij}, \tau_{ij}, \gamma_{ij}$ are nonnegative continuous ω-periodic functions. However, the verification of the important assumption that the operator $N: X \to X$ is L-compact is incomplete. The reason why the verification is incomplete is the same as that pointed out by Lu and Ge in [3]. Also, by using an existence theorem for neutral functional differential equations developed in [4,5], Fang [6] studied the existence of positive periodic solutions of (1.1). Under the transformation $N_i(t) = e^{x_i(t)}$, $i = 1,2,\ldots,n$, Fang first rewrote the above equation in the following form

$$\begin{aligned}&\left[x_i(t) + \sum_{j=1}^{n}\frac{c_{ij}(t)}{1-\tau'_{ij}(t)}e^{x_j(t-\tau_{ij}(t))}\right]' \\ &\quad = a_i(t) - \sum_{j=1}^{n}\beta_{ij}(t)e^{x_j(t)} - \sum_{j=1}^{n}\left(b_{ij}(t) - \left(\frac{c_{ij}(t)}{1-\tau'_{ij}(t)}\right)'\right)e^{x_j(t-\tau_{ij}(t))}, \\ &\quad i=1,2,\ldots,n,\end{aligned} \tag{1.2}$$

and made use of the existence theorem to obtain the existence of at least one periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^{\mathrm{T}}$ of (1.2). Then he claimed that $N^*(t) = (e^{x_1^*(t)}, e^{x_2^*(t)}, \ldots, e^{x_n^*(t)})^{\mathrm{T}}$ is a positive periodic solution of (1.1). Unfortunately, according to his proof, he only proved that (1.2) has at least one continuous periodic solution $x^*(t)$ satisfying that

$$x_i^*(t) + \sum_{j=1}^{n}\frac{c_{ij}(t)}{1-\tau'_{ij}(t)}e^{x_j^*(t-\tau_{ij}(t))}, \quad i=1,2,\ldots,n,$$

are differentiable. However, in general, $x^*(t)$ is not differentiable. So, $N^*(t)$ is not necessarily a solution of (1.1).

The main purpose of this paper is by using a fixed point theorem of strict-set-contraction to establish new criteria to guarantee the existence of positive periodic solutions of the following neutral Lotka–Volterra system with state dependent delays

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = x_i(t)\Bigg[&r_i(t) - \sum_{j=1}^{n}a_{ij}(t)x_j(t) - \sum_{j=1}^{n}b_{ij}(t)x_j\big(t-\tau_{ij}\big(t,x_1(t),\ldots,x_n(t)\big)\big) \\ &- \sum_{j=1}^{n}c_{ij}(t)x'_j\big(t-\sigma_{ij}\big(t,x_1(t),\ldots,x_n(t)\big)\big)\Bigg], \quad i=1,2,\ldots,n,\end{aligned} \tag{1.3}$$

where $r_i, a_{ij}, b_{ij}, c_{ij} \in C(\mathbb{R}, \mathbb{R}^+)$ $(i, j = 1, 2, \dots, n)$ are ω-periodic functions and $\tau_{ij}, \sigma_{ij} \in C(\mathbb{R}^{n+1}, \mathbb{R})$ $(i = 1, 2, \dots, n)$ are ω-periodic functions with respect to their first arguments, respectively. Obviously, (1.1) is a special case of system (1.3).

For convenience, we introduce the notation

$$\delta_i := e^{-\int_0^\omega r_i(s)\,\mathrm{d}s}, \quad i = 1, 2, \dots, n,$$

$$M_{ij} = \int_0^\omega \left[\delta_j a_{ij}(s) + \delta_j b_{ij}(s) - c_{ij}(s)\right] \mathrm{d}s, \quad i = 1, 2, \dots, n,\ j = 1, 2, \dots, m,$$

$$\hat{M}_{ij} = \int_0^\omega \left[a_{ij}(s) + b_{ij}(s) + c_{ij}(s)\right] \mathrm{d}s, \quad i = 1, 2, \dots, n,\ j = 1, 2, \dots, m,$$

$$f^M = \max_{t\in[0,\omega]}\left\{f(t)\right\}, \qquad f^m = \min_{t\in[0,\omega]}\left\{f(t)\right\},$$

where f is a continuous ω-periodic function.

Throughout this paper, we assume that:

(H$_1$) $$\delta_i := e^{-\int_0^\omega r_i(s)\,\mathrm{d}s} < 1, \quad i = 1, 2, \dots, n.$$

(H$_2$) $$\delta_j a_{ij}(t) + \delta_j b_{ij}(t) - c_{ij}(t) \geqslant 0, \quad i = 1, 2, \dots, n,\ j = 1, 2, \dots, m.$$

(H$_3$) $$\left(1 + r_i^m\right)\frac{\delta_i^2 M_{ij}}{1 - \delta_i} \geqslant \max_{t\in[0,\omega]}\left\{a_{ij}(t) + b_{ij}(t) + c_{ij}(t)\right\},$$
$$i = 1, 2, \dots, n,\ j = 1, 2, \dots, m.$$

(H$_4$) $$\frac{\hat{M}_{ij}(r_i^M - 1)}{\delta_i(1 - \delta_i)} \leqslant \min_{t\in[0,\omega]}\left\{\delta_j a_{ij}(t) + \delta_j b_{ij}(t) - c_{ij}(t)\right\},$$
$$i = 1, 2, \dots, n,\ j = 1, 2, \dots, m.$$

(H$_5$) $$\left[\max_{1\leqslant i\leqslant n}\left\{\frac{1 - \delta_i}{\delta_i^2 \min_{1\leqslant j\leqslant n}\{M_{ij}\}}\right\}\right]\left[\max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^n c_{ij}^M\right\}\right] < 1.$$

2. Preliminaries

In order to obtain the existence of a periodic solution of system (1.3), we first make the following preparations:

Let E be a Banach space and K be a cone in E. The semi-order induced by the cone K is denoted by "$\leqslant$." That is, $x \leqslant y$ if and only if $y - x \in K$. We also use the notations $\nleqslant$ and $\ngeqslant$ that mean $x \nleqslant y$ if and only if $y - x \notin K$ and $x \ngeqslant y$ if and only if $x - y \notin K$, respectively. In addition, for a bounded subset $A \subset E$, let $\alpha_E(A)$ denote the (Kuratowski) measure of non-compactness defined by

$$\alpha_E(A) = \inf\left\{\delta > 0\text{: there is a finite number of subsets } A_i \subset A \text{ such that } A = \bigcup_i A_i \text{ and } \operatorname{diam}(A_i) \leqslant \delta\right\},$$

where $\operatorname{diam}(A_i)$ denotes the diameter of the set A_i.

Let E, F be two Banach spaces and $D \subset E$, a continuous and bounded map $\Phi : D \to F$ is called k-set contractive if for any bounded set $S \subset D$ we have

$$\alpha_F\big(\Phi(S)\big) \leqslant k\alpha_E(S).$$

Φ is called strict-set-contractive if it is k-set-contractive for some $0 \leqslant k < 1$.

The following lemma cited from Refs. [7,8] which is useful for the proof of our main results of this paper.

Lemma 2.1. [7,8] *Let K be a cone of the real Banach space X and $K_{r,R} = \{x \in K\colon r \leqslant \|x\| \leqslant R\}$ with $R > r > 0$. Suppose that $\Phi : K_{r,R} \to K$ is strict-set-contractive such that one of the following two conditions is satisfied*:

(i) $\Phi x \nleqslant x,\ \forall x \in K,\ \|x\| = r$ *and* $\Phi x \ngeqslant x,\ \forall x \in K,\ \|x\| = R$.
(ii) $\Phi x \ngeqslant x,\ \forall x \in K,\ \|x\| = r$ *and* $\Phi x \nleqslant x,\ \forall x \in K,\ \|x\| = R$.

Then Φ has at least one fixed point in $K_{r,R}$.

In order to apply Lemma 2.1 to system (1.3), we set

$$C_\omega^0 = \big\{x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}\colon\ x \in C^0\big(\mathbb{R}, \mathbb{R}^n\big),\ x(t+\omega) = x(t)\big\}$$

with the norm defined by $\|x\| = \sum_{i=1}^n |x_i|_0$ where $|x_i|_0 = \max_{t\in[0,\omega]}\{|x_i(t)|\}$, $i = 1, 2, \ldots, n$, and

$$C_\omega^1 = \big\{x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}\colon\ x \in C^1\big(\mathbb{R}, \mathbb{R}^n\big),\ x(t+\omega) = x(t)\big\}$$

with the norm defined by $\|x\|_1 = \max\{\|x\|, \|x'\|\}$. Then C_ω^0, C_ω^1 are all Banach spaces. Define the cone K in C_ω^1 by

$$K = \big\{x\colon\ x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in C_\omega^1,\ x_j(t) \geqslant \delta_j |x_j|_1,\ t \in [0, \omega],\ j = 1, 2, \ldots, n\big\}, \tag{2.1}$$

where $|x_j|_1 = \max_{t\in[0,\omega]}\{|x_j(t)|, |x_j'(t)|\}$, $j = 1, 2, \ldots, n$.

Let the map Φ be defined by

$$(\Phi x)(t) = \big((\Phi_1 x)(t), (\Phi_2 x)(t), \ldots, (\Phi_n x)(t)\big)^{\mathrm{T}}, \tag{2.2}$$

where $x \in K$, $t \in \mathbb{R}$,

$$(\Phi_i x)(t) = \int_t^{t+\omega} G_i(t,s) x_i(s) \Bigg[\sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) x_j\big(s - \tau_{ij}\big(s, x_1(s), \ldots, x_n(s)\big)\big) + \sum_{j=1}^n c_{ij}(s) x_j'\big(s - \sigma_{ij}\big(s, x_1(s), \ldots, x_n(s)\big)\big)\Bigg]\,\mathrm{d}s$$

and

$$G_i(t,s) = \frac{e^{-\int_t^s r_i(\theta)\,\mathrm{d}\theta}}{1 - e^{-\int_0^\omega r_i(\theta)\,\mathrm{d}\theta}}, \quad s \in [t, t+\omega],\ i = 1, 2, \ldots, n.$$

It is easy to see that $G_i(t+\omega, s+\omega) = G_i(t,s)$ and

$$\frac{\delta_i}{1-\delta_i} \leqslant G_i(t,s) \leqslant \frac{1}{1-\delta_i}, \quad s \in [t, t+\omega],\ i = 1, 2, \ldots, n.$$

In the following, we will give some lemmas concerning K and Φ defined by (2.1) and (2.2), respectively.

Lemma 2.2. *Assume that* (H_1)–(H_3) *hold.*

(i) *If* $\max\{r_i^M,\ i=1,2,\dots,n\} \leqslant 1$, *then* $\Phi : K \to K$ *is well defined.*
(ii) *If* (H_4) *holds and* $\min\{r_i^M,\ i=1,2,\dots,n\} > 1$, *then* $\Phi : K \to K$ *is well defined.*

Proof. For any $x \in K$, it is clear that $\Phi x \in C^1(\mathbb{R},\mathbb{R})$. In view of (2.2), for $t \in \mathbb{R}$, we obtain

$$\begin{aligned}
(\Phi_i x)(t+\omega) &= \int_{t+\omega}^{t+2\omega} G_i(t+\omega,s)x_i(s)\Bigg[\sum_{j=1}^n a_{ij}(s)x_j(s) \\
&\quad + \sum_{j=1}^n b_{ij}(s)x_j\big(s-\tau_{ij}\big(s,x_1(s),\dots,x_n(s)\big)\big) \\
&\quad + \sum_{j=1}^n c_{ij}(s)x_j'\big(s-\sigma_{ij}\big(s,x_1(s),\dots,x_n(s)\big)\big)\Bigg]\,\mathrm{d}s \\
&= \int_{t}^{t+\omega} G_i(t+\omega,u+\omega)x_i(u+\omega)\Bigg[\sum_{j=1}^n a_{ij}(u+\omega)x_j(u+\omega) \\
&\quad + \sum_{j=1}^n b_{ij}(u+\omega)x_j\big(u+\omega-\tau_{ij}\big(u+\omega,x_1(u+\omega),\dots,x_n(u+\omega)\big)\big) \\
&\quad + \sum_{j=1}^n c_{ij}(u+\omega)x_j'\big(u+\omega-\sigma_{ij}\big(u+\omega,x_1(u+\omega),\dots,x_n(u+\omega)\big)\big)\Bigg]\,\mathrm{d}u \\
&= \int_{t}^{t+\omega} G_i(t,u)x_i(u)\Bigg[\sum_{j=1}^n a_{ij}(u)x_j(u) \\
&\quad + \sum_{j=1}^n b_{ij}(u)x_j\big(u-\tau_{ij}\big(u,x_1(u),\dots,x_n(u)\big)\big) \\
&\quad + \sum_{j=1}^n c_{ij}(u)x_j'\big(u-\sigma_{ij}\big(u,x_1(u),\dots,x_n(u)\big)\big)\Bigg]\,\mathrm{d}u \\
&= (\Phi_i x)(t), \quad i=1,2,\dots,n.
\end{aligned}$$

That is, $(\Phi x)(t+\omega) = (\Phi x)(t)$, $t \in \mathbb{R}$. So $\Phi x \in C_\omega^1$. In view of (H_2), for $x \in K$, $t \in [0,\omega]$, we have

$$\begin{aligned}
&\sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j\big(t-\tau_{ij}\big(u,x_1(t),\dots,x_n(t)\big)\big) \\
&\quad + \sum_{j=1}^n c_{ij}(t)x_j'\big(t-\sigma_{ij}\big(t,x_1(t),\dots,x_n(t)\big)\big)
\end{aligned}$$

$$\begin{aligned}
&\geqslant \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j\big(t-\tau_{ij}\big(u, x_1(t), \ldots, x_n(t)\big)\big) \\
&\quad - \sum_{j=1}^{n} c_{ij}(t)\big|x_j'\big(t-\sigma_{ij}\big(t, x_1(t), \ldots, x_n(t)\big)\big)\big| \\
&\geqslant \sum_{j=1}^{n} \delta_j a_{ij}(t)|x_j|_1 + \sum_{j=1}^{n} \delta_j b_{ij}(t)|x_j|_1 - \sum_{j=1}^{n} c_{ij}(t)|x_j|_1 \\
&= \sum_{j=1}^{n} \big[\delta_j a_{ij}(t) + \delta_j b_{ij}(t) - c_{ij}(t)\big]|x_j|_1 \\
&\geqslant 0, \quad i = 1, 2, \ldots, n. \qquad (2.3)
\end{aligned}$$

Therefore, for $x \in K$, $t \in [0, \omega]$, we find

$$\begin{aligned}
|\Phi_i x|_0 \leqslant \frac{1}{1-\delta_i} \int_0^{\omega} x_i(s)\Bigg[&\sum_{j=1}^{n} a_{ij}(s)x_j(s) + \sum_{j=1}^{n} b_{ij}(s)x_j\big(s-\tau_{ij}\big(s, x_1(s), \ldots, x_n(s)\big)\big) \\
&+ \sum_{j=1}^{n} c_{ij}(s)x_j'\big(s-\sigma_{ij}\big(s, x_1(s), \ldots, x_n(s)\big)\big)\Bigg]\,\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
(\Phi_i x)(t) &\geqslant \frac{\delta_i}{1-\delta_i} \int_t^{t+\omega} x_i(s)\Bigg[\sum_{j=1}^{n} a_{ij}(s)x_j(s) + \sum_{j=1}^{n} b_{ij}(s)x_j\big(s-\tau_{ij}\big(s, x_1(s), \ldots, x_n(s)\big)\big) \\
&\quad + \sum_{j=1}^{n} c_{ij}(s)x_j'\big(s-\sigma_{ij}\big(s, x_1(s), \ldots, x_n(s)\big)\big)\Bigg]\,\mathrm{d}s \\
&= \frac{\delta_i}{1-\delta_i} \int_0^{\omega} x_i(s)\Bigg[\sum_{j=1}^{n} a_{ij}(s)x_j(s) + \sum_{j=1}^{n} b_{ij}(s)x_j\big(s-\tau_{ij}\big(s, x_1(s), \ldots, x_n(s)\big)\big) \\
&\quad + \sum_{j=1}^{n} c_{ij}(s)x_j'\big(s-\sigma_{ij}\big(s, x_1(s), \ldots, x_n(s)\big)\big)\Bigg]\,\mathrm{d}s \\
&\geqslant \delta_i|\Phi_i x|_0, \quad i = 1, 2, \ldots, n. \qquad (2.4)
\end{aligned}$$

Now, we show that $(\Phi_i x)'(t) \geqslant \delta_i|(\Phi_i x)'|_0$, $t \in [0, \omega]$. From (2.2), we have

$$\begin{aligned}
(\Phi_i x)'(t) = G_i(t, t+\omega)x_i(t+\omega)\Bigg[&\sum_{j=1}^{n} a_{ij}(t+\omega)x_j(t+\omega) \\
&+ \sum_{j=1}^{n} b_{ij}(t)x_j\big(t+\omega-\tau_{ij}\big(t, x_1(t+\omega), \ldots, x_n(t+\omega)\big)\big) \\
&+ \sum_{j=1}^{n} c_{ij}(t)x_j'\big(t+\omega-\sigma_{ij}\big(t, x_1(t+\omega), \ldots, x_n(t+\omega)\big)\big)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&- G_i(t,t)x_i(t)\Bigg[\sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j\big(t-\tau_{ij}\big(t,x_1(t),\dots,x_n(t)\big)\big) \\
&+ \sum_{j=1}^{n} c_{ij}(t)x'_j\big(t-\sigma_{ij}\big(t,x_1(t),\dots,x_n(t)\big)\big)\Bigg] + r_i(t)(\Phi_i x)(t) \\
={}& r_i(t)(\Phi_i x)(t) - x_i(t)\Bigg[\sum_{j=1}^{n} a_{ij}(t)x_j(t) \\
&+ \sum_{j=1}^{n} b_{ij}(t)x_j\big(t-\tau_{ij}\big(t,x_1(t),\dots,x_n(t)\big)\big) \\
&+ \sum_{j=1}^{n} c_{ij}(t)x'_j\big(t-\sigma_{ij}\big(t,x_1(t),\dots,x_n(t)\big)\big)\Bigg], \quad i=1,2,\dots,n. \qquad (2.5)
\end{aligned}
$$

It follows from (2.3) and (2.5) that if $(\Phi_i x)'(t) \geqslant 0$, $i=1,2,\dots,n$, then

$$
(\Phi_i x)'(t) \leqslant r_i(t)(\Phi_i x)(t) \leqslant r_i^M(\Phi_i x)(t) \leqslant (\Phi_i x)(t), \quad i=1,2,\dots,n. \qquad (2.6)
$$

On the other hand, from (2.4), (2.5) and (H_3), if $(\Phi_i x)'(t) < 0$, $i=1,2,\dots,n$, then

$$
\begin{aligned}
&-(\Phi_i x)'(t) \\
&= x_i(t)\Bigg[\sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j\big(t-\tau_{ij}\big(t,x_1(t),\dots,x_n(t)\big)\big) \\
&\quad + \sum_{j=1}^{n} c_{ij}(t)x'_j\big(t-\sigma_{ij}\big(t,x_1(t),\dots,x_n(t)\big)\big)\Bigg] - r_i(t)(\Phi_i x)(t) \\
&\leqslant |x_i|_1 \sum_{j=1}^{n}\big[a_{ij}(t)+b_{ij}(t)+c_{ij}(t)\big]|x_j|_1 - r_i^m(\Phi_i x)(t) \\
&\leqslant \big(1+r_i^m\big)\frac{\delta_i^2}{1-\delta_i}|x_i|_1 \sum_{j=1}^{n}\left\{\int_0^{\omega}\big[\delta_j a_{ij}(s)+\delta_j b_{ij}(s)-c_{ij}(s)\big]|x_j|_1\,\mathrm{d}s\right\} - r_i^m(\Phi_i x)(t) \\
&= \big(1+r_i^m\big)\int_t^{t+\omega}\frac{\delta_i}{1-\delta_i}\delta_i|x_i|_1\sum_{j=1}^{n}\big[\delta_j a_{ij}(s)+\delta_j b_{ij}(s)-c_{ij}(s)\big]|x_j|_1\,\mathrm{d}s - r_i^m(\Phi_i x)(t) \\
&\leqslant \big(1+r_i^m\big)\int_t^{t+\omega} G_i(t,s)x_i(s)\Bigg[\sum_{j=1}^{n} a_{ij}(s)x_j(s) \\
&\quad + \sum_{j=1}^{n} b_{ij}(t)x_j\big(s-\tau_{ij}\big(s,x_1(s),\dots,x_n(s)\big)\big) \\
&\quad - \sum_{j=1}^{n} c_{ij}(t)\big|x'_j\big(s-\sigma_{ij}\big(t,x_1(s),\dots,x_n(s)\big)\big)\big|\Bigg]\,\mathrm{d}s - r_i^m(\Phi_i x)(t)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \left(1+r_i^m\right) \int_t^{t+\omega} G_i(t,s)x_i(s)\Biggl[\sum_{j=1}^n a_{ij}(s)x_j(s) \\
&\quad + \sum_{j=1}^n b_{ij}(t)x_j\bigl(s-\tau_{ij}\bigl(s,x_1(s),\ldots,x_n(s)\bigr)\bigr) \\
&\quad + \sum_{j=1}^n c_{ij}(t)x_j'\bigl(s-\sigma_{ij}\bigl(t,x_1(s),\ldots,x_n(s)\bigr)\bigr)\Biggr]\,\mathrm{d}s - r_i^m(\Phi_i x)(t) \\
&= \left(1+r_i^m\right)(\Phi_i x)(t) - r_i^m(\Phi_i x)(t) \\
&= (\Phi_i x)(t), \quad i=1,2,\ldots,n.
\end{aligned} \tag{2.7}
$$

It follows from (2.6) and (2.7) that $|(\Phi_i x)'|_0 \leqslant |\Phi_i x|_0$, $i=1,2,\ldots,n$. So $|\Phi_i x|_1 = |\Phi_i x|_0$, $i=1,2,\ldots,n$. By (2.4) we have $(\Phi_i x)(t) \geqslant \delta_i |\Phi_i x|_1$, $i=1,2,\ldots,n$. Hence, $\Phi x \in K$. The proof of (i) is complete.

(ii) In view of the proof of (i), we only need to prove that $(\Phi_i x)'(t) \geqslant 0$, $i=1,2,\ldots,n$, imply

$$
(\Phi_i x)'(t) \leqslant (\Phi_i x)(t), \quad i=1,2,\ldots,n.
$$

From (2.3), (2.5), (H_2) and (H_4), we obtain

$$
\begin{aligned}
(\Phi_i x)'(t) &\leqslant r_i(t)(\Phi_i x)(t) - \delta_i |x_i|_1 \Biggl[\sum_{j=1}^n a_{ij}(t)x_j(t) \\
&\quad + \sum_{j=1}^n b_{ij}(t)x_j\bigl(t-\tau_{ij}\bigl(t,x_1(t),\ldots,x_n(t)\bigr)\bigr) \\
&\quad - \sum_{j=1}^n c_{ij}(t)\bigl|x_j'\bigl(t-\tau_{ij}\bigl(t,x_1(t),\ldots,x_n(t)\bigr)\bigr)\bigr|\Biggr] \\
&\leqslant r_i(t)(\Phi_i x)(t) - \delta_i |x_i|_1 \sum_{j=1}^n \bigl[\delta_j a_{ij}(t) + \delta_j b_{ij}(t) - c_{ij}(t)\bigr]|x_j|_1 \\
&\leqslant r_i^M(\Phi_i x)(t) - \delta_i |x_i|_1 \frac{r_i^M - 1}{\delta_i(1-\delta_i)} \sum_{j=1}^n \int_0^\omega \bigl[a_{ij}(s) + b_{ij}(s) + c_{ij}(s)\bigr]|x_j|_1\,\mathrm{d}s \\
&\leqslant r_i^M(\Phi_i x)(t) - \left(r_i^M - 1\right) \int_t^{t+\omega} \frac{1}{1-\delta_i}|x_i|_1 \\
&\quad \times \sum_{j=1}^n \bigl[|x_j|_1 a_{ij}(s) + |x_j|_1 b_{ij}(s) + |x_j|_1 c_{ij}(s)\bigr]\,\mathrm{d}s \\
&\leqslant r_i^M(\Phi_i x)(t) - \left(r_i^M - 1\right) \int_t^{t+\omega} G_i(t,s)x_i(s)\Biggl[\sum_{j=1}^n a_{ij}(s)x_j(s) \\
&\quad + \sum_{j=1}^n b_{ij}(s)x_j\bigl(s-\tau_{ij}\bigl(s,x_1(s),\ldots,x_n(s)\bigr)\bigr)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} c_{ij}(s)\left|x_j'\left(s-\sigma_{ij}\left(s, x_1(s), \ldots, x_n(s)\right)\right)\right|\Bigg] \mathrm{d}s \\
&\leqslant r_i^M (\Phi_i x)(t) - \left(r_i^M - 1\right) \int_t^{t+\omega} G_i(t,s) x_i(s) \Bigg[\sum_{j=1}^{n} a_{ij}(s) x_j(s) \\
&\quad + \sum_{j=1}^{n} b_{ij}(s) x_j\left(s-\tau_{ij}\left(s, x_1(s), \ldots, x_n(s)\right)\right) \\
&\quad + \sum_{j=1}^{n} c_{ij}(s) x_j'\left(s-\sigma_{ij}\left(s, x_1(s), \ldots, x_n(s)\right)\right)\Bigg] \mathrm{d}s \\
&= r_i^M (\Phi_i x)(t) - \left(r_i^M - 1\right)(\Phi_i x)(t) \\
&= (\Phi_i x)(t), \quad i = 1, 2, \ldots, n.
\end{aligned}
$$

The proof of (ii) is complete. □

Lemma 2.3. *Assume that* (H_1)–(H_3) *hold and* $R \max_{1 \leqslant i \leqslant n}\{\sum_{j=1}^{n} c_{ij}^M\} < 1$.

(i) *If* $\max\{r_i^M,\ i = 1, 2, \ldots, n\} \leqslant 1$, *then* $\Phi: K \cap \bar{\Omega}_R \to K$ *is strict-set-contractive.*
(ii) *If* (H_4) *holds and* $\min\{r_i^M,\ i = 1, 2, \ldots, n\} > 1$, *then* $\Phi: K \cap \bar{\Omega}_R \to K$ *is strict-set-contractive.*

Here $\Omega_R = \{x \in C_\omega^1\colon \|x\|_1 < R\}$.

Proof. We only need to prove (i), since the proof of (ii) is similar. It is easy to see that Φ is continuous and bounded. Now we prove that $\alpha_{C_\omega^1}(\Phi(S)) \leqslant (R \max_{1 \leqslant i \leqslant n}\{\sum_{j=1}^{n} c_{ij}^M\}) \alpha_{C_\omega^1}(S)$ for any bounded set $S \subset \bar{\Omega}_R$. Let

$$\eta = \alpha_{C_\omega^1}(S).$$

Then, for any positive number $\varepsilon < (R \max_{1 \leqslant i \leqslant n}\{\sum_{j=1}^{n} c_{ij}^M\})\eta$, there is a finite family of subsets $\{S_i\}$ satisfying $S = \bigcup_i S_i$ with $\operatorname{diam}(S_i) \leqslant \eta + \varepsilon$. Therefore

$$\|x - y\|_1 \leqslant \eta + \varepsilon \quad \text{for any } x, y \in S_i. \tag{2.8}$$

As S and S_i are precompact in C_ω^0, it follows that there is a finite family of subsets $\{S_{ij}\}$ of S_i such that $S_i = \bigcup_j S_{ij}$ and

$$\|x - y\| \leqslant \varepsilon \quad \text{for any } x, y \in S_{ij}. \tag{2.9}$$

In addition, for any $x \in S$ and $t \in [0, \omega]$, we have

$$
\begin{aligned}
\left|(\Phi_i x)(t)\right| = \int_t^{t+\omega} G_i(t,s) x_i(s) \Bigg[&\sum_{j=1}^{n} a_{ij}(s) x_j(s) \\
&+ \sum_{j=1}^{n} b_{ij}(s) x_j\left(s-\tau_{ij}\left(s, x_1(s), \ldots, x_n(s)\right)\right)
\end{aligned}
$$

$$+\sum_{j=1}^{n} c_{ij}(t)x_j'\big(s-\tau_{ij}\big(s,x_1(s),\ldots,x_n(s)\big)\big)\Bigg]\,\mathrm{d}s$$

$$\leqslant \frac{R^2}{1-\delta_i}\int_0^{\omega}\sum_{j=1}^{n}\big[a_{ij}(s)+b_{ij}(s)+c_{ij}(s)\big]\,\mathrm{d}s := H_i$$

and

$$\begin{aligned}\big|(\Phi_i x)'(t)\big| = \Bigg| & r_i(t)(\Phi_i x)(t) - x_i(t)\Bigg[\sum_{j=1}^{n} a_{ij}(t)x_j(t) \\ & + \sum_{j=1}^{n} b_{ij}(t)x_j\big(t-\tau_{ij}\big(t,x_1(t),\ldots,x_n(t)\big)\big) \\ & + \sum_{j=1}^{n} c_{ij}(t)x_j'\big(t-\tau_{ij}\big(t,x_1(t),\ldots,x_n(t)\big)\big)\Bigg]\Bigg| \\ \leqslant{} & r_i^M H_i + R^2\sum_{j=1}^{n}\big(a_{ij}^M+b_{ij}^M+c_{ij}^M\big), \quad i=1,2,\ldots,n.\end{aligned}$$

Hence,

$$\big\|(\Phi x)\big\| \leqslant \sum_{i=1}^{n} H_i$$

and

$$\big\|(\Phi x)'\big\| \leqslant \sum_{i=1}^{n}\big(r_i^M H_i\big) + R^2\sum_{i=1}^{n}\sum_{j=1}^{n}\big(a_{ij}^M+b_{ij}^M+c_{ij}^M\big).$$

Applying the Arzela–Ascoli theorem, we know that $\Phi(S)$ is precompact in C_ω^0. Then, there is a finite family of subsets $\{S_{ijk}\}$ of S_{ij} such that $S_{ij} = \bigcup_k S_{ijk}$ and

$$\|\Phi x - \Phi y\| \leqslant \varepsilon \quad \text{for any } x, y \in S_{ijk}. \tag{2.10}$$

From (2.3), (2.5) and (2.8)–(2.10) and (H_2), for any $x, y \in S_{ijk}$, we obtain

$$\begin{aligned}&\big\|(\Phi_i x)' - (\Phi_i y)'\big\| \\ &\quad = \max_{t\in[0,\omega]}\Bigg\{\Bigg| r_i(t)(\Phi_i x)(t) - r_i(t)(\Phi_i y)(t) \\ &\qquad - x_i(t)\Bigg[\sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j\big(t-\tau_{ij}\big(t,x_1(t),\ldots,x_n(t)\big)\big) \\ &\qquad + \sum_{j=1}^{n} c_{ij}(t)x_j'\big(t-\tau_{ij}\big(t,x_1(t),\ldots,x_n(t)\big)\big)\Bigg] \\ &\qquad + y_i(t)\Bigg[\sum_{j=1}^{n} a_{ij}(t)y_j(t) + \sum_{j=1}^{n} b_{ij}(t)y_j\big(t-\tau_{ij}\big(t,y_1(t),\ldots,y_n(t)\big)\big)\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{j=1}^{n} c_{ij}(t) y_j'\big(t - \tau_{ij}\big(t, y_1(t), \ldots, y_n(t)\big)\big)\Bigg]\Bigg|\Bigg\}\\
&\leqslant \max_{t\in[0,\omega]} \big\{\big|r_i(t)\big[(\Phi_i x)(t) - (\Phi_i y)(t)\big]\big|\big\}\\
&\quad + \max_{t\in[0,\omega]} \Bigg\{\Bigg|x_i(t)\Bigg[\sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j\big(t - \tau_{ij}\big(t, x_1(t), \ldots, x_n(t)\big)\big)\\
&\quad + \sum_{j=1}^{n} c_{ij}(t) x_j'\big(t - \tau_{ij}\big(t, x_1(t), \ldots, x_n(t)\big)\big)\Bigg]\\
&\quad - y_i(t)\Bigg[\sum_{j=1}^{n} a_{ij}(t)y_j(t) + \sum_{j=1}^{n} b_{ij}(t)y_j\big(t - \tau_{ij}\big(t, y_1(t), \ldots, y_n(t)\big)\big)\\
&\quad + \sum_{j=1}^{n} c_{ij}(t) y_j'\big(t - \tau_{ij}\big(t, y_1(t), \ldots, y_n(t)\big)\big)\Bigg]\Bigg|\Bigg\}\\
&\leqslant r_i^M \big|(\Phi_i x) - (\Phi_i y)\big|_0\\
&\quad + \max_{t\in[0,\omega]} \Bigg\{\Bigg|x_i(t)\Bigg\{\Bigg[\sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j\big(t - \tau_{ij}\big(t, x_1(t), \ldots, x_n(t)\big)\big)\\
&\quad + \sum_{j=1}^{n} c_{ij}(t) x_j'\big(t - \tau_{ij}\big(t, x_1(t), \ldots, x_n(t)\big)\big)\Bigg]\\
&\quad - \Bigg[\sum_{j=1}^{n} a_{ij}(t)y_j(t) + \sum_{j=1}^{n} b_{ij}(t)y_j\big(t - \tau_{ij}\big(t, y_1(t), \ldots, y_n(t)\big)\big)\\
&\quad + \sum_{j=1}^{n} c_{ij}(t) y_j'\big(t - \tau_{ij}\big(t, y_1(t), \ldots, y_n(t)\big)\big)\Bigg]\Bigg\}\Bigg|\Bigg\}\\
&\quad + \max_{t\in[0,\omega]} \Bigg\{\Bigg|\Bigg[\sum_{j=1}^{n} a_{ij}(t)y_j(t) + \sum_{j=1}^{n} b_{ij}(t)y_j\big(t - \tau_{ij}\big(t, y_1(t), \ldots, y_n(t)\big)\big)\\
&\quad + \sum_{j=1}^{n} c_{ij}(t) y_j'\big(t - \tau_{ij}\big(t, y_1(t), \ldots, y_n(t)\big)\big)\Bigg]\big[x_i(t) - y_i(t)\big]\Bigg|\Bigg\}\\
&\leqslant r_i^M \varepsilon + |x_i|_0 \max_{t\in[0,\omega]} \Bigg\{\sum_{j=1}^{n}\big[a_{ij}(t)\big|x_i(t) - y_i(t)\big|\\
&\quad + b_{ij}(t)\big|x_j\big(t - \tau_{ij}\big(t, x_1(t), \ldots, x_n(t)\big)\big) - y_j\big(t - \tau_{ij}\big(t, y_1(t), \ldots, y_n(t)\big)\big)\big|\\
&\quad + c_{ij}(t)\big|x_j'\big(t - \tau_{ij}\big(t, x_1(t), \ldots, x_n(t)\big)\big) - y_j'\big(t - \tau_{ij}\big(t, y_1(t), \ldots, y_n(t)\big)\big)\big|\big]\Bigg\}\\
&\quad + \max_{t\in[0,\omega]} \Bigg\{\Bigg|\Bigg[\sum_{j=1}^{n} a_{ij}(t)y_j(t) + \sum_{j=1}^{n} b_{ij}(t)y_j\big(t - \tau_{ij}\big(t, y_1(t), \ldots, y_n(t)\big)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{j=1}^{n} c_{ij}(t) y_j'\big(t - \tau_{ij}\big(t, y_1(t), \ldots, y_n(t)\big)\big)\Bigg]\big[x_i(t) - y_i(t)\big]\Bigg|\Bigg\} \\
&\leqslant r_i^M \varepsilon + R\left(\sum_{j=1}^{n}\big[a_{ij}^M + b_{ij}^M\big]\right)\varepsilon + |x_i|_0\left(\sum_{j=1}^{n} c_{ij}^M\right)(\eta + \varepsilon) \\
&\quad + R\left(\sum_{j=1}^{n}\big[a_{ij}^M + b_{ij}^M + c_{ij}^M\big]\right)\varepsilon \\
&= |x_i|_0\left(\sum_{j=1}^{n} c_{ij}^M\right)\eta + \hat{H}_i \varepsilon,
\end{aligned}
\tag{2.11}
$$

where

$$
\hat{H}_i = r_i^M + 2R\sum_{j=1}^{n} a_{ij}^M + 2R\sum_{j=1}^{n} b_{ij}^M + 2R\sum_{j=1}^{n} c_{ij}^M, \quad i = 1, 2, \ldots, n.
$$

From (2.10) and (2.11) we have

$$
\begin{aligned}
\|\Phi x - \Phi y\|_1 &\leqslant \left(\sum_{i=1}^{n} |x_i|_0 \sum_{j=1}^{n} c_{ij}^M\right)\eta + \varepsilon\sum_{i=1}^{n} \hat{H}_i \\
&= R\max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n} c_{ij}^M\right\}\eta + \varepsilon\sum_{i=1}^{n} \hat{H}_i \quad \text{for any } x, y \in S_{ijk}.
\end{aligned}
$$

As ε is arbitrary small, it follows that

$$
\alpha_{C_\omega^1}\big(\Phi(S)\big) \leqslant \left(R\max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n} c_{ij}^M\right\}\right)\alpha_{C_\omega^1}(S).
$$

Therefore, Φ is strict-set-contractive. The proof of Lemma 2.3 is complete. □

3. Main result

Our main result of this paper is as follows:

Theorem 3.1. *Assume that* (H_1)–(H_3), (H_5) *hold.*

(i) *If* $\max\{r_i^M,\ i = 1, 2, \ldots, n\} \leqslant 1$, *then system* (1.3) *has at least one positive* ω*-periodic solution.*
(ii) *If* (H_4) *holds and* $\min\{r_i^M,\ i = 1, 2, \ldots, n\} > 1$, *then system* (1.3) *has at least one positive* ω*-periodic solution.*

Proof. We only need to prove (i), since the proof of (ii) is similar. Let $R = \max_{1\leqslant i\leqslant n}\{\frac{1-\delta_i}{\delta_i^2 \min_{1\leqslant j\leqslant n}\{M_{ij}\}}\}$ and $0 < r < \min_{1\leqslant i\leqslant n}\{\frac{\delta_i(1-\delta_i)}{\max_{1\leqslant j\leqslant n}\{\hat{M}_{ij}\}}\}$. Then we have $0 < r < R$. From Lemmas 2.2 and 2.3, we know that Φ is strict-set-contractive on $K_{r,R}$. In view of (2.5), we see that if there exists $x^* \in K$ such that $\Phi x^* = x^*$, then x^* is one positive ω-periodic solution of system (1.3). Now, we shall prove that condition (ii) of Lemma 2.1 holds.

First, we prove that $\Phi x \ngeqslant x$, $\forall x \in K$, $\|x\|_1 = r$. Otherwise, there exists $x \in K$, $\|x\|_1 = r$ such that $\Phi x \geqslant x$. So $\|x\| > 0$ and $\Phi x - x \in K$, which implies that

$$(\Phi_i x)(t) - x_i(t) \geqslant \delta_i |\Phi_i x - x_i|_1 \geqslant 0 \quad \text{for any } t \in [0,\omega],\ i = 1,2,\ldots,n. \tag{3.1}$$

Moreover, for $t \in [0,\omega]$, we have

$$\begin{aligned}(\Phi_i x)(t) &= \int_t^{t+\omega} G_i(t,s)x_i(s)\Bigg[\sum_{j=1}^n a_{ij}(s)x_j(s) + \sum_{j=1}^n b_{ij}(t)x_j\big(s-\tau_{ij}\big(s,x_1(s),\ldots,x_n(s)\big)\big)\\ &\quad + \sum_{j=1}^n c_{ij}(s)x_j'\big(s-\sigma_{ij}\big(s,x_1(s),\ldots,x_n(s)\big)\big)\Bigg]\mathrm{d}s\\ &\leqslant \frac{1}{1-\delta_i}|x_i|_0 \sum_{j=1}^n \int_0^\omega \big[a_{ij}(s)+b_{ij}(s)+c_{ij}(t)\big]|x_j|_1\,\mathrm{d}s\\ &\leqslant \frac{1}{1-\delta_i}\max_{1\leqslant j\leqslant n}\{\hat{M}_{ij}\}|x_i|_0 \sum_{j=1}^n |x_j|_1\\ &= \frac{\max_{1\leqslant j\leqslant n}\{\hat{M}_{ij}\}}{1-\delta_i} r|x_i|_0\\ &< \delta_i|x_i|_0, \quad i=1,2,\ldots,n. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we find

$$\|x\| \leqslant \|\Phi x\| < \max_{1\leqslant i\leqslant n}\{\delta_i\}\|x\| < \|x\|,$$

which is a contradiction. Finally, we prove that $\Phi x \nleqslant x$, $\forall x \in K$, $\|x\|_1 = R$ also holds. For this case, we only need to prove that

$$\Phi x \nless x, \quad x \in K,\ \|x\|_1 = R.$$

Suppose, for the sake of contradiction, that there exists $x \in K$ and $\|x\|_1 = R$ such that $\Phi x < x$. Thus $x - \Phi x \in K \setminus \{0\}$. Furthermore, for any $t \in [0,\omega]$, we have

$$x_i(t) - (\Phi_i x)(t) \geqslant \delta_i |x_i - \Phi_i x|_1 > 0, \quad i = 1,2,\ldots,n. \tag{3.3}$$

In addition, for any $t \in [0,\omega]$, we find

$$\begin{aligned}(\Phi_i x)(t) &= \int_t^{t+\omega} G_i(t,s)x_i(s)\Bigg[\sum_{j=1}^n a_{ij}(s)x_j(s) + \sum_{j=1}^n b_{ij}(s)x_j\big(s-\tau_{ij}\big(s,x_1(s),\ldots,x_n(s)\big)\big)\\ &\quad + \sum_{j=1}^n c_{ij}(s)x_j'\big(s-\sigma_{ij}\big(s,x_1(s),\ldots,x_n(s)\big)\big)\Bigg]\mathrm{d}s\\ &\geqslant \frac{\delta_i}{1-\delta_i}\delta_i|x_i|_1 \sum_{j=1}^n |x_j|_1 \int_0^\omega \big[\delta_j a_{ij}(s)+\delta_j b_{ij}(s)-c_{ij}(s)\big]\,\mathrm{d}s\\ &= \frac{\delta_i^2}{1-\delta_i}|x_i|_1 \sum_{j=1}^n M_{ij}|x_j|_1, \quad i=1,2,\ldots,n. \end{aligned}$$

Thus,

$$\begin{aligned}
\|\Phi x\| &= \sum_{i=1}^{n} \left|(\Phi_i x)\right|_0 \\
&\geqslant \sum_{i=1}^{n} \frac{\delta_i^2}{1-\delta_i} |x_i|_1 \sum_{j=1}^{n} M_{ij} |x_j|_1 \\
&\geqslant \sum_{i=1}^{n} \frac{\delta_i^2 \min_{1\leqslant j\leqslant n}\{M_{ij}\}}{1-\delta_i} |x_i|_1 \sum_{j=1}^{n} |x_j|_1 \\
&\geqslant \min_{1\leqslant i\leqslant n} \left\{ \frac{\delta_i^2 \min_{1\leqslant j\leqslant n}\{M_{ij}\}}{1-\delta_i} \right\} \sum_{i=1}^{n} |x_i|_1 \sum_{j=1}^{n} |x_j|_1 \\
&\geqslant \min_{1\leqslant i\leqslant n} \left\{ \frac{\delta_i^2 \min_{1\leqslant j\leqslant n}\{M_{ij}\}}{1-\delta_i} \right\} R^2 = R. \qquad (3.4)
\end{aligned}$$

From (3.3) and (3.4), we obtain

$$\|x\| > \|\Phi x\| \geqslant R,$$

which is a contradiction. Therefore, condition (ii) of Lemma 2.1 holds. By Lemma 2.1, we see that Φ has at least one nonzero fixed point in $K_{r,R}$. Therefore, system (1.3) has at least one positive ω-periodic solution. The proof of Theorem 3.1 is complete. □

Example. Consider the following system

$$\begin{cases}
x_1'(t) = x_1(t)\left[\dfrac{1+\cos t}{4\pi} - (5-2\sin t)x_2(t) - (2+\sin t)x_1\bigl(t-\tau_{11}(t)\bigr) \right. \\
\qquad \left. - \dfrac{1-\sin t}{20} x_2'\bigl(t-\sigma_{12}(t)\bigr)\right], \\
x_2'(t) = x_2(t)\left[\dfrac{1-\sin t}{20} - (3-2\cos t)x_1(t) - (4-\sin t)x_2\bigl(t-\tau_{21}(t)\bigr) \right. \\
\qquad \left. - \dfrac{2-\cos t}{10} x_1\bigl(t-\sigma_{21}(t)\bigr)\right],
\end{cases} \qquad (3.5)$$

where $\tau_{11}, \tau_{21}, \sigma_{12}, \sigma_{21} \in C(\mathbb{R}, \mathbb{R})$ are 2π-periodic functions. Obviously,

$$\begin{aligned}
&r_1(t) = \frac{1+\cos t}{4\pi}, \qquad r_2(t) = \frac{1-\sin t}{20}, \qquad a_{11}(t) = 0, \qquad a_{12}(t) = 5-2\sin t, \\
&a_{21}(t) = 3-2\cos t, \qquad a_{22}(t) = 0, \qquad b_{11}(t) = 2+\sin t, \qquad b_{12}(t) = 0, \\
&b_{21}(t) = 0, \qquad b_{22}(t) = 4-\sin t, \qquad c_{11}(t) = 0, \qquad c_{12}(t) = \frac{1-\sin t}{20}, \\
&c_{21}(t) = \frac{2-\cos t}{10}, \qquad c_{22}(t) = 0, \qquad r_1^m = 0, \qquad r_2^m = 0.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\delta_1 = e^{-\frac{1}{2}} < 1, \qquad \delta_2 = e^{-\frac{\pi}{10}} < 1, \qquad M_{11} = 4\pi\delta_1 = 7.6219, \\
&M_{12} = 10\pi\delta_2 - \frac{\pi}{10} = 22.946, \qquad M_{21} = 6\pi\delta_1 - \frac{2\pi}{5} = 10.176,
\end{aligned}$$

$$M_{22} = 8\pi\delta_2 = 18.357,$$
$$\min_{1\leqslant j\leqslant 2}\{M_{1j}\} = M_{11} = 7.6219, \qquad \min_{1\leqslant j\leqslant 2}\{M_{2j}\} = M_{21} = 10.176,$$
$$\delta_1 a_{11}(t) + \delta_1 b_{11}(t) - c_{11}(t) = \delta_1 b_{11}(t) = \delta_1(2 + \sin t) > 0,$$
$$\delta_1 a_{21}(t) + \delta_1 b_{21}(t) - c_{21}(t) = \delta_1(3 - 2\cos t) - \frac{2 - \cos t}{10} \geqslant e^{-\frac{1}{2}} - \frac{1}{10} > 0,$$
$$\delta_2 a_{12}(t) + \delta_2 b_{12}(t) - c_{12}(t) = 5\delta_2 - \frac{1}{20} - \left(2\delta_2 - \frac{1}{20}\right)\sin t \geqslant 3\delta_2 > 0,$$
$$\delta_2 a_{22}(t) + \delta_2 b_{22}(t) - c_{22}(t) = \delta_2(4 - \sin t) \geqslant 0,$$
$$\left(1 + r_1^m\right)\frac{\delta_1^2 M_{11}}{1 - \delta_1} > \pi > 3 = \max_{t\in[0,2\pi]}\left\{a_{11}(t) + b_{11}(t) + c_{11}(t)\right\},$$
$$\left(1 + r_1^m\right)\frac{\delta_1^2 M_{12}}{1 - \delta_1} > 3\pi > 7.1 = \max_{t\in[0,2\pi]}\left\{a_{12}(t) + b_{12}(t) + c_{12}(t)\right\},$$
$$\left(1 + r_2^m\right)\frac{\delta_2^2 M_{21}}{1 - \delta_2} > 2\pi > 5.3 = \max_{t\in[0,2\pi]}\left\{a_{21}(t) + b_{21}(t) + c_{21}(t)\right\},$$
$$\left(1 + r_2^m\right)\frac{\delta_2^2 M_{22}}{1 - \delta_2} > 5\pi > 5 = \max_{t\in[0,2\pi]}\left\{a_{22}(t) + b_{22}(t) + c_{22}(t)\right\},$$
$$\left\{\frac{1 - \delta_1}{\delta_1^2 \min_{1\leqslant j\leqslant n}\{M_{ij}\}}\right\} = \frac{1 - e^{-\frac{1}{2}}}{4\pi e^{-\frac{3}{2}}} = 0.14033,$$
$$\left\{\frac{1 - \delta_2}{\delta_2^2 \min_{1\leqslant j\leqslant n}\{M_{ij}\}}\right\} = \frac{1 - e^{-\frac{\pi}{10}}}{e^{-\frac{\pi}{5}}(6\pi e^{-\frac{1}{2}} - \frac{2\pi}{5})} = 0.04966,$$
$$c_{11}^M + c_{12}^M = c_{12}^M = \frac{1}{10} < 1, \qquad c_{21}^M + c_{22}^M = c_{22}^M = \frac{3}{10} < 1.$$

Hence

$$\left[\max_{1\leqslant i\leqslant 2}\left\{\frac{1 - \delta_i}{\delta_i^2 \min_{1\leqslant j\leqslant 2}\{M_{ij}\}}\right\}\right]\left[\max_{1\leqslant i\leqslant 2}\left\{\sum_{j=1}^{2} c_{ij}^M\right\}\right] < 1.$$

Therefore, (H_1)–(H_3), (H_5) hold and $r_i^M \leqslant 1$, $i = 1, 2$. According to Theorem 3.1, system (3.5) has at least one positive 2π-periodic solution.

References

[1] R.E. Gaines, J.L. Mawhin, Coincide Degree and Nonlinear Differential Equations, Lecture Notes in Math., vol. 586, Springer-Verlag, Berlin, 1977.

[2] Z. Yang, J. Cao, Positive periodic solutions of neutral Lotka–Volterra system with periodic delays, Appl. Math. Comput. 149 (2004) 661–687.

[3] S. Lu, W. Ge, Existence of positive periodic solutions for neutral population model with multiple delays, Appl. Math. Comput. 153 (2004) 885–902.

[4] L. Erbe, W. Krawcewicz, J. Wu, A composite coincidence degree with applications to boundary value problems of neutral equations, Trans. Amer. Math. Soc. 335 (1993) 459–478.

[5] W. Krawcewicz, J. Wu, Theory of Degrees with Applications to Bifurcations and Differential Equations, Wiley, New York, 1996.

[6] H. Fang, Positive periodic solutions of N-species neutral delay systems, Czechoslovak Math. J. 53 (128) (2003) 561–570.
[7] N.P. Các, J.A. Gatica, Fixed point theorems for mappings in ordered Banach spaces, J. Math. Anal. Appl. 71 (1979) 547–557.
[8] D. Guo, Positive solutions of nonlinear operator equations and its applications to nonlinear integral equations, Adv. Math. 13 (1984) 294–310 (in Chinese).