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Inequalities for formal power series and entire functions

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Abstract

We present several integral and exponential inequalities for formal power series and for both arbitrary entire functions of exponential type and generalized Borel transforms. They are obtained through certain limit procedures which involve the multiparameter binomial inequalities, integral inequalities for continuous functions, and weighted norm inequalities for analytic functions. Some applications to the confluent hypergeometric functions, Bessel functions, Laguerre polynomials, and trigonometric functions are discussed. Also some generalizations are given.

Keywords: Integral inequalities; Convolutions; Hadamard product; Formal power series; Entire functions of exponential type; Borel-associated functions; Integral operators; Confluent hypergeometric functions; Bessel functions; Laguerre polynomials; Exponential inequalities; Trigonometric inequalities

1. Introduction

We present several inequalities of a new type and overview the related results and applications. These inequalities are expressed in terms of the formal power series and entire functions. They are established through certain limit procedures which involve three interrelated theorems on the general inequalities (Theorems A–C in Section 2). The idea to link the classical Cauchy (or Cauchy–Schwartz) inequality to the Euler's gamma and beta functions is important for the whole subject. Applications of our inequalities comprise the generalized hypergeometric series, special functions and orthogonal polynomials, integral and coefficient convolutions, entire functions, generalized Borel transforms, fractional integrals, bi-Hermitian forms, and conformal mappings [5–9].

The results obtained in this paper can be regarded as the lim–lim and lim–lim–lim inequalities. The main of them is presented in Theorem D (Section 3). It is a non-trivial limit case of the weighted norm inequalities established in [8] (see Theorem C) and implied by the general inequality for continuous functions which is proved in [6] (the extremal functions are found in [9]; see Theorem B). In its turn, the general inequality in [6] is obtained as a limit case of the multiparameter binomial inequalities given in [5, Theorem 1] (see Theorem A). A simple limit procedure allows us to show that Theorem D leads to some lim–lim–lim inequalities for formal power series and for both arbitrary entire functions of exponential type and generalized Borel transforms. These inequalities are presented in Corollaries 1–3

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(Section 3). Then we consider some examples and applications which involve the confluent hypergeometric functions, Bessel functions, Laguerre polynomials, trigonometric functions, and Euler's divergent series (Section 4). Finally, some generalizations are discussed (Section 5).

The open disk $\{z\colon |z| < r\}$ is denoted by D_r $(r > 0)$ throughout the paper. Let $h(z) = \sum_{n=0}^{\infty} c_n z^n$ be an entire function of exponential type, i.e. $\overline{\lim}_{n\to\infty} n|c_n|^{1/n} < \infty$. The function h is of exponential type $\sigma < \infty$ if the limit above equals $e\sigma$ (if h is a polynomial or if its order is less than 1, then h is of exponential type 0 according to this definition; cf. [10, Chapter 9]). We use the standard notation for the Hadamard product or the coefficient convolution of h and any power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (see, e.g., [13])

$$(f * h)(z) = \sum_{n=0}^{\infty} a_n c_n z^n.$$

The Cauchy–Hadamard theorem implies that $f * h$ is an entire function of exponential type if f is analytic in some open disk D_r. For a fixed h, the convolution $f * h$ can be considered as a linear operator on the class of all formal power series about $z = 0$. In the sequel we use ${}_1F_1(1;\alpha;z)$ and the limit function $\lim_{\alpha\to 0} \alpha[{}_1F_1(1;\alpha;z) - 1] = ze^z$ as the fixed functions h. Here and below ${}_1F_1(a;b;z)$ stands for the confluent hypergeometric function

$${}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \cdot \frac{z^n}{n!}$$

provided that none of the shifted factorials $(b)_n = b\cdots(b+n-1)$ $(n \geqslant 1)$ are equal to 0. The basic information about the hypergeometric and related functions can be found in [1,4].

Given a power series $f(z) = a_0 + a_1 z + \cdots$, the α-convolution $f_{*\alpha}$ $(\alpha > 0)$ and the 0-convolution f_{*0} are defined by the formulas [8]

$$f_{*\alpha}(z) = f(z) * {}_1F_1(1;\alpha;z) = \sum_{n=0}^{\infty} \frac{a_n}{(\alpha)_n} z^n \tag{1}$$

and

$$f_{*0}(z) = f(z) * \left(ze^z\right) = \sum_{n=1}^{\infty} \frac{a_n}{(n-1)!} z^n. \tag{2}$$

2. Theorems on general inequalities

Theorem A presents a parametrized inequality for two complex vectors and binomial weights. It is proved by a method of recurrence relations which involves the binomial coefficients $d_n(\alpha)$ defined by the formula

$$d_n(\alpha) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!} = \frac{(\alpha)_n}{n!} \quad (n = 0, 1, \ldots). \tag{3}$$

Theorem A. *(See [5].) Let* $\mathbf{a} = (a_0, \ldots, a_n)$ *and* $\mathbf{b} = (b_0, \ldots, b_n)$ *be non-zero complex vectors* $(n = 1, 2, \ldots)$. *Then for any numbers* $\alpha, \beta > 0$, *and* $\lambda \geqslant 0$, *the following inequality holds*:

$$d_n(\lambda+\alpha+\beta) \sum_{k=0}^{n} \frac{d_{n-k}(\lambda)}{d_k(\alpha+\beta)} \left| \sum_{l=0}^{k} a_l b_{k-l} \right|^2 \leqslant \sum_{k=0}^{n} \frac{d_{n-k}(\lambda+\beta)}{d_k(\alpha)} |a_k|^2 \cdot \sum_{k=0}^{n} \frac{d_{n-k}(\lambda+\alpha)}{d_k(\beta)} |b_k|^2, \tag{4}$$

where coefficients d_k *are defined by* (3).

For $\lambda > 0$, *the equality in* (4) *holds if and only if* $a_k = \eta^k d_k(\alpha) a_0$ *and* $b_k = \eta^k d_k(\beta) b_0$ $(|\eta| = 1;\ k = 1, \ldots, n)$.

The case $\lambda = 0$ *in* (4) *corresponds to the Cauchy–Schwartz inequality* (*the equality holds if and only if* $d_{n-k}(\beta) a_k = c d_k(\alpha) \bar{b}_{n-k}$ *for all* $k \leqslant n$ *and a constant* c).

Theorem B gives an integral version of inequality (4). Its proof is based on a limit procedure as $n \to \infty$, which involves Bernstein's polynomials, and on the detailed analysis of the extremal functions.

Theorem B. *(See [6,9].) Let $\phi(t)$ and $\psi(t)$ be complex-valued continuous functions on* $[0,1]$. *Then for any numbers $\alpha, \beta, \lambda > 0$, the following inequality holds*:

$$\int_0^1 \tau^{\alpha+\beta-1}(1-\tau)^{\lambda-1}\left|\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\phi(\tau t)\psi\big(\tau(1-t)\big)\,dt\right|^2 d\tau$$
$$\leqslant \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\alpha+\beta+\lambda)}{\Gamma(\alpha+\lambda)\Gamma(\beta+\lambda)\Gamma(\alpha+\beta)}\int_0^1 \tau^{\alpha-1}(1-\tau)^{\beta+\lambda-1}\big|\phi(\tau)\big|^2\,d\tau\cdot\int_0^1 \tau^{\beta-1}(1-\tau)^{\alpha+\lambda-1}\big|\psi(\tau)\big|^2\,d\tau. \tag{5}$$

The equality in (5), *provided that ϕ and ψ are not identically* 0, *holds if and only if $\phi(t)=\phi(0)e^{i\theta t}$ and $\psi(t)=\psi(0)e^{i\theta t}$ for $t\in[0,1]$ and a real θ.*

The Cauchy–Bunyakovskii–Schwartz inequality is the limit case of inequality (5) (divided by $\Gamma(\lambda)$) as $\lambda\to 0$. Let

$$A(\alpha,\beta)[f]=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}f(t)\,dt/B(\alpha,\beta),$$

where $f(t)$ is defined on $[0,1]$ and $B(\alpha,\beta)$ is the beta function. Then inequality (5) can be presented as follows (cf. [6]):

$$A(\alpha+\beta,\lambda)\big[|G|^2\big]\leqslant A(\alpha,\beta+\lambda)\big[|\phi|^2\big]A(\beta,\alpha+\lambda)\big[|\psi|^2\big],$$

where $G(\tau)=A(\alpha,\beta)[\phi(\tau t)\psi(\tau(1-t))]$, $\tau\in[0,1]$.

Theorem C is a direct consequence of Theorem B (see Theorem 1 in [8] and the immediate remark to it). We use the notation from [8] where the following integral operators are introduced: $M(\alpha,\beta,\gamma)$ on the class of all entire functions $F(z)$

$$M(\alpha,\beta,\gamma)[F](z)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}e^{\gamma t}\big|F(zt)\big|^2\,dt/B(\alpha,\beta), \tag{6}$$

and $L(\alpha,\beta,\gamma)$ on the class of all functions $f(z)$ that are analytic in a neighbourhood of the origin

$$L(\alpha,\beta,\gamma)[f](z)=M(\alpha,\beta,\gamma)[f_{*\alpha}](z)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}e^{\gamma t}\big|f_{*\alpha}(zt)\big|^2\,dt/B(\alpha,\beta). \tag{7}$$

In (6) and (7): α and $\beta>0$, γ is any real number, z is any complex number, and $f_{*\alpha}$ is the α-convolution defined by (1). Clearly, we can use the operator M for all functions F which are analytic in some disk D_r. Note that

$$\lim_{\beta\to 0} M(\alpha,\beta,\gamma)[F](z)=e^{\gamma}\big|F(z)\big|^2.$$

Respectively, the operator L can be used for all formal power series f for which the convolutions $f_{*\alpha}$ are analytic in D_r. Then $M[F](z)$ and $L[f](z)$ are defined for $z\in D_r$.

Theorem C. *(See [8].) Let $f(z)=a_0+a_1z+\cdots$ and $g(z)=b_0+b_1z+\cdots$ be any power series and assume that the convolutions $f_{*\alpha}(z)$ and $g_{*\beta}(z)$ defined by* (1) *are analytic in some disk D_r. Then for any complex $\zeta\in D_r$, real γ, and $\alpha,\beta,\lambda>0$, the following inequality holds*:

$$L(\alpha+\beta,\lambda,\gamma)[fg](\zeta)\leqslant L(\alpha,\beta+\lambda,\gamma)[f](\zeta)\cdot L(\beta,\alpha+\lambda,\gamma)[g](\zeta), \tag{8}$$

where L is defined by (6)–(7).

The equality in (8), *provided that $\sum_{n=0}^{\infty}|a_n|$ and $\sum_{n=0}^{\infty}|b_n|$ are not equal to zero, holds if and only if*

$$f(z)=f(0)\big[1+(\gamma+i\theta)z/(2\zeta)\big]^{-\alpha}\quad\text{and}\quad g(z)=g(0)\big[1+(\gamma+i\theta)z/(2\zeta)\big]^{-\beta}$$

in a neighbourhood of the origin ($\zeta\neq 0$, θ is a real number), or $\zeta=\gamma=0$.

In addition, inequality (8) holds for $\lambda = 0$; in this case the equality holds if and only if functions $e^{\gamma t} f_{*\alpha}(\zeta t)$ and $\overline{g_{*\beta}(\zeta(1-t))}$ are proportional for $t \in [0,1]$. If f and g in Theorem C are analytic in a neighbourhood of the origin then inequality (8) can be presented in terms of two entire functions of finite exponential type

$$F(z) = \sum_{n=0}^{\infty} A_n z^n \text{ (type } \sigma_1) \quad \text{and} \quad G(z) = \sum_{n=0}^{\infty} B_n z^n \text{ (type } \sigma_2). \tag{9}$$

The corresponding theorem is given in [8]. It involves two analytic functions in a neighbourhood of ∞

$$\Phi(\alpha, z) = \sum_{n=0}^{\infty} A_n \cdot (\alpha)_n z^{-n-1} \quad \big(|z| > \sigma_1\big) \quad \text{and} \quad \Psi(\beta, z) = \sum_{n=0}^{\infty} B_n \cdot (\beta)_n z^{-n-1} \quad \big(|z| > \sigma_2\big). \tag{10}$$

Functions $\Phi(\alpha, z)$ and $\Psi(\beta, z)$ are known as the generalized Borel-associated functions to F and G (or generalized Borel transforms) by means of the functions ${}_1F_1(1;\alpha;z)$ and ${}_1F_1(1;\beta;z)$, respectively. The classical case of the Borel-associated functions to F and G corresponds to $\alpha = \beta = 1$ [2,12], i.e.

$$\Phi(z) = \Phi(1, z) = \int_0^{\infty} F(t) e^{-zt}\, dt \quad \text{and} \quad \Psi(z) = \Psi(1, z) = \int_0^{\infty} G(t) e^{-zt}\, dt$$

are the classical Borel transforms. The most general associated functions are introduced by Nachbin [11].

Remark 1. The integral convolution formula

$$(fg)_{*(\alpha+\beta)}(z) B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} f_{*\alpha}(zt) g_{*\beta}\big(z(1-t)\big)\, dt \quad (\alpha, \beta > 0)$$

used in [8] shows that the $(\alpha+\beta)$-convolution $(fg)_{*(\alpha+\beta)}$ is analytic in D_r if $f_{*\alpha}$ and $g_{*\beta}$ are analytic there. This formula implies by induction that the $(n\alpha)$-convolution $(f^n)_{*(n\alpha)}$ is analytic in D_r for any $n = 2, 3, \ldots$. Also by induction, inequality (8) of Theorem C with $g = f^m$, $\beta = m\alpha$ $(m = 1, 2, \ldots)$, and any real γ leads to the following inequality:

$$L(n\alpha, \lambda, \gamma)\big[f^n\big](\zeta) \leqslant L^n\big(\alpha, (n-1)\alpha + \lambda, \gamma\big)[f](\zeta) \quad (\alpha, \lambda > 0;\ \zeta \in D_r), \tag{11}$$

where the equality, provided that f is not identically 0, holds if and only if

$$f(z) = f(0)\big[1 + (\gamma + i\theta) z/(2\zeta)\big]^{-\alpha} \quad (\zeta \neq 0,\ \theta \text{ is a real number}), \text{ or } \zeta = \gamma = 0$$

$(n = 2, 3, \ldots)$. The case $\gamma = 0$ in (11) is used in [8] to prove some limit inequalities for convolutions, entire functions, and generalized Borel transforms. Theorem D in the next section gives more general limit inequalities than the ones in [8] as $n \to \infty$.

3. The limit inequalities

To prove the main limit result we need two lemmas.

Lemma 1. *Let $u(z) = \sum_{n=1}^{\infty} a_n z^n$ and $v(z) = \sum_{n=1}^{\infty} b_n z^n$ be power series such that the 0-convolutions $u_{*0}(z)$ and $v_{*0}(z)$ are analytic in a disk D_r. Then the 0-convolution $(uv)_{*0}(z)$ is analytic in D_r and for any $\rho < r$, the following inequality holds*:

$$\max_{|z| \leqslant \rho} \big|(uv)_{*0}(z)\big| \leqslant \max_{|z| \leqslant \rho} \big|u_{*0}(z)\big| \max_{|z| \leqslant \rho} \big|v_{*0}(z)\big|. \tag{12}$$

Proof. Note that $(uv)_{*0}$ can be presented in the integral form

$$(uv)_{*0}(z) = \int_0^1 \big[t(1-t)\big]^{-1} u_{*0}(zt) v_{*0}\big(z(1-t)\big)\, dt \quad (z \in D_r). \tag{13}$$

Indeed, by definition (2) the integral in (13) is equal to

$$\sum_{n,m=1}^{\infty} \frac{a_n b_m z^{n+m}}{(n-1)!(m-1)!} \int_0^1 t^{n-1}(1-t)^{m-1}\,dt,$$

and the double sum can be written as

$$\sum_{k=2}^{\infty} \frac{z^k}{(k-1)!} \sum_{n=1}^{k-1} a_n b_{k-n} = (uv)_{*0}(z).$$

Formula (13) shows that $(uv)_{*0}$ is analytic in D_r. Also (13) implies inequality (12) for $\rho < r$, since functions $u_{*0}(z)/z$ and $v_{*0}(z)/z$ are analytic in D_r. $\square$

Lemma 2. *Let $u(z) = a_1 z + a_2 z^2 + \cdots$ be a power series such that the 0-convolution $u_{*0}(z)$ is analytic in a disk D_r. Then for any entire function*

$$V(w) = \sum_{k=0}^{\infty} b_k w^k,$$

the 0-convolution $(V \circ u)_{*0}(z)$ is analytic in D_r.

Proof. Lemma 1 implies that $(u^k)_{*0}$ is analytic in D_r for any $k = 2, 3, \ldots$ and

$$\max_{|z|\leqslant\rho} \left|\left(u^k\right)_{*0}(z)\right| \leqslant \left(\max_{|z|\leqslant\rho} \left|u_{*0}(z)\right|\right)^k \quad (\rho < r). \tag{14}$$

Let

$$U_N(z) = \sum_{k=0}^{N} b_k u^k(z) \quad (N = 1, 2, \ldots).$$

It is easy to see that $(U_N)_{*0}$ can be presented as

$$(U_N)_{*0}(z) = \sum_{k=1}^{N} b_k \left(u^k\right)_{*0}(z) \tag{15}$$

and therefore $(U_N)_{*0}$ is analytic in D_r for any $N \geqslant 1$. Formula (15) and inequality (14) for $k \geqslant 2$ imply that the limit function

$$(V \circ u)_{*0}(z) = \lim_{N\to\infty} (U_N)_{*0}(z)$$

is analytic in D_r. Here we use Weierstrass' M-test and theorem on uniformly convergent series of analytic functions. $\square$

Theorem D. *Let $f(z) = 1 + a_1 z + \cdots$ be a power series such that the 0-convolution $(\log f)_{*0}(z)$ defined by (2) is analytic in a disk D_r. Then for any $\alpha, \lambda > 0$ and real γ, the α-convolution $f_{*\alpha}(z)$ defined by (1) is analytic in D_r and the following inequality holds*:

$$\int_0^1 t^{\alpha-1}(1-t)^{\lambda-1} e^{\gamma t} \left|f_{*\alpha}(\zeta t)\right|^2 dt$$
$$\leqslant B(\alpha, \lambda) \exp\left\{\alpha \int_0^1 t^{-1}(1-t)^{\alpha+\lambda-1}\left[e^{\gamma t}\left|\Omega(\zeta t)\right|^2 - 1\right] dt\right\} \quad (\zeta \in D_r), \tag{16}$$

where

$$\Omega(z) = 1 + \alpha^{-1}(\log f)_{*0}(z) = 1 + \left(\alpha^{-1}\log f(z)\right) * \left(ze^z\right). \tag{17}$$

The equality in (16) *holds if and only if*

$$f(z) = \left[1 + (\gamma + i\theta)z/(2\zeta)\right]^{-\alpha} \quad (\zeta \neq 0,\ \theta \text{ is a real number}),\ \text{or } \zeta = \gamma = 0.$$

Proof. Let n be a natural number, and let the formal power series presentation for $f^{1/n}$ be as follows: $f^{1/n}(z) = 1 + \sum_{k=1}^{\infty} a_{k,n} z^k$.

Lemma 2 with $u = (\log f)/n$ and $V(w) = e^w$ implies that $(f^{1/n})_{*0}$ is analytic in D_r. By definition (2) and the Cauchy–Hadamard theorem, we have

$$\overline{\lim}_{k\to\infty}\left|\frac{a_{k,n}}{(k-1)!}\right|^{1/k} \leqslant \frac{1}{r}.$$

Hence the inequality

$$\overline{\lim}_{k\to\infty}\left|\frac{a_{k,n}}{(\alpha/n)_k}\right|^{1/k} \leqslant \frac{1}{r}$$

holds for any $\alpha > 0$. By definition (1) and the Cauchy–Hadamard theorem, we conclude that for any $n = 1, 2, \ldots$ and $\alpha > 0$, the (α/n)-convolution $(f^{1/n})_{*(\alpha/n)}$ is analytic in D_r.

Now we replace f by $f^{1/n}$ and α by α/n in (11) for even n. Then according to definition (6)–(7), we obtain

$$\begin{aligned} L(\alpha,\lambda,\gamma)[f](\zeta) &\leqslant L^2(\alpha/2, \alpha/2+\lambda, \gamma)\left[f^{1/2}\right](\zeta) \\ &\leqslant \left[\frac{\int_0^1 t^{\alpha/n-1}(1-t)^{(n-1)\alpha/n+\lambda-1} e^{\gamma t} |(f^{1/n})_{*(\alpha/n)}(\zeta t)|^2\,dt}{B(\alpha/n, (n-1)\alpha/n+\lambda)}\right]^n \end{aligned} \tag{18}$$

for $n = 4, 6, \ldots$. Here again we use definition (1)

$$\left(f^{1/n}\right)_{*(\alpha/n)}(z) = 1 + \frac{1}{\alpha}\sum_{k=1}^{\infty}\frac{n a_{k,n} z^k}{(\alpha/n+1)\cdots(\alpha/n+k-1)}.$$

Let $\sum_{k=1}^{\infty} c_k z^k$ be the formal power series presentation for $\log f(z)$. We have

$$\lim_{n\to\infty}\frac{n a_{k,n}}{(\alpha/n+1)\cdots(\alpha/n+k-1)} = \frac{c_k}{(k-1)!} \quad (k = 1, 2, \ldots).$$

The expression in the brackets on the right-hand side of (18) equals

$$1 + \frac{\alpha\Gamma(\alpha+\lambda)}{n\Gamma(\alpha/n+1)\Gamma((n-1)\alpha/n+\lambda)}\int_0^1 t^{\alpha/n-1}(1-t)^{(n-1)\alpha/n+\lambda-1}\left[e^{\gamma t}\left|\left(f^{1/n}\right)_{*(\alpha/n)}(\zeta t)\right|^2 - 1\right]dt.$$

It follows that this expression can be written in the form

$$1 + n^{-1}\int_0^1 \alpha t^{-1}(1-t)^{\alpha+\lambda-1}\left[e^{\gamma t}\left|1 + \frac{1}{\alpha}\sum_{k=1}^{\infty}\frac{c_k\zeta^k}{(k-1)!}t^k\right|^2 - 1\right]dt + o\left(n^{-1}\right), \quad \text{as } n \to \infty. \tag{19}$$

We use (18), (19), and (17) to obtain inequality (16). Inequalities (18) and the equality condition in (11) imply the equality condition in (16). $\square$

Corollary 1. *Let $f(z) = 1 + a_1 z + \cdots$ be a power series such that the* 0*-convolution* $(\log f)_{*0}(z)$ *is analytic in a disk D_r. Then for any $\alpha > 0$, real γ, and complex $\zeta \in D_r$, the following inequality holds*:

$$\left|f_{*\alpha}(\zeta)\right| \leqslant \exp\left\{\frac{\alpha}{2}\int_0^1 t^{-1}(1-t)^{\alpha-1}\left[\left|e^{\gamma t}\Omega(\zeta t)\right|^2 - 1\right]dt - \gamma\right\}, \tag{20}$$

where $f_{*\alpha}$ and Ω are defined by (1) *and* (17), *respectively.*

The equality in (20) *holds if and only if*

$$f(z) = \left[1 + (\gamma + i\theta)z/\zeta\right]^{-\alpha} \quad (\zeta \neq 0,\ \theta \text{ is a real number}),\ \text{or } \zeta = \gamma = 0.$$

Proof. As $\lambda(1-t)^{\lambda-1}$ and $\lambda\Gamma(\lambda)$ for $\lambda = 0$ give the delta function $\delta(1-t)$ and 1, respectively, we multiply both sides of (16) by λ and then we obtain the limit inequality as $\lambda \to 0$. Also we replace γ by 2γ. The equality condition in (20) is implied by (18) as $\lambda \to 0$ and $n \to \infty$, and the equality condition in (16). $\square$

Corollary 2. *Let function f, $f(0) = 1$, be analytic in a neighbourhood of the origin. Then for any $\alpha, \lambda > 0$, real γ, and complex ζ, the following inequalities hold*:

$$\int_0^1 t^{\alpha-1}(1-t)^{\lambda-1}e^{\gamma t}\left|f_{*\alpha}(\zeta t)\right|^2 dt \leqslant B(\alpha,\lambda)\exp\left\{\alpha\int_0^1 t^{-1}(1-t)^{\alpha+\lambda-1}\left[e^{\gamma t}\left|\Omega(\zeta t)\right|^2 - 1\right]dt\right\} \tag{21}$$

and

$$\left|f_{*\alpha}(\zeta)\right| \leqslant \exp\left\{\frac{\alpha}{2}\int_0^1 t^{-1}(1-t)^{\alpha-1}\left[\left|e^{\gamma t}\Omega(\zeta t)\right|^2 - 1\right]dt - \gamma\right\}, \tag{22}$$

where $f_{*\alpha}$ and Ω are defined by (1) *and* (17), *respectively.*

The equality in (21) *and* (22) *holds for the same functions $f(z)$ as in* (16) *and* (20), *respectively.*

Corollary 3. *Let F, $F(0) = 1$, be an entire function of the finite exponential type. Then for any $\alpha, \lambda > 0$, real γ, and complex ζ, the following inequalities hold*:

$$\int_0^1 t^{\alpha-1}(1-t)^{\lambda-1}e^{\gamma t}\left|F(\zeta t)\right|^2 dt \leqslant B(\alpha,\lambda)\exp\left\{\alpha\int_0^1 t^{-1}(1-t)^{\alpha+\lambda-1}\left[e^{\gamma t}\left|\omega(\zeta t)\right|^2 - 1\right]dt\right\} \tag{23}$$

and

$$\left|F(\zeta)\right| \leqslant \exp\left\{\frac{\alpha}{2}\int_0^1 t^{-1}(1-t)^{\alpha-1}\left[\left|e^{\gamma t}\omega(\zeta t)\right|^2 - 1\right]dt - \gamma\right\}, \tag{24}$$

where $\omega(z) = 1 + (\alpha^{-1}\log[z^{-1}\Phi(\alpha, z^{-1})]) * (ze^z)$ and Φ is the generalized Borel-associated function to F defined by (9)–(10).

The equality in (23) *and* (24) *holds if and only if*

$$F(z) = \exp\left[-(\gamma + i\theta)z/(2\zeta)\right] \quad \text{and} \quad F(z) = \exp\left[-(\gamma + i\theta)z/\zeta\right],$$

respectively ($\zeta \neq 0$, θ *is a real number*), *or* $\zeta = \gamma = 0$.

Remark 2. For any function f, $f(0) = 1$, which is analytic in a neighbourhood of the origin, any $\alpha > 0$ and complex ζ, inequality (22) and (17) imply that $|f_{*\alpha}(\zeta)| \leqslant e^{\gamma}$, provided that $|1 + \alpha^{-1}(\log f)_{*0}(\zeta t)| \leqslant e^{\gamma t}$ for $t \in [0, 1]$ and some real γ. As a function of γ the right-hand side of inequality (22) attains its minimum at $\gamma = \gamma_0$, where γ_0 is defined by the equation

$$\alpha\int_0^1 (1-t)^{\alpha-1}\left|e^{\gamma_0 t}\left[1 + \alpha^{-1}(\log f)_{*0}(\zeta t)\right]\right|^2 dt = 1.$$

For example, if $f(\zeta) = (1-\zeta)^{-\alpha}$ then $\gamma_0 = -\Re\zeta$. The similar statements are true for inequalities (20) and (24).

4. Examples and applications

We consider several straightforward consequences of the limit inequalities from Section 3. Some of them are expressed in terms of the confluent hypergeometric and confluent hypergeometric limit functions, i.e. ${}_1F_1(a;b;z)$ and

$$ {}_0F_1(-;b;z) = \lim_{a\to\infty} {}_1F_1(a;b;z/a) = \sum_{n=0}^{\infty} \frac{z^n}{(b)_n n!}. $$

(1) We use the divergent (for $z \neq 0$) power series

$$ g(z) = \sum_{n=1}^{\infty} (n-1)! z^n $$

first discussed by Euler (1754). Note that $g(z)$ may be obtained as that solution of the differential equation $z^2 g'(z) = g(z) - z$ which vanishes as $z = 0$ [3]. Let the power series f in Theorem D be defined by the formal expansion

$$ f(z) = \exp\{g(z)\} = 1 + z + \frac{3}{2}z^2 + \cdots, $$

and let $\alpha = 1$. We obtain that $(\log f)_{*0}(z) = g_{*0}(z) = z/(1-z)$ and $\Omega(z) = (1-z)^{-1}$. It follows from this theorem and Corollary 1 that the 1-convolution

$$ f_{*1}(z) = f(z) * e^z = 1 + z + \frac{3}{4}z^2 + \cdots $$

is analytic in the unit disk D_1 and satisfies the inequality

$$ |f_{*1}(\zeta)| \leqslant \exp\left\{\frac{1}{2}\left[\int_0^1 \left(\frac{e^{\gamma t}}{|1-\zeta t|^2} - 1\right)\frac{dt}{t} - \gamma\right]\right\} $$

for any real γ and $\zeta \in D_1$. In particular, for $\gamma = 0$ this inequality allows us to give the simple estimates for $|f_{*1}|$. Namely, we have that

$$ |f_{*1}(\zeta)| \leqslant \exp\left\{-\frac{\Im[\zeta \log(1-\zeta)]}{2\Im\zeta}\right\} $$

if $\Im\zeta \neq 0$; also for $x \in (-1,1)$ we obtain that

$$ |f_{*1}(ix)| \leqslant (1+x^2)^{-1/4} \quad \text{and} \quad |f_{*1}(x)| \leqslant (1-x)^{-1/2} \exp\{x/[2(1-x)]\}. $$

(2) Let $f(z)$ in Corollary 2 be equal to e^z. Then

$$ f_{*\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{(\alpha)_n n!} = {}_0F_1(-;\alpha;z) \quad \text{and} \quad \Omega(z) = 1 + z/\alpha. $$

For any real γ, complex ζ, and $\alpha,\ \lambda > 0$, inequalities (21) and (22) imply that

$$ \int_0^1 t^{\alpha-1}(1-t)^{\lambda-1} e^{\gamma t} |{}_0F_1(-;\alpha;\zeta t)|^2\, dt \leqslant B(\alpha,\lambda) \exp\left\{\frac{2\Re\zeta}{\alpha+\lambda} + \frac{|\zeta|^2}{\alpha(\alpha+\lambda)_2} + \sum_{n=1}^{\infty} \gamma^n Q_n(\alpha+\lambda)\right\} \tag{25} $$

and

$$ |{}_0F_1(-;\alpha;\zeta)| \leqslant \exp\left\{\frac{\Re\zeta}{\alpha}\left(1 + \frac{\gamma}{\alpha+1}\right) + \frac{|\zeta|^2}{\alpha(\alpha)_2}\left(\frac{1}{2} + \frac{\gamma}{\alpha+2}\right) + \frac{1}{2}\sum_{n=2}^{\infty} \gamma^n Q_n(\alpha)\right\}, \tag{26} $$

where

$$ Q_n(x) = \frac{\alpha}{n(x)_n} + \frac{2\Re\zeta}{(x)_{n+1}} + \frac{|\zeta|^2(n+1)}{\alpha(x)_{n+2}} \quad (n = 1, 2, \ldots). $$

A particular case of inequality (25), when $\gamma = 0$, is discussed in [7]. If $\gamma = 0$ in (26) we obtain that

$$\left|{}_0F_1(-;\alpha;\zeta)\right| \leqslant \exp\left\{\frac{\Re\zeta}{\alpha} + \frac{|\zeta|^2}{2\alpha^2(\alpha+1)}\right\} \tag{27}$$

for any complex ζ and $\alpha > 0$. One can use the real and purely imaginary values of ζ to see that the coefficients for $\Re\zeta$ and $|\zeta|^2$ in (27) are best possible for any $\alpha > 0$. Some consequences of (27) with $|\zeta|$ depending on α may be useful. For example, given the argument of ζ, one can minimize the right-hand side of (27) with respect to $|\zeta|$. A more effective result is generated by inequality (22) (or (26)) if its right-hand side is minimized by $\gamma = 0$ (see Remark 2). Here is the corresponding condition for ζ and α: $\Re\zeta + |\zeta|^2/[\alpha(\alpha+2)] = 0$. It follows that for any $\theta \in [-\pi/2, \pi/2]$,

$$\left|{}_0F_1\bigl(-;\alpha;\zeta_1(\theta)\bigr)\right| \leqslant \exp\left\{-\frac{(\alpha+1)}{2}\cos^2\theta\right\} \tag{28}$$

and

$$\left|{}_0F_1\bigl(-;\alpha;\zeta_2(\theta)\bigr)\right| \leqslant \exp\left\{-\frac{\alpha(\alpha+2)}{2(\alpha+1)}\cos^2\theta\right\}, \tag{29}$$

where $\zeta_1(\theta) = -\alpha(\alpha+1)\cos\theta \cdot e^{i\theta}$ and $\zeta_2(\theta) = -\alpha(\alpha+2)\cos\theta \cdot e^{i\theta}$.

Inequality (27) can be improved for some real $\zeta = x$ and $\alpha > 0$. Namely, inequality (22) with $\gamma = -x/\alpha$ and the elementary inequality $e^y \geqslant |1+y|$ $(y \geqslant -1.27)$ imply that

$$\left|{}_0F_1(-;\alpha;x)\right| \leqslant e^{x/\alpha} \quad (x \geqslant -1.27\alpha). \tag{30}$$

Inequalities (25)–(30) imply some inequalities for trigonometric and special functions. For example, from (27) we have that

$$|\sin\zeta| = \left|\zeta\,{}_0F_1\bigl(-;3/2;-\zeta^2/4\bigr)\right| \leqslant |\zeta|\exp\bigl\{-\Re\bigl(\zeta^2\bigr)/6 + |\zeta|^4/180\bigr\}$$

and

$$|\cos\zeta| = \left|{}_0F_1\bigl(-;1/2;-\zeta^2/4\bigr)\right| \leqslant \exp\bigl\{-\Re\bigl(\zeta^2\bigr)/2 + |\zeta|^4/12\bigr\}$$

for any complex ζ.

Here is another example: since the Bessel function of the first kind of order α can be presented in the form

$$J_\alpha(z) = (z/2)^\alpha\,{}_0F_1\bigl[-;\alpha+1;-(z/2)^2\bigr]/\Gamma(\alpha+1),$$

from (27) we obtain that

$$\left|J_\alpha(\zeta)\right| \leqslant \left|\frac{\zeta}{2}\right|^\alpha \cdot \frac{1}{\Gamma(\alpha+1)}\exp\left\{-\frac{\Re(\zeta^2)}{4(\alpha+1)} + \frac{|\zeta|^4}{32(\alpha+1)(\alpha+1)_2}\right\}$$

for any complex ζ and $\alpha > -1$ (see also [7]).

For any $\theta \in [-\pi/2, \pi/2]$, inequality (29) implies that

$$|\sin\zeta| \leqslant |\zeta|e^{-1.05\cos^2\theta} \quad \bigl(\zeta = \pm\sqrt{21\cos\theta}\cdot e^{i\theta/2}\bigr), \qquad |\cos\zeta| \leqslant e^{-5\cos^2\theta/12} \quad \bigl(\zeta = \pm\sqrt{5\cos\theta}\cdot e^{i\theta/2}\bigr),$$

and

$$\left|J_\alpha(\zeta)\right| \leqslant \left|\frac{\zeta}{2}\right|^\alpha \cdot \frac{1}{\Gamma(\alpha+1)}\exp\left\{-\frac{(\alpha+1)(\alpha+3)}{2(\alpha+2)}\cos^2\theta\right\},$$

where $\zeta = \pm 2\sqrt{(\alpha+1)(\alpha+3)\cos\theta}\cdot e^{i\theta/2}$ and $\alpha > -1$.

For real x inequality (30) implies that

$$|\sin x| \leqslant |x|e^{-x^2/6} \quad \bigl(|x| \leqslant 2.76\bigr), \qquad |\cos x| \leqslant e^{-x^2/2} \quad \bigl(|x| \leqslant 1.59\bigr),$$

and

$$\left|J_\alpha(x)\right| \leqslant \left|\frac{x}{2}\right|^\alpha \cdot \frac{1}{\Gamma(\alpha+1)}\exp\left\{-\frac{x^2}{4(\alpha+1)}\right\} \quad \bigl(|x| \leqslant 2.25\sqrt{\alpha+1},\ \alpha > -1\bigr).$$

Certainly the ranges for x in the last three inequalities can be improved; in particular, 2.76 and 1.59 can be replaced by 3.578... and 1.778..., respectively.

(3) If $f(z) = (1+z^2)^{-1}$ in Corollary 2 then

$$f_{*\alpha}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(\alpha)_{2n}} \quad \text{and} \quad (\log f)_{*0}(z) = 2(\cos z - 1).$$

Hence

$$f_{*1}(z) = \cos z \quad \text{and} \quad f_{*2}(z) = \frac{\sin z}{z}.$$

Applying (22) we obtain the following inequalities:

$$2\log|\cos\zeta| \leqslant \int_0^1 \bigl[e^{\gamma t}\bigl|2\cos(\zeta t) - 1\bigr|^2 - 1\bigr]\frac{dt}{t} - \gamma$$

and

$$\log\left|\frac{\sin\zeta}{\zeta}\right| \leqslant \int_0^1 \bigl(t^{-1} - 1\bigr)\bigl[e^{\gamma t}\bigl|\cos(\zeta t)\bigr|^2 - 1\bigr]\,dt - \frac{\gamma}{2}$$

for any complex ζ and real γ.

(4) Now let $f(z)$ in Corollary 2 be equal to $(1-z)^{-c}$, where c is a complex number. We have that

$$f_{*\alpha}(z) = \sum_{n=0}^{\infty} \frac{(c)_n z^n}{(\alpha)_n n!} = {}_1F_1(c;\alpha;z) \quad \text{and} \quad \Omega(z) = 1 + \frac{c}{\alpha}\bigl(e^z - 1\bigr).$$

From (21) and (22), we obtain that

$$\int_0^1 t^{\alpha-1}(1-t)^{\lambda-1}e^{\gamma t}\bigl|{}_1F_1(c;\alpha;\zeta t)\bigr|^2\,dt \leqslant B(\alpha,\lambda)\exp\left\{\sum_{n=1}^{\infty}\frac{T_n}{(\alpha+\lambda)_n}\right\} \tag{31}$$

and

$$\bigl|{}_1F_1(c;\alpha;\zeta)\bigr| \leqslant \exp\left\{\frac{\Re(c\zeta)}{\alpha} + \frac{1}{2}\sum_{n=2}^{\infty}\frac{T_n}{(\alpha)_n}\right\} \tag{32}$$

for any α and $\lambda > 0$, real γ, complex c and ζ, and

$$T_n = \bigl\{\bigl(\alpha^2 + |c|^2\bigr)\gamma^n + 2\Re\bigl[\bigl(\alpha c - |c|^2\bigr)(\zeta+\gamma)^n - \alpha c\gamma^n\bigr] + |c|^2(2\Re\zeta + \gamma)^n\bigr\}/(\alpha n) \quad (n = 1, 2, \ldots).$$

For real $\zeta = x$ and $\alpha > 0$, inequality (22) with $\gamma = -cx/\alpha$ and the elementary inequality

$$e^{yb} \geqslant \bigl|1 + y\bigl(e^b - 1\bigr)\bigr|,$$

where $y < 0$, $b \leqslant \log(1 - 1.27y^{-1})$ or $y > 1$, $b \geqslant \log(1 - y^{-1})$, imply that

$$\bigl|{}_1F_1(c;\alpha;x)\bigr| \leqslant e^{cx/\alpha}, \tag{33}$$

where $c < 0$, $x \leqslant \log(1 - 1.27\alpha/c)$ or $c > \alpha$, $x \geqslant \log(1 - \alpha/c)$. Also inequality (22) with $\gamma = 0$ and the trivial inequality

$$\bigl|1 + y\bigl(e^b - 1\bigr)\bigr| \leqslant 1 \quad \bigl(y < 0,\ 0 \leqslant b \leqslant \log\bigl(1 - 2y^{-1}\bigr)\bigr)$$

imply that

$$\bigl|{}_1F_1(c;\alpha;x)\bigr| \leqslant 1 \quad \bigl(c < 0,\ 0 \leqslant x \leqslant \log(1 - 2\alpha/c)\bigr). \tag{34}$$

If $c = \alpha$, then inequality (31) implies that

$$ {}_1F_1(\alpha;\alpha+\lambda;x) \leqslant \exp\left\{\alpha\sum_{n=1}^{\infty}\frac{x^n}{n(\alpha+\lambda)_n}\right\} \tag{35}$$

for any real x and $\alpha, \lambda > 0$. If $c = \alpha$ in (32) or $\lambda \to 0$ in (35), we obtain that

$$\sum_{n=0}^{\infty}\frac{x^n}{(n+2)(\alpha)_n} \geqslant 0$$

for any real x and $\alpha \geqslant 2$. The case $\alpha = 2$ gives the trivial inequality $e^x \geqslant 1 + x$.

Finally, we give some estimates for the generalized Laguerre polynomials which can be defined as

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1(-n;\alpha+1;x) \quad (n = 0, 1, \ldots;\ \alpha > -1).$$

For all n and $\alpha > -1$, inequality (33) implies that

$$\left|L_n^\alpha(x)\right| \leqslant \frac{(\alpha+1)_n}{n!}e^{-nx/(\alpha+1)}, \quad x \leqslant \log\left[\frac{n+1.27(\alpha+1)}{n}\right],$$

and inequality (34) results in

$$\left|L_n^\alpha(x)\right| \leqslant \frac{(\alpha+1)_n}{n!}, \quad 0 \leqslant x \leqslant \log\left[\frac{n+2\alpha+2}{n}\right].$$

5. Some generalizations

Theorem E which presents an integral inequality for four continuous functions and four positive parameters can be regarded as a generalization of Theorem B. Also the statement of Theorem E can be viewed as an integral version of the discrete inequality for four complex vectors and binomial weights established in [5, Theorem 2]. It is proved there that this discrete result is both a generalization and consequence of Theorem A which is known to be the discrete predecessor of Theorem B. Now we show that Theorems E and B are equivalent in the same way as their discrete predecessors, i.e. they can be obtained from one another. A similar approach can be used, for example, to generalize Theorem C.

Theorem E. *Let $f(t), g(t), u(t)$, and $v(t)$ be complex-valued continuous functions on* $[0,1]$. *Then for any numbers $\alpha, \beta, \lambda > 0$, and $\mu \geqslant 0$, the following inequality holds*:

$$\begin{aligned}
&\frac{2\Gamma(\alpha+\beta+\mu)}{\Gamma(\lambda)\Gamma(\alpha+\beta+\mu+\lambda)}\int_0^1 \tau^{\alpha+\beta+\mu-1}(1-\tau)^{\lambda-1}\left|\int_0^1 t_1^{\alpha+\mu-1}(1-t_1)^{\beta-1}f(\tau t_1)g\big(\tau(1-t_1)\big)\,dt_1\right| \\
&\quad\times\left|\int_0^1 t_2^{\alpha-1}(1-t_2)^{\beta+\mu-1}u(\tau t_2)v\big(\tau(1-t_2)\big)\,dt_2\right|d\tau \leqslant \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\lambda)\Gamma(\beta+\lambda)} \\
&\quad\times\int_0^1 t^{\alpha+\mu-1}(1-t)^{\beta+\lambda-1}\left|f(t)\right|^2dt\int_0^1 t^{\beta+\mu-1}(1-t)^{\alpha+\lambda-1}\left|v(t)\right|^2dt + \frac{\Gamma(\alpha+\mu)\Gamma(\beta+\mu)}{\Gamma(\alpha+\mu+\lambda)\Gamma(\beta+\mu+\lambda)} \\
&\quad\times\int_0^1 t^{\alpha-1}(1-t)^{\beta+\mu+\lambda-1}\left|u(t)\right|^2dt\int_0^1 t^{\beta-1}(1-t)^{\alpha+\mu+\lambda-1}\left|g(t)\right|^2dt.
\end{aligned}\tag{36}$$

The equality in (36), *provided that f, g, u, and v are not identically* 0, *holds if and only if $f(t) = f(0)e^{i\theta_1 t}$, $g(t) = g(0)e^{i\theta_1 t}$, $u(t) = u(0)e^{i\theta_2 t}$, $v(t) = v(0)e^{i\theta_2 t}$ for $t \in [0,1]$ and some real θ_1, θ_2, and $|f(0)v(0)| = |g(0)u(0)|$.*

Proof. Using the Cauchy–Bunyakovskii–Schwartz inequality we estimate the triple integral on the left-hand side of (36) from above by the product

$$Q^{\frac{1}{2}} \cdot T^{\frac{1}{2}},$$

where

$$Q = \int_0^1 \tau^{\alpha+\beta+\mu-1}(1-\tau)^{\lambda-1} \left| \int_0^1 t^{\alpha+\mu-1}(1-t)^{\beta-1} f(\tau t) g\big(\tau(1-t)\big)\, dt \right|^2 d\tau$$

and

$$T = \int_0^1 \tau^{\alpha+\beta+\mu-1}(1-\tau)^{\lambda-1} \left| \int_0^1 t^{\alpha-1}(1-t)^{\beta+\mu-1} u(\tau t) v\big(\tau(1-t)\big)\, dt \right|^2 d\tau. \tag{37}$$

Applying inequality (5) of Theorem A we estimate Q and T in (37). Namely, we use (5) with $\phi(t) = f(t)$, $\psi(t) = g(t)$, and with $\alpha + \mu$ instead of α to estimate Q. Then we use (5) with $\phi(t) = u(t)$, $\psi(t) = v(t)$, and with $\beta + \mu$ instead of β to estimate T. We obtain that the product $Q^{\frac{1}{2}} \cdot T^{\frac{1}{2}}$ is not greater than

$$\begin{aligned} &\frac{\Gamma(\lambda)\Gamma(\alpha+\beta+\mu+\lambda)}{\Gamma(\alpha+\beta+\mu)} \left[\frac{\Gamma(\alpha)\Gamma(\alpha+\mu)\Gamma(\beta)\Gamma(\beta+\mu)}{\Gamma(\alpha+\lambda)\Gamma(\alpha+\mu+\lambda)\Gamma(\beta+\lambda)\Gamma(\beta+\mu+\lambda)} \right]^{\frac{1}{2}} \\ &\quad \times \left[\int_0^1 t^{\alpha+\mu-1}(1-t)^{\beta+\lambda-1} \big|f(t)\big|^2 dt \int_0^1 t^{\beta+\mu-1}(1-t)^{\alpha+\lambda-1} \big|v(t)\big|^2 dt \right]^{\frac{1}{2}} \\ &\quad \times \left[\int_0^1 t^{\alpha-1}(1-t)^{\beta+\mu+\lambda-1} \big|u(t)\big|^2 dt \int_0^1 t^{\beta-1}(1-t)^{\alpha+\mu+\lambda-1} \big|g(t)\big|^2 dt \right]^{\frac{1}{2}}. \end{aligned} \tag{38}$$

Now we normalize each integral in (38) with the corresponding value of the beta function and then we use the arithmetic mean-geometric mean inequality to get (36). The equality statement in Theorem E is implied by the one in Theorem B. □

Remark 3. The statement of Theorem B corresponds to that of Theorem E when $f(t) = u(t) = \phi(t)$ and $g(t) = v(t) = \psi(t)$ on $[0, 1]$, and $\mu = 0$. The limiting case of (36) as $\lambda \to 0$ is based on the Cauchy–Bunyakovskii–Schwartz inequality. It is easy to see that in this case the equality in (36) holds if and only if $g(t) = c_1 \overline{f}(1-t)$ and $v(t) = c_2 \bar{u}(1-t)$ for $t \in [0, 1]$ and some constants c_1 and c_2, $|c_1| = |c_2|$.

As an example of a straightforward application of Theorem E, we give a general inequality for the confluent hypergeometric functions. We use inequality (36) with $f(t) = g(t) = e^{xt/2}$ and $u(t) = v(t) = e^{yt/2}$ for some real x and y. Also for $\lambda = 0$ we take the equality condition in (36) into account (see Remark 2).

Corollary 4. *For any $\alpha, \beta, \mu, \lambda \geqslant 0$ and any real x and y, the following inequality holds*

$$\begin{aligned} &2\,{}_1F_1\big(\alpha+\beta+\mu; \alpha+\beta+\mu+\lambda; (x+y)/2\big) \\ &\quad \leqslant {}_1F_1(\alpha+\mu; \alpha+\beta+\mu+\lambda; x)\,{}_1F_1(\beta+\mu; \alpha+\beta+\mu+\lambda; y) \\ &\qquad + {}_1F_1(\alpha; \alpha+\beta+\mu+\lambda; y)\,{}_1F_1(\beta; \alpha+\beta+\mu+\lambda; x). \end{aligned} \tag{39}$$

For $\alpha, \beta > 0$, the equality in (39) *holds if and only if $x = y = 0$.*

Inequality (39) with $\mu = 0$ and $x = y$ is discussed in [6,7].

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