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On the predictability of discrete dynamical systems III

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Abstract

Given an integer $n \geqslant 2$, a metrizable compact topological n-manifold X with boundary and a finite positive Borel measure μ on X, we prove that "most" homeomorphisms $f : X \to X$ are non-sensitive μ-almost everywhere on X. Moreover, we also prove that for "most" homeomorphisms $f : X \to X$ the non-wandering set Ω_f has μ-measure zero.
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1. Introduction

Throughout this paper we fix an integer $n \geqslant 2$, a metrizable compact topological n-manifold X with (or without) boundary [6], and a metric d which is compatible with the topology of X. We denote by $i(X)$ (respectively $b(X)$) the interior (respectively the boundary) of the manifold X and by $H(X)$ (respectively $C(X)$) the set of all homeomorphisms from X onto X (respectively of all continuous functions from X into X) endowed with the supremum metric

$$\tilde{d}(f, g) = \sup_{x \in X} d\big(f(x), g(x)\big).$$

If M is a Baire space, we say that "most elements of M" satisfy a certain property P if the set of all $x \in M$ that do not satisfy property P is of the first category in M. The terms "typical" and "generic" are often used instead of "most."

Given a function $f : X \to X$ and a point $a \in X$, recall [1,2] that f is non-sensitive at a if for every $\epsilon > 0$ there is $\delta > 0$ such that for any choice of points $a_0 \in B(a; \delta)$, $a_1 \in B(f(a_0); \delta)$, $a_2 \in B(f(a_1); \delta)$, …, we have that

$$d\big(a_m, f^m(a)\big) < \epsilon \quad \text{for every } m \geqslant 0.$$

Intuitively, if f is non-sensitive at a then we can predict the future evolution of a in the discrete dynamical system (X, f) forever as accurately as we want, provided we can compute the initial condition a and the values of f precisely enough.

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Although this notion is very strong, we proved in [2] that most functions in $H(B^n)$, where B^n is the closed unit ball of $\mathbb{R}^n$, are non-sensitive almost everywhere on B^n with respect to Lebesgue measure. Later, in [3], we established a very general result in this direction for the space $C(X)$, namely: *if μ is any finite positive Borel measure on X, then most functions in $C(X)$ are non-sensitive μ-almost everywhere on X.*

The goal of the present paper is to complete our work on non-sensitivity by considering homeomorphisms on an arbitrary X. Our main result is the following version for $H(X)$ of the above-mentioned result from [3]:

Theorem 1. *If μ is any finite positive Borel measure on X, then most functions in $H(X)$ are non-sensitive μ-almost everywhere on X.*

This theorem will be proved in Section 3. The main strategy of its proof consists in considering functions that have a certain "well-controlled behaviour" on arbitrarily small subsets of X that cover arbitrarily large portions of X (in measure), such behaviour naturally organizing these small subsets of X in trees. This strategy, which was introduced in [2], has been shown very useful in establishing results for most functions that are true almost everywhere with respect to a given measure (see [3] and [4]). However, we point out that there are many differences in the proof presented here for homeomorphisms in a very general setting compared to the proofs of the main results in [2] and [3], specially in the construction of the trees, in the way we make small perturbations in the given function and in the way we deal with the boundary of the manifold.

Recall that a point $x \in X$ is a non-wandering point of a function $f : X \to X$ if for every neighborhood U of x, $f^k(U) \cap U \neq \emptyset$ for infinitely many integers $k \geqslant 1$. We denote by Ω_f the set of all non-wandering points of f.

By making a small modification in the proof of Theorem 1 we shall obtain the following interesting fact:

Theorem 2. *If μ is any finite positive Borel measure on X, then for most functions $f \in H(X)$, $\mu(\Omega_f) = 0$.*

We shall prove this theorem in Section 4.

2. Some preliminary results

Given $A \subset X$, we denote by $\overline{A}$, $\operatorname{Int} A$, $\operatorname{Bd} A$ and $\operatorname{diam} A$ the closure, the interior, the boundary and the diameter of A in X, respectively. For each $\delta > 0$,

$$N_\delta(A) = \big\{x \in X;\ d(x, A) < \delta\big\}$$

is the δ-neighborhood of A. We denote by B^n and D^n the closed unit ball and the open unit ball of $\mathbb{R}^n$ with respect to the euclidean norm, respectively, and we define

$$H^n = \big\{(x_1, \ldots, x_n) \in \mathbb{R}^n;\ x_n \geqslant 0\big\}.$$

Moreover, $\mathcal{U}_X$ (respectively $\mathcal{V}_X$) denotes the set of all subsets A of $i(X)$ (respectively A of X) for which there is a homeomorphism $\psi : A \to B^n$ (respectively $\psi : A \to B^n \cap H^n$) with $\psi(\operatorname{Int} A) = D^n$ (respectively $\psi(\operatorname{Int} A) = D^n \cap H^n$). Finally, $\mathcal{W}_X$ (respectively $\mathcal{Z}_X$) denotes the set of all $A \in \mathcal{U}_X$ (respectively $A \in \mathcal{V}_X$) such that A has a fundamental system of neighborhoods that belong to $\mathcal{U}_X$ (respectively to $\mathcal{V}_X$). Note that each point $a \in i(X)$ (respectively $a \in b(X)$) has a fundamental system of neighborhoods that belong to $\mathcal{W}_X$ (respectively to $\mathcal{Z}_X$).

The results below, which are variants of Lemma 4 of [3], will be very important for our purposes:

Lemma 3. *Suppose μ is a finite positive Borel measure on X with $\mu(b(X)) = 0$. For every $\delta > 0$, there are pairwise disjoint sets $B_1, \ldots, B_s \in \mathcal{W}_X$ such that*

$$\mu\big(X - (B_1 \cup \cdots \cup B_s)\big) < \delta \quad \textit{and} \quad \operatorname{diam} B_i < \delta \quad \textit{for every } 1 \leqslant i \leqslant s.$$

Proof. By a box in $\mathbb{R}^n$ we mean a set of the form

$$\big\{(x_1, \ldots, x_n) \in \mathbb{R}^n;\ a_i \leqslant x_i < a_i + \epsilon \text{ for } 1 \leqslant i \leqslant n\big\},$$

where $(a_1, \ldots, a_n) \in \mathbb{R}^n$ and $\epsilon > 0$. A closed box in $\mathbb{R}^n$ is the closure of a box in $\mathbb{R}^n$.

Since X is a compact topological n-manifold, there are sets $V_1, \ldots, V_\ell, \ldots, V_t \subset X$ such that:

1. For $1 \leqslant j \leqslant \ell$, there is a homeomorphism $\psi_j : V_j \to B^n$ with $\psi_j(\operatorname{Int} V_j) = D^n$;
2. For $\ell < j \leqslant t$, there is a homeomorphism $\psi_j : V_j \to B^n \cap H^n$ with $\psi_j(\operatorname{Int} V_j) = D^n \cap H^n$;
3. The sets $\psi_1^{-1}(\frac{1}{2}B^n), \ldots, \psi_\ell^{-1}(\frac{1}{2}B^n), \psi_{\ell+1}^{-1}(\frac{1}{2}B^n \cap H^n), \ldots, \psi_t^{-1}(\frac{1}{2}B^n \cap H^n)$ cover X.

For each $1 \leqslant j \leqslant \ell$ (respectively $\ell < j \leqslant t$), choose $r_j \in [1/2, 1)$ such that the boundary of the set $A_j = \psi_j^{-1}(r_j B^n)$ (respectively $A_j = \psi_j^{-1}(r_j B^n \cap H^n)$) has μ-measure zero.

Fix $\eta > 0$. Write the set $\psi_1((\operatorname{Int} A_1) - b(X))$ as a countable union of disjoint boxes $C_{1,i}$ ($i \in \mathbb{N}^*$) with diameters less than η and put $B_{1,i} = \psi_1^{-1}(C_{1,i})$ ($i \in \mathbb{N}^*$). Then

$$(\operatorname{Int} A_1) - b(X) = \bigcup_{i=1}^{\infty} B_{1,i}.$$

Now, write the set $\psi_2((\operatorname{Int} A_2) - (b(X) \cup A_1))$ as a countable union of disjoint boxes $C_{2,i}$ ($i \in \mathbb{N}^*$) with diameters less than η and put $B_{2,i} = \psi_2^{-1}(C_{2,i})$ ($i \in \mathbb{N}^*$). Then

$$(\operatorname{Int} A_2) - \big(b(X) \cup A_1\big) = \bigcup_{i=1}^{\infty} B_{2,i}.$$

Now, consider the set $\psi_3((\operatorname{Int} A_3) - (b(X) \cup A_1 \cup A_2))$ and continue this process. By choosing η small enough we can guarantee that each $B_{j,i}$ has diameter less than δ. Since

$$X - \bigcup_{j=1}^{t} \bigcup_{i=1}^{\infty} B_{j,i} \subset b(X) \cup \operatorname{Bd} A_1 \cup \cdots \cup \operatorname{Bd} A_t,$$

there is a finite set $\{B_{j_1,i_1}, \ldots, B_{j_s,i_s}\}$ of the $B_{j,i}$ such that $\mu(X - (B_{j_1,i_1} \cup \cdots \cup B_{j_s,i_s})) < \delta$. Since each $C_{j,i}$ is the union of an increasing sequence of closed boxes, there are closed boxes $C_1 \subset C_{j_1,i_1}, \ldots, C_s \subset C_{j_s,i_s}$ such that

$$\mu\big(X - \big(\psi_{j_1}^{-1}(C_1) \cup \cdots \cup \psi_{j_s}^{-1}(C_s)\big)\big) < \delta.$$

Hence, the sets $B_1 = \psi_{j_1}^{-1}(C_1), \ldots, B_s = \psi_{j_s}^{-1}(C_s)$ have all the desired properties. □

Lemma 4. *Suppose $b(X) \neq \emptyset$ and μ is a finite positive Borel measure on X with $\mu(i(X)) = 0$. For every $\delta > 0$, there are pairwise disjoint sets $B_1, \ldots, B_s \in \mathcal{Z}_X$ such that*

$$\mu\big(X - (B_1 \cup \cdots \cup B_s)\big) < \delta \quad \textit{and} \quad \operatorname{diam} B_i < \delta \quad \textit{for every } 1 \leqslant i \leqslant s.$$

Proof. By compactness, there are sets $V_1, \ldots, V_t \subset X$ such that:

1. For each $1 \leqslant j \leqslant t$, there is a homeomorphism $\psi_j : V_j \to B^n \cap H^n$ with $\psi_j(\operatorname{Int} V_j) = D^n \cap H^n$;
2. The sets $\psi_1^{-1}(\frac{1}{2}B^n \cap H^n), \ldots, \psi_t^{-1}(\frac{1}{2}B^n \cap H^n)$ cover $b(X)$.

For each $1 \leqslant j \leqslant t$, choose $r_j \in [1/2, 1)$ such that the boundary of the set $A_j = \psi_j^{-1}(r_j B^n \cap H^n)$ has μ-measure zero.

Fix $\eta > 0$. Write the set $\psi_1(\operatorname{Int} A_1)$ as a countable union of disjoint boxes $C_{1,i}$ ($i \in \mathbb{N}^*$) with diameters less than η and put $B_{1,i} = \psi_1^{-1}(C_{1,i})$ ($i \in \mathbb{N}^*$). Then

$$\operatorname{Int} A_1 = \bigcup_{i=1}^{\infty} B_{1,i}.$$

Now, write the set $\psi_2((\operatorname{Int} A_2) - A_1)$ as a countable union of disjoint boxes $C_{2,i}$ ($i \in \mathbb{N}^*$) with diameters less than η and put $B_{2,i} = \psi_2^{-1}(C_{2,i})$ ($i \in \mathbb{N}^*$). Then

$$(\operatorname{Int} A_2) - A_1 = \bigcup_{i=1}^{\infty} B_{2,i}.$$

Now, consider the set $\psi_3((\operatorname{Int} A_3) - (A_1 \cup A_2))$ and continue this process. By choosing η small enough we can guarantee that each $B_{j,i}$ has diameter less than δ. Since

$$X - \bigcup_{j=1}^{t} \bigcup_{i=1}^{\infty} B_{j,i} \subset i(X) \cup \operatorname{Bd} A_1 \cup \cdots \cup \operatorname{Bd} A_t,$$

there is a finite set $\{B_{j_1,i_1}, \ldots, B_{j_s,i_s}\}$ of the $B_{j,i}$ such that $\mu(X - (B_{j_1,i_1} \cup \cdots \cup B_{j_s,i_s})) < \delta$. Moreover, we may assume that no B_{j_k,i_k} is contained in $i(X)$ (because $\mu(i(X)) = 0$), so that each C_{j_k,i_k} meets the hyperplane $W = \{(x_1, \ldots, x_n) \in \mathbb{R}^n;\ x_n = 0\}$. Since each C_{j_k,i_k} is the union of an increasing sequence of closed boxes that intersect W, there are closed boxes $C_1 \subset C_{j_1,i_1}, \ldots, C_s \subset C_{j_s,i_s}$ that intersect W such that

$$\mu\big(X - \big(\psi_{j_1}^{-1}(C_1) \cup \cdots \cup \psi_{j_s}^{-1}(C_s)\big)\big) < \delta.$$

Thus, the sets $B_1 = \psi_{j_1}^{-1}(C_1), \ldots, B_s = \psi_{j_s}^{-1}(C_s)$ have all the desired properties. □

3. Proof of Theorem 1

Given a collection $\mathcal{C}$ of sets, recall [3] that a $\mathcal{C}$-tree is a pair (T, φ), where T is a finite rooted tree [5] and φ is a bijective correspondence between the set $V(T)$ of all vertices of T and a collection of pairwise disjoint sets in $\mathcal{C}$. If (T, φ) is a $\mathcal{C}$-tree, we usually omit the correspondence φ and speak just of the $\mathcal{C}$-tree T; moreover, we identify each vertex of T with its corresponding set of $\mathcal{C}$ (under φ). If T is a $\mathcal{C}$-tree and $V_1, V_2 \in V(T)$, we write "$V_2 < V_1$" or "$V_1 > V_2$" to mean that V_1 and V_2 are adjacent and that the unique path connecting V_2 to the root of T passes through V_1. Two $\mathcal{C}$-trees T_1 and T_2 are said to be disjoint if $A \cap B = \emptyset$ whenever $A \in V(T_1)$ and $B \in V(T_2)$.

It is enough to prove Theorem 1 in the case $\mu(b(X)) = 0$ and in the case $b(X) \neq \emptyset$ and $\mu(i(X)) = 0$. So, we divide the proof in these two cases.

Case I. $\mu(b(X)) = 0$.

For each integer $k \geqslant 1$, let $\mathcal{O}_k$ be the set of all $f \in H(X)$ for which there are finitely many pairwise disjoint $\mathcal{W}_X$-trees $T_1, \ldots, T_s$ such that:

(i) $\operatorname{diam} A < 1/k$ for all $A \in V(T_1) \cup \cdots \cup V(T_s)$;
(ii) If $A, B \in V(T_i)$ and $B < A$, then $f(B) \subset \operatorname{Int} A$;
(iii) If R_i is the root of T_i, then there is $S_i \in V(T_i)$ such that $f(R_i) \subset \operatorname{Int} S_i$;
(iv) $\mu(X - \bigcup\{A;\ A \in V(T_i) \text{ for some } 1 \leqslant i \leqslant s\}) < 1/k$.

Clearly, each $\mathcal{O}_k$ is open in $H(X)$. Let $f \in \bigcap_{k=1}^{\infty} \mathcal{O}_k$. Then, for each $k \geqslant 1$, there are pairwise disjoint $\mathcal{W}_X$-trees $T_{k,1}, \ldots, T_{k,s_k}$ so that (i) through (iv) hold with $T_{k,1}, \ldots, T_{k,s_k}$ in place of $T_1, \ldots, T_s$. Put

$$Q_k = \bigcup\big\{A;\ A \in V(T_{k,i}) \text{ for some } 1 \leqslant i \leqslant s_k\big\} \quad (k \geqslant 1)$$

and

$$Q = \bigcap_{r=1}^{\infty} \bigcup_{k=r}^{\infty} Q_k.$$

By property (iv), $\mu(X - Q_k) < 1/k$ for every $k \geqslant 1$, which implies $\mu(Q) = \mu(X)$. Moreover, by properties (ii) and (iii), for each $k \geqslant 1$, there exists $0 < \delta_k < 1/k$ such that

$$f\big(N_{\delta_k}(B)\big) \subset \operatorname{Int} A$$

whenever $A, B \in V(T_{k,i})$ and $B < A$ $(1 \leqslant i \leqslant s_k)$, and such that

$$f\big(N_{\delta_k}(R_{k,i})\big) \subset \operatorname{Int} S_{k,i},$$

where $R_{k,i}$ and $S_{k,i}$ are related to $T_{k,i}$ $(1 \leqslant i \leqslant s_k)$ as R_i and S_i are related to T_i in property (iii). If $a \in Q_k$ and if $a_0 \in B(a; \delta_k), a_1 \in B(f(a_0); \delta_k), a_2 \in B(f(a_1); \delta_k), \ldots$, then

$$d\big(a_m, f^m(a)\big) < \delta_k + 1/k < 2/k \quad \text{for every } m \geqslant 0.$$

This shows that f is non-sensitive at every point of Q.

It remains to show that each $\mathcal{O}_k$ is dense in $H(X)$. Fix $k \geqslant 1$, $f \in H(X)$ and $\epsilon > 0$. Let $0 < \delta < \min\{1/k, \epsilon/4\}$ be such that

$$d\big(f(x), f(y)\big) < \epsilon/4 \quad \text{whenever } d(x, y) < \delta \quad (x, y \in X).$$

By Lemma 3, there is a finite collection $\mathcal{C}$ of pairwise disjoint sets of $\mathcal{W}_X$ such that

$$\mu\left(X - \bigcup_{A \in \mathcal{C}} A\right) < 1/k \quad \text{and} \quad \operatorname{diam} A < \delta \quad \text{for every } A \in \mathcal{C}.$$

Let c be the cardinality of $\mathcal{C}$. Let $\mathcal{M}$ be a finite collection of sets in $\mathcal{U}_X \cup \mathcal{V}_X$ whose diameters are less than $\epsilon/(16c)$ and whose interiors cover X, and let $0 < \eta < \epsilon/(8c)$ be a Lebesgue number for the open covering $\{\operatorname{Int} M;\ M \in \mathcal{M}\}$ of X. We shall construct pairwise disjoint $\mathcal{W}_X$-trees $T_1, \ldots, T_s$ satisfying the following properties:

(A) $\mathcal{C} \subset V(T_1) \cup \cdots \cup V(T_s)$;
(B) $\operatorname{diam} A < \delta$ for every $A \in V(T_1) \cup \cdots \cup V(T_s)$;
(C) if $A, B \in V(T_i)$ and $B < A$, then $f(B) \cap A \neq \emptyset$;
(D) for each $1 \leqslant i \leqslant s$, let R_i be the root of T_i. Put

$$I = \big\{i \in \{1, \ldots, s\};\ f(R_i) \cap A \neq \emptyset \text{ for some } A \in V(T_i)\big\}$$

and $J = \{1, \ldots, s\} - I$. For each $i \in J$, the following properties hold:
(1) $\operatorname{diam} f(R_i) < \eta/2$ and $d(f(R_i), A) < \eta/2$ for some $A \in V(T_i)$;
(2) $f(R_i) \cap A = \emptyset$ for every $A \in V(T_1) \cup \cdots \cup V(T_s)$.

In the construction of the $\mathcal{W}_X$-trees $T_1, \ldots, T_s$ we shall need a variable $\mathcal{B}$, which will denote the set of all elements of $\mathcal{W}_X$ that have been used as a vertex of some tree up to the current step. We begin by choosing $A_1 \in \mathcal{C}$ and by putting it as a vertex of T_1 (hence $\mathcal{B} = \{A_1\}$). Suppose that in a certain moment T_1 consists of the vertices $A_1 < A_2 < \cdots < A_j$ (so that $\mathcal{B} = \{A_1, \ldots, A_j\}$). We look at the set $f(A_j)$. There are three possibilities:

Case 1: $f(A_j) \cap A \neq \emptyset$ for some $A \in \mathcal{B}$.

We stop the construction of T_1 for the time being (so A_j is the root R_1 of T_1).

Case 2: $f(A_j) \cap A = \emptyset$ for every $A \in \mathcal{B}$ and $f(A_j) \cap A \neq \emptyset$ for some $A \in \mathcal{C} - \mathcal{B}$.

Let $A_{j+1} \in \mathcal{C} - \mathcal{B}$ be such that $f(A_j) \cap A_{j+1} \neq \emptyset$ and put A_{j+1} as a new vertex of T_1 satisfying $A_j < A_{j+1}$.

Case 3: $f(A_j) \cap A = \emptyset$ for every $A \in \mathcal{B} \cup \mathcal{C}$.

Choose $A_{j+1} \in \mathcal{W}_X$ disjoint from each element of $\mathcal{B} \cup \mathcal{C}$ such that $\operatorname{diam} A_{j+1} < \delta/j$ and $f(A_j) \cap A_{j+1} \neq \emptyset$, and put A_{j+1} as a new vertex of T_1 satisfying $A_j < A_{j+1}$.

We will stop the construction of T_1 as soon as Case 1 happens or we obtain an A_m such that $f(A_m) \cap A = \emptyset$ for every $A \in \mathcal{C}$, $\operatorname{diam} f(A_m) < \eta/2$ and $d(f(A_m), A_k) < \eta/2$ for some $1 \leqslant k \leqslant m$ (in this case A_m will be the root R_1 of T_1). A simple compactness argument shows that if Case 1 does not happen then we will obtain such an A_m in a finite number of steps.

Now, suppose that we have already constructed $T_1, \ldots, T_{i-1}$. If $\mathcal{B} \supset \mathcal{C}$, we are done. If this is not the case, we choose $A'_1 \in \mathcal{C} - \mathcal{B}$ and put it as a vertex of T_i. If in a certain moment T_i consists of the vertices $A'_1 < A'_2 < \cdots < A'_j$, then we look at $f(A'_j)$. We have the following possibilities:

Case 1a: $f(A'_j) \cap A \neq \emptyset$ for some $A \in \mathcal{B} - V(T_i)$.

Let $\tilde{A} \in \mathcal{B} - V(T_i)$ be such that $f(A'_j) \cap \tilde{A} \neq \emptyset$. Then $\tilde{A}$ is a vertex of a previous tree; say $\tilde{A} \in V(T_{i_0})$, where $1 \leqslant i_0 < i$. In this case we will have no tree T_i for the time being. We will just enlarge T_{i_0} by putting the chain $A'_1 < A'_2 < \cdots < A'_j$ as a new branch of it, satisfying the relation $A'_j < \tilde{A}$.

Case 1b: $f(A'_j) \cap A = \emptyset$ for every $A \in \mathcal{B} - V(T_i)$ and $f(A'_j) \cap A \neq \emptyset$ for some $A \in V(T_i)$.

We stop the construction of T_i for the time being (so A'_j is the root R_i of T_i).

Case 2: $f(A'_j) \cap A = \emptyset$ for every $A \in \mathcal{B}$ and $f(A'_j) \cap A \neq \emptyset$ for some $A \in \mathcal{C} - \mathcal{B}$.

Let $A'_{j+1} \in \mathcal{C} - \mathcal{B}$ be such that $f(A'_j) \cap A'_{j+1} \neq \emptyset$ and put A'_{j+1} as a new vertex of T_i satisfying $A'_j < A'_{j+1}$.

Case 3: $f(A'_j) \cap A = \emptyset$ for every $A \in \mathcal{B} \cup \mathcal{C}$.

Choose $A'_{j+1} \in \mathcal{W}_X$ disjoint from each element of $\mathcal{B} \cup \mathcal{C} \cup \{f(R_1), \ldots, f(R_{i-1})\}$ such that $\operatorname{diam} A'_{j+1} < \delta/j$ and $f(A'_j) \cap A'_{j+1} \neq \emptyset$, and put A'_{j+1} as a new vertex of T_i satisfying $A'_j < A'_{j+1}$.

We will stop the construction of T_i as soon as Cases 1a or 1b happen or we obtain an A'_m such that $f(A'_m) \cap A = \emptyset$ for every $A \in \mathcal{C}$, $\operatorname{diam} f(A'_m) < \eta/2$ and $d(f(A'_m), A'_k) < \eta/2$ for some $1 \leqslant k \leqslant m$ (in this case A'_m will be the root R_i of T_i).

The $\mathcal{W}_X$-trees $T_1, \ldots, T_s$ so constructed clearly satisfy properties (A) through (D). Moreover, $s \leqslant c$.

For each $i \in I$, choose $S_i \in V(T_i)$ such that $f(R_i) \cap S_i \neq \emptyset$; for $i \in J$ the set S_i will be chosen later. For each $A \in V(T_1) \cup \cdots \cup V(T_s)$, choose a neighborhood V_A of A which belongs to $\mathcal{U}_X$ and has diameter less than δ, so that the family $\{V_A\}_{A \in V(T_1) \cup \cdots \cup V(T_s)}$ is pairwise disjoint and, for each $i \in J$,

$$\operatorname{diam} f(V_{R_i}) < \frac{\eta}{2} \tag{1}$$

and

$$f(V_{R_i}) \cap V_A = \emptyset \quad \text{for every } A \in V(T_1) \cup \cdots \cup V(T_s). \tag{2}$$

We shall now make some small perturbations in our function f in order to obtain a function $g \in H(X)$ such that:

(α) $\tilde{d}(g, f) < \epsilon/2$;
(β) $g(B) \subset \operatorname{Int} V_A$ whenever $A, B \in V(T_i)$ and $B < A$ $(1 \leqslant i \leqslant s)$;
(γ) $g(R_i) \subset \operatorname{Int} V_{S_i}$ for $1 \leqslant i \leqslant s$.

We remark that during the construction of g it may be necessary to make changes in our trees, so that the trees $T_1, \ldots, T_s$ in (β) and (γ) may actually be different from the original ones (the number s may also change). Nevertheless, we shall use the same notation for the trees to avoid introducing more notation here. We also remark that all small perturbations made in the sequel can be performed by applying the following elementary fact: *if $K \subset D^n$ is compact and if $V \subset D^n$ is open and non-empty, then there is a homeomorphism $\phi : B^n \to B^n$ such that $\phi(K) \subset V$ and $\phi(x) = x$ for all points x in the unit sphere of $\mathbb{R}^n$.*

For each $B \in V(T_i) - \{R_i\}$ $(1 \leqslant i \leqslant s)$, let A_B be the unique element of $V(T_i)$ such that $B < A_B$, choose $b_B \in B$ such that $f(b_B) \in A_B$ and let U_B be an open neighborhood of b_B contained in $\operatorname{Int} V_B$ such that $f(U_B) \subset \operatorname{Int} V_{A_B}$. Let $\varphi_B : V_B \to V_B$ be a homeomorphism such that $\varphi_B(B) \subset U_B$ and $\varphi_B(x) = x$ for all $x \in \operatorname{Bd} V_B$, and define

$$g_0(x) = f\big(\varphi_B(x)\big) \quad \text{for all } x \in V_B.$$

For each $i \in I$, choose $b_i \in R_i$ such that $f(b_i) \in S_i$ and let U_i be an open neighborhood of b_i contained in $\operatorname{Int} V_{R_i}$ such that $f(U_i) \subset \operatorname{Int} V_{S_i}$. Let $\varphi_i : V_{R_i} \to V_{R_i}$ be a homeomorphism such that $\varphi_i(R_i) \subset U_i$ and $\varphi_i(x) = x$ for all $x \in \operatorname{Bd} V_{R_i}$, and define

$$g_0(x) = f\big(\varphi_i(x)\big) \quad \text{for all } x \in V_{R_i}.$$

Define $g_0(x) = f(x)$ for all the remaining values of x in X. Then $g_0 \in H(X)$, $\tilde{d}(g_0, f) < \epsilon/4$, g_0 satisfies (β) and g_0 satisfies (γ) for $i \in I$. Moreover, by (1), (D) and (2), for each $i \in J$,

$$\operatorname{diam} g_0(V_{R_i}) < \frac{\eta}{2}, \tag{3}$$

$$d\big(g_0(R_i), A\big) < \frac{\eta}{2} \quad \text{for some } A \in V(T_i) \tag{4}$$

and

$$g_0(V_{R_i}) \cap V_A = \emptyset \quad \text{for every } A \in V(T_1) \cup \cdots \cup V(T_s). \tag{5}$$

Now, we need to change g_0 a little bit in order to obtain (γ) also for $i \in J$. Suppose j is the smallest element of J. By (3), (4) and the definition of η, there is $M \in \mathcal{M}$ such that

$$g_0(V_{R_j}) \subset \operatorname{Int} M \quad \text{and} \quad A \cap \operatorname{Int} M \neq \emptyset \quad \text{for some } A \in V(T_j).$$

We identify M with B^n endowed with the euclidean metric d' in such a way that $\operatorname{Int} M = D^n$ or we identify M with $B^n \cap H^n$ endowed with d' in such a way that $\operatorname{Int} M = D^n \cap H^n$, depending on whether $M \in \mathcal{U}_X$ or $M \in \mathcal{V}_X$. Let $A' \in V(T_1) \cup \cdots \cup V(T_s)$ be such that

$$d'\big(g_0(R_j), V_{A'} \cap \operatorname{Int} M\big) = \min d'\big(g_0(R_j), V_A \cap \operatorname{Int} M\big),$$

where the minimum is taken over all $A \in V(T_1) \cup \cdots \cup V(T_s)$ with $V_A \cap \operatorname{Int} M \neq \emptyset$. It follows from (5) that this minimum is non-zero. Let $x \in R_j$ be such that

$$d'\big(g_0(x), V_{A'} \cap \operatorname{Int} M\big) = d'\big(g_0(R_j), V_{A'} \cap \operatorname{Int} M\big)$$

and choose $y \in \operatorname{Int} V_{A'} \cap \operatorname{Int} M$ such that the line segment L joining $g_0(x)$ to y in $M = B^n$ (respectively $M = B^n \cap H^n$) does not meet V_A for any $A \neq A'$, does not meet $g_0(B)$ for any $B < A'$, and does not meet $g_0(R_i)$ if $A' = S_i$ for some $i \in I$. Let

$$K = \big\{i \in J;\ g_0(V_{R_i}) \cap L \neq \emptyset\big\};$$

note that $j \in K$. Cover L by a finite number of open balls $B_1, \ldots, B_\ell$ in $\mathbb{R}^n$ which have centers on L and are contained in $\operatorname{Int} M = D^n$ (respectively $\operatorname{Int} M = D^n \cap H^n$), so that $B_1 \cup \cdots \cup B_\ell$ is disjoint from V_A for $A \neq A'$, from $g_0(B)$ for $B < A'$, from $g_0(R_i)$ if $A' = S_i$ for some $i \in I$, and from $g_0(V_{R_i})$ for $i \in J - K$. Let $\phi : X \to X$ be a homeomorphism such that

$$\phi(L) \subset \operatorname{Int} V_{A'} \quad \text{and} \quad \phi(x) = x \quad \text{if } x \notin B_1 \cup \cdots \cup B_\ell.$$

Put $g_1' = \phi \circ g_0 \in H(X)$. Then $\tilde{d}(g_1', g_0) < \epsilon/(16c)$, $\operatorname{diam} g_1'(V_{R_i}) < 3\epsilon/(16c)$ for every $i \in K$, and

$$g_1'(V_{R_i}) \cap \operatorname{Int} V_{A'} \neq \emptyset \quad \text{for every } i \in K.$$

For each $i \in K$, choose $b_i \in \operatorname{Int} V_{R_i}$ such that $g_1'(b_i) \in \operatorname{Int} V_{A'}$ and let U_i be an open neighborhood of b_i contained in $\operatorname{Int} V_{R_i}$ such that $g_1'(U_i) \subset \operatorname{Int} V_{A'}$. Let $\varphi_i : V_{R_i} \to V_{R_i}$ be a homeomorphism such that $\varphi_i(R_i) \subset U_i$ and $\varphi_i(x) = x$ for all $x \in \operatorname{Bd} V_{R_i}$, and define

$$g_1(x) = g_1'\big(\varphi_i(x)\big) \quad \text{for all } x \in V_{R_i}.$$

Put $g_1(x) = g_1'(x)$ for all the remaining values of x in X. Then $g_1 \in H(X)$, $\tilde{d}(g_1, g_1') < 3\epsilon/(16c)$ (and so $\tilde{d}(g_1, g_0) < \epsilon/(4c)$) and

$$g_1(R_i) \subset \operatorname{Int} V_{A'} \quad \text{for every } i \in K.$$

Let $w \in \{1, \ldots, s\}$ be such that $A' \in V(T_w)$. For each $i \in K$, $i \neq w$, we eliminate the tree T_i by putting it as a new branch of T_w, satisfying the relation $R_i < A'$, and if $w \in K$ then we define $S_w = A'$. If $K = \{w\}$ then the trees remained unchanged and our function g_1 satisfies (β) and satisfies (γ) for $i \in I \cup \{j\}$. If this is not the case then the trees, the number s and the sets I and J have changed, but we shall use the same notation anyway. In this case, note that g_1 satisfies (β) and satisfies (γ) for $i \notin J$, but now the number of elements of J has decreased at least one.

Let J' be the set of all $i \in J$ (the new J) for which S_i has not been defined yet. In other words, J' is our old J minus K. Note that, for each $i \in J'$,

$$\operatorname{diam} g_1(V_{R_i}) < \frac{\eta}{2},$$

$$d\big(g_1(R_i), A\big) < \frac{\eta}{2} \quad \text{for some } A \in V(T_i)$$

and

$$g_1(V_{R_i}) \cap V_A = \emptyset \quad \text{for every } A \in V(T_1) \cup \cdots \cup V(T_s).$$

Hence, we can repeat the process. In this way we will finally obtain a function $g = g_{c'}$, for some $c' \leqslant c$, which satisfies (α), (β) and (γ).

For each $A \in V(T_1) \cup \cdots \cup V(T_s)$, let W_A be a closed neighborhood of A contained in $\operatorname{Int} V_A$ such that $g(B) \subset W_A$ whenever $B < A$ and $g(R_i) \subset W_A$ if $A = S_i$. Let $\psi \in H(X)$ be such that $\tilde{d}(\psi, \mathrm{id}_X) < \epsilon/2$ (where id_X is the identity mapping of X) and $\psi(W_A) \subset \operatorname{Int} A$ for every $A \in V(T_1) \cup \cdots \cup V(T_s)$. Put $h = \psi \circ g$. Then $h \in \mathcal{O}_k$ and $\tilde{d}(h, f) < \epsilon$, as desired.

Case II. $b(X) \neq \emptyset$ and $\mu(i(X)) = 0$.

For each integer $k \geqslant 1$, let $\mathcal{O}_k$ be the set of all $f \in H(X)$ for which there are finitely many pairwise disjoint $\mathcal{Z}_X$-trees $T_1, \ldots, T_s$ satisfying properties (i) through (iv) as in Case I. Then, each $\mathcal{O}_k$ is open in $H(X)$ and each $f \in \bigcap_{k=1}^{\infty} \mathcal{O}_k$ is non-sensitive μ-almost everywhere on X. In order to prove that each $\mathcal{O}_k$ is dense in $H(X)$, we fix $k \geqslant 1$, $f \in H(X)$ and $\epsilon > 0$. Let $0 < \delta < \min\{1/k, \epsilon/4\}$ be such that

$$d\big(f(x), f(y)\big) < \epsilon/4 \quad \text{whenever } d(x, y) < \delta \ (x, y \in X).$$

By Lemma 4, there is a finite collection $\mathcal{C}$ of pairwise disjoint sets of $\mathcal{Z}_X$ such that

$$\mu\left(X - \bigcup_{A \in \mathcal{C}} A\right) < 1/k \quad \text{and} \quad \operatorname{diam} A < \delta \quad \text{for every } A \in \mathcal{C}.$$

Let c, $\mathcal{M}$ and η be as in Case I. We shall construct pairwise disjoint $\mathcal{Z}_X$-trees $T_1, \ldots, T_s$ such that:

(A′) $\mathcal{C} \subset V(T_1) \cup \cdots \cup V(T_s)$;
(B′) $\operatorname{diam} A < \delta$ for every $A \in V(T_1) \cup \cdots \cup V(T_s)$;
(C′) If $A, B \in V(T_i)$ and $B < A$, then $f(B) \cap A \cap b(X) \neq \emptyset$;
(D′) For each $1 \leqslant i \leqslant s$, let R_i be the root of T_i. Put

$$I = \big\{i \in \{1, \ldots, s\};\ f(R_i) \cap A \cap b(X) \neq \emptyset \text{ for some } A \in V(T_i)\big\}$$

and $J = \{1, \ldots, s\} - I$. For each $i \in J$, the following properties hold:
(1) $\operatorname{diam} f(R_i) < \eta/2$ and $d(f(R_i) \cap b(X), A \cap b(X)) < \eta/2$ for some $A \in V(T_i)$;
(2) $f(R_i) \cap A = \emptyset$ for every $A \in V(T_1) \cup \cdots \cup V(T_s)$.

In the construction of the $\mathcal{Z}_X$-trees $T_1, \ldots, T_s$, the variable $\mathcal{B}$ will denote the set of all elements of $\mathcal{Z}_X$ that have been used as a vertex of some tree up to the current step. We begin by choosing $A_1 \in \mathcal{C}$ and by putting it as a vertex of T_1. If in a certain moment T_1 consists of the vertices $A_1 < A_2 < \cdots < A_j$, we look at the set $f(A_j)$. There are three possibilities:

Case 1: $f(A_j) \cap A \cap b(X) \neq \emptyset$ for some $A \in \mathcal{B}$.

We stop the construction of T_1 for the time being (so A_j is the root R_1 of T_1).

Case 2: $f(A_j) \cap A \cap b(X) = \emptyset$ for every $A \in \mathcal{B}$ and $f(A_j) \cap A \cap b(X) \neq \emptyset$ for some $A \in \mathcal{C} - \mathcal{B}$.

Let $A_{j+1} \in \mathcal{C} - \mathcal{B}$ be such that $f(A_j) \cap A_{j+1} \cap b(X) \neq \emptyset$ and put A_{j+1} as a new vertex of T_1 satisfying $A_j < A_{j+1}$.

Case 3: $f(A_j) \cap A \cap b(X) = \emptyset$ for every $A \in \mathcal{B} \cup \mathcal{C}$.

Choose $A_{j+1} \in \mathcal{Z}_X$ disjoint from each element of $\mathcal{B} \cup \mathcal{C}$ such that $\operatorname{diam} A_{j+1} < \delta/j$ and $f(A_j) \cap A_{j+1} \cap b(X) \neq \emptyset$, and put A_{j+1} as a new vertex of T_1 satisfying $A_j < A_{j+1}$.

We will stop the construction of T_1 as soon as Case 1 happens or we obtain an A_m such that $A_m \notin \mathcal{C}$, $f(A_m) \cap A \cap b(X) = \emptyset$ for every $A \in \mathcal{C}$, $\operatorname{diam} f(A_m) < \eta/2$ and $d(f(A_m) \cap b(X), A_k \cap b(X)) < \eta/2$ for some $1 \leqslant k \leqslant m$ (in this case A_m will be the root R_1 of T_1).

Now, suppose that we have already constructed $T_1, \ldots, T_{i-1}$. If $\mathcal{B} \supset \mathcal{C}$, we stop. If this is not the case, we choose $A'_1 \in \mathcal{C} - \mathcal{B}$ and put it as a vertex of T_i. If in a certain moment T_i consists of the vertices $A'_1 < A'_2 < \cdots < A'_j$, then we look at $f(A'_j)$. We have the following possibilities:

Case 1a: $f(A'_j) \cap A \cap b(X) \neq \emptyset$ for some $A \in \mathcal{B} - V(T_i)$.

Let $\tilde{A} \in \mathcal{B} - V(T_i)$ be such that $f(A'_j) \cap \tilde{A} \cap b(X) \neq \emptyset$. Then $\tilde{A}$ is a vertex of a previous tree; say $\tilde{A} \in V(T_{i_0})$, where $1 \leqslant i_0 < i$. In this case we will have no tree T_i for the time being. We will just enlarge T_{i_0} by putting the chain $A'_1 < A'_2 < \cdots < A'_j$ as a new branch of it, satisfying the relation $A'_j < \tilde{A}$.

Case 1b: $f(A'_j) \cap A \cap b(X) = \emptyset$ for every $A \in \mathcal{B} - V(T_i)$ and $f(A'_j) \cap A \cap b(X) \neq \emptyset$ for some $A \in V(T_i)$.

We stop the construction of T_i for the time being (so A'_j is the root R_i of T_i).

Case 2: $f(A'_j) \cap A \cap b(X) = \emptyset$ for every $A \in \mathcal{B}$ and $f(A'_j) \cap A \cap b(X) \neq \emptyset$ for some $A \in \mathcal{C} - \mathcal{B}$.

Let $A'_{j+1} \in \mathcal{C} - \mathcal{B}$ be such that $f(A'_j) \cap A'_{j+1} \cap b(X) \neq \emptyset$ and put A'_{j+1} as a new vertex of T_i satisfying $A'_j < A'_{j+1}$.

Case 3: $f(A'_j) \cap A \cap b(X) = \emptyset$ for every $A \in \mathcal{B} \cup \mathcal{C}$.

Choose $A'_{j+1} \in \mathcal{Z}_X$ disjoint from each element of $\mathcal{B} \cup \mathcal{C} \cup \{f(R_1), \ldots, f(R_{i-1})\}$ such that $\operatorname{diam} A'_{j+1} < \delta/j$ and $f(A'_j) \cap A'_{j+1} \cap b(X) \neq \emptyset$, and put A'_{j+1} as a new vertex of T_i satisfying $A'_j < A'_{j+1}$.

We will stop the construction of T_i as soon as Cases 1a or 1b happen or we obtain an A'_m such that $A'_m \notin \mathcal{C}$, $f(A'_m) \cap A \cap b(X) = \emptyset$ for every $A \in \mathcal{C}$, $\operatorname{diam} f(A'_m) < \eta/2$ and $d(f(A'_m) \cap b(X), A'_k \cap b(X)) < \eta/2$ for some $1 \leqslant k \leqslant m$ (in this case A'_m will be the root R_i of T_i).

The $\mathcal{Z}_X$-trees $T_1, \ldots, T_s$ so constructed clearly satisfy properties (A′) through (D′), with the possible exception of part (2) of (D′). For each $i \in J$, starting with the smallest one and going up to the largest one, we do the following procedure: since $f(R_i) \cap A \cap b(X) = \emptyset$ for every $A \in V(T_1) \cup \cdots \cup V(T_s)$ and since $R_i \in \mathcal{Z}_X$, there is a set $R'_i \in \mathcal{Z}_X$ such that $\operatorname{diam} R'_i < \delta$, $R'_i \cap A = \emptyset$ for every $A \in [V(T_1) \cup \cdots \cup V(T_s)] - \{R_i\}$, $R'_i \cap b(X) \supset R_i \cap b(X)$, $\operatorname{diam} f(R'_i) < \eta/2$ and $f(R'_i) \cap A = \emptyset$ for every $A \in V(T_1) \cup \cdots \cup V(T_s)$; replace R_i by R'_i. In this way we will obtain all the desired properties. Note that, $s \leqslant c$.

For each $i \in I$, choose $S_i \in V(T_i)$ such that $f(R_i) \cap S_i \cap b(X) \neq \emptyset$; for $i \in J$ the set S_i will be chosen later. For each $A \in V(T_1) \cup \cdots \cup V(T_s)$, choose a neighborhood V_A of A which belongs to $\mathcal{V}_X$ and has diameter $< \delta$, so that the family $\{V_A\}_{A \in V(T_1) \cup \cdots \cup V(T_s)}$ is pairwise disjoint and, for each $i \in J$,

$$\operatorname{diam} f(V_{R_i}) < \frac{\eta}{2} \tag{1′}$$

and

$$f(V_{R_i}) \cap V_A = \emptyset \quad \text{for every } A \in V(T_1) \cup \cdots \cup V(T_s). \tag{2′}$$

We shall construct a function $g \in H(X)$ such that:

(α) $\tilde{d}(g, f) < \epsilon/2$;
(β) $g(B) \subset \operatorname{Int} V_A$ whenever $A, B \in V(T_i)$ and $B < A$ $(1 \leqslant i \leqslant s)$;
(γ) $g(R_i) \subset \operatorname{Int} V_{S_i}$ for $1 \leqslant i \leqslant s$.

As in Case I, during the construction of g it may be necessary to make changes in our trees $T_1, \ldots, T_s$, but we shall use the same notation anyway. Moreover, we shall often use the following elementary fact: *if $K \subset D^n \cap H^n$ is compact and if $V \subset D^n \cap H^n$ is open and meets the hyperplane $\{(x_1, \ldots, x_n) \in \mathbb{R}^n;\ x_n = 0\}$, then there is a homeomorphism $\phi : B^n \cap H^n \to B^n \cap H^n$ such that $\phi(K) \subset V$ and $\phi(x) = x$ for all $x \in H^n$ with $\|x\| = 1$.*

For each $B \in V(T_i) - \{R_i\}$ $(1 \leqslant i \leqslant s)$, let A_B be the unique element of $V(T_i)$ such that $B < A_B$, choose $b_B \in B \cap b(X)$ such that $f(b_B) \in A_B$ and let U_B be an open neighborhood of b_B contained in $\operatorname{Int} V_B$ such that $f(U_B) \subset \operatorname{Int} V_{A_B}$. Let $\varphi_B : V_B \to V_B$ be a homeomorphism such that $\varphi_B(B) \subset U_B$ and $\varphi_B(x) = x$ for all $x \in \operatorname{Bd} V_B$, and define

$$g_0(x) = f\big(\varphi_B(x)\big) \quad \text{for all } x \in V_B.$$

For each $i \in I$, choose $b_i \in R_i \cap b(X)$ such that $f(b_i) \in S_i$ and let U_i be an open neighborhood of b_i contained in $\operatorname{Int} V_{R_i}$ such that $f(U_i) \subset \operatorname{Int} V_{S_i}$. Let $\varphi_i : V_{R_i} \to V_{R_i}$ be a homeomorphism such that $\varphi_i(R_i) \subset U_i$ and $\varphi_i(x) = x$ for all $x \in \operatorname{Bd} V_{R_i}$, and define

$$g_0(x) = f\big(\varphi_i(x)\big) \quad \text{for all } x \in V_{R_i}.$$

Define $g_0(x) = f(x)$ for all the remaining values of x in X. Then $g_0 \in H(X)$, $\tilde{d}(g_0, f) < \epsilon/4$, g_0 satisfies (β) and g_0 satisfies (γ) for $i \in I$. Moreover, by $(1')$, (D') and $(2')$, for each $i \in J$,

$$\operatorname{diam} g_0(V_{R_i}) < \frac{\eta}{2}, \tag{3'}$$

$$d\big(g_0(R_i) \cap b(X), A \cap b(X)\big) < \frac{\eta}{2} \quad \text{for some } A \in V(T_i) \tag{4'}$$

and

$$g_0(V_{R_i}) \cap V_A = \emptyset \quad \text{for every } A \in V(T_1) \cup \cdots \cup V(T_s). \tag{5'}$$

Now, we need to change g_0 a little bit in order to obtain (γ) also for $i \in J$. Suppose j is the smallest element of J. By $(3')$, $(4')$ and the definition of η, there is $M \in \mathcal{M}$ such that

$$g_0(V_{R_j}) \subset \operatorname{Int} M \quad \text{and} \quad A \cap b(X) \cap \operatorname{Int} M \neq \emptyset \quad \text{for some } A \in V(T_j).$$

Note that M necessarily belongs to $\mathcal{V}_X$. We identify M with $B^n \cap H^n$ endowed with the euclidean metric d' in such a way that $\operatorname{Int} M = D^n \cap H^n$. Let $A' \in V(T_1) \cup \cdots \cup V(T_s)$ be such that

$$d'\big(g_0(R_j) \cap b(X), V_{A'} \cap b(X) \cap \operatorname{Int} M\big) = \min d'\big(g_0(R_j) \cap b(X), V_A \cap b(X) \cap \operatorname{Int} M\big),$$

where the minimum is taken over all $A \in V(T_1) \cup \cdots \cup V(T_s)$ with $V_A \cap b(X) \cap \operatorname{Int} M \neq \emptyset$. It follows from $(5')$ that this minimum is non-zero. Let $x \in R_j \cap b(X)$ be such that

$$d'\big(g_0(x), V_{A'} \cap b(X) \cap \operatorname{Int} M\big) = d'\big(g_0(R_j) \cap b(X), V_{A'} \cap b(X) \cap \operatorname{Int} M\big)$$

and choose $y \in b(X) \cap \operatorname{Int} V_{A'} \cap \operatorname{Int} M$ such that the line segment L joining $g_0(x)$ to y in $M = B^n \cap H^n$ does not meet V_A for any $A \neq A'$, does not meet $g_0(B)$ for any $B < A'$, and does not meet $g_0(R_i)$ if $A' = S_i$ for some $i \in I$. Let

$$K = \big\{i \in J;\ g_0(V_{R_i}) \cap L \neq \emptyset\big\};$$

note that $j \in K$. Cover L by a finite number of sets $\tilde{B}_1 = B_1 \cap H^n, \ldots, \tilde{B}_\ell = B_\ell \cap H^n$, where $B_1, \ldots, B_\ell$ are open balls in $\mathbb{R}^n$ which have centers on L and are contained in D^n, so that $\tilde{B}_1 \cup \cdots \cup \tilde{B}_\ell$ is disjoint from V_A for $A \neq A'$, from $g_0(B)$ for $B < A'$, from $g_0(R_i)$ if $A' = S_i$ for some $i \in I$, and from $g_0(V_{R_i})$ for $i \in J - K$. Let $\phi : X \to X$ be a homeomorphism such that

$$\phi(L) \subset \operatorname{Int} V_{A'} \quad \text{and} \quad \phi(x) = x \quad \text{if } x \notin \tilde{B}_1 \cup \cdots \cup \tilde{B}_\ell.$$

Put $g_1' = \phi \circ g_0 \in H(X)$. Then $\tilde{d}(g_1', g_0) < \epsilon/(16c)$, $\operatorname{diam} g_1'(V_{R_i}) < 3\epsilon/(16c)$ for every $i \in K$, and

$$g_1'(V_{R_i}) \cap b(X) \cap \operatorname{Int} V_{A'} \neq \emptyset \quad \text{for every } i \in K.$$

For each $i \in K$, choose $b_i \in b(X) \cap \operatorname{Int} V_{R_i}$ such that $g_1'(b_i) \in \operatorname{Int} V_{A'}$ and let U_i be an open neighborhood of b_i contained in $\operatorname{Int} V_{R_i}$ such that $g_1'(U_i) \subset \operatorname{Int} V_{A'}$. Let $\varphi_i : V_{R_i} \to V_{R_i}$ be a homeomorphism such that $\varphi_i(R_i) \subset U_i$ and $\varphi_i(x) = x$ for all $x \in \operatorname{Bd} V_{R_i}$, and define

$$g_1(x) = g_1'\big(\varphi_i(x)\big) \quad \text{for all } x \in V_{R_i}.$$

Put $g_1(x) = g_1'(x)$ for all the remaining values of x in X. Then $g_1 \in H(X)$, $\tilde{d}(g_1, g_1') < 3\epsilon/(16c)$ (and so $\tilde{d}(g_1, g_0) < \epsilon/(4c)$) and

$$g_1(R_i) \subset \operatorname{Int} V_{A'} \quad \text{for every } i \in K.$$

Let $w \in \{1, \dots, s\}$ be such that $A' \in V(T_w)$. For each $i \in K$, $i \neq w$, we eliminate the tree T_i by putting it as a new branch of T_w, satisfying the relation $R_i < A'$, and if $w \in K$ then we define $S_w = A'$.

If J' is the set of all $i \in J$ (the new J) for which S_i has not been defined yet, then, for each $i \in J'$,

$$\operatorname{diam} g_1(V_{R_i}) < \frac{\eta}{2},$$

$$d\big(g_1(R_i) \cap b(X), A \cap b(X)\big) < \frac{\eta}{2} \quad \text{for some } A \in V(T_i)$$

and

$$g_1(V_{R_i}) \cap V_A = \emptyset \quad \text{for every } A \in V(T_1) \cup \cdots \cup V(T_s).$$

Hence, we can repeat the process and eventually obtain the desired function g.

Now, by arguing exactly as in the last paragraph of Case I we obtain a function $h \in \mathcal{O}_k$ such that $\tilde{d}(h, f) < \epsilon$, which completes the proof.

4. Proof of Theorem 2

We also divide this proof in two cases.

Case I. $\mu(b(X)) = 0$.

For each integer $k \geqslant 1$, let $\mathcal{O}_k$ be the set of all functions $f \in H(X)$ such that there are finitely many pairwise disjoint $\mathcal{W}_X$-trees $T_1, \dots, T_s$ with the following properties:

(i) If $A, B \in V(T_i)$ and $B < A$, then $f(B) \subset \operatorname{Int} A$;
(ii) If R_i is the root of T_i, then there is $S_i \in V(T_i)$ such that $f(R_i) \subset \operatorname{Int} S_i$;
(iii) Let $R_i = A_{i,1} > A_{i,2} > \cdots > A_{i,t_i} = S_i$ be the chain of successive elements of $V(T_i)$ connecting R_i to S_i. If

$$Y_1 = \bigcup_s \big\{A;\ A \in V(T_i) - \{A_{i,1}, \dots, A_{i,t_i}\} \text{ for some } 1 \leqslant i \leqslant s\big\} \quad \text{and}$$

$$Y_2 = \bigcup_{i=1}^{s} \big[\big(A_{i,t_i} - f(A_{i,1})\big) \cup \big(A_{i,t_i-1} - f^2(A_{i,1})\big) \cup \cdots \cup \big(A_{i,1} - f^{t_i}(A_{i,1})\big)\big],$$

then $\mu(X - (Y_1 \cup Y_2)) < 1/k$.

Since μ is regular, each $\mathcal{O}_k$ is open in $H(X)$. It is also clear that $\mu(\Omega_f) = 0$ for every $f \in \bigcap_{k=1}^{\infty} \mathcal{O}_k$. Let us show that each $\mathcal{O}_k$ is dense in $H(X)$. Fix $k \geqslant 1$, $f \in H(X)$ and $\epsilon > 0$. Let $0 < \delta < \min\{1/k, \epsilon/4\}$ be such that $d(f(x), f(y)) < \epsilon/4$ whenever $d(x, y) < \delta$ $(x, y \in X)$. By Lemma 3, there are pairwise disjoint sets $B_1, \dots, B_r \in \mathcal{W}_X$ such that

$$\mu\big(X - (B_1 \cup \cdots \cup B_r)\big) < 1/k \quad \text{and} \quad \operatorname{diam} B_i < \delta \quad \text{for every } 1 \leqslant i \leqslant r.$$

By arguing as in the proof of Theorem 1 (Case I) with $\mathcal{C} = \{B_1, \dots, B_r\}$, we can construct pairwise disjoint $\mathcal{W}_X$-trees $T_1, \dots, T_s$ and a function $h \in H(X)$ so that $\tilde{d}(h, f) < \epsilon$, $\{B_1, \dots, B_r\} \subset V(T_1) \cup \cdots \cup V(T_s)$ and (i) and (ii) hold with h in place of f. Since the set $Z = \{x \in X;\ \mu(\{x\}) > 0\}$ is countable, the set $X - Z$ is dense in X. Thus, the function ψ at the end of the proof of Theorem 1 (Case I) can be constructed so that $\psi(W_A)$ is contained in a very small neighborhood of a point in $(\operatorname{Int} A) - Z$, for each $A \in V(T_1) \cup \cdots \cup V(T_s)$. By choosing these neighborhoods small enough, we will also obtain (iii) with h in place of f.

Case II. $b(X) \neq \emptyset$ and $\mu(i(X)) = 0$.

It is enough to argue as in the proof of Case I, but we need to replace the collection $\mathcal{W}_X$ by the collection $\mathcal{Z}_X$, to use Lemma 4 instead of Lemma 3 and to refer to Case II of Theorem 1 instead of Case I. Moreover, we need to observe that $b(X) - Z$ is dense in $b(X)$, because Z is countable and $n \geqslant 2$, and so the function ψ can be constructed so that $\psi(W_A)$ is contained in a very small neighborhood of a point in $[b(X) \cap \operatorname{Int} A] - Z$, for each $A \in V(T_1) \cup \cdots \cup V(T_s)$.

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