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Elements of harmonic analysis related to the third basic zero order Bessel function

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Abstract

This paper is devoted to the study of some q-harmonic analysis related to the third q-Bessel function of order zero. We establish a product formula leading to a q-translation with some positive kernel. As an application, we provide a q-analogue of the continuous wavelet transform related to this harmonic analysis.

Keywords: Basic Bessel functions; q-Analysis; q-Bessel transforms

1. Introduction

In literature, harmonic analysis related to the classical Bessel function is well developed (see [13] and references therein). The generalized translation associated with the Bessel operator introduced by J. Delsarte in 1938 is related to the product formula for the normalized Bessel function. It plays together with its positivity a central role in the development of the classical harmonic analysis.

However, there exist only few papers devoted to the harmonic analysis related to the basic Bessel functions (see [1,2,10]). None of these papers treated a product formula for the q-Bessel functions or the positivity of the associated q-generalized translation. Some of them defined the q-generalized translation using a q-transmutation operator.

In this paper, we are concerned with $J_0(.;q^2)$, the third Jackson q-Bessel function of order zero. We construct a product formula for this function leading to a positive q-translation which is necessary and constructive for some applications, such as the q-wavelet transform which is the aim of Section 5.

This paper is organized as follows: In Section 2, we present some preliminary results and notations that will be useful in the sequel. In Section 3, we define and study the q-generalized translation associated with the q-Bessel

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operator. In particular we prove that it is positive and by using a q-analogue of Graf's addition formula, we prove that it is related to a product formula for the zero order q-Bessel function. Section 4, is devoted to the study of the q-Bessel transform of order zero and its related convolution product. Finally, as an application, we provide a q-analogue of the continuous wavelet transform related to this harmonic analysis, we prove for this q-wavelet transform a Plancherel theorem and we give an inversion formula.

2. Notations and preliminaries

We recall some usual notions and notations used in the q-theory (see [3] and [5]). We refer to the book by G. Gasper and M. Rahman [3] for the definitions, notations and properties of the q-shifted factorials and the q-hypergeometric functions. Throughout this paper, we assume $q \in]0,1[$ and we write

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C},$$

$\mathbb{R}_q = \{\pm q^n\colon n \in \mathbb{Z}\}$, $\mathbb{R}_{q,+} = \{q^n\colon n \in \mathbb{Z}\}$ and $\widetilde{\mathbb{R}}_{q,+} = \mathbb{R}_{q,+} \cup \{0\}$.

The q-derivatives $D_q f$ and $D_q^+ f$ of a function f are given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad \big(D_q^+ f\big)(x) = \frac{f(q^{-1}x) - f(x)}{(1-q)x}, \quad \text{if } x \neq 0, \tag{1}$$

$(D_q f)(0) = f'(0)$ and $(D_q^+ f)(0) = q^{-1} f'(0)$ provided $f'(0)$ exists.

Using these two q-derivatives, we put

$$\Delta_q = \frac{(1-q)^2}{x} D_q\big[x D_q^+\big]. \tag{2}$$

The q-Jackson integrals from 0 to a and from 0 to ∞ are defined by (see [4])

$$\int_0^a f(x)\, d_q x = (1-q)a \sum_{n=0}^{\infty} f\big(aq^n\big) q^n, \tag{3}$$

$$\int_0^\infty f(x)\, d_q x = (1-q) \sum_{n=-\infty}^{\infty} f\big(q^n\big) q^n \tag{4}$$

provided the sums converge absolutely.

The q-Jackson integral in a generic interval $[a,b]$ is given by (see [4])

$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x. \tag{5}$$

We recall that the q-hypergeometric function ${}_1\varphi_1$ satisfies the following properties (see [2] or [9]):

(1) For all $w, z \in \mathbb{C}$, we have

$$(w,q)_\infty\, {}_1\varphi_1(0; w; q; z) = (z,q)_\infty\, {}_1\varphi_1(0; z; q; w). \tag{6}$$

(2) For $n \in \mathbb{Z}$ and $z \in \mathbb{C}$, we have

$$\big(q^{1-n};q\big)_\infty\, {}_1\varphi_1\big(0; q^{1-n}; q; z\big) = (-1)^n q^{\frac{n(n-1)}{2}} z^n \big(q^{n+1};q\big)_\infty\, {}_1\varphi_1\big(0; q^{n+1}; q; q^n z\big). \tag{7}$$

(3) Both sides of (6) are majorized by

$$(-z;q)_\infty(-w;q)_\infty \quad \text{and by} \quad q^{\frac{n(n-1)}{2}} |z|^n \big(-|z|;q\big)_\infty (-q;q)_\infty \tag{8}$$

if $w = q^{1-n}$ $(n \in \mathbb{N})$.

In [9], T.H. Koornwinder and R. Swarttouw, in order to study a q-analogue of the Hankel transform and to give its inversion formula and a Plancherel formula, defined the third Jackson's q-Bessel function using the q-hypergeometric function ${}_1\varphi_1$, as follows

$$J_\alpha\big(z;q^2\big)=\frac{z^\alpha(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty}\,{}_1\varphi_1\big(0;q^{2\alpha+2};q^2,q^2z^2\big). \tag{9}$$

They proved the following orthogonality relation

$$\sum_{k=-\infty}^{\infty}q^{2k}q^{n+m}J_{n+k}\big(x;q^2\big)J_{m+k}\big(x;q^2\big)=\delta_{n,m},\quad |x|<q^{-1},\ n,m\in\mathbb{Z}. \tag{10}$$

In [6] and more generally in [7], the authors gave the following q-analogue of Graf's addition formula, by the use of an analytic approach:

$$J_\upsilon\big(Rq^{(y+z+\upsilon)};q^2\big)J_{x-\upsilon}\big(q^z;q^2\big)=\sum_{k=-\infty}^{\infty}J_k\big(Rq^{(y+x+k)};q^2\big)J_{\upsilon+k}\big(Rq^{(y+\upsilon+k)};q^2\big)J_x\big(q^{(z-k)};q^2\big), \tag{11}$$

where $z\in\mathbb{Z}$, $R,x,y,\upsilon\in\mathbb{C}$ satisfying $q^{2(1+\Re(x)+\Re(y))}|R|^2<1$, $\Re(x)>-1$ and $R\neq 0$.

We have the following behavior:

Lemma 1. *For $\alpha\geqslant-\frac{1}{2}$ and $x\in\mathbb{R}_{q,+}$, we have*

(1) $$\big|J_\alpha(x;q^2)\big|\leqslant\frac{(-q^2;q^2)_\infty(-q^{2(\alpha+1)};q^2)_\infty}{(q^2;q^2)_\infty}\begin{cases}x^\alpha, & \text{if } x\leqslant 1,\\ q^{(\frac{\operatorname{Log}(x)}{\operatorname{Log} q})^2}, & \text{if } x\geqslant 1.\end{cases}$$

(2) *For all $\nu\in\mathbb{R}$, we have $J_\alpha(x;q^2)=o(x^{-\nu})$ as $x\to+\infty$.*

In particular, we have $\lim_{x\to+\infty}J_\alpha(x;q^2)=0$.

(3) $D_q^+(x^{-\alpha}J_\alpha(x;q^2))=-(1-q)^{-1}x^{-\alpha}J_{\alpha+1}(x;q^2)$.

Proof. (1) From (6)–(8), we have

- For $x=q^n\in\mathbb{R}_{q,+}$, $n\in\mathbb{N}$,

$$\begin{aligned}\big|x^{-\alpha}J_\alpha\big(x;q^2\big)\big|&=\frac{1}{(q^2;q^2)_\infty}\big|\big(q^{2\alpha+2};q^2\big)_\infty\,{}_1\varphi_1\big(0;q^{2\alpha+2};q^2,q^{2n+2}\big)\big|\\&\leqslant\frac{1}{(q^2;q^2)_\infty}\big(-q^{2(n+1)};q^2\big)_\infty\big(-q^{2\alpha+2};q^2\big)_\infty\\&\leqslant\frac{1}{(q^2;q^2)_\infty}\big(-q^2;q^2\big)_\infty\big(-q^{2\alpha+2};q^2\big)_\infty.\end{aligned}$$

- For $x=q^{-n}\in\mathbb{R}_{q,+}$, $n\in\mathbb{N}$,

$$\begin{aligned}\big|x^{-\alpha}J_\alpha\big(x;q^2\big)\big|&=\frac{1}{(q^2;q^2)_\infty}\big|\big(q^{2(1-n)};q^2\big)_\infty\,{}_1\varphi_1\big(0;q^{2(1-n)};q^2,q^{2\alpha+2}\big)\big|\\&\leqslant\frac{1}{(q^2;q^2)_\infty}q^{n(n+2\alpha+1)}\big(-q^2;q^2\big)_\infty\big(-q^{2\alpha+2};q^2\big)_\infty.\end{aligned}$$

So,

$$\big|J_\alpha\big(q^n;q^2\big)\big|\leqslant\frac{(-q^2;q^2)_\infty(-q^{2(\alpha+1)};q^2)_\infty}{(q^2;q^2)_\infty}\begin{cases}q^{n\alpha}, & \text{if } n\geqslant 0,\\ q^{n(n-\alpha-1)}, & \text{if } n\leqslant 0.\end{cases} \tag{12}$$

Hence, since $\alpha > -1$, we get

$$\left|J_\alpha\left(q^n; q^2\right)\right| \leqslant \frac{(-q^2;q^2)_\infty(-q^{2(\alpha+1)};q^2)_\infty}{(q^2;q^2)_\infty}\begin{cases} q^{n\alpha}, & \text{if } n \geqslant 0,\\ q^{n^2}, & \text{if } n \leqslant 0,\end{cases}$$

which is equivalent to

$$\left|J_\alpha\left(x; q^2\right)\right| \leqslant \frac{(-q^2;q^2)_\infty(-q^{2(\alpha+1)};q^2)_\infty}{(q^2;q^2)_\infty}\begin{cases} x^{\alpha}, & \text{if } x \leqslant 1,\\ q^{(\frac{\operatorname{Log}(x)}{\operatorname{Log} q})^2}, & \text{if } x \geqslant 1.\end{cases}$$

Relation (2) is a direct consequence of (1) and by simple calculus we obtain the relation (3). $\square$

It follows from [2, Proposition 1], that for all $\lambda \in \mathbb{C}$, the function $x \mapsto J_0(\lambda x; q^2)$ is the solution of the q-problem

$$\begin{cases} \Delta_q y(x) + \lambda^2 y(x) = 0,\\ y(0) = 1, \quad y'(0) = 0.\end{cases}$$

We need the following spaces and sets:

- $\mathcal{S}_{*q}(\mathbb{R}_q)$ is the space of all functions f on $\mathbb{R}_{q,+}$ such that for all $m, n \in \mathbb{N}$, we have $\sup_{x\in\mathbb{R}_{q,+}} |x^{2m}\Delta_q^n(f)(x)| < \infty$ and for all $n \in \mathbb{N}$, we have $(D_q^+(\Delta_q^n f))(x) \to 0$ as $x \downarrow 0$ in $\mathbb{R}_{q,+}$.
- $\mathcal{D}_{*q}(\mathbb{R}_q)$ is the space of all functions f on $\mathbb{R}_{q,+}$ with bounded support such that for all $n \in \mathbb{N}$, we have $(D_q^+(\Delta_q^n f))(x) \to 0$ as $x \downarrow 0$ in $\mathbb{R}_{q,+}$.
- $\mathcal{C}_{*q,0}(\mathbb{R}_q)$ is the space of all functions f on $\widetilde{\mathbb{R}}_{q,+}$ for which $f(x) \to 0$ as $x \to \infty$ in $\mathbb{R}_{q,+}$ and $f(x) \to f(0)$ as $x \downarrow 0$ in $\mathbb{R}_{q,+}$.
- $L_q^p(\mathbb{R}_{q,+}, x^{2\alpha+1}\,d_qx)$, $p > 0$, the set of all functions defined on $\mathbb{R}_{q,+}$ such that

$$\|f\|_{p,q,\alpha} = \left\{\int_0^\infty \left|f(x)\right|^p x^{2\alpha+1}\,d_qx\right\}^{\frac{1}{p}} < \infty. \tag{13}$$

In particular, for $\alpha = 0$, we write $\|.\|_{p,q,0} = \|.\|_{p,q}$.

Remark 1. Using Lemma 1 and the unnumbered formula after Lemma 1, we can see that for all $\lambda \in \mathbb{R}_{q,+}$,

$$\left(x \mapsto J_0\left(\lambda x; q^2\right)\right) \in \mathcal{S}_{*q}(\mathbb{R}_q), \tag{14}$$

since

$$\forall n \in \mathbb{N}, \quad \Delta_q^n\left(J_0\left(\lambda x; q^2\right)\right) = (-1)^n\lambda^{2n}J_0\left(\lambda x; q^2\right) \quad \text{and} \quad D_q^+\left(J_0\left(\lambda x; q^2\right)\right) = -\frac{\lambda}{1-q}J_1\left(\lambda x; q^2\right).$$

3. A q-generalized translation

We begin by introducing the following positive kernel defined for $m, n, k \in \mathbb{Z}$ by

$$K\left(q^m, q^n, q^k\right) = \left[J_{m-k}\left(q^{n-k}; q^2\right)\right]^2. \tag{15}$$

It satisfies the following properties:

Proposition 1. *For $m, n, k \in \mathbb{Z}$, we have*

$$\text{(1)} \qquad 0 \leqslant K\left(q^m, q^n, q^k\right) \leqslant \frac{(-q^{2(1+n-k)}, -q^2; q^2)_\infty^2}{(q^2;q^2)_\infty^2}\begin{cases} q^{2(m-k)(n-k)}, & \text{if } m \geqslant k,\\ q^{2(m-k)(m-n-1)}, & \text{if } m \leqslant k.\end{cases} \tag{16}$$

(2) $$K\left(q^m, q^n, q^k\right) = K\left(q^n, q^m, q^k\right). \tag{17}$$

(3) $$K\left(q^m, q^n, q^k\right) = q^{2(k-n)} K\left(q^m, q^k, q^n\right). \tag{18}$$

(4) $q^{2m+2n} K\left(q^m, q^n, q^k\right)$ *is symmetric in* n, m, k.

(5) $$\sum_{n=-\infty}^{\infty} q^{2(n-k)} K\left(q^n, q^m, q^k\right) = 1. \tag{19}$$

(6) $$\forall m, n, k \in \mathbb{Z}, \quad 0 \leqslant K\left(q^m, q^n, q^k\right) \leqslant \min\left\{q^{2|k-n|}, q^{2|k-m|}, q^{2|n-m|}\right\}. \tag{20}$$

(7) $$K\left(q^{m+r}, q^{n+r}, q^{k+r}\right) = K\left(q^m, q^n, q^k\right), \quad r \in \mathbb{Z}. \tag{21}$$

Proof. (1) is a direct consequence of (12), with $\alpha = n - k$.

(2) follows from (6) and the definition of the kernel K.

(3) By application of (6) and twice (7), we can write, for $m, n, k \in \mathbb{Z}$,

$$\begin{aligned}
\left(q^2; q^2\right)_\infty J_{m-k}\left(q^{n-k}; q^2\right) &= q^{(n-k)(m-k)} \left(q^{2(m-k+1)}; q^2\right)_\infty {}_1\varphi_1\left(0; q^{2(m-k+1)}; q^2, q^{2(n-k+1)}\right) \\
&= (-1)^{k-m} q^{(n-m+1)(k-m)} \left(q^{2(k-m+1)}; q^2\right)_\infty {}_1\varphi_1\left(0; q^{2(k-m+1)}; q^2, q^{2(n-m+1)}\right) \\
&= (-1)^{k-m} q^{(n-m+1)(k-m)} \left(q^{2(n-m+1)}; q^2\right)_\infty {}_1\varphi_1\left(0; q^{2(n-m+1)}; q^2, q^{2(k-m+1)}\right) \\
&= (-1)^{k-n} q^{k-n} q^{(m-n)(k-n)} \left(q^{2(m-n+1)}; q^2\right)_\infty {}_1\varphi_1\left(0; q^{2(m-n+1)}; q^2, q^{2(k-n+1)}\right),
\end{aligned}$$

thus

$$K\left(q^m, q^n, q^k\right) = q^{2(k-n)} \left[J_{m-n}\left(q^{k-n}; q^2\right)\right]^2 = q^{2(k-n)} K\left(q^m, q^k, q^n\right).$$

(4) follows from (17) and (18).

(5) If $m \geqslant k$, the relation follows from the orthogonality relation (10). If $k \geqslant m$, it suffices to use the relation (18), and we obtain

$$\sum_{n=-\infty}^{\infty} q^{2(n-k)} K\left(q^n, q^m, q^k\right) = \sum_{n=-\infty}^{\infty} q^{2(n-m)} K\left(q^n, q^k, q^m\right) = 1.$$

(6) follows from (17)–(19).

(7) is by definition of the kernel K. □

With the help of the kernel K, we define the q-generalized translation as:

Definition 1. Let f be a function defined on $\mathbb{R}_{q,+}$, the q-generalized translation of f is given by

$$T_{x,q} f(y) = \sum_{k=-\infty}^{\infty} K\left(x, y, q^k\right) f\left(q^k\right), \quad x, y \in \mathbb{R}_{q,+}, \tag{22}$$

provided the sum absolutely converges and

$$T_{0,q} f = f.$$

We give some properties of the q-generalized translation in the two following results:

Proposition 2. *The following properties*:

(1) *the q-generalized translation is positive*;
(2) $T_{x,q} f(y) = T_{y,q} f(x)$, $x, y \in \mathbb{R}_{q,+}$;

(3) *for* $f \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$,

$$T_{0,q}f(y) = \lim_{n\to+\infty} T_{q^n,q}f(y), \quad y \in \mathbb{R}_{q,+};$$

(4) $$T_{x,q}J_0(.;q^2)(y) = J_0(x;q^2)J_0(y;q^2), \quad x, y \in \mathbb{R}_{q,+}, \tag{23}$$

hold.

Proof. (1) Let f be a positive function defined on $\mathbb{R}_{q,+}$, then for all $x, y \in \mathbb{R}_{q,+}$, we have

$$T_{x,q}f(y) = \sum_{k=-\infty}^{\infty} K(x, y, q^k)f(q^k) \geqslant 0.$$

(2) is a consequence of (17).

(3) Let $f \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$ and $y = q^m$, $m \in \mathbb{Z}$. On the one hand, we have from (7) and (9),

$$\forall p \in \mathbb{Z},\ \forall x \in \mathbb{R}_{q,+}, \quad J_{-p}(x;q^2) = (-1)^p q^p J_p(q^p x;q^2).$$

So, $\forall p \in \mathbb{Z}$, $\forall x \in \mathbb{R}_{q,+}$, $\lim_{n\to+\infty} J_p(q^n x;q^2) = \delta_{p,0}$.

Hence, $\forall p \in \mathbb{Z}$, $\lim_{n\to+\infty} K(q^m, q^n, q^k) = \delta_{m,k}$.

On the other hand, we have from (20),

$$\forall n, k \in \mathbb{Z}, \quad \left|K(q^m, q^n, q^k)f(q^k)\right| \leqslant q^{-2m}.q^{2k}\left|f(q^k)\right|.$$

Finally, since $f \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$, we obtain the result by the Lebesgue theorem.

(4) In (11), take $x = \upsilon = 0$, $R = 1$, we obtain for all $z \in \mathbb{Z}$ and $y \geqslant 0$,

$$J_0(q^{y+z};q^2)J_0(q^z;q^2) = \sum_{k=-\infty}^{\infty} \left(J_k(q^{y+k};q^2)\right)^2 J_0(q^{z-k};q^2) = \sum_{k=-\infty}^{\infty} \left(J_{z-k}(q^{y+z-k};q^2)\right)^2 J_0(q^k;q^2).$$

Then for all y and z integers, such that $y \geqslant z$, we have

$$\begin{aligned} J_0(q^y;q^2)J_0(q^z;q^2) &= \sum_{k=-\infty}^{\infty} \left(J_{z-k}(q^{y-k};q^2)\right)^2 J_0(q^k;q^2) = \sum_{k=-\infty}^{\infty} K(q^z, q^y, q^k)J_0(q^k;q^2) \\ &= T_{q^z,q}J_0(.;q^2)(q^y) = T_{q^y,q}J_0(.;q^2)(q^z). \qquad \square \end{aligned}$$

Remark 2. (1) Note that for $r, y, z \in \mathbb{Z}$, we have

$$\begin{aligned} J_0(q^{y+r};q^2)J_0(q^{z+r};q^2) &= \sum_{k=-\infty}^{\infty} \left(J_{z+r-k}(q^{y+r-k};q^2)\right)^2 J_0(q^k;q^2) \\ &= \sum_{k=-\infty}^{\infty} \left(J_{z-(k-r)}(q^{y-(k-r)};q^2)\right)^2 J_0(q^k;q^2) \\ &= \sum_{k=-\infty}^{\infty} \left(J_{z-k}(q^{y-k};q^2)\right)^2 J_0(q^{k+r};q^2) \\ &= T_{q^z,q}\left(x \mapsto J_0(q^r x;q^2)\right)(q^y) = T_{q^y,q}\left(x \mapsto J_0(q^r x;q^2)\right)(q^z). \end{aligned}$$

(2) From the product formula (23), the positivity of the kernel K and the relations (17) and (18), one can prove easily that

$$\forall x \in \mathbb{R}_{q,+}, \quad \left|J_0(x;q^2)\right| \leqslant 1. \tag{24}$$

Proposition 3. *For $f, g \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$, we have for all $x \in \mathbb{R}_{q,+}$,*

(1) $\int_0^\infty T_{x,q}f(y)y\,d_qy = \int_0^\infty f(y)y\,d_qy$,
(2) $\int_0^\infty T_{x,q}f(y)g(y)y\,d_qy = \int_0^\infty f(y)T_{x,q}g(y)y\,d_qy$.

Proof. (1) Let $x = q^m \in \mathbb{R}_{q,+}$. Since $f \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$ and K is positive, we have using (19),

$$\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left|q^{2n}K\left(q^m, q^n, q^k\right)f\left(q^k\right)\right| = \sum_{k=-\infty}^{\infty}q^{2k}\left|f\left(q^k\right)\right|\sum_{n=-\infty}^{\infty}q^{2(n-k)}K\left(q^m, q^n, q^k\right) = \frac{1}{1-q}\|f\|_{1,q}.$$

So, Fubini's theorem together with (19) give

$$\begin{aligned}\int_0^\infty T_{x,q}f(y)y\,d_qy &= (1-q)\sum_{n=-\infty}^{\infty}q^{2n}T_{x,q}f\left(q^n\right)\\ &= (1-q)\sum_{n=-\infty}^{\infty}q^{2n}\sum_{k=-\infty}^{\infty}K\left(q^m, q^n, q^k\right)f\left(q^k\right)\\ &= (1-q)\sum_{k=-\infty}^{\infty}q^{2k}f\left(q^k\right)\sum_{n=-\infty}^{\infty}q^{2(n-k)}K\left(q^m, q^n, q^k\right)\\ &= (1-q)\sum_{k=-\infty}^{\infty}q^{2k}f\left(q^k\right) = \int_0^\infty f(y)y\,d_qy.\end{aligned}$$

(2) From Proposition 1, one can easily see that for $x = q^m \in \mathbb{R}_{q,+}$,

$$\begin{aligned}\int_0^\infty T_{x,q}f(y)g(y)y\,d_qy &= (1-q)\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}q^{2n}g\left(q^n\right)K\left(q^m, q^n, q^k\right)f\left(q^k\right)\\ &= (1-q)\sum_{k=-\infty}^{\infty}q^{2k}f\left(q^k\right)\sum_{n=-\infty}^{\infty}q^{2(n-k)}K\left(q^m, q^n, q^k\right)g\left(q^n\right)\\ &= (1-q)\sum_{k=-\infty}^{\infty}q^{2k}f\left(q^k\right)\sum_{n=-\infty}^{\infty}K\left(q^m, q^k, q^n\right)g\left(q^n\right)\\ &= \int_0^\infty f(y)T_{x,q}g(y)y\,d_qy.\end{aligned}$$

The interchange of summations is legitimated by the Fubini's theorem. In fact, since $f, g \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$, we have from (20),

$$\begin{aligned}\int_0^\infty\left|T_{x,q}f(y)g(y)\right|y\,d_qy &= (1-q)\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}q^{2n}\left|g\left(q^n\right)\right|\left|K\left(q^m, q^n, q^k\right)\right|\left|f\left(q^k\right)\right|\\ &\leqslant (1-q)q^{-2m}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}q^{2n}\left|g\left(q^n\right)\right|q^{2k}\left|f\left(q^k\right)\right|\\ &= \frac{1}{(1-q)q^{2m}}\|f\|_{1,q}\|g\|_{1,q}. \qquad \square\end{aligned}$$

4. q-Bessel Fourier transform

We define for $f \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$, the q-Bessel Fourier transform by

$$\mathcal{F}_q(f)(\lambda) = \frac{1}{1-q}\int_0^\infty f(x)J_0\big(\lambda x; q^2\big)x\,d_qx, \quad \lambda \in \widetilde{\mathbb{R}}_{q,+}. \tag{25}$$

In the following proposition, we summarize some of its properties which are easily deduced from the results shown before.

Proposition 4.

(1) *For $f \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$, we have*

$$\mathcal{F}_q(f) \in \mathcal{C}_{*q,0}(\mathbb{R}_q) \tag{26}$$

and

$$\big|\mathcal{F}_q(f)(\lambda)\big| \leqslant \frac{1}{1-q}\|f\|_{1,q}, \quad \lambda \in \widetilde{\mathbb{R}}_{q,+}. \tag{27}$$

(2) *For $f \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$, we have*

$$\mathcal{F}_q(T_{x,q}f)(\lambda) = J_0\big(\lambda x; q^2\big)\mathcal{F}_q(f)(\lambda), \quad x, \lambda \in \widetilde{\mathbb{R}}_{q,+}. \tag{28}$$

(3) *If f, $D_q^+ f$, $\Delta_q f \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$ and $xD_q^+f(x) \to 0$ as $x \downarrow 0$ in $\mathbb{R}_{q,+}$, then*

$$\mathcal{F}_q(\Delta_q f)(\lambda) = -\lambda^2\mathcal{F}_q(f)(\lambda), \quad \lambda \in \mathbb{R}_{q,+}. \tag{29}$$

(4) *If $f, x^2 f \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$, then*

$$\mathcal{F}_q\big(x^2 f\big) = -\Delta_q\big(\mathcal{F}_q(f)\big). \tag{30}$$

Proof. (1) Let $f \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$. From the relation (24) and Lemma 1, we get

$$\forall \lambda, x \in \mathbb{R}_{q,+}, \quad \big|f(x)J_0\big(\lambda x; q^2\big)\big| \leqslant \big|f(x)\big|,$$

$$\lim_{\lambda\to\infty}\big(f(x)J_0\big(\lambda x; q^2\big)\big) = 0 \quad \text{and} \quad \lim_{\lambda\to 0}\big(f(x)J_0\big(\lambda x; q^2\big)\big) = f(x) \quad (\text{in } \mathbb{R}_{q,+}).$$

Then, the result follows from the Lebesgue theorem.

(2) is a simple deduction of the product formula (23) and Propositions 2 and 3.

(3) Observe that if $g \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$ then $\lim_{x\to 0}(x^2 g(x)) = \lim_{x\to\infty}(x^2 g(x)) = 0$ in $\mathbb{R}_{q,+}$. So, in the conditions of the theorem, Lemma 1 gives for $\lambda \in \mathbb{R}_{q,+}$ and $i = 0, 1$,

$$\big[xf\big(xq^{-1}\big)J_i\big(\lambda x; q^2\big)\big]_0^\infty = \lim_{x\to\infty}\big[xf\big(xq^{-1}\big)J_i\big(\lambda x; q^2\big)\big] - \lim_{x\to 0}\big[xf\big(xq^{-1}\big)J_i\big(\lambda x; q^2\big)\big] = 0$$

and

$$\big[xD_q^+f(x)J_i\big(\lambda xq^{-1}; q^2\big)\big]_0^\infty = 0.$$

Hence, by application of the q-integration by parts rule, we obtain

$$\begin{aligned}\mathcal{F}_q(\Delta_q f)(\lambda) &= (1-q)^2\int_0^\infty D_q\big(xD_q^+\big)(f)(x)J_0\big(\lambda x; q^2\big)\,d_qx \\ &= (1-q)^2\big[xD_q^+f(x)J_0\big(\lambda xq^{-1}; q^2\big)\big]_0^\infty - (1-q)^2\int_0^\infty xD_q^+(f)(x)D_q^+J_0\big(\lambda x; q^2\big)\,d_qx\end{aligned}$$

$$= -(1-q)^2 \int_0^\infty D_q^+(f)(x).x D_q^+ J_0(\lambda x; q^2)\, d_q x$$

$$= (1-q)\lambda\big[xf(xq^{-1}) J_1(\lambda x; q^2)\big]_0^\infty + (1-q)^2 \int_0^\infty f(x) D_q(x D_q^+) J_0(\lambda x; q^2)\, d_q x$$

$$= \int_0^\infty x f(x) \Delta_q J_0(\lambda x; q^2)\, d_q x = -\lambda^2 \int_0^\infty f(x) J_0(\lambda x; q^2) x\, d_q x = -\lambda^2 \mathcal{F}_q(f)(\lambda).$$

(4) follows from the fact $\Delta_q[\lambda \mapsto J_0(\lambda x; q^2)] = -x^2 J_0(\lambda x; q^2)$, $x \in \mathbb{R}_{q,+}$. □

Theorem 1. *For $f \in L_q^1(\mathbb{R}_{q,+}, x\, d_q x)$, we have*

$$f(x) = \frac{1}{1-q} \int_0^\infty \mathcal{F}_q(f)(\lambda) J_0(\lambda x; q^2) \lambda\, d_q \lambda, \quad x \in \mathbb{R}_{q+}. \tag{31}$$

Proof. Let $f \in L_q^1(\mathbb{R}_{q,+}, x\, d_q x)$. From the orthogonality relation (10) together with $J_k(q^m; q^2) = J_m(q^k; q^2)$, we have

$$\int_0^\infty J_0(\lambda x; q^2) J_0(\lambda t; q^2) \lambda\, d_q \lambda = \frac{(1-q)}{xt} \delta_{x,t}, \quad x, t \in \mathbb{R}_{q,+}.$$

On the other hand, for $x \in \mathbb{R}_{q,+}$, we have from (14) and (24),

$$\int_0^\infty \int_0^\infty \big|f(t) J_0(\lambda t; q^2) J_0(\lambda x; q^2)\big| t\lambda\, d_q\lambda\, d_q t \leqslant \int_0^\infty \big|f(t)\big| t\, d_q t \int_0^\infty \big|J_0(\lambda x; q^2)\big| \lambda\, d_q \lambda$$
$$= \|f\|_{1,q} \big\|\big(\lambda \mapsto J_0(\lambda x; q^2)\big)\big\|_{1,q} < \infty.$$

So, by the use of the Fubini's theorem, we obtain

$$\int_0^\infty \mathcal{F}_q(f)(\lambda) J_0(\lambda x; q^2) \lambda\, d_q \lambda = \frac{1}{1-q} \int_0^\infty \int_0^\infty f(t) J_0(\lambda t; q^2) J_0(\lambda x; q^2) t\lambda\, d_q t\, d_q \lambda$$
$$= \frac{1}{1-q} \int_0^\infty t f(t) \left(\int_0^\infty J_0(\lambda x; q^2) J_0(\lambda t; q^2) \lambda\, d_q \lambda \right) d_q t$$
$$= \frac{1}{x} \int_0^\infty f(t) \delta_{x,t}\, d_q t = (1-q) f(x). \qquad \square$$

Theorem 2 *(Plancherel's formula).*

(1) *$\mathcal{F}_q$ is an isomorphism from $\mathcal{S}_{*q}(\mathbb{R}_q)$ onto itself, and we have*:
  (a) *$\mathcal{F}_q^{-1} = \mathcal{F}_q$,*
  (b) *for all $f \in \mathcal{S}_{*q}(\mathbb{R}_q)$, $\|\mathcal{F}_q(f)\|_{2,q} = \|f\|_{2,q}$.*

(2) *$\mathcal{F}_q$ can be uniquely extended to an isometric isomorphism on $L_q^2(\mathbb{R}_{q,+}, x\, d_q x)$.*

Proof. (1) By simple calculus, one can prove that for $g \in \mathcal{S}_{*q}(\mathbb{R}_q)$, we have for all $n \in \mathbb{N}$, the functions $(x \mapsto x^{2n} g(x))$ and $\Delta_q^n g$ belong to $\mathcal{S}_{*q}(\mathbb{R}_q) \subset L_q^1(\mathbb{R}_{q,+}, x^{2\alpha+1}\, d_q x)$, $\alpha \geqslant 0$.

Now, let $f \in \mathcal{S}_{*q}(\mathbb{R}_q)$. From (29), (30) and (27), we have for all $n, m \in \mathbb{N}$,

$$\forall \lambda \in \mathbb{R}_{q,+}, \quad \left|\lambda^{2m} \Delta_q^n \mathcal{F}_q(f)(\lambda)\right| = \left|(-1)^{n+m} \mathcal{F}_q\left(\Delta_q^m\left(x^{2n} f\right)\right)(\lambda)\right| \leqslant \frac{1}{1-q} \left\|\Delta_q^m\left(x^{2n} f\right)\right\|_{1,q} < \infty.$$

Moreover, from (30) and Lemma 1, we have for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}_{q,+}$,

$$\begin{aligned} \left|D_q^+\left(\Delta_q^n \mathcal{F}_q(f)\right)(\lambda)\right| &= \left|(-1)^n D_q^+\left(\mathcal{F}_q\left(x^{2n} f\right)\right)(\lambda)\right| = (1-q)^{-2}\left|\int_0^\infty x^{2n+1} f(x) J_1\left(\lambda x; q^2\right) x\, d_q x\right| \\ &\leqslant \lambda \frac{(-q^2;q^2)_\infty(-q^4;q^2)_\infty}{(1-q)^2(q^2;q^2)_\infty} \int_0^\infty \left|x^{2n} f(x)\right| x^3\, d_q x = \lambda \frac{(-q^2;q^2)_\infty(-q^4;q^2)_\infty}{(1-q)^2(q^2;q^2)_\infty} \left\|x^{2n} f\right\|_{1,q,1}, \end{aligned}$$

so, for all $n \in \mathbb{N}$, $D_q^+(\Delta_q^n \mathcal{F}_q(f))(\lambda) \to 0$ as $\lambda \to 0$ in $\mathbb{R}_{q,+}$.

Thus $\mathcal{F}_q(f) \in \mathcal{S}_{*q}(\mathbb{R}_q)$. This proves that $\mathcal{S}_{*q}(\mathbb{R}_q)$ is stable by $\mathcal{F}_q$.

The previous theorem achieves the proof of (a) and the same technique as in its proof gives (b).

(2) If we consider functions of bounded support, we can prove that $\mathcal{S}_{*q}(\mathbb{R}_q)$ is dense in $L_q^2(\mathbb{R}_{q,+}, x\, d_q x)$. This gives the result. □

Remark 3. (1) Using the previous theorem and the relations (28) and (14), one can see that, for $f \in \mathcal{S}_{*q}(\mathbb{R}_q)$, we have for all $x \in \widetilde{\mathbb{R}}_{q,+}$, $T_{x,q} f \in \mathcal{S}_{*q}(\mathbb{R}_q)$.

(2) By Proposition 3, we have for all $x \in \widetilde{\mathbb{R}}_{q,+}$ and all $f \in L_q^1(\mathbb{R}_{q,+}, x\, d_q x)$,

$$T_{x,q} f \in L_q^1(\mathbb{R}_{q,+}, x\, d_q x) \quad \text{and} \quad \|T_{x,q} f\|_{1,q} \leqslant \|f\|_{1,q}. \tag{32}$$

(3) Similarly, we have for all $x \in \widetilde{\mathbb{R}}_{q,+}$ and all $f \in L_q^2(\mathbb{R}_{q,+}, x\, d_q x)$,

$$T_{x,q} f \in L_q^2(\mathbb{R}_{q,+}, x\, d_q x) \quad \text{and} \quad \|T_{x,q} f\|_{2,q} \leqslant \|f\|_{2,q}. \tag{33}$$

Indeed, from the properties (17)–(20) of the kernel K, we have

$$\forall x, y, z \in \mathbb{R}_{q,+}, \quad 0 \leqslant K(x,y,z) \leqslant \frac{z^2}{y^2} \quad \text{and} \quad \sum_{k=-\infty}^{\infty} K\left(x,y,q^k\right) = 1.$$

Then, using the Cauchy–Schwartz inequality, we get for $x, y \in \mathbb{R}_{q,+}$,

$$\begin{aligned} \left|T_{x,q} f(y)\right|^2 &\leqslant \left(\sum_{k=-\infty}^{\infty} K\left(x,y,q^k\right)\left|f\left(q^k\right)\right|\right)^2 = \left(\sum_{k=-\infty}^{\infty} \sqrt{K\left(x,y,q^k\right)}.\sqrt{K\left(x,y,q^k\right)}\left|f\left(q^k\right)\right|\right)^2 \\ &\leqslant \sum_{k=-\infty}^{\infty} K\left(x,y,q^k\right). \sum_{k=-\infty}^{\infty} K\left(x,y,q^k\right)\left|f\left(q^k\right)\right|^2 = \sum_{k=-\infty}^{\infty} K\left(x,y,q^k\right)\left|f\left(q^k\right)\right|^2 \\ &\leqslant \frac{1}{y^2} \sum_{k=-\infty}^{\infty} q^{2k}\left|f\left(q^k\right)\right|^2 < \infty. \end{aligned}$$

So, by the use of the Fubini's theorem, we obtain for $x \in \mathbb{R}_{q,+}$,

$$\begin{aligned} \int_0^\infty \left|T_{x,q} f(y)\right|^2 y\, d_q y &= (1-q) \sum_{m=-\infty}^{\infty} \left|T_{x,q} f\left(q^m\right)\right|^2 q^{2m} \leqslant (1-q) \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q^{2m} K\left(x,q^m,q^k\right)\left|f\left(q^k\right)\right|^2 \\ &= (1-q) \sum_{k=-\infty}^{\infty} \left|f\left(q^k\right)\right|^2 q^{2k} \sum_{m=-\infty}^{\infty} q^{2(m-k)} K\left(x,q^m,q^k\right) = \|f\|_{2,q}^2. \end{aligned}$$

Now, we define the q-convolution product of two suitable functions f and g by

$$f *_B g(x) = \frac{1}{1-q} \int_0^\infty T_{x,q} f(y) g(y) y \, d_q y, \quad x \in \mathbb{R}_{q,+}. \tag{34}$$

It satisfies the following properties:

Proposition 5. *For $f, g, h \in L_q^1(\mathbb{R}_{q,+}, x \, d_q x)$, we have*

(1) $f *_B g = g *_B f \in L_q^1(\mathbb{R}_{q,+}, x \, d_q x)$ *and* $\|f *_B g\|_{1,q} \leqslant \frac{1}{1-q} \|f\|_{1,q} \|g\|_{1,q}$;
(2) $\mathcal{F}_q(f *_B g) = \mathcal{F}_q(f) \mathcal{F}_q(g)$;
(3) $(f *_B g) *_B h = f *_B (g *_B h)$.

Proof. Let $f, g, h \in L_q^1(\mathbb{R}_{q,+}, x \, d_q x)$.

(1) From Remark 3, Proposition 2 and the Fubini's theorem, we have

$$\begin{aligned}
\int_0^\infty \left| f *_B g(x) \right| x \, d_q x &\leqslant \frac{1}{1-q} \int_0^\infty \int_0^\infty \left| T_{y,q} f(x) \right| \left| g(y) \right| y x \, d_q y \, d_q x = \frac{1}{1-q} \int_0^\infty \left| g(y) \right| y \left(\int_0^\infty \left| T_{y,q} f(x) \right| x \, d_q x \right) d_q y \\
&\leqslant \frac{1}{1-q} \|f\|_{1,q} \|g\|_{1,q}.
\end{aligned}$$

The commutativity of the q-convolution product follows from Proposition 3.

(2) For $\lambda \in \mathbb{R}_{q,+}$, we have

$$\mathcal{F}_q(f *_B g)(\lambda) = \frac{1}{(1-q)^2} \int_0^\infty \left(\int_0^\infty T_{x,q} f(y) g(y) y \, d_q y \right) J_0\big(\lambda x; q^2\big) x \, d_q x. \tag{35}$$

By (24), we have

$$\int_0^\infty \int_0^\infty \left| T_{x,q} f(y) g(y) J_0\big(\lambda x; q^2\big) \right| y x \, d_q y \, d_q x \leqslant \int_0^\infty \int_0^\infty \left| T_{x,q} f(y) g(y) \right| y x \, d_q y \, d_q x \leqslant \|f\|_{1,q} \|g\|_{1,q}.$$

So, by the Fubini's theorem, we can exchange the order of the two q-integral signs in (35). Furthermore, using the product formula (23), we obtain

$$\begin{aligned}
\mathcal{F}_q(f *_B g)(\lambda) &= \frac{1}{(1-q)^2} \int_0^\infty \left(\int_0^\infty T_{y,q} f(x) J_0\big(\lambda x; q^2\big) x \, d_q x \right) g(y) y \, d_q y \\
&= \frac{1}{(1-q)^2} \int_0^\infty \left(\int_0^\infty f(x) T_{y,q} J_0\big(\lambda x; q^2\big) x \, d_q x \right) g(y) y \, d_q y \\
&= \frac{1}{(1-q)^2} \int_0^\infty \left(\int_0^\infty f(x) J_0\big(\lambda x; q^2\big) x \, d_q x \right) g(y) J_0\big(\lambda y; q^2\big) y \, d_q y \\
&= \mathcal{F}_q(f)(\lambda) \mathcal{F}_q(g)(\lambda).
\end{aligned}$$

(3) The same arguments as in (1) give the result. □

Proposition 6. *For $f \in L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$ and $g \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$, we have*

$$f *_B g \in L_q^2(\mathbb{R}_{q,+}, x\,d_qx), \quad \|f *_B g\|_{2,q} \leqslant \frac{1}{1-q}\|f\|_{2,q}\|g\|_{1,q}$$

and

$$\mathcal{F}_q(f *_B g) = \mathcal{F}_q(f)\mathcal{F}_q(g). \tag{36}$$

Proof. Let $f \in L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$ and $g \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$. Using (33) and the Minkowski's integral inequality (see [11]),

$$\|f *_B g\|_{2,q} \leqslant \frac{1}{1-q}\int_0^\infty \big\|x \mapsto T_{y,q}f(x)g(y)\big\|_{2,q}\,y\,d_qy \leqslant \frac{1}{1-q}\int_0^\infty \|f\|_{2,q}\big|g(y)\big|\,y\,d_qy = \frac{1}{1-q}\|f\|_{2,q}\|g\|_{1,q}.$$

On the other hand, the functions $f_p = f.\chi_{[q^p,q^{-p}]}$, $p \in \mathbb{N}$, $\chi_{[q^p,q^{-p}]}$ is the characteristic function of $[q^p, q^{-p}]$, are in $L_q^1(\mathbb{R}_{q,+}, x\,d_qx) \cap L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$ and $(f_p)_p$ converges in $L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$ to f. Moreover, using the previous inequality, we get for all $p \in \mathbb{N}$,

$$\|f *_B g - f_p *_B g\|_{2,q} = \big\|(f - f_p) *_B g\big\|_{2,q} \leqslant \frac{1}{1-q}\|f - f_p\|_{2,q}\|g\|_{1,q}.$$

So,

$$f_p *_B g \to f *_B g \quad \text{as } p \to \infty \text{ in } L_q^2(\mathbb{R}_{q,+}, x\,d_qx).$$

Finally, from the continuity of $\mathcal{F}_q$ on $L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$ (Theorem 2) and Proposition 5, we obtain

$$\mathcal{F}_q(f *_B g) = \lim_{p\to+\infty}\mathcal{F}_q(f_p *_B g) = \lim_{p\to+\infty}\mathcal{F}_q(f_p)\mathcal{F}_q(g) = \mathcal{F}_q(f)\mathcal{F}_q(g). \qquad \square$$

5. q-Wavelet transforms associated with the operator Δ_q

In [12], K. Trimèche generalized the theory of continuous wavelet transforms as presented by T.H. Koornwinder in [8] and studied the generalized wavelets and the generalized continuous wavelet transforms associated with a class of singular differential operators. This class contains, in particular, the so-called Bessel operator, which was studied extensively in [13]. In this section, we shall provide the q-analogues of the generalized wavelets and the generalized continuous wavelet transforms associated with the zero order Bessel operator.

Definition 2. A q-wavelet associated with the q-Bessel operator Δ_q is an even function $g \in L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$ satisfying the following admissibility condition:

$$0 < C_g = \int_0^\infty \big|\mathcal{F}_q(g)(a)\big|^2\frac{d_qa}{a} < \infty. \tag{37}$$

Remarks. (1) For all $\lambda \in \mathbb{R}_{q,+}$, we have

$$C_g = \int_0^\infty \big|\mathcal{F}_q(g)(a\lambda)\big|^2\frac{d_qa}{a}.$$

(2) Let f be a nonzero function in $\mathcal{S}_{*q}(\mathbb{R}_q)$ (respectively $\mathcal{D}_{*q}(\mathbb{R}_q)$). Then $g = \Delta_q f$ is a q-wavelet associated with the q-Bessel operator Δ_q, in $\mathcal{S}_{*q}(\mathbb{R}_q)$ (respectively $\mathcal{D}_{*q}(\mathbb{R}_q)$) and we have

$$C_g = \int_0^\infty a^3\big|\mathcal{F}_q(f)(a)\big|^2\,d_qa.$$

Indeed: The change of variable $u = a\lambda$ leads to:

(1) $$\int_0^\infty \left|\mathcal{F}_q(g)(a\lambda)\right|^2 \frac{d_q a}{a} = \int_0^\infty \left|\mathcal{F}_q(g)(u)\right|^2 \frac{d_q u}{u}.$$

(2) $$C_g = \int_0^\infty \left|\mathcal{F}_q(g)(a)\right|^2 \frac{d_q a}{a} = \int_0^\infty \left|\mathcal{F}_q(\Delta_q f)(a)\right|^2 \frac{d_q a}{a} = \int_0^\infty a^4 \left|\mathcal{F}_q(f)(a)\right|^2 \frac{d_q a}{a} = \int_0^\infty a^3 \left|\mathcal{F}_q(f)(a)\right|^2 d_q a.$$

Proposition 7. *Let $g \neq 0$ be a function in $L_q^2(\mathbb{R}_{q,+}, x\,d_q x)$ satisfying*:

(1) *$\mathcal{F}_q(g)$ is continuous at* 0.
(2) *$\exists \beta > 0$ such that*

$$\mathcal{F}_q(g)(x) - \mathcal{F}_q(g)(0) = O\left(x^\beta\right) \quad \text{as } x \to 0.$$

Then, (37) *is equivalent to*

$$\mathcal{F}_q(g)(0) = 0. \tag{38}$$

Proof. • We suppose that (37) is satisfied.
If $\mathcal{F}_q(g)(0) \neq 0$, then from the condition (1) there exist $p_0 \in \mathbb{N}$ and $M > 0$, such that

$$\forall n \geqslant p_0, \quad \left|\mathcal{F}_q(g)\left(q^n\right)\right| \geqslant M.$$

Then, the q-integral in (37) would be equal to ∞.
• Conversely, we suppose that $\mathcal{F}_q(g)(0) = 0$.
As $g \neq 0$, we deduce from Theorem 2, that the first inequality in (37) is true.
On the other hand, from the condition (2), there exist $n_0 \in \mathbb{N}$ and $\epsilon > 0$, such that for all $n \geqslant n_0$,

$$\left|\mathcal{F}_q(g)\left(q^n\right)\right| \leqslant \epsilon q^{n\beta}.$$

Then using the definition of the q-integral and Theorem 2, we obtain

$$\begin{aligned}
\int_0^\infty \left|\mathcal{F}_q(g)(a)\right|^2 \frac{d_q a}{a} &= (1-q) \sum_{n=-\infty}^{\infty} \left|\mathcal{F}_q(g)\left(q^n\right)\right|^2 \\
&= (1-q) \sum_{n=-\infty}^{n_0} \left|\mathcal{F}_q(g)\left(q^n\right)\right|^2 + (1-q) \sum_{n=n_0+1}^{\infty} \left|\mathcal{F}_q(g)\left(q^n\right)\right|^2 \\
&\leqslant \frac{(1-q)}{q^{2n_0}} \sum_{n=-\infty}^{\infty} q^{2n} \left|\mathcal{F}_q(g)\left(q^n\right)\right|^2 + (1-q)\epsilon^2 \sum_{n=0}^{\infty} q^{2n\beta} \\
&\leqslant \frac{\|\mathcal{F}_q(g)\|_{2,q}^2}{q^{2n_0}} + \frac{1-q}{1-q^{2\beta}} \epsilon^2 \\
&= (1-q)^2 \frac{\|g\|_{2,q}^2}{q^{2n_0}} + \frac{1-q}{1-q^{2\beta}} \epsilon^2.
\end{aligned}$$

This proves the second inequality of (37). □

Remark 4. Owing to (26), the continuity assumption in the previous proposition will certainly hold if $f \in L_q^1(\mathbb{R}_{q,+}, x\,d_q x) \cap L_q^2(\mathbb{R}_{q,+}, x\,d_q x)$. Then (38) can be equivalently written as

$$\int_0^\infty g(x) x\,d_q x = 0.$$

Theorem 3. *Let $a \in \mathbb{R}_{q,+}$ and $g \in L^1_q(\mathbb{R}_{q,+}, x\,d_qx)$ (respectively $L^2_q(\mathbb{R}_{q,+}, x\,d_qx)$). Then, the function g_a defined for $x \in \mathbb{R}_{q,+}$ by*

$$g_a(x) = \frac{1}{a^2} g\left(\frac{x}{a}\right) \tag{39}$$

satisfies:

(i) *the function g_a belongs to $L^1_q(\mathbb{R}_{q,+}, x\,d_qx)$ (respectively $L^2_q(\mathbb{R}_{q,+}, x\,d_qx)$) and we have*

$$\|g_a\|_{1,q} = \|g\|_{1,q} \quad \left(\text{respectively } \|g_a\|_{2,q} = \frac{1}{a}\|g\|_{2,q}\right); \tag{40}$$

(ii) *for all $\lambda \in \mathbb{R}_{q,+}$, we have*

$$\mathcal{F}_q(g_a)(\lambda) = \mathcal{F}_q(g)(a\lambda). \tag{41}$$

Proof. The change of variable $u = \frac{x}{a}$ leads to:

$$\int_0^\infty |g_a(x)| x\,d_qx = \frac{1}{a^2}\int_0^\infty \left|g\left(\frac{x}{a}\right)\right| x\,d_qx = \int_0^\infty |g(u)| u\,d_qu,$$

$$\int_0^\infty |g_a(x)|^2 x\,d_qx = \frac{1}{a^4}\int_0^\infty \left|g\left(\frac{x}{a}\right)\right|^2 x\,d_qx = \frac{1}{a^2}\int_0^\infty |g(u)|^2 u\,d_qu$$

and for $\lambda \in \mathbb{R}_{q,+}$,

$$\mathcal{F}_q(g_a)(\lambda) = \frac{1}{a^2(1-q)}\int_0^\infty g\left(\frac{x}{a}\right) J_0\big(\lambda x; q^2\big) x\,d_qx = \frac{1}{1-q}\int_0^\infty g(u) J_0\big(a\lambda u; q^2\big) u\,d_qu = \mathcal{F}_q(g)(a\lambda). \qquad \square$$

Proposition 8. *Let g be in $\mathcal{S}_{*q}(\mathbb{R}_q)$ (respectively $\mathcal{D}_{*q}(\mathbb{R}_q)$). Then for all $a \in \mathbb{R}_{q,+}$ the function g_a given by the relation* (39) *belongs to $\mathcal{S}_{*q}(\mathbb{R}_q)$ (respectively $\mathcal{D}_{*q}(\mathbb{R}_q)$).*

Theorem 4. *Let $g \in L^2_q(\mathbb{R}_{q,+}, x\,d_qx)$ be a q-wavelet associated with the operator Δ_q. Then for all $a \in \mathbb{R}_{q,+}$ and $b \in \widetilde{\mathbb{R}}_{q,+}$, the function*

$$g_{a,b} = \sqrt{a}\,T_{b,q}(g_a) \tag{42}$$

is a q-wavelet associated with the operator Δ_q in $L^2_q(\mathbb{R}_{q,+}, x\,d_qx)$ and we have

$$C_{g_{a,b}} = a\int_0^\infty \left(J_0\left(\frac{xb}{a}; q^2\right)\right)^2 \left|\mathcal{F}_q(g)(x)\right|^2 \frac{d_qx}{x}. \tag{43}$$

Here $T_{b,q}$, $b \in \widetilde{\mathbb{R}}_{q,+}$, are the q-generalized translations defined by the relation (22).

Proof. As g_a is in $L^2_q(\mathbb{R}_{q,+}, x\,d_qx)$, Remark 3 shows that the relation (42) defines an element of $L^2_q(\mathbb{R}_{q,+}, x\,d_qx)$. On the other hand, we have for all $a \in \mathbb{R}_{q,+}$ and all $b \in \widetilde{\mathbb{R}}_{q,+}$,

$$\begin{aligned} C_{g_{a,b}} &= \int_0^\infty \left|\mathcal{F}_q(g_{a,b})(x)\right|^2 \frac{d_qx}{x} = \int_0^\infty a\left|\mathcal{F}_q\big(T_{b,q}(g_a)\big)(x)\right|^2 \frac{d_qx}{x} = a\int_0^\infty \big(J_0\big(bx; q^2\big)\big)^2 \left|\mathcal{F}_q(g_a)(x)\right|^2 \frac{d_qx}{x} \\ &= a\int_0^\infty \big(J_0\big(bx; q^2\big)\big)^2 \left|\mathcal{F}_q(g)(ax)\right|^2 \frac{d_qx}{x} = a\int_0^\infty \left(J_0\left(\frac{xb}{a}; q^2\right)\right)^2 \left|\mathcal{F}_q(g)(x)\right|^2 \frac{d_qx}{x}. \end{aligned}$$

This relation implies (43).

Now, we shall prove that the function $g_{a,b}$ satisfies the admissibility relation (37).

As $g \neq 0$, we deduce from (43) and Theorem 1 that $C_{g_{a,b}} \neq 0$. On the other hand, from the relations (37) and (24), we deduce that

$$C_{g_{a,b}} \leqslant aC_g < \infty,$$

which gives the result. □

Definition 3. Let g be a q-wavelet associated with the q-Bessel operator Δ_q in $L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$. We define the continuous q-wavelet transform associated with the q-Bessel operator by

$$\Psi_{q,g}(f)(a,b) = \frac{1}{1-q}\int_0^\infty f(x)\overline{g_{a,b}}(x)x\,d_qx, \quad a \in \mathbb{R}_{q,+},\ b \in \widetilde{\mathbb{R}}_{q,+} \text{ and } f \in L_q^2(\mathbb{R}_{q,+}, x\,d_qx). \tag{44}$$

Remark 5. If $g \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx) \cap L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$, then from Theorem 2 and the relation (36), the relation (44) can also be written in the form

$$\begin{aligned}\Psi_{q,g}(f)(a,b) &= \sqrt{a}\, f * \overline{g_a}(b) = \sqrt{a}\mathcal{F}_q\big(\mathcal{F}_q(f *_B \overline{g_a})\big)(b) = \sqrt{a}\mathcal{F}_q\big(\mathcal{F}_q(f).\big(\mathcal{F}_q\overline{(g_a)}\big)\big)(b)\\ &= \frac{\sqrt{a}}{1-q}\int_0^\infty \mathcal{F}_q(f)(x)\mathcal{F}_q(\bar{g})(ax)J_0\big(bx;q^2\big)x\,d_qx.\end{aligned}$$

We give some properties of $\Psi_{q,g}$ in the following proposition.

Proposition 9. *Let g be a q-wavelet associated with the q-Bessel operator Δ_q in $L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$, then:*

(i) *For $f \in L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$, $a \in \mathbb{R}_{q,+}$ and $b \in \widetilde{\mathbb{R}}_{q,+}$, we have*

$$\big|\Psi_{q,g}(f)(a,b)\big| \leqslant \frac{1}{(1-q)a^{1/2}}\|f\|_{2,q}\|g\|_{2,q}. \tag{45}$$

(ii) *For $f \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx) \cap L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$ and $a \in \mathbb{R}_{q,+}$, the function $b \mapsto \Psi_{q,g}(f)(a,b)$ is continuous on $\widetilde{\mathbb{R}}_{q,+}$ and we have*

$$\lim_{b\to\infty} \Psi_{q,g}(f)(a,b) = 0. \tag{46}$$

(iii) *For f in $L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$ and $a \in \mathbb{R}_{q,+}$, the function $b \mapsto \Psi_{q,g}(f)(a,b)$ is in $L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$.*

Proof. (i) For $a \in \mathbb{R}_{q,+}$ and $b \in \widetilde{\mathbb{R}}_{q,+}$, we have

$$\begin{aligned}\big|\Psi_{q,g}(f)(a,b)\big| &= \frac{1}{1-q}\left|\int_0^\infty f(x)\overline{g_{a,b}}(x)x\,d_qx\right| \leqslant \frac{\sqrt{a}}{1-q}\int_0^\infty \big|f(x)\big|\big|T_{q,b}g_a(x)\big|x\,d_qx\\ &\leqslant \frac{1}{(1-q)a^{1/2}}\|f\|_{2,q}\|g\|_{2,q},\end{aligned}$$

by using the relations (33) and (40).

(ii) For the induced topology on $\widetilde{\mathbb{R}}_{q,+}$ by that of $\mathbb{R}$, the function $b \mapsto \Psi_{q,g}(f)(a,b)$ is continuous on $\mathbb{R}_{q,+}$, since every element of $\mathbb{R}_{q,+}$ is an isolated point of $\widetilde{\mathbb{R}}_{q,+}$. So, it suffices to prove the continuity at 0. For $b \in \widetilde{\mathbb{R}}_{q,+}$, we have

$$\Psi_{q,g}(f)(a,b) = \sqrt{a}\mathcal{F}_q\big[\mathcal{F}_q(f).\mathcal{F}_q(\overline{g_a})\big](b) = \frac{\sqrt{a}}{1-q}\int_0^\infty \mathcal{F}_q(f)(x).\mathcal{F}_q(\overline{g_a})(x)J_0\big(bx;q^2\big)x\,d_qx$$

and

$$\forall x \in \mathbb{R}_{q,+}, \quad \left|J_0\left(bx; q^2\right)\right| \leqslant 1.$$

Since $f, g \in L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$, then by Theorem 2, $\mathcal{F}_q(f)$ and $\mathcal{F}_q(\overline{g_a})$ are in $L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$.

So, the product $\mathcal{F}_q(f).\mathcal{F}_q(\overline{g_a})$ is in $L_q^1(\mathbb{R}_{q,+}, x\,d_qx)$. Thus, by application of the Lebesgue theorem, we obtain

$$\lim_{\substack{b \to 0 \\ b \in \widetilde{\mathbb{R}}_{q,+}}} \Psi_{q,g}(f)(a,b) = \lim_{\substack{b \to 0 \\ b \in \widetilde{\mathbb{R}}_{q,+}}} \sqrt{a} \int_0^\infty \mathcal{F}_q(f)(x).\mathcal{F}_q(\overline{g_a})(x) J_0\left(bx; q^2\right) x\,d_qx = \Psi_{q,g}(f)(a,0),$$

which proves the continuity of $\Psi_{q,g}(f)(a,.)$ at 0.

Finally (26) implies that

$$\Psi_{q,g}(a,b) = \sqrt{a}\mathcal{F}_q\left[\mathcal{F}_q(f).\mathcal{F}_q(\overline{g_a})\right](b)$$

tends to 0 as b tends to ∞.

(iii) is an immediate consequence of the relation

$$\Psi_{q,g}(f)(a,b) = \sqrt{a} f *_B \overline{g_a}(b)$$

and the properties of the q-Bessel convolution product. □

Theorem 5. *Let $g \in L_q^1(\mathbb{R}_{q,+}, x\,d_qx) \cap L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$ be a q-wavelet associated with the q-Bessel operator Δ_q.*

(i) (Plancheral formula for $\Psi_{q,g}$) *For $f \in L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$, we have*

$$\frac{1}{C_g} \int_0^\infty \int_0^\infty \left|\Psi_{q,g}(f)(a,b)\right|^2 b \frac{d_qb\,d_qa}{a^2} = \|f\|_{2,q}^2. \tag{47}$$

(ii) (Parseval formula for $\Psi_{q,g}$) *For $f_1, f_2 \in L_q^2(\mathbb{R}_{q,+}, x\,d_qx)$, we have*

$$\int_0^\infty f_1(x)\overline{f}_2(x)x\,d_qx = \frac{1}{C_g} \int_0^\infty \int_0^\infty \Psi_{q,g}(f_1)(a,b)\overline{\Psi_{q,g}}(f_2)(a,b) b \frac{d_qa\,d_qb}{a^2}. \tag{48}$$

Proof. (i) By using Fubini's theorem, Theorem 2 and the relations (41) and (36), we get

$$\begin{aligned}
\int_0^\infty \int_0^\infty \left|\Psi_{q,g}(f)(a,b)\right|^2 b \frac{d_qa\,d_qb}{a^2} &= \int_0^\infty \left(\int_0^\infty |f *_B \overline{g_a}|^2(b) b\,d_qb \right) \frac{d_qa}{a} \\
&= \int_0^\infty \left(\int_0^\infty \left|\mathcal{F}_q(f)(x)\right|^2 \left|\mathcal{F}_q(\overline{g_a})\right|^2(x) x\,d_qx \right) \frac{d_qa}{a} \\
&= \int_0^\infty \left|\mathcal{F}_q(f)(x)\right|^2 \left(\int_0^\infty \left|\mathcal{F}_q(g)(ax)\right|^2 \frac{d_qa}{a} \right) x\,d_qx \\
&= C_g \int_0^\infty \left|\mathcal{F}_q(f)(x)\right|^2 x\,d_qx = C_g \|f\|_{2,q}^2.
\end{aligned}$$

The relation (47) is then proved.

(ii) The result is easily deduced from (47). □

Theorem 6. *Let $g \in L^1_q(\mathbb{R}_{q,+}, x\, d_qx) \cap L^2_q(\mathbb{R}_{q,+}, x\, d_qx)$ be a q-wavelet associated with the q-Bessel operator Δ_q, then for all $f \in L^2_q(\mathbb{R}_{q,+}, x\, d_qx)$, we have*

$$f(x) = \frac{1}{(1-q)C_g} \int_0^\infty \int_0^\infty \Psi_{q,g}(f)(a,b) g_{a,b}(x) b \frac{d_qa\, d_qb}{a^2}, \quad x \in \mathbb{R}_{q,+}. \tag{49}$$

Proof. For $x \in \mathbb{R}_{q,+}$, we have $h = \delta_x$ belongs to $L^2_q(\mathbb{R}_{q,+}, x\, d_qx)$. On the other hand, according to the relation (48) of the previous theorem, the definition of $\Psi_{q,g}$ and the definition of the q-Jackson integral, we have

$$\begin{aligned}
(1-q)x^2 f(x) &= \int_0^\infty f(t)\bar{h}(t)t\, d_qt = \frac{1}{C_g} \int_0^\infty \int_0^\infty \Psi_{q,g}(f)(a,b)\overline{\Psi_{q,g}}(h)(a,b) b \frac{d_qa\, d_qb}{a^2} \\
&= \frac{1}{(1-q)C_g} \int_0^\infty \int_0^\infty \Psi_{q,g}(f)(a,b) \left(\int_0^\infty \bar{h}(t) g_{a,b}(t) t\, d_qt \right) b \frac{d_qa\, d_qb}{a^2} \\
&= x^2 \frac{1}{C_g} \int_0^\infty \int_0^\infty \Psi_{q,g}(f)(a,b) g_{a,b}(x) b \frac{d_qa\, d_qb}{a^2},
\end{aligned}$$

thus

$$f(x) = \frac{1}{(1-q)C_g} \int_0^\infty \int_0^\infty \Psi_{q,g}(f)(a,b) g_{a,b}(x) b \frac{d_qa\, d_qb}{a^2},$$

which completes the proof. □

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