J. Math. Anal. Appl. 344 (2008) 898–905

Contents lists available at ScienceDirect

J. Math. Anal. Appl.

www.elsevier.com/locate/jmaa

# Non-collision periodic solutions of second order singular dynamical systems

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ARTICLE INFO

*Article history:*
Received 22 November 2007
Available online 28 March 2008
Submitted by J.J. Nieto

*Keywords:*
Non-collision periodic solutions
Second order dynamical systems
Strong singularity
Weak singularity
Fixed point theorem in cones

ABSTRACT

We study the existence of non-collision periodic solutions for second order singular dynamical systems. The repulsive case and the attractive case are dealt with using a unified topological approach. The proof is based on a well-known fixed point theorem for completely continuous operators, involving a new type of cone. We do not need to consider so-called strong force conditions. Moreover, for the repulsive case, the critical case can be covered. Recent results in the literature, even in the scalar case, are complemented, generalized and improved.



## 1. Introduction

We study the existence of non-collision periodic solutions of the second order non-autonomous dynamical system

$$\ddot{x} + a(t)x = f(t, x), \tag{1.1}$$

or

$$-\ddot{x} + a(t)x = f(t, x), \tag{1.2}$$

where $a$ is a continuous, 1-periodic scalar function, in short $a \in \mathbb{C}(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, and $f \in \mathbb{C}((\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$ is a continuous vector-valued function. By a non-collision periodic solution, we mean a function $x \in \mathbb{C}^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$ solving (1.1) and such that $x(t) \neq 0$ for all $t$.

Given $x = (x_1, \dots, x_N)$, $y = (y_1, \dots, y_N) \in \mathbb{R}^N$, the usual scalar product is denoted by $\langle x, y\rangle$, that is

$$\langle x, y\rangle = \sum_{i=1}^{N} x_i y_i.$$

The usual Euclidean norm is denoted by $|x|_2 = \sqrt{\langle x, x\rangle}$. We are mainly interested in systems with a singularity at $x = 0$, which means, there exists a vector $v \in \mathbb{R}^N$, $|v|_2 = 1$ such that

$$\lim_{x \to 0,\, x \in C} \langle v, f(t, x)\rangle = +\infty, \tag{1.3}$$

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1 D. Franco was partially supported by MEC (Spain) and FEDER, grant MTM2007-60679.

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2008.03.041

here $\mathcal{C}$ denotes the following cone in $\mathbb{R}^N$

$$\mathcal{C} = \mathcal{C}_{\gamma,v} = \{x \in \mathbb{R}^N\colon\ \langle v, x\rangle \geqslant \gamma |x|_*\}, \tag{1.4}$$

where $\gamma \in (0, 1]$ is some fixed number and $|\cdot|_*$ is a norm in $\mathbb{R}^N$. Then (1.1) presents a singularity of repulsive type whereas (1.2) has an attractive singularity.

We remark that the cone (1.4) was first employed in [9] in connection with fixed point theorems in cones and we believe that it is a natural setting when dealing with systems. We also use the norm $|x|_1 = \sum_{i=1}^N |x_i|$ for $x \in \mathbb{R}^N$. Note that $\mathcal{C}$ is just the cone $\overline{\mathbb{R}^N_+}$ if we take

$$v = \left(\frac{1}{\sqrt{N}}, \dots, \frac{1}{\sqrt{N}}\right), \qquad \gamma = \frac{1}{\sqrt{N}}, \qquad |\cdot|_* = |\cdot|_1. \tag{1.5}$$

The question of existence of non-collision periodic solutions for scalar equations and dynamical systems has attracted much attention during the last two decades [1,7,8,16,23,28]. Usually, in the literature, the proof is based on variational methods [2,19–22], or topological methods, which were started with the pioneering paper of Lazer and Solimini [15]. In particular, the method of upper and lower solutions [3], degree theory [27], some fixed point theorems in cones for completely continuous operators [25], Schauder's fixed point theorem [10] and a nonlinear alternative principle of Leray–Schauder type [5] are the most relevant tools. Usually, the proof requires some strong force condition, which was first introduced with this name by Gordon in [13]. For example, if we consider the system

$$\ddot{x} + \nabla_x V(t, x) = p(t) \tag{1.6}$$

with $V(t, x) = -\frac{1}{(|x|_2)^\alpha}$, the strong force condition corresponds to the case $\alpha \geqslant 2$. There are also some works concerning the existence of periodic solutions under the presence of weak singularities [5,8–10]. Here we remark that, even in the scalar case, the existence of periodic solutions for singular problems has commanded much attention in recent years [4,15,17,18, 24,26].

Among those interesting results obtained in the literature, we recall several very recent results for (1.1) or (1.2), which motivated our study. In [9], using a fixed point theorem in cones, it was proved that (1.1) or (1.2) has at least one non-collision periodic solution assuming the nonlinearity $f$ satisfies suitable properties in one direction, which imply that $f$ neither needs to be positive nor to have a constant sign behavior. In [5], by employing a nonlinear alternative principle of Leray–Schauder, it was proved that the system

$$x_i'' + a(t)x_i = \frac{1}{(|x|_2)^\alpha} + e_i(t), \quad i = 1, 2, \dots, N, \tag{1.7}$$

has at least one positive periodic solution, where $x = (x_1, \dots, x_N) \in \mathbb{C}^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$, $\alpha > 0$, and for each $i = 1, 2, \dots, N$, $e_i$ satisfies $\int_0^1 e_i(t)\,dt \geqslant 0$. See [5, Theorem 3.1].

The results in [5,9] can be applicable to the case of a strong singularity as well as the case of a weak singularity and they complement those in [18], but they are not comparable. The assumptions in [18] involve a uniform lower bound on function $e$ in (1.7) without imposing any restriction from above, but it does not include some rather natural cases. On the other hand, the assumptions in [9,24] cannot handle unbounded forcing terms, but impose some kind of restriction over the oscillation of $e$. We also notice that the positivity of the Green function plays a very important role in [5,9], and therefore they cannot cover the critical case, such as $k = \pi$ when $a(t) \equiv k^2$, whereas the result in [18] covers such a case.

In this paper, we generalize and improve the above known results. The repulsive case and the attractive case are dealt with using a unified topological approach. The new results can cover both a strong singularity and a weak singularity. We do not need the positivity of the Green function, and therefore, for the repulsive case, the critical case can be covered as in [10,18,26]. On the other hand, we shed some new light on an open question stated in [26] (see point (4) in Remark 3.5), which was only partially answered very recently in [5]. The proof of our results is based on a fixed point theorem in cones. Some fixed point theorems in cones have been extensively applied recently [11,24].

The remaining part of the paper is organized as follows. In Section 2, some preliminary results are given. In Section 3, the main results are stated and proved. The periodically forced Lagrangian systems (1.6) are also considered. Some illustrating examples are also given.

## 2. Preliminaries

Let $a(t)$ be taken in $\mathbb{C}(\mathbb{R}/\mathbb{Z}, \mathbb{R})$. Throughout this paper, we assume that the Hill equation

$$x'' + a(t)x = 0 \tag{2.1}$$

associated with the periodic boundary conditions

$$x(0) = x(1), \qquad x'(0) = x'(1) \tag{2.2}$$

satisfies the following hypothesis:

(A) The Hill equation (2.1) is non-resonant and the Green function $G(t,s)$, associated with (2.1)–(2.2) verifies $\int_0^1 G(t,s)\,ds > 0$ for all $t$.

In other words, the anti-maximum principle holds for (2.1)–(2.2). When $a(t) \equiv k^2$, condition (A) is equivalent to $0 < k^2 \leqslant \lambda_1 = \pi^2$. In this case, we have

$$G(t,s) = \begin{cases} \frac{\sin k(t-s)+\sin k(1-t+s)}{2k(1-\cos k)}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \frac{\sin k(s-t)+\sin k(1-s+t)}{2k(1-\cos k)}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

and

$$0 \leqslant G(t,s) \leqslant \frac{1}{2k\sin\frac{k}{2}}, \qquad \int_0^1 G(t,s)\,ds = \frac{1}{k^2}.$$

See [24]. For a non-constant function $a(t)$, there is not an explicit expression of the Green function, but there is an $L^p$-criterion proved in [24], which has been used in the related literature, see for instance [4,5,10,26]. Here we omit the statement.

Under hypothesis (A), we always denote

$$\tau = \left(\min_{0\leqslant t\leqslant 1} \int_0^1 G(t,s)\,ds\right)^{-1}, \qquad \nu = \left(\max_{0\leqslant t\leqslant 1} \int_0^1 G(t,s)\,ds\right)^{-1}, \tag{2.3}$$

and

$$M = \max_{0\leqslant s,t\leqslant 1} G(t,s), \qquad \sigma = \frac{1}{\tau M}. \tag{2.4}$$

One may readily see that $0 < \sigma \leqslant 1$. When $a(t) \equiv k^2$ and $0 < k \leqslant \pi$, we have

$$\tau = \nu = k^2, \qquad M = \frac{1}{2k\sin\frac{k}{2}}, \qquad \sigma = \frac{2}{k}\sin\frac{k}{2}.$$

The proof of the main results in this paper is based on the following well-known fixed point theorem in cones, which can be found in [12]. Let $K$ be a cone in $X$ and $D$ a subset of $X$, we write $D_K = D \cap K$ and $\partial_K D = (\partial D) \cap K$.

**Theorem 2.1.** *Let $X$ be a Banach space and $K$ a cone in $X$. Assume $\Omega^1$, $\Omega^2$ are open bounded subsets of $X$ with $\Omega^1_K \neq \emptyset$, $\overline{\Omega}^1_K \subset \Omega^2_K$. Let*

$$T : \overline{\Omega}^2_K \setminus \Omega^1_K \to K$$

*be a continuous and completely continuous operator such that either*

- *$Tx \neq \lambda x$ for $\lambda > 1$ and $x \in \partial_K \Omega^1$,*
- *there exists $\hat{e} \in K \setminus \{0\}$ such that $x \neq Tx + \lambda\hat{e}$ for all $x \in \partial_K \Omega^2$ and all $\lambda > 0$;*

*or*

- *$Tx \neq \lambda x$ for $\lambda > 1$ and $x \in \partial_K \Omega^2$,*
- *there exists $\hat{e} \in K \setminus \{0\}$ such that $x \neq Tx + \lambda\hat{e}$ for all $x \in \partial_K \Omega^1$ and all $\lambda > 0$.*

*Then $T$ has a fixed point in $\overline{\Omega}^2_K \setminus \Omega^1_K$.*

In the applications below, we shall denote by $\mathcal{E} = \mathcal{BC}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$ the Banach space of bounded continuous periodic functions $x : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^N$ with the norm $\|x\| = \max_t |x(t)|_*$. For a fixed vector $v \in \mathbb{R}^N$, $|v|_2 = 1$, let $\mathcal{E}_{\mathcal{C}} = \mathcal{BC}(\mathbb{R}/\mathbb{Z}, \mathcal{C})$, here $\mathcal{C}$ is the cone given as (1.4). We will use the related new cone in $\mathcal{E}$ that is defined by

$$K = \left\{ x \in \mathcal{E}_{\mathcal{C}}:\ \int_0^1 \langle v, x(t)\rangle\,dt \geqslant \sigma \max_t \langle v, x(t)\rangle \right\}. \tag{2.5}$$

**Lemma 2.2.** *The set $K$ defined by* (2.5) *is a cone in the Banach space $\mathcal{E}$.*

**Proof.** Clearly $K$ is closed. Since $0 < \sigma \leqslant 1$, $x(t) \equiv v$ belongs to $K$, and thus $K$ is nonempty. Moreover, for $x, y \in K$ and $a, b \in \mathbb{R}^+$, we have

$$\int_0^1 \langle v, ax(t) + by(t)\rangle\, dt = a\int_0^1 \langle v, x(t)\rangle\, dt + b\int_0^1 \langle v, y(t)\rangle\, dt \geqslant a\sigma \max_t \langle v, x(t)\rangle + b\sigma \max_t \langle v, y(t)\rangle \geqslant \sigma \max_t \langle v, ax(t) + by(t)\rangle.$$

This inequality together with the fact that $\mathcal{C}$ is a cone guarantees that $ax + by \in K$, and therefore $K$ is a cone. $\square$

For each $r > 0$, we define the following two open sets

$$\Omega^r = \left\{ x \in \mathcal{E}\colon \int_0^1 \langle v, x(t)\rangle\, dt < \sigma r \right\},$$

$$B^r = \left\{ x \in \mathcal{E}\colon \max_t \langle v, x(t)\rangle < r \right\}.$$

**Lemma 2.3.** *For each $r > 0$, $\Omega^r$ and $B^r$ defined above have the following properties*:

(a) *$\Omega^r_K$ and $B^r_K$ are open relative to $K$.*
(b) *$B^{\sigma r}_K \subset \Omega^r_K \subset B^r_K$.*
(c) *$x \in \partial_K \Omega^r$ if and only if $x \in K$ and $\int_0^1 \langle v, x(t)\rangle = \sigma r$.*
(d) *For each $\delta > r$, we have $\Omega^r_K = (\Omega^r \cap B^\delta)_K$, $\overline{\Omega^r}_K = (\overline{\Omega^r \cap B^\delta})_K$.*

**Proof.** (a) is trivial because $\Omega^r$ and $B^r$ are open sets. (c) is clear since, for each $x \in K$, we have

$$\int_0^1 \langle v, x(t)\rangle\, dt \geqslant \sigma \max_t \langle v, x(t)\rangle.$$

For each $x \in B^{\sigma r}_K$, $\max_t \langle v, x(t)\rangle < \sigma r$. Then, since $x \in K$,

$$\int_0^1 \langle v, x(t)\rangle\, dt \leqslant \int_0^1 \max_t \langle v, x(t)\rangle\, dt < \sigma r$$

and $x \in \Omega^r_K$.

For each $x \in \Omega^r_K$, $\int_0^1 \langle v, x(t)\rangle\, dt < \sigma r$. Then

$$\max_t \langle v, x(t)\rangle \leqslant \frac{1}{\sigma} \int_0^1 \langle v, x(t)\rangle\, dt < r,$$

and therefore $x \in B^r_K$. So (b) has been proved.

Next we prove (d). The first equality follows immediately from (b). For the second let $x \in \overline{\Omega^r}_K$, then from (c) we have

$$\sigma \max_t \langle v, x(t)\rangle \leqslant \int_0^1 \langle v, x(t)\rangle\, dt \leqslant \sigma r < \sigma \delta.$$

Therefore, $x \in (\overline{\Omega^r} \cap B^\delta) \cap K$. Now, since $\Omega^r$ and $B^\delta$ are open sets we have $\overline{\Omega^r} \cap B^\delta \subset \overline{\Omega^r \cap B^\delta}$. Thus $x \in (\overline{\Omega^r \cap B^\delta})_K$, and therefore $\overline{\Omega^r}_K \subseteq (\overline{\Omega^r \cap B^\delta})_K$. The reverse inclusion is trivial. $\square$

**Remark 2.4.** For each $r > 0$, although the sets $\Omega^r$ and $B^r$ are unbounded, we can use Theorem 2.1 with $\Omega^r$ and $B^r$ taking into account (d) in Lemma 2.3, because $B^R_K$ is bounded for each $R > 0$. To see this, we only need to remember that $\langle v, x(t)\rangle \geqslant \gamma |x(t)|_*$ for each $t$. Therefore, we can choose adequate open bounded sets.

## 3. Main results

In this section, we state and prove the main results. First we recall that $\mathcal{C}$ denotes the cone defined by (1.4).

**Theorem 3.1.** *Assume that there exists a vector $v \in \mathbb{R}^N$, $|v|_2 = 1$ such that* (1.3) *holds. Furthermore, suppose that there exists $R > 0$ such that*

($H_1$) *$f(t,x) \in \mathcal{C}$ for each $t \in \mathbb{R}$ and $x \in \mathcal{C}$ with $0 < \langle v, x\rangle \leqslant R$;*
($H_2$) *$\langle v, f(t,x)\rangle \leqslant \nu\langle v, x\rangle$ for each $t \in \mathbb{R}$ and $x \in \mathcal{C}$ with $\sigma R \leqslant \langle v, x\rangle \leqslant R$.*

*Then* (1.1) *has at least one non-collision periodic solution $x \in \mathcal{E}_{\mathcal{C}}$.*

**Proof.** By the singularity condition (1.3), we can choose a positive number $r$ small enough such that $r < \sigma R$ and

$$\big\langle v, f(t,x)\big\rangle \geqslant \tau\langle v, x\rangle \quad \text{for each } t \in \mathbb{R} \text{ and } x \in \mathcal{C} \text{ with } \sigma r \leqslant \langle v, x\rangle \leqslant r. \tag{3.1}$$

It is well known that finding a solution for (1.1) in $\mathcal{E}_{\mathcal{C}}$ is equivalent to finding a fixed point for the operator $T : \mathcal{E}_{\mathcal{C}} \to \mathcal{E}_{\mathcal{C}}$ defined by

$$Tx(t) = \int_0^1 G(t,s) f\big(s, x(s)\big)\, ds. \tag{3.2}$$

Define the open sets $\Omega^1, \Omega^2$ by $\Omega^1 = \Omega^r$ and $\Omega^2 = B^R$. One may readily verify that $T : \overline{\Omega}^2_K \setminus \Omega^1_K \to \mathcal{E}_{\mathcal{C}}$ is well defined. In fact, for $t \in \mathbb{R}$ and $x \in \mathcal{E}_{\mathcal{C}}$ with $\sigma r \leqslant \langle v, x\rangle \leqslant R$, since ($H_1$) holds, we have

$$\langle v, Tx(t)\rangle = \int_0^1 G(t,s)\big\langle v, f\big(s,x(s)\big)\big\rangle ds \geqslant \gamma \int_0^1 G(t,s)\big|f\big(s,x(s)\big)\big|_* ds \geqslant \gamma \left| \int_0^1 G(t,s) f\big(s,x(s)\big)\, ds \right|_* \geqslant \gamma \big|Tx(t)\big|_*.$$

Thus $Tx(t) \in \mathcal{C}$ for all $t$ and $T : \overline{\Omega}^2_K \setminus \Omega^1_K \to \mathcal{E}_{\mathcal{C}}$ is well defined.

Next we show that $T(\overline{\Omega}^2_K \setminus \Omega^1_K) \subset K$. In fact, for $x \in \overline{\Omega}^2_K \setminus \Omega^1_K$, we have

$$\big\langle v, Tx(t)\big\rangle = \int_0^1 G(t,s)\big\langle v, f\big(s,x(s)\big)\big\rangle ds \leqslant M \int_0^1 \big\langle v, f\big(s,x(s)\big)\big\rangle ds.$$

Therefore,

$$\max_t \big\langle v, Tx(t)\big\rangle \leqslant M \int_0^1 \big\langle v, f\big(s,x(s)\big)\big\rangle ds,$$

and we obtain

$$\begin{aligned}\int_0^1 \big\langle v, Tx(t)\big\rangle dt &= \int_0^1 \int_0^1 G(t,s)\big\langle v, f\big(s,x(s)\big)\big\rangle ds\, dt = \int_0^1 \left( \int_0^1 G(t,s)\, dt \right) \big\langle v, f\big(s,x(s)\big)\big\rangle ds \\ &\geqslant \tau^{-1} \int_0^1 \big\langle v, f\big(s,x(s)\big)\big\rangle ds \geqslant \sigma \max_t \big\langle v, Tx(t)\big\rangle.\end{aligned}$$

Therefore $T(\overline{\Omega}^2_K \setminus \Omega^1_K) \subset K$. One may readily verify that $T : \overline{\Omega}^2_K \setminus \Omega^1_K \to \mathcal{E}_{\mathcal{C}}$ is completely continuous since $f$ is continuous for $t \in \mathbb{R}$ and $x \in \mathcal{C}$ with $\sigma r \leqslant \langle v, x\rangle \leqslant R$.

We claim that:

(i) $Tx \neq \lambda x$ for $\lambda > 1$ and $x \in \partial_K \Omega^2$,
(ii) there exists $\hat{e} \in K \setminus \{0\}$ such that $x \neq Tx + \lambda\hat{e}$ for all $x \in \partial_K \Omega^1$ and all $\lambda > 0$.

We start with (i). Suppose that there exist $x \in \partial_K \Omega^2$ and $\lambda > 1$ such that $Tx = \lambda x$. Since $x \in \partial_K \Omega^2$, we have $\sigma R \leqslant \langle v, x(t)\rangle \leqslant R$ for all $t$ and $\langle v, x(t^*)\rangle = R$ for some $t^*$. Following from ($H_2$), we have

$$\big\langle v, f(t,x)\big\rangle \leqslant \nu\langle v, x\rangle, \quad \text{for each } t \in \mathbb{R}.$$

Therefore,

$$R = \langle v, x(t^*)\rangle < \lambda\langle v, x(t^*)\rangle = \langle v, Tx(t^*)\rangle = \int_0^1 G(t^*, s)\langle v, f(s, x(s))\rangle\, ds \leqslant \nu \int_0^1 G(t^*, s)\langle v, x(s)\rangle\, ds$$

$$\leqslant \nu \max_t \langle v, x(t)\rangle \int_0^1 G(t^*, s)\, ds \leqslant \max_t \langle v, x(t)\rangle \leqslant R.$$

This is a contradiction, and therefore claim (i) holds.

Next we consider part (ii). Let $\hat{e} \equiv v$, then $\hat{e} \in K$. Suppose that there exist $x \in \partial_K \Omega^1$ and $\lambda > 0$ such that $x = Tx + \lambda\hat{e}$. From Lemma 2.3(c) we have

$$\sigma r = \int_0^1 \langle v, x(t)\rangle\, dt.$$

Using (3.1), we obtain

$$\int_0^1 \langle v, x(t)\rangle\, dt = \int_0^1 \langle v, Tx(t)\rangle\, dt + \int_0^1 \langle v, \lambda v\rangle\, dt = \int_0^1 \int_0^1 G(t, s)\langle v, f(s, x(s))\rangle\, ds\, dt + \lambda$$

$$\geqslant \tau \int_0^1 \left( \int_0^1 G(t, s)\, dt \right) \langle v, x(s)\rangle\, ds + \lambda \geqslant \int_0^1 \langle v, x(t)\rangle\, dt + \lambda.$$

This implies $\sigma r \geqslant \sigma r + \lambda$, which is a contradiction, and therefore (ii) holds.

It follows from Theorem 2.1 that $T$ has a fixed point $x \in \overline{\Omega}_K^2 \setminus \Omega_K^1$. Clearly, this fixed point is a non-collision periodic solution of (1.1). □

Now we apply Theorem 3.1 to (1.6). Assume that $V \in \mathbb{C}((\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$ satisfies

$$\lim_{x \to 0} V(t, x) = +\infty \tag{3.3}$$

and there exists a fixed vector $v \in \mathbb{R}^N$, $|v|_2 = 1$ such that

$$\lim_{x \to 0,\, x \in \mathcal{C}} \langle v, \nabla V(t, x)\rangle = -\infty. \tag{3.4}$$

The following theorem is a direct consequence of Theorem 3.1, and it improves those in [9].

**Theorem 3.2.** *Assume that* (3.3)–(3.4) *are satisfied. Then* (1.6) *has at least one non-collision periodic solution if there exist* $0 < k \leqslant \pi$, $R > 0$, *such that*

($H_3$) $p(t) + k^2 x - \nabla V(t, x) \in \mathcal{C}$ *for each* $t \in \mathbb{R}$ *and* $x \in \mathcal{C}$ *with* $0 < \langle v, x\rangle \leqslant R$;
($H_4$) $\langle v, p(t)\rangle \leqslant \langle v, \nabla V(t, x)\rangle$ *for each* $t \in \mathbb{R}$ *and* $x \in \mathcal{C}$ *with* $\sigma R \leqslant \langle v, x\rangle \leqslant R$.

Using Theorem 3.2, we can easily obtain the following result for the system

$$\ddot{x} + k^2 x = \nabla\left(\frac{1}{|x|_2^\alpha}\right) + p(t). \tag{3.5}$$

**Corollary 3.3.** *Assume that there exists* $v \in \mathbb{R}^N$, $|v|_2 = 1$ *such that* $\langle v, p(t)\rangle$ *as a function of* $t$ *does not change sign. Then* (3.5) *has at least one non-collision periodic solution for any* $0 < k \leqslant \pi$ *and* $\alpha > 0$.

**Corollary 3.4.** *Assume that* $a(t) \equiv k^2$, $0 < k \leqslant \pi$, *and each component of* $f$ *is given by*

$$f_i(t, x) = b\left(\sum_{i=1}^N x_i\right)^{-\lambda} + e(t), \quad i = 1, 2, \ldots, N,$$

*here* $b, \lambda > 0$ *and* $e$ *is a continuous scalar function on* $[0, 1]$. *Let* $e^* = \max_t e(t)$, $e_* = \min_t e(t)$. *Then* (1.1) *has at least one non-collision periodic solution if one of the following two conditions holds*:

(i) $e_* \geqslant 0$,
(ii) $e_* < 0$ *and* $e^* \leqslant \frac{k^\lambda}{2^\lambda \sin^\lambda \frac{k}{2}} e_* + \frac{2k}{N}(\frac{b}{|e_*|})^{\frac{1}{\lambda}} \sin\frac{k}{2}$.

**Proof.** Take $v, \gamma$ and $|\cdot|_*$ as in (1.5). Then $f(t,x) \in \mathcal{C}$ for each $t \in \mathbb{R}$ and $x \in \mathcal{C}$ with $0 < \langle v, x\rangle < +\infty$ because each component of $f$ is nonnegative.

In our case, we have

$$\langle v, f\rangle = \sqrt{N} b \left(\sum_{i=1}^{N} x_i\right)^{-\lambda} + \sqrt{N} e(t), \qquad \langle v, x\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i.$$

Conditions of Theorem 3.1 reduce to finding $R > 0$ such that

$$b\left(\sum_{i=1}^{N} x_i\right)^{-\lambda} + e_* \geqslant 0, \qquad 0 < \sum_{i=1}^{N} x_i \leqslant \sqrt{N} R, \tag{3.6}$$

and

$$Nb\left(\sum_{i=1}^{N} x_i\right)^{-\lambda} + Ne^* \leqslant k^2 \sum_{i=1}^{N} x_i, \qquad \sqrt{N}\sigma R \leqslant \sum_{i=1}^{N} x_i \leqslant \sqrt{N} R. \tag{3.7}$$

(i) is clear since $\lambda > 0$ and $e_* \geqslant 0$. Next we prove the result (ii). It is easy to see that (3.6) holds if we fix

$$R = \frac{1}{\sqrt{N}} \left(\frac{b}{|e_*|}\right)^{\frac{1}{\lambda}}.$$

Since the function $k^2 s - \frac{b}{s^\lambda}$, $s > 0$, is nondecreasing, (3.7) holds if (ii) is satisfied. $\square$

**Remark 3.5.** Consider the following scalar equation

$$x'' - \frac{b}{x^\lambda} + k^2 x = e(t) \tag{3.8}$$

with $b, \lambda > 0$, $0 < k \leqslant \pi$, and $e \in \mathbb{C}[0,1]$. Using Corollary 3.4, (3.8) has at least one positive periodic solution if $e_* \geqslant 0$ or $e_* < 0$ and satisfying:

$$e^* \leqslant \frac{k^\lambda}{2^\lambda \sin^\lambda \frac{k}{2}} e_* + 2k\left(\frac{b}{|e_*|}\right)^{\frac{1}{\lambda}} \sin\frac{k}{2}.$$

Such a result improves those in the literature in the following three directions:

(1) It improves those in [9,24] because it can cover the critical case.
(2) It improves the result in [18] because it can cover the case $e_* = 0$ when dealing with the critical case.
(3) It complements the result in [18] because the condition imposes some kind of restriction over the difference en $e^* - e_*$ in which $e_*$ can be under the bound given in [18].
(4) Related to (3.8), Torres posed an open problem in [26], which can be stated as "whether (3.8) has periodic solutions when $b > 0, \lambda \geqslant 1$, $0 < k \leqslant \pi$ and $\min_t \int_0^1 G(t,s)e(s)\,ds = 0$." Now we have given a partial positive answer because we can cover the case $e_* = 0$, and in some cases $e^* < 0$, under a strong singularity.

Finally, we consider the system with an attractive singularity (1.2). When $a(t)$ is positive, then the linear problem

$$-x'' + a(t)x = e(t)$$

with periodic boundary conditions (2.2) has a positive Green's function [6,14]. In other words, the hypothesis (A) holds. Then, the problem of finding a periodic solution of system (1.2) is expressed as a fixed point problem for the same operator defined in (3.2). This means that all the results obtained in this section are automatically valid for the system (1.2). For instance, the counterpart of Theorem 3.1 for the attractive case is as follows.

**Theorem 3.6.** *Assume that $a(t) \succ 0$ and there exists a vector $v \in \mathbb{R}^N$, $|v|_2 = 1$ such that* (1.3) *holds. Then* (1.2) *has at least one non-collision periodic solution if there exists $R > 0$ such that* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$ *hold.*


## Acknowledgments

The authors express their thanks to the referees for their valuable suggestions.

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