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The pointwise estimates of solutions for semilinear dissipative wave equation in multi-dimensions

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ABSTRACT

In this paper, we study the global-in-time existence and the pointwise estimates of solutions to the Cauchy problem for the dissipative wave equation in multi-dimensions. Using the fixed point theorem, we obtain the global existence of the solution. In addition, the pointwise estimates of the solution are obtained by the method of the Green function. Furthermore, we obtain the L^p, $1 \leqslant p \leqslant \infty$, convergence rate of the solution.

1. Introduction

In this paper, we study the pointwise estimates of the solution $u(x,t)$ to the Cauchy problem for the dissipative wave equation in multi-dimensions

$$\begin{cases} \partial_t^2 u - \Delta_x u + \partial_t u = f(u), & x \in \mathbb{R}^n,\ t > 0, \\ (u, \partial_t u)(x, 0) = (u_0, u_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geqslant 3$, $f(u) = O(u^{\theta+1})$ and θ is a positive integer. Eq. (1.1) is often called the semilinear dissipative wave equation or semilinear telegraph equation.

Let us first review some of the works in this field. There are many results on (1.1) corresponding to different forms of the nonlinear term. For $f(u) = -|u|^\theta u$, Kawashima, Nakao and Ono [8] studied the decay properties of solutions to (1.1) by using the energy method combined with L^p–L^q estimates. Ono [19] derived sharp decay rates in the subcritical case of solutions in unbounded domains in $\mathbb{R}^n$ without the smallness condition on the initial data. Besides, by employing the weighted L^2 energy method, Nishihara and Zhao [18] obtained that the behavior of solutions to (1.1) as $t \to \infty$ is expected to be the same as that for the corresponding heat equation, and Nishihara [17] studied the global asymptotic behaviors in three and four dimensions. When $n \geqslant 1$, Ikehata, Nishihara and Zhao [6] obtained the decay properties of solutions to the Cauchy problem (1.1). This work extended the initial data class in [18] to a wider class. In [11], the pointwise estimates of solutions to (1.1) are obtained by Liu.

As referred to above, it is worth to mentioning a recent result due to Liu in [11]. She considered the global existence and the pointwise estimates of the solution to (1.1) for $f(u) = -|u|^\theta u$. In her paper, the existence of the solution was obtained by using the energy method. However, if $f(u) = |u|^\theta u$ or $f(u) = \pm|u|^{\theta+1}$, the usual energy method does not work well in proving the existence of the solution. The reason is that we cannot control the lower order term in making energy estimates

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(see [11]). In this paper, to overcome this difficulty, we employ a different approach which is showed later. Furthermore, for $f(u) = O(u^{\theta+1})$, we obtain the pointwise estimates of the solution by using the Green function. In addition, for $f(u) = |u|^\theta u$ or $f(u) = \pm|u|^{\theta+1}$, there are also some other methods in obtaining the global existence of the solution to (1.1). In [16], with the explicit formula of the fundamental solution to (1.1), Nishihara showed the global existence of the weak solution and its decay estimates in the three-dimensional space. Nakao and Ono in [14] proved the global existence and decay of weak solutions for (1.1) under the assumption that $\|u_1\| + \|\nabla u_0\|$ is small by using the potential well method. They also obtained the classical solution of (1.1) when $n = 3$ and $\theta = 2$. By the way, the constant θ in this paper satisfying $1 + \theta \geqslant 2 > 1 + \frac{4}{n}$, and $1 + \frac{4}{n}$ was proposed in [14] by them.

In particular, for the case $f(u) = |u|^\theta u$, the semilinear Cauchy problem (1.1) has been investigated by many authors. Ikehata, Miyaoka and Nakatake [5] have obtained the global existence of the weak solution to (1.1) and its decay for $\frac{2}{n} < \theta \leqslant \frac{n}{[n-2]^+} - 1$ ($n = 1, 2, 3$ and $[a]^+ = \max\{a, 0\}$). Furthermore, Hosono and Ogawa [3] obtained the L^p–L^q type estimate of the difference between the solution to (1.1) and the solutions of corresponding heat and wave equations in the two-dimensional space. They employed the Fourier transform and observed the detailed asymptotic behavior of the fundamental solution to (1.1). Meanwhile, when $2 \leqslant n \leqslant 5$, the same type estimate has been studied by Narazaki in [15].

There also have been a lot of investigations for those cases. For their results, please refer to [4,7,10,20,21,23,25].

The main purpose of this paper is to study the existence of the global classical solution of (1.1) and its pointwise estimates in the multi-dimensional space. First, no matter what kind of $f(u) = O(u^{\theta+1})$ is, we obtain the solution by using a unified method. We use the Green function to express the solution to (1.1) and obtain the pointwise estimates of the Green function, the same as in [13]. Compared with the methods in [3,15,16], the method in this paper is more useful to show a clear structure of the solution. Besides, we bypass the difficulty in energy estimating and obtain the global solution directly without proving the existence of the local solution (also see [9]). Then, we give the pointwise estimates of the solution to (1.1) by using the method of the Green function. Finally, as a corollary of the pointwise estimates, the optimal L^p $(1 \leqslant p \leqslant \infty)$ convergence rate can be obtained easily.

The following is the main theorem in this paper.

Theorem 1.1. *Let s and θ be integers such that $s > n$, $\theta > 2/n$, let δ be a positive constant, let $(u_0, u_1) \in (W^{s,1} \cap H^{s+1}) \times (W^{s,1} \cap H^s)$, and let $f = f(v)$ be a function of class C^s. Assume that*

$$\left|\partial_v^k f(v)\right| \leqslant C_{k,\delta}|v|^{\theta+1-k}, \quad |v| \leqslant \delta,\ 0 \leqslant k \leqslant s,\ k < \theta + 1,$$
$$\left|\partial_v^k f(v)\right| \leqslant C_{k,\delta}, \quad |v| \leqslant \delta,\ \text{if}\ \theta + 1 \leqslant k \leqslant s,$$

and

$$\left|\partial_v^s f(v_1) - \partial_v^s f(v_2)\right| \leqslant C_{k,\delta}|v_1 - v_2|, \quad |v_1|, |v_2| \leqslant \delta$$

when $s \geqslant \theta$. If $\|u_0\|_{s+1} + \|u_1\|_s + \|u_0\|_{s,1} + \|u_1\|_{s,1}$ is sufficiently small, then (1.1) admits a unique, global, classical solution $u(x,t)$. Moreover, if $s \geqslant 2n$, for any multi-index α, $|\alpha| < s - n/2$, there exists some constant $r > n/2$ such that

$$\left|D_x^\alpha u_0(x)\right| + \left|D_x^\alpha u_1(x)\right| \leqslant C\left(1 + |x|^2\right)^{-r},$$

then for $|\alpha| < n$, the solution to Eq. (1.1) has the following estimates

$$\left|D_x^\alpha u(x,t)\right| \leqslant C(1+t)^{-(n+|\alpha|)/2} B_r\left(|x|, t\right),$$

where $B_r(|x|, t) = (1 + |x|^2/(1+t))^{-r}$.

Notations. In what follows, we denote generic positive constants by C. $W^{m,p}(\mathbb{R}^n)$, $m \in \mathbb{Z}_+$, $p \in [1, \infty]$, denotes the usual Sobolev space with its norm

$$\|f\|_{W^{m,p}(\mathbb{R}^n)} := \sum_{|k|=0}^{m} \left\|\partial_x^k f\right\|_{L^p}.$$

In particular, we use $W^{m,2}(\mathbb{R}^n) = H^m(\mathbb{R}^n)$, $\|\cdot\| = \|\cdot\|_{L^2}$, $\|\cdot\|_{m,p} = \|\cdot\|_{W^{m,p}}$ and $\|\cdot\|_m = \|\cdot\|_{H^m}$.

The rest of paper is organized as follows. In the next section, we show the estimates of the Green function. Then the global existence of solutions is proved in Section 3. Furthermore, the pointwise estimates of solutions for nonlinear equations are obtained in Section 4.

2. Pointwise estimates of Green function

Now we study the Green function for (1.1), i.e. we consider the solution to the following initial value problem:

$$\begin{cases} (\partial_t^2 - \Delta_x + \partial_t)G(x,t) = 0, & x \in \mathbb{R}^n,\ t > 0, \\ G(x,0) = 0, & x \in \mathbb{R}^n, \\ \partial_t G(x,0) = \delta(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

By Fourier transform with respect to the variable x, we deduce that

$$\begin{cases} (\partial_t^2 + \partial_t)\hat{G}(\xi,t) + |\xi|^2\hat{G}(\xi,t) = 0, & \xi \in \mathbb{R}^n,\ t > 0, \\ \hat{G}(\xi,0) = 0, & \xi \in \mathbb{R}^n, \\ \partial_t \hat{G}(\xi,0) = 1, & \xi \in \mathbb{R}^n. \end{cases} \tag{2.2}$$

The symbol of the operator for Eq. (2.1) is

$$\tau^2 + \tau + |\xi|^2 = 0. \tag{2.3}$$

Here, τ and $\xi = (\xi_1, \ldots, \xi_n)$ correspond to $\frac{\partial}{\partial t}$ and $(D_{x_1}, \ldots, D_{x_n})$ respectively, where $D_{x_i} = (1/\sqrt{-1})(\partial/\partial x_j)$, $j = 1, \ldots, n$. It is easy to see that

$$\tau = \lambda_\pm(\xi) = \frac{1}{2}\left(-1 \pm \sqrt{1 - 4|\xi|^2}\right). \tag{2.4}$$

By a direct calculation, we have

$$\hat{G}(\xi,t) = \left(1 - 4|\xi|^2\right)^{-\frac{1}{2}}\left(e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}\right). \tag{2.5}$$

In the following, we are going to obtain some properties of the Green function $G(x,t)$.
For convenience we decompose $\hat{G}(\xi,t) = \hat{G}^+(\xi,t) + \hat{G}^-(\xi,t)$, where

$$\hat{G}^\pm(\xi,t) = \pm\lambda_0^{-1}e^{\lambda_\pm(\xi,t)}, \quad \lambda_0(\xi) = \left(1 - 4|\xi|^2\right)^{\frac{1}{2}}.$$

Let

$$\chi_1(\xi) = \begin{cases} 1, & |\xi| < \varepsilon, \\ 0, & |\xi| > 2\varepsilon, \end{cases} \qquad \chi_3(\xi) = \begin{cases} 1, & |\xi| > R, \\ 0, & |\xi| < R - 1, \end{cases}$$

be the smooth cut-off functions, where ε and R are any fixed positive numbers satisfying $2\varepsilon < R - 1$.
Set

$$\chi_2(\xi) = 1 - \chi_1(\xi) - \chi_3(\xi),$$

and

$$\hat{G}_i^\pm(\xi,t) = \chi_i(\xi)\hat{G}^\pm(\xi,t), \quad i = 1,2,3. \tag{2.6}$$

We are going to study $G_i(x,t)$, which is the inverse Fourier transform corresponding to $\hat{G}_i(\xi,t)$.
Before estimating $G_i(x,t)$, we give three lemmas which will be used later. The proofs of those lemmas can be found in [12,13,24], so we omit them here.

Lemma 2.1. *If supp $\hat{g}(\xi) \subset O_R := \{\xi;\ |\xi| > R\}$, and $\hat{g}(\xi)$ satisfies*

$$\left|\hat{g}(\xi)\right| \leqslant C, \qquad \left|D_\xi^\beta \hat{g}(\xi)\right| \leqslant C|\xi|^{-1-|\beta|}, \quad |\beta| \geqslant 1, \tag{2.7}$$

then there exist distributions $g_1(x)$, $g_2(x)$ and a constant C_0 such that

$$g(x) = g_1(x) + g_2(x) + C_0\delta(x), \tag{2.8}$$

where $\delta(x)$ is the Dirac function. Furthermore, for a positive integer $2N > n + |\alpha|$,

$$\left|D_x^\alpha g_1(x)\right| \leqslant C\left(1 + |x|^2\right)^{-N}, \tag{2.9}$$

$$\|g_2\|_{L_1} \leqslant C, \qquad supp\, g_2(x) \subset \left\{x;\ |x| < 2\varepsilon_1\right\}, \tag{2.10}$$

with ε_1 being sufficiently small.

Lemma 2.2. *For any $N > 0$, $\tau \geqslant 0$, and multi-index α, we have that*

$$\int_{|z|\leqslant 1} \frac{(1+\frac{|x+tz|^2}{1+\tau})^{-N}}{\sqrt{1+\tau}}\, dV_z \leqslant C(1+t)^{2N}\left(1+\frac{|x|^2}{1+\tau}\right)^{-N}. \tag{2.11}$$

Lemma 2.3. *Assume that $h(x) \in C^\infty(\mathbb{R}^n)$, and for any multi-index α,*

$$\left|D_x^\alpha h(x)\right| \leqslant C\left(1+|x|^2\right)^{-N},$$

then

$$\left|\int_{|z|\leqslant 1} \frac{D^\alpha h(x+tz)z^\alpha}{\sqrt{1-|z|^2}}\, dz\right| \leqslant C(1+t)^{2N+1}\left(1+\frac{|x|^2}{1+t}\right)^{-N}. \tag{2.12}$$

Now we consider $G_i(x,t)$. Firstly, we give two propositions regarding to $G_1(x,t)$ and $G_2(x,t)$. The proofs of them can be found in [13].

Proposition 2.1. *For sufficiently small ε, there exist a constant $C > 0$, and $N > n$ such that*

$$\left|\partial_t^l D_x^\alpha G_1(x,t)\right| \leqslant C(1+t)^{-(n+|\alpha|+2l)/2} B_N\big(|x|,t\big), \quad l = 0, 1. \tag{2.13}$$

Proposition 2.2. *For fixed ε and R, there exist positive numbers b, C and $N > n$ such that*

$$\left|\partial_t^l D_x^\alpha G_2(x,t)\right| \leqslant Ce^{-bt} B_N\big(|x|,t\big), \quad l = 0, 1. \tag{2.14}$$

The following Kirchhoff formulas can be seen in [1,2].

Lemma 2.4. *Assume that $\omega(x,t)$ is the fundamental solution of the following wave equation with $c = 1$,*

$$\begin{cases} \omega_{tt} - c^2\Delta\omega = 0, \\ \omega|_{t=0} = 0, \\ \partial_t\omega|_{t=0} = \delta(x). \end{cases}$$

There are constants a_α, b_α depending only on the spatial dimension $n \geqslant 1$ such that, if $h \in C^\infty(\mathbb{R}^n)$, then

$$(\omega * h)(x,t) = \sum_{0\leqslant|\alpha|\leqslant\frac{n-3}{2}} a_\alpha t^{|\alpha|+1} \int_{|z|=1} D^\alpha h(x+tz)z^\alpha\, dS_z,$$
$$(\omega_t * h)(x,t) = \sum_{0\leqslant|\alpha|\leqslant\frac{n-1}{2}} b_\alpha t^{|\alpha|} \int_{|z|=1} D^\alpha h(x+tz)z^\alpha\, dS_z,$$

for odd n, and

$$(\omega * h)(x,t) = \sum_{0\leqslant|\alpha|\leqslant\frac{n-2}{2}} a_\alpha t^{|\alpha|+1} \int_{|z|\leqslant 1} \frac{D^\alpha h(x+tz)z^\alpha}{\sqrt{1-|z|^2}}\, dz,$$
$$(\omega_t * h)(x,t) = \sum_{0\leqslant|\alpha|\leqslant\frac{n}{2}} b_\alpha t^{|\alpha|} \int_{|z|\leqslant 1} \frac{D^\alpha h(x+tz)z^\alpha}{\sqrt{1-|z|^2}}\, dz,$$

for even n. Here dS_z denotes surface measure on the unit sphere in $\mathbb{R}^n$.

By Lemmas 2.2–2.4, we have the following lemma.

Lemma 2.5. *Assume that $h \in C^\infty(\mathbb{R}^n)$, and for any multi-index α,*

$$\left|D_x^\alpha h(x)\right| \leqslant C\left(1+|x|^2\right)^{-N},$$

then we have that

$$|\omega * h| \leqslant C(1+t)^{2N+1+\frac{n}{2}}\left(1+\frac{|x|^2}{1+t}\right)^{-N}, \tag{2.15}$$

and

$$|\omega_t * h| \leqslant C(1+t)^{2N+1+\frac{n}{2}}\left(1+\frac{|x|^2}{1+t}\right)^{-N}, \tag{2.16}$$

where ω is the fundamental solution of the wave equation as in Lemma 2.4.

Now, for $G_3(x,t)$, when $|\xi|$ is sufficiently large, we have

$$\begin{aligned}\lambda_\pm(\xi) &= \frac{1}{2}\big(-1 \pm |\xi|\sqrt{|\xi|^{-2}-4}\big)\\ &= \frac{1}{2}\left(-1 \pm 2\sqrt{-1}|\xi| \pm \left(\sum_{j=1}^{m-1} a_j|\xi|^{1-2j}\right)\right) + O\big(|\xi|^{1-2m}\big),\end{aligned}$$

and

$$\lambda_0^{-1}(\xi) = \big(1-4|\xi|^2\big)^{-\frac{1}{2}} = |\xi|^{-1}\left(-\frac{\sqrt{-1}}{2} + O\big(|\xi|^{-2}\big)\right).$$

This implies that

$$e^{\lambda_\pm(\xi)t} = e^{-t/2}e^{\pm\sqrt{-1}|\xi|t}\left(1+\left(\sum_{j=1}^{m-1}(\pm a_j)|\xi|^{1-2j}\right)t + \cdots + \frac{1}{m!}\left(\sum_{j=1}^{m-1}(\pm a_j)|\xi|^{1-2j}\right)^m t^m + R^\pm(\xi,t)\right),$$

where $R^\pm(\xi,t) \leqslant (1+t)^{m+1}(1+|\xi|)^{1-2m}$.

On the other hand, we denote that

$$\hat{\omega}(\xi,t) = (2\pi)^{-n/2}\sin\big(|\xi|t\big)/|\xi|, \qquad \hat{\omega}_t(\xi,t) = (2\pi)^{-n/2}\cos\big(|\xi|t\big).$$

By a direct and a little tedious calculation, we get that

$$\begin{aligned}\hat{G}_3(\xi,t) &= e^{-t/2}\hat{\omega}_t\left(\sum_{j=1}^{2m-2} p_{1j}^0(t)q_j(\xi) + \hat{R}_1^0(\xi,t)\right)\\ &\quad + e^{-t/2}\hat{\omega}\left(C_1\chi_3(\xi) + \sum_{j=1}^{2m-2} p_{2j}^0(t)q_j(\xi) + \hat{R}_2^0(\xi,t)\right),\end{aligned}$$

and

$$\begin{aligned}\partial_t\hat{G}_3(\xi,t) &= e^{-t/2}\hat{\omega}_t\left(C_2\chi_3(\xi) + \sum_{j=1}^{2m-2} p_{1j}^1(t)q_j(\xi) + \hat{R}_1^1(\xi,t)\right)\\ &\quad + e^{-t/2}\hat{\omega}\left(p_0^1(t)\chi_3(\xi) + C_3\chi_3(\xi)|\xi| + \sum_{j=1}^{2m-2} p_{2j}^1(t)q_j(\xi) + \hat{R}_2^1(\xi,t)\right),\end{aligned}$$

here

$$\begin{aligned}&p_{kj}^l(t) \leqslant C(1+t)^j, \qquad q_j(\xi) = \chi_3(\xi)|\xi|^{-j}, \quad 1 \leqslant j \leqslant 2m-2;\ l=0,1;\ k=1,2;\\ &p_0^1(t) \leqslant C(1+t); \quad C_1, C_2, C_3 \text{ are some constants};\\ &\big|\hat{R}_1^0(\xi,t)\big|, \big|\hat{R}_2^0(\xi,t)\big| \leqslant C(1+t)^{m+1}\big(1+|\xi|\big)^{1-2m},\\ &\big|\hat{R}_1^1(\xi,t)\big|, \big|\hat{R}_2^1(\xi,t)\big| \leqslant C(1+t)^{m+1}\big(1+|\xi|\big)^{2-2m}.\end{aligned}$$

Since $|\xi| > R$, it is observed that

$$\big|\hat{G}_3(\xi,t)\big| + \big|\partial_t\hat{G}_3(\xi,t)\big| \leqslant Ce^{-t/4}. \tag{2.17}$$

In the following, we use $q_0(D_x), q_0^1(D_x), q_j(D_x)$ $(1 \leqslant j \leqslant 2m-2)$, $\omega(D_x,t), \omega_t(D_x,t)$, $R_1^0(D_x,t), R_2^0(D_x,t), R_1^1(D_x,t)$, $R_2^1(D_x,t)$ to denote the pseudo-differential operators with symbols $\chi_3(\xi), \chi_3(\xi)|\xi|, q_j(\xi)$ $(1 \leqslant j \leqslant 2m-2)$, $\hat{\omega}(\xi,t), \hat{\omega}_t(\xi,t)$, $\hat{R}_1^0(\xi,t), \hat{R}_2^0(\xi,t), \hat{R}_1^1(\xi,t), \hat{R}_2^1(\xi,t)$ respectively.

Lemma 2.6. *For R being sufficiently large, there exist distributions $\bar{q}_0(x), \tilde{q}_0(x)$; $\bar{q}_0^1(x), \tilde{q}_0^1(x)$; $\bar{q}_j(x), \tilde{q}_j(x)$, $1 \leqslant j \leqslant 2m-2$ and a constant C_0 such that*

$$q_0(D_x)\delta(x) = \bar{q}_0(x) + \tilde{q}_0(x) + C_0\delta(x),$$
$$q_0^1(D_x)\delta(x) = (-\Delta)^{\frac{1}{2}}\big(\bar{q}_0^1(x) + \tilde{q}_0^1(x) + C_0\delta(x)\big),$$
$$q_j(D_x)\delta(x) = \bar{q}_j(x) + \tilde{q}_j(x) + C_0\delta(x), \quad 1 \leqslant j \leqslant 2m-2,$$

and

$$\big|D_x^\alpha \bar{q}_0(x)\big|, \big|D_x^\alpha \bar{q}_0^1(x)\big|, \big|D_x^\alpha \bar{q}_j(x)\big| \ (1 \leqslant j \leqslant 2m-2) \leqslant C\big(1+|x|^2\big)^{-N},$$
$$\|\tilde{q}_0\|_{L^1}, \big\|\tilde{q}_0^1\big\|_{L^1}, \|\tilde{q}_j\|_{L^1} \ (1 \leqslant j \leqslant 2m-2) \leqslant C,$$
$$\operatorname{supp} \tilde{q}_0, \operatorname{supp} \tilde{q}_0^1, \operatorname{supp} \tilde{q}_j \ (1 \leqslant j \leqslant 2m-2) \subset \big\{x;\ |x| \leqslant 2\varepsilon_1\big\},$$

with ε_1 being sufficiently small.

Proof. It is easy to get that

$$\big|D_\xi^\beta \chi_3(\xi)\big|, \big|D_\xi^\beta\big(|\xi|^{-1}\chi_3(\xi)\big)\big|, \big|D_\xi^\beta q_j(\xi)\big| \ (1 \leqslant j \leqslant 2m-2) \leqslant C|\xi|^{-1-|\beta|},$$
$$\operatorname{supp} q_j(\xi) \ (1 \leqslant j \leqslant 2m-2) \subset \big\{\xi;\ |\xi| > R\big\}.$$

By using Lemma 2.1, we complete the proof of Lemma 2.6. □

Let

$$Q_0(x) = \tilde{q}_0(x) + C_0\delta(x),$$
$$Q_0^1(x) = \tilde{q}_0^1(x) + C_0\delta(x),$$
$$Q_j(x) = \tilde{q}_j(x) + C_0\delta(x), \quad 1 \leqslant j \leqslant 2m-2,$$

and

$$L_0^0(x,t) = C_1\omega(D_x,t)Q_0(x),$$
$$L_0^1(x,t) = C_2\omega_t(D_x,t)Q_0(x) + p_0^1(t)\omega(D_x,t)Q_0(x) + C_3\omega(D_x,t)(-\Delta)^{\frac{1}{2}}Q_0^1(x),$$
$$L_j^0(x,t) = p_{1j}^0(t)\omega_t(D_x,t)Q_j(x) + p_{2j}^0(t)\omega_t(D_x,t)Q_j(x), \quad 1 \leqslant j \leqslant 2m-2,$$
$$L_j^1(x,t) = p_{1j}^1(t)\omega_t(D_x,t)Q_j(x) + p_{2j}^1(t)\omega_t(D_x,t)Q_j(x), \quad 1 \leqslant j \leqslant 2m-2.$$

We have the following proposition. The proof of it is omitted here. The interested reader can found it in [13].

Proposition 2.3. *For R sufficiently large, there exists distribution*

$$K_m^l(x,t) = e^{-t/2}\left(L_0^l(x,t) + \sum_{j=1}^{2m-2} L_j^l(x,t)\right), \quad l = 0, 1, \tag{2.18}$$

such that $m \geqslant [\frac{|\alpha|+n+4}{2}]$, we have that

$$\big|D_x^\alpha\big(\partial_t^l G_3 - K_m^l\big)(x,t)\big| \leqslant Ce^{-t/4}B_N\big(|x|,t\big), \quad l = 0, 1. \tag{2.19}$$

Then, by Propositions 2.1–2.3, we obtain the following estimates of the Green function.

Theorem 2.1. *For any multi-index α, and $m \geqslant [\frac{|\alpha|+n+4}{2}]$, we have that*

$$\big|D_x^\alpha\big(\partial_t^l G - K_m^l\big)(x,t)\big| \leqslant C(1+t)^{-\frac{n+|\alpha|+2l}{2}}B_N\big(|x|,t\big), \quad l = 0, 1. \tag{2.20}$$

3. The global existence of solution

3.1. Basic lemmas

First of all, we show some useful lemmas. The proof of the following lemma can be found in [22]. We omit it.

Lemma 3.1. *Let $1 \leqslant r,\ p \leqslant \infty$, $m \in \mathbb{N}$. Then there is a constant $c > 0$ such that for all $w \in W^{m,p} \cap L^r$ the inequality*

$$\left\| D^j w \right\|_{L^q} \leqslant c \left\| D^m w \right\|_{L^p}^{j/m} \| w \|_{L^r}^{1-j/m}$$

holds, where $j \in \{0, 1, \dots, m\}$ and

$$\frac{1}{q} = \frac{j}{m}\frac{1}{p} + \left(1 - \frac{j}{m}\right)\frac{1}{r}.$$

Lemma 3.2. *Let s and θ be positive integers, let $\delta > 0$, $p, q, r \in [1, \infty]$ be such that $1/r = 1/p + 1/q$, and let $k \in \{0, 1, 2, \dots, s\}$. Let $F = F(v)$ be a function of class C^s that satisfies*

$$\left|\partial_v^l F(v)\right| \leqslant C_{l,\delta} |v|^{\theta+1-l}, \quad |v| \leqslant \delta,\ 0 \leqslant l \leqslant s,\ l < \theta + 1,$$

and

$$\left|\partial_v^l F(v)\right| \leqslant C_{l,\delta}, \quad |v| \leqslant \delta,\ l \leqslant s,\ \text{if}\ \theta + 1 \leqslant l.$$

If $v \in W^{k,q} \cap L^p \cap L^\infty$ and $\|v\|_{L^\infty} \leqslant \delta$, then

$$\left\| F(v) \right\|_{k,r} \leqslant C_{k,\delta} \|v\|_{k,q} \|v\|_{L^p} \|v\|_{L^\infty}^{\theta-1}. \tag{3.1}$$

Proof. Leibnitz's formula shows that

$$\partial_x^\alpha F(v) = \sum_{\alpha_1 + \cdots + \alpha_m = \alpha} c_{\alpha_1,\dots,\alpha_m} F^{(m)}(v) \partial_x^{\alpha_1} v \cdots \partial_x^{\alpha_m} v.$$

Consider the case where $m \geqslant 2$. Let

$$\frac{1}{r_j} = \frac{|\alpha_j|}{k} \cdot \frac{1}{q} + \left(1 - \frac{|\alpha_j|}{k}\right) \frac{1}{(m-1)p} \quad (j = 1, \dots, m).$$

Since $1/r_1 + \cdots + 1/r_m = 1/r$, Hölder's inequality and Lemma 3.1 show that

$$\begin{aligned}
\left\| F^{(m)}(v) \partial_x^{\alpha_1} v \cdots \partial_x^{\alpha_m} v \right\|_{L^r} &\leqslant C \|v\|_{L^\infty}^{\max\{\theta+1-m,0\}} \left\| \partial_x^{\alpha_1} v \right\|_{L^{r_1}} \cdots \left\| \partial_x^{\alpha_m} v \right\|_{L^{r_m}} \\
&\leqslant C \|v\|_{L^\infty}^{\max\{\theta+1-m,0\}} \|v\|_{k,q}^{|\alpha_1|/k} \|v\|_{L^{(m-1)p}}^{(1-|\alpha_1|/k)} \cdots \|v\|_{k,q}^{|\alpha_m|/k} \|v\|_{L^{(m-1)p}}^{(1-|\alpha_m|/k)} \\
&= C \|v\|_{L^\infty}^{\max\{\theta+1-m,0\}} \|v\|_{k,q} \|v\|_{L^{(m-1)p}}^{(m-1)} \\
&\leqslant C \|v\|_{L^\infty}^{\theta-1} \|v\|_{k,q} \|v\|_{L^p}.
\end{aligned} \tag{3.2}$$

When $m = 1$, Hölder's inequality again shows that

$$\left\| F'(v) \partial_x^\alpha v \right\|_r \leqslant C \left\| |v|^{\theta-1} |v| \partial_x^\alpha v \right\|_r \leqslant C \|v\|_{L^\infty}^{\theta-1} \|v\|_{k,q} \|v\|_p. \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$\left\| F(v) \right\|_{k,r} \leqslant C \sum_{\alpha_1 + \cdots + \alpha_m = \alpha} \left\| F^{(m)}(v) \partial_x^{\alpha_1} v \cdots \partial_x^{\alpha_m} v \right\|_{L^r} \leqslant C \|v\|_{L^\infty}^{\theta-1} \|v\|_{k,q} \|v\|_p.$$

Thus, the proof of Lemma 3.2 is completed. □

Lemma 3.3. *Let s and θ be positive integers, let $\delta > 0$, $p, q, r \in [1, \infty]$ be such that $1/r = 1/p + 1/q$, and let $k \in \{0, 1, 2, \dots, s\}$. Let $F = F(v)$ be a function that satisfies the assumptions of Lemma* 3.1. *Moreover assume that*

$$\left|\partial_v^s F(v_1) - \partial_v^s F(v_2)\right| \leqslant C_\delta \left(|v_1| + |v_2|\right)^{\max\{\theta-s,0\}} |v_1 - v_2|, \quad |v_1|, |v_2| \leqslant \delta.$$

If $v_1, v_2 \in W^{k,q} \cap L^p \cap L^\infty$ satisfy $\|v_1\|_{L^\infty} \leqslant \delta$ and $\|v_2\|_{L^\infty} \leqslant \delta$, then

$$\begin{aligned}
\left\| F(v_1) - F(v_2) \right\|_{k,r} \leqslant C_{k,\delta} \big\{ &\left(\|v_1\|_{k,q} + \|v_2\|_{k,q} \right) \|v_1 - v_2\|_{L^p} \\
&+ \left(\|v_1\|_{L^p} + \|v_2\|_{L^p} \right) \|v_1 - v_2\|_{k,q} \big\} \left(\|v_1\|_{L^\infty} + \|v_2\|_{L^\infty} \right)^{\theta-1}.
\end{aligned} \tag{3.4}$$

Since the proof of Lemma 3.3 is similar to that of Lemma 3.1, we omit it here.
By using Lemma 2.4, we obtain the following lemma.

Lemma 3.4. *Assume that* $0 \leqslant \tau \leqslant t$ *and* $h(\cdot,\tau) \in W^{|\alpha|,\infty}(\mathbb{R}^n)$, *for any multi-index* α *with* $|\alpha| \geqslant n/2$ *and* $2 < p \leqslant +\infty$, *then we have*

$$\left|\omega(\cdot,t-\tau) * h(\cdot,\tau)\right| \leqslant C \sum_{0\leqslant|\alpha|\leqslant\frac{n-2}{2}} a_\alpha (t-\tau)^{|\alpha|+1} \left\|D^\alpha h(\cdot,\tau)\right\|_{L^\infty}, \tag{3.5}$$

$$\left|\omega_t(\cdot,t-\tau) * h(\cdot,\tau)\right| \leqslant C \sum_{0\leqslant|\alpha|\leqslant\frac{n}{2}} b_\alpha (t-\tau)^{|\alpha|} \left\|D^\alpha h(\cdot,\tau)\right\|_{L^\infty}. \tag{3.6}$$

Lemma 3.5. *Assume that* $0 \leqslant \tau \leqslant t$ *and* $g(\cdot,\tau) \in W^{s-[\frac{n}{2}],\infty}(\mathbb{R}^n)$, *then we have*

$$\left\|K_m^l(t-\tau) * g(\tau)\right\|_{s-n-1,\infty} \leqslant Ce^{-(t-\tau)/4}\left\|g(\tau)\right\|_{s-[\frac{n}{2}]-1+l,\infty}, \quad l=0,1. \tag{3.7}$$

Proof. Firstly, by using (2.18) and the triangle inequality, we have

$$\begin{aligned}\left\|K_m^0(t-\tau) * g(\tau)\right\|_{s-n-1,\infty} &\leqslant Ce^{-(t-\tau)/2}\left(\left\|L_0^0(t-\tau) * g(\tau)\right\|_{s-n-1,\infty} + \sum_{j=1}^{2m-2}\left\|L_j^0(t-\tau) * g(\tau)\right\|_{s-n-1,\infty}\right)\\ &:= D_1 + D_2.\end{aligned}$$

By using the Young inequality, Lemmas 2.4, 2.6 and 3.4, we have

$$\begin{aligned}D_1 &\leqslant Ce^{-(t-\tau)/2}\left\|\omega(D_x,t-\tau)Q_0 * g(\tau)\right\|_{s-n-1,\infty}\\ &\leqslant Ce^{-(t-\tau)/2}\sum_{0\leqslant|\beta|\leqslant\frac{n-2}{2}}(t-\tau)^{|\beta|+1}\left\|Q_0 * D_x^\beta g(\tau)\right\|_{s-n-1,\infty}\\ &\leqslant Ce^{-(t-\tau)/2}\sum_{0\leqslant|\beta|\leqslant\frac{n-2}{2}}(t-\tau)^{|\beta|+1}\left(\left\|\tilde{q}_0 * D_x^\beta g(\tau)\right\|_{s-n-1,\infty} + \left\|D_x^\beta g(\tau)\right\|_{s-n-1,\infty}\right)\\ &\leqslant Ce^{-(t-\tau)/2}\sum_{0\leqslant|\beta|\leqslant\frac{n-2}{2}}(t-\tau)^{|\beta|+1}\left(\|\tilde{q}_0\|_{L^1}\left\|D_x^\beta g(\tau)\right\|_{s-n-1,\infty} + \left\|D_x^\beta g(\tau)\right\|_{s-n-1,\infty}\right)\\ &\leqslant Ce^{-(t-\tau)/2}(t-\tau)^{n/2}\left\|g(\tau)\right\|_{s-[\frac{n}{2}]-1,\infty}\\ &\leqslant Ce^{-(t-\tau)/4}\left\|g(\tau)\right\|_{s-[\frac{n}{2}]-1,\infty}.\end{aligned}$$

Similarly, we get the same estimate for D_2. Thus, we obtain

$$\left\|K_m^0(t-\tau) * g(\tau)\right\|_{s-n-1,\infty} \leqslant Ce^{-(t-\tau)/4}\|g\|_{s-[\frac{n}{2}]-1,\infty}.$$

Finally, for $l=1$, it is more straightforward to carry out the estimates on $\|K_m^1(t-\tau) * g(\tau)\|_{s-n-1,\infty}$ in the similar way. The details are omitted. Thus, this completes the proof of Lemma 3.5. $\square$

3.2. Global existence for the Cauchy problem

Here, we use the fixed point theorem of Banach to obtain a theorem about the global-in-time existence of the solution to (1.1).

We consider the following Cauchy problem

$$\begin{cases}\partial_t^2 u - \Delta_x u + \partial_t u = f(v), & x \in \mathbb{R}^n,\ t > 0,\\ (u,\partial_t u)(x,0) = (u_0,u_1)(x), & x \in \mathbb{R}^n.\end{cases} \tag{3.8}$$

By the Duhamel principle, the solution to (3.8) can be expressed as follows:

$$u(x,t) = G(t) * (u_0+u_1) + \partial_t G(t) * u_0 + \int_0^t G(t-\tau) * f(v)(\tau)\,d\tau. \tag{3.9}$$

To apply the contraction mapping theorem, we denote the function on the right-hand side of (3.9) by $Tv(x,t)$. Define a space $X_{t,E}$ as follows:

$$X_{t,E} = \{ v = v(x,t) \mid D_s(v) \leqslant E \},$$

where E is a positive constant and

$$D_s(v) = \sup_{t\geqslant 0}(1+t)^{\frac{n}{2}} \| v(\cdot,t)\|_{W^{s-n-1,\infty}(\mathbb{R}^n)} + \sup_{t\geqslant 0}(1+t)^{\frac{n}{4}} \| v(\cdot,t)\|_{H^s(\mathbb{R}^n)}.$$

Then $(X_{s,E}, D_s(\cdot))$ is a Banach space.

In the following lemma, we show that T is a map from $X_{s,E}$ to itself.

Lemma 3.6. *If E and $\|u_0\|_s + \|u_1\|_s + \|u_0\|_{s,1} + \|u_1\|_{s,1}$ are sufficiently small, then T is a map from $X_{s,E}$ to $X_{s,E}$.*

Proof. Firstly, we write

$$\begin{aligned}
\|Tv\|_{s-n-1,\infty} \leqslant{} & \big\| \big(G - K_m^0\big)(t) * (u_0+u_1)\big\|_{s-n-1,\infty} + \big\| \big(\partial_t G - K_m^1\big)(t) * u_0\big\|_{s-n-1,\infty} \\
& + \big\| K_m^0(t) * (u_0+u_1)\big\|_{s-n-1,\infty} + \big\| K_m^1(t) * u_0\big\|_{s-n-1,\infty} \\
& + \int_0^t \big\| \big(G - K_m^0\big)(t-\tau) * f(v)(\tau)\big\|_{s-n-1,\infty}\, d\tau \\
& + \int_0^t \big\| K_m^0(t-\tau) * f(v)(\tau)\big\|_{s-n-1,\infty}\, d\tau \\
={} & \sum_{i=1}^6 I_i.
\end{aligned}$$

For I_1, it follows from the Young inequality and Theorem 2.1 that

$$I_1 \leqslant \big\| \big(G - K_m^0\big)(t)\big\|_{L^\infty} \|u_0+u_1\|_{s-n-1,1} \leqslant C(1+t)^{-n/2}\|u_0+u_1\|_{s-n-1,1}. \tag{3.10}$$

Similarly to the estimates of I_1, we obtain

$$I_2 \leqslant C(1+t)^{-n/2}\|u_0\|_{s-n-1,1}. \tag{3.11}$$

From Lemma 3.5, it follows that

$$I_3 \leqslant C(1+t)^{-n/2}\|u_0+u_1\|_{s-[\frac{n}{2}]-1,\infty} \quad \text{and} \quad I_4 \leqslant C(1+t)^{-n/2}\|u_0+u_1\|_{s-[\frac{n}{2}],\infty}. \tag{3.12}$$

By using the Young inequality and Lemma 3.2 and noticing $n > \frac{2}{\theta}$, for I_5, we have

$$\begin{aligned}
I_5 &\leqslant \int_0^t \big\| \big(G - K_m^0\big)(t-\tau)\big\|_{L^\infty} \big\| f(v)(\tau)\big\|_{s-n-1,1}\, d\tau \\
&\leqslant C\int_0^t (1+t-\tau)^{-n/2} \big\| f(v)(\tau)\big\|_{s-n-1,1}\, d\tau \\
&\leqslant C\int_0^t (1+t-\tau)^{-n/2} \big\| v(\tau)\big\|_{s-n-1} \big\| v(\tau)\big\| \big\| v(\tau)\big\|_{L^\infty}^{\theta-1}\, d\tau \\
&\leqslant CE^{\theta+1}\int_0^t (1+t-\tau)^{-n/2}(1+\tau)^{-n\theta/2}\, d\tau \\
&\leqslant C(1+t)^{-n/2}E^{\theta+1}.
\end{aligned} \tag{3.13}$$

For I_6, it follows from Lemmas 3.2, 3.5 and the Sobolev inequality that

$$\begin{aligned}
I_6 &\leqslant C \int_0^t e^{-(t-\tau)/4} \left\| f(v)(\tau) \right\|_{s-[\frac{n}{2}]-1,\infty} d\tau \\
&\leqslant C \int_0^t e^{-(t-\tau)/4} \left\| f(v)(\tau) \right\|_s d\tau \\
&\leqslant C \int_0^t e^{-(t-\tau)/4} \left\| v(\tau) \right\|_s \left\| v(\tau) \right\|_{L^\infty}^{\theta} d\tau \\
&\leqslant C E^{\theta+1} \int_0^t e^{-(t-\tau)/4} (1+\tau)^{-n/4-(n\theta)/2} d\tau \\
&\leqslant C (1+t)^{-n/2} E^{\theta+1}.
\end{aligned} \tag{3.14}$$

Thus, the combination of (3.10)–(3.14) gives

$$\|Tv\|_{s-n-1,\infty} \leqslant C(1+t)^{-\frac{n}{2}} \big(\|u_0\|_{s-[\frac{n}{2}],\infty} + \|u_1\|_{s-[\frac{n}{2}],\infty} + \|u_0\|_{s-n-1,1} + \|u_1\|_{s-n-1,1} + E^{\theta+1} \big). \tag{3.15}$$

Now, we consider

$$\begin{aligned}
\|Tv\|_s \leqslant\; & \left\| (G-G_3)(t) * (u_0+u_1) \right\|_s + \left\| \partial_t (G-G_3)(t) * u_0 \right\|_s \\
& + \left\| G_3(t) * (u_0+u_1) \right\|_s + \left\| \partial_t G_3(t) * u_0 \right\|_s \\
& + \int_0^t \left\| (G-G_3)(t-\tau) * f(v)(\tau) \right\|_s d\tau + \int_0^t \left\| G_3(t-\tau) * f(v)(\tau) \right\|_s d\tau \\
:=\; & \sum_{i=1}^{6} J_i.
\end{aligned}$$

By using Propositions 2.1–2.2 and the Young inequality, for J_1, we obtain

$$J_1 \leqslant \left\| (G-G_3)(t) \right\| \|u_0+u_1\|_{s,1} \leqslant C(1+t)^{-n/4} \|u_0+u_1\|_{s,1}. \tag{3.16}$$

For J_3, it follows from the Plancherel theorem and (2.17) that

$$\begin{aligned}
J_3 &\leqslant \sum_{|\alpha|=0}^{s} \left\| G_3(t) * D_x^\alpha (u_0+u_1) \right\| = \sum_{|\alpha|=0}^{s} \left\| \hat{G}_3(t) D_x^\alpha \widehat{(u_0+u_1)} \right\| \\
&\leqslant C e^{-t/4} \|u_0+u_1\|_s.
\end{aligned} \tag{3.17}$$

Similarly to the estimates of J_1 and J_3, we obtain

$$J_2 \leqslant C(1+t)^{-n/4} \|u_0\|_{s,1} \quad \text{and} \quad J_4 \leqslant C e^{-t/4} \|u_0\|_s. \tag{3.18}$$

Using Lemma 3.2, the Young inequality and noticing $n > \frac{2}{\theta}$, we have

$$\begin{aligned}
J_5 &\leqslant C \int_0^t \left\| (G-G_3)(t-\tau) \right\| \left\| f(v)(\tau) \right\|_{s,1} d\tau \\
&\leqslant C \int_0^t (1+t-\tau)^{-n/4} \left\| v(\tau) \right\|_s \left\| v(\tau) \right\| \left\| v(\tau) \right\|_{L^\infty}^{\theta-1} d\tau \\
&\leqslant C E^{\theta+1} \int_0^t (1+t-\tau)^{-n/4} (1+\tau)^{-n\theta/2} d\tau \\
&\leqslant C (1+t)^{-n/4} E^{\theta+1}.
\end{aligned} \tag{3.19}$$

For J_6, by using the Plancherel theorem and (2.17), we get

$$J_6 \leqslant C \int_0^t e^{-(t-\tau)/4} \left\| f(v)(\tau) \right\|_s d\tau \leqslant C(1+t)^{-n/4} E^{\theta+1}. \tag{3.20}$$

Thus, we obtain

$$\|Tv\|_s \leqslant C(1+t)^{-n/4} \big(\|u_0\|_s + \|u_1\|_s + \|u_0\|_{s,1} + \|u_1\|_{s,1} + E^{\theta+1} \big). \tag{3.21}$$

By using (3.15), (3.21) and the small assumptions of E, u_0, u_1, we get $D_s(Tv) \leqslant E$. Thus, the proof of Lemma 3.6 is completed. $\square$

In the next lemma, we prove that T is a contraction map.

Lemma 3.7. *Assume $\bar{v}, \bar{\bar{v}} \in X_{s,E}$ and $E > 0$ is sufficiently small, then there exists a constant γ with $0 < \gamma < 1$, such that*

$$D_s(T\bar{v} - T\bar{\bar{v}}) \leqslant \gamma D_s(\bar{v} - \bar{\bar{v}}). \tag{3.22}$$

Proof. Let $u = T\bar{v} - T\bar{\bar{v}}$, it follows from (1.1) that

$$\begin{cases} \partial_t^2 u - \Delta_x u + \partial_t u = f(\bar{v}) - f(\bar{\bar{v}}), & x \in \mathbb{R}^n,\ t > 0, \\ (u, \partial_t u)(x, 0) = (0, 0), & x \in \mathbb{R}^n. \end{cases} \tag{3.23}$$

By the Duhamel principle and the triangle inequality, we have

$$\begin{aligned} \|T\bar{v} - T\bar{\bar{v}}\|_{s-n-1,\infty} &\leqslant \int_0^t \left\| \big(G - K_m^0\big)(t-\tau) * \big(f(\bar{v}) - f(\bar{\bar{v}})\big)(\tau) \right\|_{s-n-1,\infty} d\tau \\ &\quad + \int_0^t \left\| K_m^0(t-\tau) * \big(f(\bar{v}) - f(\bar{\bar{v}})\big)(\tau) \right\|_{s-n-1,\infty} d\tau \\ &:= H_1 + H_2. \end{aligned} \tag{3.24}$$

For H_1, by using the Young inequality and Lemma 3.3 and noticing $n > \frac{2}{\theta}$, we have

$$\begin{aligned} H_1 &\leqslant \int_0^t \left\| \big(G - K_m^0\big)(t-\tau) \right\|_{L^\infty} \left\| \big(f(\bar{v}) - f(\bar{\bar{v}})\big)(\tau) \right\|_{s-n-1,1} d\tau \\ &\leqslant C \int_0^t (1+t-\tau)^{-n/2} \left\| \big(f(\bar{v}) - f(\bar{\bar{v}})\big)(\tau) \right\|_{s-n-1,1} d\tau \\ &\leqslant C \int_0^t (1+t-\tau)^{-n/2} \big\{ \|\bar{v} - \bar{\bar{v}}\| \big(\|\bar{v}\|_{s-n-1} + \|\bar{\bar{v}}\|_{s-n-1} \big) \\ &\quad + \|\bar{v} - \bar{\bar{v}}\|_{s-n-1} \big(\|\bar{v}\| + \|\bar{\bar{v}}\| \big) \big\} \cdot \big(\|\bar{v}\|_{L^\infty} + \|\bar{\bar{v}}\|_{L^\infty} \big)^{\theta-1} d\tau \\ &\leqslant CE^\theta \int_0^t (1+t-\tau)^{-n/2} (1+\tau)^{-n\theta/2+n/4} \big(\|\bar{v} - \bar{\bar{v}}\| + \|\bar{v} - \bar{\bar{v}}\|_{s-n-1} \big) d\tau \\ &\leqslant CE^\theta (1+t)^{-n/2} D_s(\bar{v} - \bar{\bar{v}}). \end{aligned} \tag{3.25}$$

For H_2, it follows from Lemmas 3.3, 3.5 and the Sobolev inequality that

$$H_2 \leqslant C \int_0^t e^{-(t-\tau)/4} \left\| \big(f(\bar{v}) - f(\bar{\bar{v}})\big)(\tau) \right\|_{s-n-1,\infty} d\tau$$

$$
\begin{aligned}
&\leqslant C\int_0^t e^{-(t-\tau)/4}\big\|\big(f(\bar v)-f(\bar{\bar v})\big)(\tau)\big\|_{s-[\frac n2]}\,d\tau\\
&\leqslant C\int_0^t e^{-(t-\tau)/4}\big\{\|\bar v-\bar{\bar v}\|_{L^\infty}\big(\|\bar v\|_{s-[\frac n2]}+\|\bar{\bar v}\|_{s-[\frac n2]}\big)\\
&\quad+\|\bar v-\bar{\bar v}\|_{s-[\frac n2]}\big(\|\bar v\|_{L^\infty}+\|\bar{\bar v}\|_{L^\infty}\big)\big\}\cdot\big(\|\bar v\|_{L^\infty}+\|\bar{\bar v}\|_{L^\infty}\big)^{\theta-1}\,d\tau\\
&\leqslant CE^\theta\int_0^t e^{-(t-\tau)/4}(1+\tau)^{-n\theta/2-n/4}D_s(\bar v-\bar{\bar v})\,d\tau\\
&\leqslant CE^\theta(1+t)^{-n/2}D_s(\bar v-\bar{\bar v}).
\end{aligned}
$$

Then, we obtain

$$\|T\bar v-T\bar{\bar v}\|_{s-n-1,\infty}\leqslant CE^\theta(1+t)^{-n/2}D_s(\bar v-\bar{\bar v}). \tag{3.26}$$

On the other hand, by using the Young inequality, the Plancherel theorem, (2.17) and Lemma 3.3, we have

$$
\begin{aligned}
\|T\bar v-T\bar{\bar v}\|_s&\leqslant\int_0^t\big\|(G-G_3)(t-\tau)*\big(f(\bar v)-f(\bar{\bar v})\big)(\tau)\big\|_s\,d\tau\\
&\quad+\int_0^t\big\|G_3(t-\tau)(t-\tau)*\big(f(\bar v)-f(\bar{\bar v})\big)(\tau)\big\|_s\,d\tau\\
&\leqslant C\int_0^t(1+t-\tau)^{-n/4}\big(\big\|f(\bar v)-f(\bar{\bar v})\big\|_{s,1}+\big\|f(\bar v)-f(\bar{\bar v})\big\|_s\big)\,d\tau\\
&\leqslant C\int_0^t(1+t-\tau)^{-n/4}\big\{\big(\|\bar v-\bar{\bar v}\|+\|\bar v-\bar{\bar v}\|_{L^\infty}\big)\big(\|\bar v\|_s+\|\bar{\bar v}\|_s\big)\\
&\quad+\|\bar v-\bar{\bar v}\|_s\big(\|\bar v\|+\|\bar v\|_{L^\infty}+\|\bar{\bar v}\|+\|\bar{\bar v}\|_{L^\infty}\big)\big\}\big(\|\bar v\|_{L^\infty}+\|\bar{\bar v}\|_{L^\infty}\big)^{\theta-1}\,d\tau\\
&\leqslant CE^\theta(1+t)^{-n/4}D_s(\bar v-\bar{\bar v}).
\end{aligned}
$$

Then, we get

$$\|T\bar v-T\bar{\bar v}\|_s\leqslant CE^\theta(1+t)^{-n/4}D_s(\bar v-\bar{\bar v}). \tag{3.27}$$

Combining (3.26) with (3.27), we obtain $D_s(T\bar v-T\bar{\bar v})\leqslant CE^\theta D_s(\bar v-\bar{\bar v})$. Using the smallness assumption of E, we complete the proof of Lemma 3.7. □

By Lemmas 3.6–3.7 and using the fixed point theorem, there exists a unique global solution $u(x,t)$ to (1.1) and $u\in X_{s,E}$. For the time derivatives of the solution $u(x,t)$, we obtain some estimates by the energy method.

For any multi-index α, $0\leqslant|\alpha|\leqslant s$, by multiplying D_x^α $(1.1)_1$ with $D_x^\alpha u_t$ and integrating on $\mathbb{R}^n\times(0,t)$ with respect to (x,t) and noticing $u\in X_{s,E}$, we get

$$\big\|D_x^\alpha u_t(t)\big\|^2+\big\|D_x^\alpha\nabla u(t)\big\|^2+\int_0^t\big\|D_x^\alpha u_\tau(\tau)\big\|^2\,d\tau\leqslant C\big(\|u_0\|_{s+1}^2+\|u_1\|_s^2+CE^{\theta+1}\big).$$

By taking sum, it yields that

$$\|u\|_{s+1}^2+\|u_t\|_s^2+\int_0^t\|u_\tau\|_s^2\,d\tau\leqslant C\big(\|u_0\|_{s+1}^2+\|u_1\|_s^2+E^{\theta+1}\big). \tag{3.28}$$

Then, we have the following theorem about the global-in-time existence of solution.

Theorem 3.1. *Under the assumptions of Theorem* 1.1, *if* $\|u_0\|_{s+1} + \|u_1\|_s + \|u_0\|_{s,1} + \|u_1\|_{s,1}$ *is sufficiently small, then* (1.1) *admits a unique, global, classical solution* $u(x,t)$.

4. Pointwise estimate

In this section, we study the pointwise estimates to the nonlinear system. First, we give three lemmas for later use. We omit their proofs since they can be found in [11,13,24].

Lemma 4.1.

(1) *When* $\tau \in [0,t]$ *and* $A^2 \geqslant t$, *we have*

$$\left(1 + \frac{A^2}{1+\tau}\right)^{-n} \leqslant 3^n \left(\frac{1+\tau}{1+t}\right)^n \left(1 + \frac{A^2}{1+t}\right)^{-n}. \tag{4.1}$$

(2) *When* $A^2 \leqslant t$, *we have*

$$1 \leqslant 2^n \left(1 + \frac{A^2}{1+t}\right)^{-n}. \tag{4.2}$$

(3) *When* $n_1, n_2 > \frac{n}{2}$ *and* $n_3 = \min\{n_1, n_2\}$, *we have*

$$\int_{\mathbb{R}^n} \left(1 + \frac{|x-y|^2}{1+t}\right)^{-n_1} \left(1 + |y|^2\right)^{-n_2} dy \leqslant C \left(1 + \frac{|x|^2}{1+t}\right)^{-n_3}. \tag{4.3}$$

Lemma 4.2. *If the functions* $H(x,t)$ *and* $S(x,t)$ *satisfy*

$$\left|D_x^\alpha H(x,t)\right| \leqslant C(1+t)^{-(n+|\alpha|)/2} B_{n_1}\left(|x|,t\right),$$

and

$$\left|D_x^\alpha S(x,t)\right| \leqslant C(1+t)^{-(2n+|\alpha|)/2} B_{n_2}\left(|x|,t\right),$$

then we have

$$\left|D_x^\alpha \int_0^t \left(H(t-\tau) * S(\tau)\right) d\tau\right| \leqslant C(1+t)^{-(n+|\alpha|)/2} B_{n_3}\left(|x|,t\right),$$

where $n_1, n_2 > \frac{n}{2}$ *and* $n_3 = \min\{n_1, n_2\}$.

Lemma 4.3. *Assume that* $0 \leqslant \tau \leqslant t$ *and* $h(x,\tau)$ *satisfies*

$$D_x^\alpha h(x,\tau) \leqslant C(1+\tau)^{-\frac{\theta n+|\alpha|}{2}} \left(1 + \frac{|x|^2}{1+\tau}\right)^{-r},$$

then we have that

$$(1) \quad \int_{|z|=1} \left|D_x^\alpha h(x+tz,\tau)\right| dS_z \leqslant C(1+\tau)^{-\frac{\theta n+|\alpha|}{2}} (1+t)^{2r} \left(1 + \frac{|x|^2}{1+\tau}\right)^{-r},$$

$$(2) \quad \int_{|z|\leqslant 1} \frac{|D_x^\alpha h(x+tz,\tau)|}{\sqrt{1-|z|^2}} dV_z \leqslant C(1+\tau)^{-\frac{\theta n+|\alpha|-1}{2}} (1+t)^{2r} \left(1 + \frac{|x|^2}{1+\tau}\right)^{-r}.$$

Applying the Duhamel principle, the solution to (1.1) can be expressed as follows:

$$\begin{aligned} u(x,t) &= \left(G(t) * (u_0 + u_1) + \partial_t G(t) * u_0\right) + \int_0^t G(t-\tau) * f(u)(\tau)\, d\tau \\ &= \bar{u} + \bar{\bar{u}}. \end{aligned} \tag{4.4}$$

From Theorem 2.1 in [13], we know that for any multi-index α, $|\alpha| < s - \frac{n}{2}$, if there exists some constant $r > \frac{n}{2}$ such that $|D_x^\alpha u_0| + |D_x^\alpha u_1| \leqslant C(1+|x|^2)^{-r}$, then, for $|\alpha| < s - n$, $\bar{u}(x,t)$ has the following estimate

$$\left|D_x^\alpha \bar{u}(x,t)\right| \leqslant C(1+t)^{-\frac{n+|\alpha|}{2}}\left(1+\frac{|x|^2}{1+t}\right)^{-r}. \tag{4.5}$$

On the other hand, we consider $\bar{\bar{u}}$ by applying D_x^α on $\bar{\bar{u}}$ and rewriting it as follows

$$\begin{aligned} D_x^\alpha \bar{\bar{u}}(x,t) &= \int_0^t \left(G - K_m^0\right)(t-\tau) * D_x^\alpha f(u)(\tau)\,d\tau + \int_0^t K_m^0(t-\tau) * D_x^\alpha f(u)(\tau)\,d\tau \\ &= R_1^\alpha + R_2^\alpha. \end{aligned}$$

Set

$$\varphi_\alpha(x,t) = (1+t)^{\frac{n+|\alpha|}{2}}\left(B_r\left(|x|,t\right)\right)^{-1}, \quad r > \frac{n}{2}, \tag{4.6}$$

and

$$M(t) = \sup_{(x,\tau)\in\mathbb{R}^n\times[0,t),\ |\alpha|<n} \left|D_x^\alpha u(x,\tau)\right|\varphi_\alpha(x,\tau). \tag{4.7}$$

Proposition 4.1. *If $|\alpha| < n$, then*

$$\left|R_1^\alpha\right| \leqslant CM(t)^{\theta+1}(1+t)^{-\frac{n+|\alpha|}{2}} B_r\left(|x|,t\right). \tag{4.8}$$

Proof. Firstly, by noticing the hypothesis (4.7), we have

$$\left|D_y^\alpha f(u)(y,\tau)\right| \leqslant CM(t)^{\theta+1}(1+\tau)^{-\frac{\theta n+n+|\alpha|}{2}} B_r\left(|y|,\tau\right).$$

By Theorem 2.1, we have

$$\left|D_x^\alpha\left(G-K_m^0\right)(x,t)\right| \leqslant C(1+t)^{-\frac{n+|\alpha|}{2}} B_N\left(|x|,t\right).$$

Using Lemma 4.2 and noticing $r > \frac{n}{2}$ and θ is a positive integer, we know that (4.8) is valid. □

Proposition 4.2. *If $|\alpha| < n$, then*

$$\left|R_2^\alpha\right| \leqslant CM(t)^{\theta+1}(1+t)^{-\frac{n+|\alpha|}{2}} B_r\left(|x|,t\right). \tag{4.9}$$

Proof. Firstly, if $|x-y| \leqslant 2\varepsilon_1$, then we have

$$B_r\left(|y|,\tau\right) \leqslant B_r\left(|x|,\tau\right).$$

By using Lemmas 2.4 and 4.3, we have

$$\begin{aligned} \left|R_2^\alpha\right| &= \Bigg|\int_0^t \int_{\mathbb{R}^n} e^{-(t-\tau)/2}\Bigg(C_1\omega(D_x,t-\tau)(\tilde{q}_0 + C_0\delta)(x-y) \\ &\quad + \sum_{j=1}^{2m-2}\left(p_{1j}^0 + p_{2j}^0\right)(t-\tau)\omega_t(D_x,t-\tau)(\tilde{q}_j + C_0\delta)(x-y)\Bigg) D_y^\alpha f(u)(y,\tau)\,dy\,d\tau\Bigg| \\ &\leqslant \Bigg|\int_0^t \int_{\mathbb{R}^n} e^{-(t-\tau)/2}\Bigg(C_1\omega(D_x,t-\tau)\tilde{q}_0(x-y) \\ &\quad + \sum_{j=1}^{2m-2}\left(p_{1j}^0 + p_{2j}^0\right)(t-\tau)\omega_t(D_x,t-\tau)\tilde{q}_j(x-y)\Bigg) D_y^\alpha f(u)(y,\tau)\,dx\,d\tau\Bigg| \\ &\quad + \Bigg|\int_0^t \int_{\mathbb{R}^n} e^{-(t-\tau)/2}\Bigg(C_1\omega(D_x,t-\tau) \end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{j=1}^{2m-2} \big(p_{1j}^0 + p_{2j}^0\big)(t-\tau)\omega_t(D_x, t-\tau)\Bigg) C_0 \delta(x-y) D_y^\alpha f(u)(y,\tau)\, dy\, d\tau \Bigg| \\
&\leqslant C \int_0^t \int_{\mathbb{R}^n} e^{-(t-\tau)/2} \Bigg(|\tilde{q}_0| + \sum_{j=1}^{2m-2} |\tilde{q}_j| \Bigg)(x-y) \\
&\quad \times (1+t-\tau)^{2m-2+2r+\frac{n}{2}} M^{\theta+1}(t)(1+\tau)^{-\frac{\theta n+n+|\alpha|-1}{2}} \left(1 + \frac{|y|^2}{1+\tau}\right)^{-r} dy\, d\tau \\
&\quad + C \int_0^t e^{-(t-\tau)/2} (1+t-\tau)^{2m-2+2r+\frac{n}{2}} M^{\theta+1}(t)(1+\tau)^{-\frac{\theta n+n+|\alpha|-1}{2}} \left(1 + \frac{|x|^2}{1+\tau}\right)^{-r} d\tau \\
&\leqslant C M^{\theta+1}(t)(1+t)^{-\frac{n+|\alpha|}{2}} B_r\big(|x|, t\big).
\end{aligned}
$$

Thus, the proof of Proposition 4.2 is completed. □

Combining Propositions 4.1–4.2 and noticing (4.5), it yields that

$$
\big|D_x^\alpha u(x,t)\big| \leqslant C\big(\varepsilon + M^{\theta+1}\big)\big(\varphi_\alpha(x,t)\big)^{-1}. \tag{4.10}
$$

Since ε is sufficiently small, by continuity we have $M(t) \leqslant C\varepsilon$ with $\varepsilon << 1$. Then we obtain

$$
\big|D_x^\alpha u(x,t)\big| \leqslant C\varepsilon(1+t)^{-(n+|\alpha|)/2} B_r\big(|x|, t\big). \tag{4.11}
$$

Thus, we complete the proof of Theorem 1.1.

Corollary 4.1. *Under the assumptions of Theorem* 1.1, *for* $p \in [1, \infty]$, $|\alpha| < n$, *we have*

$$
\big\|D_x^\alpha u(\cdot, t)\big\|_{L^p} \leqslant C_p(1+t)^{-n/2(1-1/p)-|\alpha|/2}.
$$

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