J. Math. Anal. Appl. 371 (2010) 266–281

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Multiparameter weights with connections to Schauder bases

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ARTICLE INFO

Article history:
Received 15 February 2010
Available online 20 May 2010
Submitted by M. Milman

Keywords:
Weighted inequalities
Principle shift invariant systems
Gabor systems

ABSTRACT

We give several characterizations multiparameter $A_p(\mathbb{T}^d)$ in terms of strong maximal functions, rectangular conjugate functions, and rectangular partial sums. We also define a way to enumerate $\mathbb{Z}^d$ that corresponds to rectangles. We then use these enumerations to give another characterization of rectangular $A_p(\mathbb{T}^d)$ in terms of trigonometric Schauder bases. Finally, we present some applications to principle shift invariant spaces and Gabor systems.

1. Introduction

Let X be a Banach space. Recall that a sequence $\{x_n\}_{n=1}^{\infty}$ is a Schauder basis for X if for every $x \in X$ there exists a unique sequence of scalars $\{a_n\}_{n=1}^{\infty}$ such that

$$x = \sum_{n=1}^{\infty} a_n x_n \tag{1}$$

where the sum converges in the norm. The convergence in (1) is often conditional in the sense that rearrangements may not converge (see [18] and the example at the end of Section 3). Not every Banach space possesses a Schauder basis (see Enflo [2]) however if $X = \mathcal{H}$ is a (separable) Hilbert space then any orthonormal basis is a Schauder basis.

Since the scalars $a_n = a_n(x)$ are unique, they are well defined as linear functionals. In fact one can show that these linear functionals $x \mapsto a_n(x)$ are continuous and hence belong to the dual space of X, which we denote by X^*. It is for this reason that we adopt the notation $a_n(x) = \langle x, a_n\rangle$. Also notice that $\langle x_m, a_n\rangle = 0$ if $m \neq n$ and $\langle x_m, a_n\rangle = 1$ if $m = n$. The linear functionals $\{a_n\} \subset X^*$ are an example of a biorthogonal sequence to $\{x_n\}_{n\in\mathbb{N}}$.

Definition 1.1. Given a sequences $\{x_n\} \subset X$ and $\{a_n\} \subset X^*$ we say that $\{a_n\}$ is biorthogonal to $\{x_n\}$ or $(\{x_n\}, \{a_n\})$ is a biorthogonal system if

$$\langle x_m, a_n\rangle = \delta_{m,n} = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

Biorthogonal sequences play an important role in the theory of Schauder bases. One of the main theorems concerning Schauder bases is the following, which can be found in the online reference of Heil [7], or the book of Singer [18].

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.05.035

Theorem 1.2. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a Banach space X. The following statements are equivalent*:

(1) $\{x_n\}_{n=1}^{\infty}$ *is a Schauder basis for X.*
(2) $\overline{\operatorname{span}\{x_n\}_{n\in\mathbb{N}}} = X$ *and there exists a biorthogonal sequence $\{a_n\}_{n=1}^{\infty} \subset X^*$ such that the partial sum operators*

$$S_N x = \sum_{n=1}^{N} \langle x, a_n\rangle x_n$$

are uniformly bounded, i.e.

$$C = \sup_N \|S_N\|_{X\to X} < \infty. \tag{2}$$

The constant C in (2) is called the *basis constant* of $\{x_n\}$.

In this work we are concerned with the trigonometric system $\{e^{2\pi i k\cdot x}\}_{k\in\mathbb{Z}^d} = \{e_k\}_{k\in\mathbb{Z}^d}$, and when it forms a Schauder basis of weighted Lebesgue spaces, $L^p(\mathbb{T}^d, w)$ (see below for pertinent definitions). When $\mathbb{Z} = \{0, -1, 1, -2, 2, \ldots\}$, $\{e_k\}_{k\in\mathbb{Z}}$ is known to be a Schauder basis for $L^p(\mathbb{T}, w)$ exactly when w belongs to the class $A_2(\mathbb{T})$. This result is often attributed to Hunt, Muckenhoupt and Wheeden [11] but is clarified in Nielsen [13]. In fact, Nielsen [14] has characterized vector valued trigonometric Schauder bases for $L^p(\mathbb{T}\to\mathbb{C}^d, W)$ in terms of matrix A_p weights defined in Nazarov and Treil [12].

Our aim of this paper is to characterize when $\{e_k\}_{k\in\mathbb{Z}^d}$ forms a Schauder basis of $L^p(\mathbb{T}^d, w)$ for $d > 1$. In higher dimensions, one immediately encounters a problem. How should $\mathbb{Z}^d$ be enumerated so that $\{e_k\}_{k\in\mathbb{Z}^d}$ forms a Schauder basis for $L^p(\mathbb{T}^d, w)$? A natural way to enumerate $\mathbb{Z}^d$ to utilize the one-dimensional results of [11] is to enumerate corresponding to rectangles. It is for this reason that we need the notion of multiparameter A_p or rectangular A_p and a notion of rectangular enumerations.

Fefferman and Stein [4] defined multiparameter A_p, or $A_{p,\mathcal{R}}$, to be weights w such that

$$[w]_{A_{p,\mathcal{R}}} = \sup_R \left(\frac{1}{|R|}\int_R w\,dx\right)\left(\frac{1}{|R|}\int_R w^{1-p'}\,dx\right)^{p-1} < \infty,$$

where the supremum is over all rectangles R with sides parallel to the axes. They showed, in $\mathbb{R}^2$ for instance, that $A_{p,\mathcal{R}}$ characterizes the $L^p(w)$ boundedness of multiparameter operators such as the strong maximal function

$$M_S f(x) = \sup_{R\ni x} \frac{1}{|R|}\int_R \left|f(y)\right| dy$$

and the double Hilbert transform

$$Hf(x, y) = pv \int_{\mathbb{R}^2} \frac{f(s,t)}{(x-s)(y-t)}\,ds\,dt.$$

A nice account of weighted inequalities for these rectangular operators is given in the book by Garcia-Cuerva and Rubio de Francia [5, p. 450] and further applications can be found in Chang and Fefferman [1].

The organization of this paper is as follows. In Section 2 we develop the theory of rectangular weights on the Torus. Our main result of Section 2 is Theorem 2.1. Section 3 is devoted to characterizing when the trigonometric system forms a Schauder basis of $L^2(\mathbb{T}^d, w)$. We develop the theory of rectangular enumerations, and use them to give a new characterization of multiparameter $A_p(\mathbb{T}^d)$ weights. Our main result is Theorem 3.6. Finally, we give some applications of the weighted theory to principle shift invariant spaces and Gabor systems in Section 4. Theorems 4.2 and 4.3 extend the results in [8,14,16] to higher dimensions and/or more general enumerations. In Section 5 we discuss some possible extensions of the work and further questions.

2. Weighted Lebesgue spaces

In this section we develop the theory of multiparameter or rectangular A_p weights on the Torus $\mathbb{T}^d = [-1/2, 1/2]^d$. Given a function w, and $1 \leqslant p \leqslant \infty$, we say a function f is in $L^p(\mathbb{T}^d, w)$ if it is 1-periodic and the norm

$$\|f\|_{L^p(\mathbb{T}^d,w)} = \left(\int_{\mathbb{T}^d} \left|f(x)\right|^p w(x)\,dx\right)^{1/p}$$

is finite, with use the usual modification when $p = \infty$. When $w \equiv 1$ we simply write $L^p(\mathbb{T}^d)$. A weight is a positive $L^1(\mathbb{T}^d)$ function. Define $A_{p,\mathcal{R}}(\mathbb{T}^d)$ as all weights defined on $\mathbb{T}^d$ with

$$[w]_{A_{p,\mathcal{R}}(\mathbb{T}^d)} = \sup_{R\subset\mathbb{T}^d}\left(\frac{1}{|R|}\int_R w\,dx\right)\left(\frac{1}{|R|}\int_R w^{1-p'}\,dx\right) < \infty, \tag{3}$$

where the supremum is over all rectangles contained in $\mathbb{T}^d$ whose sides are parallel to the axes. Equivalently we could define $A_{p,\mathcal{R}}(\mathbb{T}^d)$ as weights on $\mathbb{R}^d$ that are 1-periodic and satisfy (3) except with the supremum over all rectangles in $\mathbb{R}^d$.

We now define rectangular versions of classical operators associated to the Torus. As is custom with weighted inequalities we start with a maximal function. Define the (local) strong Hardy–Littlewood maximal operator associated to $\mathbb{T}^d$ by

$$M_{\mathcal{R},\mathbb{T}^d} f(x) = \sup_{R\subset\mathbb{T}^d:\ x\in R}\frac{1}{|R|}\int_R \bigl|f(y)\bigr|\,dy.$$

We also define a rectangular conjugate function

$$C_{\mathcal{R}} f(x) = \lim_{\epsilon_1,\dots,\epsilon_d\to 0}\int_{\epsilon_1\leqslant|y_1|\leqslant 1/2}\cdots\int_{\epsilon_d\leqslant|y_d|\leqslant 1/2}\frac{f(x-y)}{\tan(\pi y_1)\cdots\tan(\pi y_d)}\,dy$$

and finally the rectangular partial Fourier sum operators

$$S_{\mathcal{R},N} f(x) = \sum_{k\in\mathbb{Z}^d:\ |k_i|\leqslant N_i}\langle f, e_k\rangle e_k$$

where $N = (N_1,\dots,N_d)\in\mathbb{N}^d$ and $e_k(x) = e^{2\pi i k\cdot x}$. Thus,

$$\langle f, e_k\rangle = \int_{\mathbb{T}^d} f\,\overline{e_k}\,dx = \int_{\mathbb{T}^d} f(x)e^{-2\pi i k\cdot x}\,dx = \hat{f}(k)$$

are the Fourier coefficients of f.

We start with a crucial lemma that illustrates the difficulty of the multiparameter situation. The following lemma says that we only need to check the $A_{p,\mathcal{R}}(\mathbb{T}^d)$ condition over rectangles with small sides. This lemma is trivial in the case of cubes, since cubes with large side-length have measure bounded away from zero. However, rectangles may have a large side yet still have small measure.

Lemma 2.1. *Suppose* $0 < c < 1$ *and* w *satisfies*

$$\sup_{\substack{R=I_1\times\cdots\times I_d\subset\mathbb{T}^d\\ |I_i|\leqslant c}}\left(\frac{1}{|R|}\int_R w\,dx\right)\left(\frac{1}{|R|}\int_R w^{1-p'}\,dx\right)^{p-1} < \infty, \tag{4}$$

then $w\in A_{p,\mathcal{R}}(\mathbb{T}^d)$.

Proof. For notational simplicity we prove this in the case $d=2$ and the same techniques can be used to prove the lemma when $d>2$. Without loss of generality we assume that $0<c<1/2$, otherwise 1 will play the role of $2c$ in our proof. Notice that since the quantity in (4) is finite, given any $R = I\times J$ with $|I|, |J|\leqslant c$ and $f\geqslant 0$ by Hölder's inequality we have

$$\left(\frac{1}{|R|}\int_R f\right)^p \leqslant C\frac{1}{w(R)}\int_R f^p w.$$

Thus, if $f = \chi_{\tilde{R}}$ where $\tilde{R}\subset R$ we have

$$w(R)\leqslant C\left(\frac{|R|}{|\tilde{R}|}\right)^p w(\tilde{R}), \tag{5}$$

and likewise for $\sigma = w^{1-p'}$,

$$\sigma(R)\leqslant C\left(\frac{|R|}{|\tilde{R}|}\right)^p \sigma(\tilde{R}). \tag{6}$$

Notice that (5) and (6) hold only for $R = I\times J$ with $|I|, |J|\leqslant c$.

Now let $R = I\times J\subset\mathbb{T}^2$ be a larger rectangle on one side, say $|I|\leqslant c$ and $c<|J|\leqslant 2c$. Split $R = R_1\cup R_2\cup R_3\cup R_4$ with $R_{\text{top}} = R_1\cup R_2$, $R_{\text{mid}} = R_2\cup R_3$ and $R_{\text{bot}} = R_3\cup R_4$ as in Fig. 1.

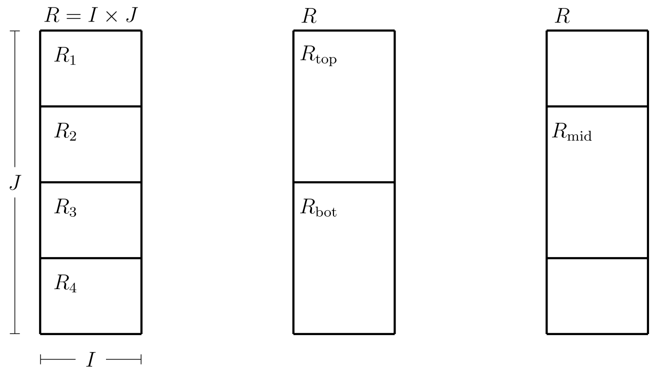

Fig. 1. R written as the union of smaller rectangles.

Now R_{top}, R_{mid} and R_{bot} have side-lengths bounded by c. Hence by (5)

$$w(R_{\text{top}}) \leqslant C w(R_2) \quad \text{and} \quad w(R_{\text{bot}}) \leqslant C w(R_3).$$

Thus

$$w(R) = w(R_{\text{top}}) + w(R_{\text{bot}}) \leqslant C\big(w(R_2) + w(R_3)\big) = C w(R_{\text{mid}}).$$

Similarly by (6)

$$\sigma(R) \leqslant C\sigma(R_{\text{mid}}).$$

Thus for such rectangles R

$$\left(\frac{1}{|R|}\int_R w\,dx\right)\left(\frac{1}{|R|}\int_R w^{1-p'}\,dx\right)^{p-1} = \frac{w(R)\sigma(R)^{p-1}}{|R|^p}$$
$$\leqslant C\left(\frac{|R_{\text{mid}}|}{|R|}\right)^p \frac{w(R_{\text{mid}})\sigma(R_{\text{mid}})^{p-1}}{|R_{\text{mid}}|^p} \leqslant C.$$

Similarly we may extend this to all rectangles with $R = I \times J$ with $|I| \leqslant 2c$ and $|J| \leqslant 2c$, showing that

$$\sup_{\substack{R = I\times J \subset \mathbb{T}^2 \\ |I|,|J| \leqslant 2c}} \left(\frac{1}{|R|}\int_R w\,dx\right)\left(\frac{1}{|R|}\int_R w^{1-p'}\,dx\right)^{p-1} < \infty.$$

From here, using the same process we may expand the rectangles side-length to $\min\{4c, 1\}$, and if necessary to $\min\{8c, 1\}$, and so on. This process must stop in finitely many steps since we are taking the supremum over $R \subset [-1/2, 1/2]^n$. We may continue expanding the sides of the rectangles until we have

$$\sup_{R\subset\mathbb{T}^2} \left(\frac{1}{|R|}\int_R w\,dx\right)\left(\frac{1}{|R|}\int_R w^{1-p'}\,dx\right)^{p-1} < \infty,$$

showing $w \in A_{p,\mathcal{R}}(\mathbb{T}^2)$. $\square$

Theorem 2.2. *Suppose w is a weight on $\mathbb{T}^d$ and $1 < p < \infty$. The following statements are equivalent.*

(1) $w \in A_{p,\mathcal{R}}(\mathbb{T}^d)$.
(2) *For each $i = 1, \ldots, d$ the functions $w_i = w(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_d)$ are in $A_p(\mathbb{T})$ uniformly for almost every $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \in \mathbb{T}^{d-1}$.*
(3) *$M_{\mathcal{R},\mathbb{T}^d}$ is a bounded operator on $L^p(\mathbb{T}^d, w)$.*
(4) *$C_{\mathcal{R}}$ is a bounded operator on $L^p(\mathbb{T}^d, w)$.*
(5) *The rectangular partial sum operators $S_{\mathcal{R},N}$ are uniformly bounded in N on $L^p(\mathbb{T}^d, w)$.*
(6) *If $f \in L^p(\mathbb{T}^d, w)$ then*

$$\lim_{N_1,\ldots,N_d\to\infty} \int_{\mathbb{T}^d} \big|f(x) - S_{\mathcal{R},N} f(x)\big|^p w\,dx = 0.$$

Remark 2.3. The proof of some implications of Theorem 2.2 will follow Theorem 6.2 in [5, p. 453], but we give the details for the convenience of the reader. The case $d = 1$ is contained in [11]. When $p = 2$ the equivalence (1) $\Leftrightarrow$ (5) is contained in Nielsen [13] and Heil and Powell [8]. In fact when $p = 2$, the papers [14,15] contain more general results about matrix A_p weights found in [12].

Proof. Once again we give the proof when $d = 2$ as the case $d > 2$ presents no additional difficulties. Throughout the proof we will use (x, y) and (s,t) to be points in $\mathbb{T}^2$. The following schematic diagram will be helpful for the implications of the proof.

$$\begin{tikzcd}
(3) \arrow[r] & (1) \arrow[d] & (6) \arrow[d, leftrightarrow] \\
(4) \arrow[ur] & (2) \arrow[ul] \arrow[l] \arrow[r] & (5) \arrow[ul]
\end{tikzcd}$$

(1) $\Rightarrow$ (2). If $w \in A_{p,\mathcal{R}}(\mathbb{T}^2)$, then

$$\left(\frac{1}{|I||J|}\int_I\int_J w(x,y)\,dx\,dy\right)\left(\frac{1}{|I||J|}\int_I\int_J w(x,y)^{1-p'}\,dx\,dy\right)^{p-1} \leqslant [w]_{A_{p,\mathcal{R}}(\mathbb{T}^2)}$$

for all $I, J \subset \mathbb{T}$. Fix $I \subset \mathbb{T}$ with rational end points, by the Lebesgue differentiation theorem we get

$$\left(\frac{1}{|I|}\int_I w(x,y)\,dy\right)\left(\frac{1}{|I|}\int_I w(x,y)^{1-p'}\,dy\right)^{p-1} \leqslant [w]_{A_{p,\mathcal{R}}(\mathbb{T}^2)} \tag{7}$$

for a.e. $x \in \mathbb{T}$. Thus for each $I \subset \mathbb{T}$ with rational endpoints we have inequality (7). Taking the supremum overall I with rational endpoints, we have

$$\sup_{I=[p,q]\subset\mathbb{T}:\ p,q\in\mathbb{Q}}\left(\frac{1}{|I|}\int_I w(x,y)\,dy\right)\left(\frac{1}{|I|}\int_I w(x,y)^{1-p'}\,dy\right)^{p-1} \leqslant [w]_{A_{p,\mathcal{R}}(\mathbb{T}^2)} \tag{8}$$

for almost every $x \in \mathbb{T}$. But the left side of inequality (8) is equal to $[w(x,\cdot)]_{A_p(\mathbb{T})}$. The proof that $[w(\cdot,y)]_{A_p(\mathbb{T})} \leqslant [w]_{A_{p,\mathcal{R}}(\mathbb{T}^2)}$ for a.e. y is similar.

(2) $\Rightarrow$ (3). Let M_i, $i = 1, 2$ be the one-dimensional Hardy–Littlewood maximal function applied to the ith variable of $f(x, y)$, i.e.

$$M_1 f(x,y) = \sup_{\substack{[a,b]\subset\mathbb{T}\\ x\in[a,b]}} \frac{1}{b-a}\int_a^b \left|f(s,y)\right|ds$$

and

$$M_2 f(x,y) = \sup_{\substack{[a,b]\subset\mathbb{T}\\ y\in[a,b]}} \frac{1}{b-a}\int_a^b \left|f(x,t)\right|dt$$

so that

$$M_{\mathcal{R},\mathbb{T}^2} f(x,y) \leqslant M_1 \circ M_2 f(x,y).$$

By the uniform $A_p(\mathbb{T})$ bounds on $w(\cdot, y)$ and $w(x,\cdot)$ and Fubini's theorem, we have

$$\begin{aligned}
\int_{\mathbb{T}^2} M_{\mathcal{R},\mathbb{T}^2} f^p w\,dx\,dy &\leqslant \int_{\mathbb{T}}\int_{\mathbb{T}} M_1 \circ M_2 f(x,y)^p w\,dx\,dy \\
&\leqslant C\int_{\mathbb{T}^2} \left|f(x,y)\right|^p w(x,y)\,dx\,dy.
\end{aligned}$$

(3) $\Rightarrow$ (1). Let R be a rectangle in $\mathbb{T}^2$, then for each $(x, y) \in R$

$$M(f\chi_R)(x,y) \geqslant \frac{1}{|R|}\int_R \left|f(s,t)\right|ds\,dt.$$

Hence,

$$\left(\frac{1}{|R|}\int_R |f|\right)^p \int_R w \leqslant \int_R M(f\chi_R)^p w \leqslant C\int_R |f|^p w$$

holds for any f. Setting $f = f_n = \min\{w^{1-p'}, n\}$ and taking limits shows the implication.

(2) $\Rightarrow$ (5). Let $N, M \in \mathbb{N}$, then notice

$$S_{\mathcal{R},(N,M)} f(x,y) = \sum_{|k|\leqslant N}\sum_{|j|\leqslant M} \langle f, e_{(k,j)}\rangle e_{(k,j)}$$
$$= \sum_{|k|\leqslant N}\int_{\mathbb{T}}\Bigl(\sum_{|j|\leqslant M}\bigl\langle f(\xi,\cdot), e_j\bigr\rangle e_j\Bigr) e_{-k}(\xi)\,d\xi\, e_k$$

where $e_{(j,k)}(x,y) = e^{2\pi i(jx+ky)}$ and e_n is the one variable function $e_n(\xi) = e^{2\pi i n\xi}$. It follows that $S_{\mathcal{R},(N,M)} f = S_N(S_M f)$, hence by the one-dimensional result (see [11]) we have

$$\int_{\mathbb{T}^2} (S_{\mathcal{R},(N,M)} f)^p w\,dx\,dy \leqslant C\int_{\mathbb{T}^2} |f|^p w\,dx\,dy.$$

(5) $\Rightarrow$ (1). This proof closely follows [8, Theorem 5.6] for a general $1 < p < \infty$. By Lemma 2.1 it suffices to show that

$$\sup_{\substack{R=I\times J\subset\mathbb{T}^2\\ |I|,|J|\leqslant 1/16}} \left(\frac{1}{|R|}\int_R w\right)\left(\frac{1}{|R|}\int_R w^{1-p'}\right)^{p-1} < \infty.$$

Suppose that $S_{\mathcal{R},(M,N)}$ are uniformly bounded on $L^p(\mathbb{T}^d, w)$ and write

$$S_{\mathcal{R},(M,N)} f(x,y) = \sum_{|k|\leqslant N}\sum_{|j|\leqslant M} \langle f, e_{(k,j)}\rangle e_{(k,j)} = \int_{\mathbb{T}^2} f(s,t) D_N(s-x) D_M(t-y)\,ds\,dt$$

where D_K is the Dirichlet kernel

$$D_K(z) = \frac{\sin(\pi(2K+1)z)}{\sin \pi z}.$$

Let $R = I\times J\subset\mathbb{T}^2$ be a rectangle with $|I|, |J| \leqslant 1/16$ and let N, M be the greatest integers less than or equal to $1/(16|I|)$ and $1/(16|J|)$ respectively. Notice that for $|z| \leqslant 1/(16K)$

$$D_K(z) \geqslant \frac{2}{\pi}(2K+1).$$

Hence for $(s,t), (x,y) \in R = I\times J$ so $|s-x| \leqslant |I| \leqslant 1/(16N)$ and $|t-y| \leqslant |J| \leqslant 1/(16M)$

$$D_N(s-x)D_M(t-y) \geqslant C(2N+1)(2M+1) \geqslant \frac{C}{|I||J|}.$$

If $f \geqslant 0$ is supported in R and $(x,y)\in R$ we have

$$S_{\mathcal{R},(N,M)} f(x,y) \geqslant C\frac{1}{|R|}\int_R f(s,t)\,ds\,dt.$$

Using the boundedness of $S_{\mathcal{R},(N,M)}$ we have

$$\left(\frac{1}{|R|}\int_R f(s,t)\,ds\,dt\right)^p w(R) \leqslant C\int_R S_{\mathcal{R},(N,M)} f(x,y)^p w(x,y)$$
$$\leqslant C\int_{\mathbb{T}^2} S_{\mathcal{R},(N,M)} f(x,y)^p w(x,y) \leqslant C\int_R f^p w.$$

Next, set $f = w^{1-p'}\chi_R$ (or an approximation) to arrive at

$$\left(\frac{1}{|R|}\int_R w\right)\left(\frac{1}{|R|}\int_R w^{1-p'}\right)^{p-1} \leqslant C.$$

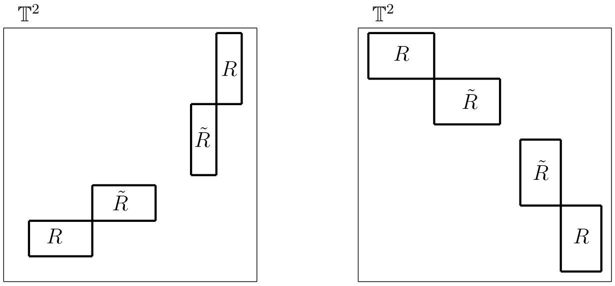

Fig. 2. Rectangles R, with their possible reflections $\tilde R$ about a corner.

(2) $\Rightarrow$ (4). Let

$$C_1 f(x,y) = pv \int_{\mathbb{T}} \frac{f(x-s,y)}{\tan(\pi s)}\,ds$$

and

$$C_2 f(x,y) = pv \int_{\mathbb{T}} \frac{f(x,y-t)}{\tan(\pi t)}\,dt.$$

Then $C_{\mathcal{R}} f = C_1(C_2 f)$ and since $w(\cdot,z), w(z,\cdot)$ are uniformly in $A_p(\mathbb{T})$, using the one-dimensional results in [11] we have

$$\begin{aligned}\int_{\mathbb{T}^2} \left|C_{\mathcal{R}} f(x,y)\right|^p w(x,y)\,dx\,dy &\leqslant \int_{\mathbb{T}^2} \left|C_1 \circ C_2 f(x,y)\right|^p w(x,y)\,dx\,dy\\ &\leqslant C \int_{\mathbb{T}^2} \left|f(x,y)\right|^p w(x,y)\,dx\,dy.\end{aligned}$$

(4) $\Rightarrow$ (1). Once again by Lemma 2.1 it suffices to consider rectangles with small sides. Let $R = I \times J$ be a rectangle in $\mathbb{T}^2$ with $|I|, |J| \leqslant 1/8$, and let $\tilde R = \tilde I \times \tilde J$ be a rectangle in $\mathbb{T}^2$ that is obtained by reflecting R about a corner as in Fig. 2.

If $\tilde R$ is obtained by reflection about the upper right or lower left corners, then for $(x,y) \in \tilde R$ and $(s,t) \in R$

$$\tan\bigl(\pi(x-s)\bigr)\tan\bigl(\pi(y-t)\bigr) \leqslant C\pi^2(x-s)(y-t) \leqslant C2|I|\cdot 2|J| = C|R|$$

and if $\tilde R$ is obtained by reflection about the upper left or lower right corners, then

$$-\tan\bigl(\pi(x-s)\bigr)\tan\bigl(\pi(y-t)\bigr) \leqslant C\pi^2(s-x)(y-t) \leqslant C|R|.$$

Hence for $f \geqslant 0$ supported in R and all $(x,y) \in \tilde R$,

$$\left|C_{\mathcal{R}} f(x,y)\right| \geqslant C\frac{1}{|R|}\int_R f(x,y)\,dx\,dy.$$

Using the boundedness of $C_{\mathcal{R}}$ on $L^p(\mathbb{T}^2, w)$ we have

$$\left(\frac{1}{|R|}\int_R f(x,y)\,dx\,dy\right)^p \int_{\tilde R} w\,dx\,dy \leqslant C\int_R f(x,y)^p w(x,y)\,dx\,dy. \tag{9}$$

Let $f = w^{-1/(p-1)}\chi_R$ (or an approximation) we have

$$\left(\frac{1}{|R|}\int_R w(x,y)^{1-p'}\,dx\,dy\right)^p \int_{\tilde R} w(x,y)\,dx\,dy \leqslant C\int_R w(x,y)^{1-p'}\,dx\,dy. \tag{10}$$

But we may also interchange the roles of R and $\tilde R$ in (9) and take $f = \chi_{\tilde R}$ to obtain

$$w(R) \leqslant C\,w(\tilde R).$$

Combining everything we have

$$\left(\frac{1}{|R|}\int_R w(x,y)\,dx\,dy\right)\left(\frac{1}{|R|}\int_R w(x,y)^{1-p'}\,dx\,dy\right)^{p-1}$$
$$\leqslant C\left(\frac{1}{|\tilde R|}\int_{\tilde R} w(x,y)\,dx\,dy\right)\left(\frac{1}{|R|}\int_R w(x,y)^{1-p'}\,dx\,dy\right)^{p-1}\leqslant C.$$

(5) $\Leftrightarrow$ (6). This is a routine computation. (6) $\Rightarrow$ (5) follows from the principle of uniform boundedness and (6) $\Leftarrow$ (5) follows from the density of the trigonometric system in $L^p(\mathbb{T}^2, w)$. $\square$

We also note that we do not need to use symmetric partial sums. For $a, b \in \mathbb{Z}^d$ with $a = (a_1, \ldots, a_d)$, $b = (b_1, \ldots, b_d)$ and $a_i \leqslant b_i$ for $1 \leqslant i \leqslant d$, define the operator

$$P_{a,b} f(x) = \sum_{k\in\mathbb{Z}^d,\, a_i\leqslant k_i\leqslant b_i} \langle f, e_k\rangle e_k.$$

Corollary 2.4. *The operators $P_{a,b}$ are bounded on $L^p(\mathbb{T}^d, w)$ uniformly in a, b if and only if $S_{N,\mathcal{R}}$ are bounded on $L^p(\mathbb{T}^d, w)$ in $N \in \mathbb{N}^d$.*

One direction of this corollary is easy since $S_{N,\mathcal{R}} = P_{-N,N}$, the other direction follows from the one-dimensional formula

$$P_{a,b} f(x) = \sum_{k=a}^{b} \langle f, e_k\rangle e_k(x)$$
$$= \begin{cases} e^{\pi i(b+a)x} \sum_{k=-(b-a)/2}^{(b-a)/2} \langle f e^{-\pi i(b+a)(\cdot)}, e_k\rangle e_k(x) & \text{if } b-a \text{ is even},\\ P_{a,b-1} f(x) + \langle f, e_b\rangle e_b(x) & \text{if } b-a \text{ is odd},\end{cases}$$

and

$$\big\|\langle f, e_b\rangle e_b\big\|_{L^p(\mathbb{T}^d,w)} \leqslant \left(\int_{\mathbb{T}^d} w\right)^{1/p} \int_{\mathbb{T}^d} |f| w^{1/p} w^{-1/p}$$
$$\leqslant \left(\int_{\mathbb{T}^d} w\right)^{1/p} \left(\int_{\mathbb{T}^d} w^{1-p'}\right)^{1/p'} \left(\int_{\mathbb{T}^d} |f|^p w\right)^{1/p}.$$

3. Schauder basis for $L^p(\mathbb{T}^d, w)$

We now turn to characterizing the trigonometric Schauder bases for $L^p(\mathbb{T}^d, w)$. It is known that $\{e_k\}_{k\in\mathbb{Z}}$ is a Schauder basis for $L^p(\mathbb{T}, w)$ if and only if $w \in A_p(\mathbb{T})$. Since Schauder bases can be conditionally convergent it is assumed that $\mathbb{Z}$ is ordered as $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$. Notice that this ordering corresponds to the symmetric partial sum operator. Specifically, if $\{\tilde e_k\}_{k\in\mathbb{Z}}$ is a dual basis to $\{e_k\}_{k\in\mathbb{Z}}$ and S_N is the partial sum operator associated to the dual basis, then

$$S_{2N+1} f = \sum_{k=-N}^{N} \langle f, \tilde e_k\rangle e_k$$

and

$$S_{2N} f = \sum_{k=-(N-1)}^{N-1} \langle f, \tilde e_k\rangle e_k + \langle f, \tilde e_N\rangle e_N.$$

This ordering is crucial, when $p \neq 2$ even when $w \equiv 1$, $\{e^{2\pi ikx}\}_{k\in\mathbb{Z}}$ is a conditional basis (see [17] and [18, p. 428]). For dimensions $d > 1$, one immediately encounters an obstacle. How should $\mathbb{Z}^d$ be enumerated so that the associated partial sum operators are uniformly bounded? One way to enumerate $\mathbb{Z}^d$ would be to correspond to spherical partials sum operators,

$$S_{\mathcal{S},M} f = \sum_{\substack{k\in\mathbb{Z}^d\\ |k|\leqslant M}} \langle f, e_k\rangle e_k, \quad M \in \mathbb{N},$$

where $|k|^2 = |k_1|^2 + \cdots + |k_d|^2$. However, these partial are unbounded on $L^p(\mathbb{T}^d)$ when $p \neq 2$ and $d \geqslant 2$. The continuous version of this is due to Fefferman [3] who showed that the unit ball is not a multiplier on $L^p(\mathbb{R}^d)$ when $p \neq 2$ and $d \geqslant 2$. In order to utilize Theorem 2.2 we would like the enumeration to correspond to rectangular partial sums,

$$S_{\mathcal{R},N} f = \sum_{k \in \mathbb{Z}^d,\, |k_i| \leqslant N_i} \langle f, e_k \rangle e_k, \quad N = (N_1, \ldots, N_d) \in \mathbb{N}^d.$$

It is for this reason that we need the notion of rectangular enumerations of $\mathbb{Z}^d$.

We say σ is an enumeration of $\mathbb{Z}^d$ if it is a one-to-one map from $\mathbb{N}$ onto $\mathbb{Z}^d$. Rectangular enumerations of $\mathbb{Z}^d$ were defined in [8] and [14], but we define a larger set of rectangular enumerations of $\mathbb{Z}^d$, denoted by $\Lambda_{\mathcal{R}}(\mathbb{Z}^d)$. Before we define the set of rectangular enumerations we clear up some notation issues. Given $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ in $\mathbb{Z}^d$ we write $a \leqslant b$ if $a_i \leqslant b_i$ for $i = 1, \ldots, d$. For $a \leqslant b$ in $\mathbb{Z}^d$ we refer to the rectangle, $R_{a,b}$, to be the set of all integer points $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ with $a_i \leqslant k_i \leqslant b_i$ for $i = 1, \ldots, d$. Occasionally we will write $R_{a,b} = \{a_1, \ldots, b_1\} \times \cdots \times \{a_d, \ldots, b_d\}$.

Definition 3.1. Let σ be an enumeration of $\mathbb{Z}$, $a, b \in \mathbb{Z}$ with $a \leqslant b$, and $J \in \mathbb{N}$. We say that σ is *consecutive* to the interval of points $\{a, a+1, \ldots, b-1, b\}$, on $\{J, J+1, \ldots, J+n\}$ (notice $n = b - a$) if for each $0 \leqslant i \leqslant n$ the set

$$\bigl\{\sigma(J), \sigma(J+1), \ldots, \sigma(J+i)\bigr\}$$

forms a consecutive set of integers between a and b. This means that for each $1 \leqslant i \leqslant n$ there exists k_i with

$$\sigma\bigl(\{J, \ldots, J+i\}\bigr) = \{k_i, k_i+1, \ldots, k_i+i\} \subset \{a, \ldots, b\}.$$

We now inductively define what it means for an enumeration of $\mathbb{Z}^d$ to be *consecutive* to a rectangle $R_{a,b} \subset \mathbb{Z}^d$. Roughly speaking, an enumeration consecutive to a rectangle $R_{a,b} \subset \mathbb{Z}^d$ fills out each element of the $(d-1)$-dimensional hyperplanes of $R_{a,b}$ consecutively, while filling out the hyperplanes themselves in a consecutive manner. Given a rectangle $R_{a,b} \subset \mathbb{Z}^d$, we define the following numbers which will be useful in our definition. Fix $1 \leqslant i \leqslant d$,

$$\begin{aligned}
N &= \#\text{ of elements in } R_{a,b} = \prod_{j=1}^{d} (b_j - a_j + 1), \\
N_i &= \#\text{ of elements in a hyperplane of } R_{a,b} \text{ perpendicular to the } i\text{th coordinate axis} \\
&= \prod_{j \neq i} (b_j - a_j + 1), \\
n_i &= \#\text{ of hyperplanes in } R_{a,b} \text{ perpendicular to the } i\text{th coordinate axis} \\
&= (b_i - a_i + 1).
\end{aligned}$$

Notice that $n_i N_i = N$. Given J, we partition the set $\{J+1, \ldots, J+N\}$ according to the $(d-1)$-hyperplanes of $R_{a,b}$ perpendicular to the ith coordinate axis. Let

$$\begin{aligned}
[J_1] &= \{J+1, \ldots, J+N_i\}, \\
[J_2] &= \{J+N_i+1, \ldots, J+2N_i\}, \\
&\vdots \\
[J_{n_i}] &= \bigl\{J+(n_i-1)N_i+1, \ldots, J+n_i N_i\bigr\}.
\end{aligned}$$

We say that an enumeration σ is consecutive to $R_{a,b}$ on $\{J+1, J+2, \ldots, J+N\}$ if there exist $1 \leqslant i \leqslant d$, such that the following hold.

(1) There exist n_i enumerations of $\mathbb{Z}^{d-1}$

$$\tau_1 = \bigl(\tau_1^{(1)}, \ldots, \tau_{d-1}^{(1)}\bigr), \quad \tau_2 = \bigl(\tau_1^{(2)}, \ldots, \tau_{d-1}^{(2)}\bigr), \quad \ldots, \quad \tau_{n_i} = \bigl(\tau_1^{(n_i)}, \ldots, \tau_{d-1}^{(n_i)}\bigr),$$

that are consecutive to the $(d-1)$-dimensional rectangle

$$\{a_1, \ldots, b_1\} \times \cdots \times \{a_{i-1}, \ldots, b_{i-1}\} \times \{a_{i+1}, \ldots, b_{i+1}\} \times \cdots \times \{a_d, \ldots, b_d\}$$

on $[J_1], [J_2], \ldots, [J_{n_i}]$ respectively.

(2) There exists an enumeration λ of $\mathbb{Z}$ that is consecutive to $\{a_i, \ldots, b_i\}$ on $\{1, \ldots, n_i\}$.

$$\begin{array}{ccccc}
\sigma(10) & \sigma(9) & \sigma(8) & \sigma(7) & \sigma(6) \\
\sigma(5) & \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) \\
\sigma(15) & \sigma(13) & \sigma(11) & \sigma(12) & \sigma(14)
\end{array}$$

Fig. 3. A consecutive enumeration to the horizontal hyperplanes of a rectangle in $\mathbb{Z}^2$.

$$\begin{array}{ccccc}
\sigma(15) & \sigma(10) & \sigma(5) & \sigma(3) & \sigma(9) \\
\sigma(13) & \sigma(11) & \sigma(4) & \sigma(2) & \sigma(7) \\
\sigma(14) & \sigma(12) & \sigma(6) & \sigma(1) & \sigma(8)
\end{array}$$

Fig. 4. A consecutive enumeration to the vertical hyperplanes of a rectangle in $\mathbb{Z}^2$.

(3) The enumeration σ has the following form

$$\begin{aligned}
\sigma(k) &= \left(\tau_1^{(1)}(k), \dots, \tau_{i-1}^{(1)}(k), \lambda(1), \tau_i^{(1)}(k), \dots, \tau_{d-1}^{(1)}(k)\right) \quad \text{for } k \in [J_1], \\
\sigma(k) &= \left(\tau_1^{(2)}(k), \dots, \tau_{i-1}^{(2)}(k), \lambda(2), \tau_i^{(2)}(k), \dots, \tau_{d-1}^{(2)}(k)\right) \quad \text{for } k \in [J_2], \\
&\vdots \\
\sigma(k) &= \left(\tau_1^{(n_i)}(k), \dots, \tau_{i-1}^{(n_i)}(k), \lambda(n_i), \tau_i^{(n_i)}(k), \dots, \tau_{d-1}^{(n_i)}(k)\right) \quad \text{for } k \in [J_{n_i}].
\end{aligned}$$

The above definition is admittedly rather technical and we illustrate some examples of a consecutive enumeration below in Fig. 3 and Fig. 4.

Definition 3.2. Let $a, b \in \mathbb{Z}^d$ with $a \leqslant b$, N be the number of elements of $R_{a,b}$ and σ be an enumeration of $\mathbb{Z}^d$. We say that σ is *adapted* to the rectangle

$$R_{a,b} = \{a_1, \dots, b_1\} \times \cdots \times \{a_d, \dots, b_d\}$$

if the following hold.

(1) The enumeration σ is consecutive to the rectangle $R_{a,b}$ on $\{1, \dots, N\}$.
(2) There exists a sequence of rectangles in $\mathbb{Z}^d$,

$$R_{a,b} = R^{(0)} \subsetneq R^{(1)} \subsetneq \cdots \subsetneq R^{(i)} \subsetneq R^{(i+1)} \subsetneq \cdots$$

with $\bigcup_i R^{(i)} = \mathbb{Z}^d$ and $R^{(i+1)} \backslash R^{(i)}$ is a rectangle in $\mathbb{Z}^d$.
(3) If N_i is the number of elements in $R^{(i+1)} \backslash R^{(i)}$, then σ is consecutive to $R^{(i+1)} \backslash R^{(i)}$ on

$$\left\{N + \sum_{m=1}^{i-1} N_m + 1, \dots, N + \sum_{m=1}^{i} N_m\right\}$$

for each $i = 0, 1, \dots$.

Remark 3.3. The important property of an enumeration adapted to a rectangle is that given any $N \in \mathbb{N}$, we can find $K \in \mathbb{N}$ with $K \leqslant N$ and rectangles $R^{(i)}, R^{(i+1)} \subset \mathbb{Z}^d$ such that

$$\sigma\left(\{1, \dots, K\}\right) = R^{(i)}$$

and $\sigma(\{K, \dots, N\})$ is the disjoint union of at most d rectangles in $R^{(i+1)} \backslash R^{(i)}$.

Definition 3.4. Given $a, b \in \mathbb{Z}^d$ with $a \leqslant b$ let

$$\mathcal{R}_{a,b} = \{\sigma \colon \sigma \text{ is adapted to } R_{a,b}\}$$

and define the set of all rectangular enumerations to be

$$\Lambda_{\mathcal{R}}(\mathbb{Z}^d) = \bigcup_{\substack{a,b\in\mathbb{Z}^d \\ a\leqslant b}} \mathcal{R}_{a,b}.$$

Remark 3.5. We remark that the set of all rectangular enumerations defined in Definition 3.4 is a larger class of enumerations than those used in [8, Definition 5.8] and [14].

We are now ready to give an alternative characterization of $A_{p,\mathcal{R}}(\mathbb{T}^d)$ in terms of Schauder bases. Our main theorem is the following.

Theorem 3.6. *Suppose* $1 < p < \infty$ *and* w *is a weight. Then the following are equivalent.*

(1) *For each* $\sigma \in \Lambda_{\mathcal{R}}(\mathbb{Z}^d)$, $\{e_{\sigma(n)}\}_{n=1}^{\infty}$ *is a Schauder basis of* $L^p(\mathbb{T}^d, w)$ *with basis constant* C_σ *and* $\sup_{\sigma\in\Lambda(\mathbb{Z}^d)} C_\sigma < \infty$.
(2) $w \in A_{p,\mathcal{R}}(\mathbb{T}^d)$.

Proof. $(1) \Rightarrow (2)$. We will show that the rectangular partial sum operations

$$S_{\mathcal{R},N} f = \sum_{\substack{k\in\mathbb{Z}^d \\ |k_i|\leqslant N_i}} \langle f, e_k\rangle e_k$$

are uniformly bounded on $L^p(\mathbb{T}^d, w)$. Set

$$C = \sup_{\sigma\in\Lambda_{\mathcal{R}}(\mathbb{Z}^d)} C_\sigma < \infty$$

where C_σ is the basis constant from the Schauder basis $\{e_{\sigma(n)}\}_{n=1}^{\infty}$. Let $N \in \mathbb{N}^d$, then there exists $\sigma \in \Lambda_{\mathcal{R}}(\mathbb{Z}^d)$ that is adapted to $R_{-N,N}$. Since $\{e_{\sigma(n)}\}_{n=1}^{\infty}$ is a Schauder basis for $L^p(\mathbb{T}^d, w)$ there exist a sequence $\{f_n\}_{n=1}^{\infty} \subset (L^p(\mathbb{T}^d, w))^* = L^{p'}(\mathbb{T}^d, w)$ such that

$$\langle e_{\sigma(m)}, f_n\rangle_w = \int_{\mathbb{T}^d} e_{\sigma(m)} \overline{f_n} w\, dx = \delta_{m,n}.$$

Since $f_n \in L^{p'}(\mathbb{T}^d, w)$ and $w \in L^1(\mathbb{T}^d)$ we have $\overline{f_n} w \in L^1(\mathbb{T}^d)$. Furthermore $\overline{f_n} w$ has the same Fourier coefficients as $e_{-\sigma(n)}$, hence

$$f_n = \frac{e_{\sigma(n)}}{w}$$

and

$$\langle f, f_n\rangle_w = \int_{\mathbb{T}^d} f e_{-\sigma(n)} w^{-1} w\, dx = \langle f, e_{\sigma(n)}\rangle. \tag{11}$$

We also have the partial sum operators associated to σ,

$$T_M^\sigma f = \sum_{n=1}^{N} \langle f, f_n\rangle_w e_{\sigma(n)}$$

uniformly bounded on $L^p(\mathbb{T}^d, w)$ with

$$\sup_M \| T_M^\sigma \| \leqslant C.$$

Since σ is adapted to $R_{-N,N}$ there exists $K \in \mathbb{N}$ with $\sigma(\{1, \dots, K\}) = R_{-N,N}$ and using (11) we have

$$T_K^\sigma f = \sum_{n=1}^{K} \langle f, f_n\rangle_w e_{\sigma(n)} = \sum_{|k_i|\leqslant N} \langle f, e_k\rangle e_k = S_{\mathcal{R},N} f$$

hence,

$$\|S_{\mathcal{R},N} f\|_{L^p(\mathbb{T}^d,w)} = \big\| T_K^\sigma f \big\|_{L^p(\mathbb{T}^d,w)} \leqslant C\|f\|_{L^p(\mathbb{T}^d,w)}.$$

(2) $\Rightarrow$ (1). Let $w \in A_{p,\mathcal{R}}(\mathbb{T}^d)$ and $\sigma \in \Lambda_{\mathcal{R}}(\mathbb{Z}^d)$. Since $w \in A_{p,\mathcal{R}}(\mathbb{T}^d)$ the operators

$$P_{a,b} f = \sum_{\substack{k \in \mathbb{Z}^d \\ a_i \leqslant k_i \leqslant b_i}} \langle f, e_k \rangle e_k$$

are bounded on $L^p(\mathbb{T}^d, w)$ uniformly in $a, b \in \mathbb{Z}^d$. The span of $\{e_{\sigma(n)}\}_{n\in\mathbb{N}}$ is dense in $L^p(\mathbb{T}^d, w)$ since $w \in L^1(\mathbb{T}^d)$. Furthermore, $w^{-1} \in L^{p'}(\mathbb{T}^d, w)$ so

$$\{f_n\}_{n\in\mathbb{N}} = \left\{ \frac{e_{\sigma(n)}}{w} \right\}_{n\in\mathbb{N}}$$

is biorthogonal to $\{e_{\sigma(n)}\}_{n\in\mathbb{N}}$ in $L^p(\mathbb{T}^d, w)$. Let T_N^σ be the partial sum operators associated to σ,

$$T_N^\sigma f = \sum_{n=1}^{N} \langle f, f_n \rangle_w e_{\sigma(n)} = \sum_{n=1}^{N} \langle f, e_{\sigma(n)} \rangle e_{\sigma(n)}.$$

Let $K \leqslant N$ be the largest integer such that σ is adapted to a rectangle $R_{a,b}$ on $\{1, \dots, K\}$. Then we may write

$$\begin{aligned} T_N^\sigma f &= \sum_{n=1}^{K} \langle f, e_{\sigma(n)} \rangle e_{\sigma(n)} + \sum_{n=K+1}^{N} \langle f, e_{\sigma(n)} \rangle e_{\sigma(n)} \\ &= P_{a,b} f + \text{Remainder terms.} \end{aligned}$$

In light of Remark 3.3 we have

$$\begin{aligned} \text{Remainder terms} = {} & \sum_{k\in R_1} \langle f, e_k \rangle e_k \\ & + \dots + \sum_{k \in R_d} \langle f, e_k \rangle e_k \end{aligned}$$

for some rectangles $R_1, \dots, R_d$. Thus

$$\|\text{Remainder terms}\|_{L^p(\mathbb{T}^d, w)} \leqslant dc \|f\|_{L^p(\mathbb{T}^d, w)}$$

and

$$\|P_{a,b} f\|_{L^p(\mathbb{T}^d,w)} \leqslant c \|f\|_{L^p(\mathbb{T}^d,w)}.$$

Thus $\|T_N^\sigma f\|_{L^p(\mathbb{T}^d,w)} \leqslant (d+1)c\|f\|_{L^p(\mathbb{T}^d,w)}$ and c is independent of σ and N. $\square$

Remark 3.7. Careful examination of the proof yields that we only need uniform bounds on the basis constant of enumerations adapted to symmetric rectangles of the form $R_{-N,N}$ for $N \in \mathbb{N}^d$. These symmetric rectangular enumerations correspond to the enumerations used in [8] and [14].

Remark 3.8. When $d = 1$ we have $A_{p,\mathcal{R}}(\mathbb{T}) = A_p(\mathbb{T})$. Thus Theorem 3.6 extends the original results of [11] to more general enumerations.

Definition 3.9. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is an unconditional basis of X if it is a Schauder basis and

$$\sum_{n=1}^{\infty} \langle x, a_{\sigma(n)} \rangle x_{\sigma(n)} = x$$

for every one-to-one map σ from $\mathbb{N}$ onto $\mathbb{N}$.

Definition 3.10. In a Hilbert space $\mathcal{H}$, a sequence is a *Riesz basis* if it is a Schauder basis $(\{x_n\}, \{a_n\})$ and there are constants $c, C > 0$ such that

$$c \sum_{n\in\mathbb{N}} \left|\langle x, a_n \rangle\right|^2 \leqslant \left\| \sum_{n\in\mathbb{N}} \langle x, a_n \rangle x_n \right\|_{\mathcal{H}}^2 \leqslant C \sum_{n\in\mathbb{N}} \left|\langle x, a_n \rangle\right|^2.$$

It is a well-known fact that in Hilbert spaces unconditional bases and Riesz bases coincide (see [7]).

Example 3.11. The power weight $|\cdot|^\delta$ is in $A_p(\mathbb{T})$ if and only if $-1 < \delta < p-1$. Thus for a fixed $1 \leqslant i \leqslant d$ and $\delta \in (-1, p-1)$, the weight

$$w_\delta(x) = w_\delta(x_1, \dots, x_d) = |x_i|^\delta, \quad x \in \mathbb{T}^d, \tag{12}$$

is in $A_{p,\mathcal{R}}(\mathbb{T}^d)$ by (2) of Theorem 2.2. When $p = 2$, $L^2(\mathbb{T}^d, w)$ is a Hilbert space and $\{e_k\}_{k\in\mathbb{Z}^d}$ is a Riesz basis if and only if $0 < c \leqslant w \leqslant C < \infty$ a.e. This follows because it is equivalent to the operator $g \mapsto \sqrt{w}g$ being a bounded invertible operator on $L^2(\mathbb{T}^d)$. The weight w_δ from (12) is not bounded above (below) if $\delta < 0$ ($\delta > 0$). Hence $\{e_{\sigma(n)}\}_{n=1}^\infty$ is a Schauder basis of $L^2(\mathbb{T}^d, w_\delta)$ for every $\sigma \in \Lambda_\mathcal{R}(\mathbb{Z}^d)$, however $\{e_k\}_{k\in\mathbb{Z}^d}$ is not an unconditional basis of $L^2(\mathbb{T}^d, w_\delta)$.

4. Applications

We now present some applications of the weighted theory to principle shift invariant spaces and Gabor systems. Versions of these applications can be found in [8] (Gabor systems) and [14,16] (shift invariant systems). In fact [14] contains a characterization of finitely generated shift invariant systems in terms of matrix A_p. However, our results are either in higher dimensions and/or contain more general enumerations than those found in [8] and [14,16]. First we introduce the modulation and shift operators,

$$M_y f(x) = e^{2\pi i x\cdot y} f(x), \qquad \tau_y f(x) = f(x-y).$$

We define the Fourier transform as

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi}\, dx$$

and the Zak transform

$$Zf(x,\xi) = \sum_{k\in\mathbb{Z}^d} f(x+k) e^{2\pi i k\cdot\xi}.$$

Both are initially defined on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and extended to $L^2(\mathbb{R}^d)$ by density. Notice the relationships

$$(\tau_y f)\hat{} = M_{-y}\hat{f} \quad \text{and} \quad (M_y f)\hat{} = \tau_y \hat{f}$$

and

$$Z(M_z \tau_y f)(x,\xi) = e^{2\pi i z\cdot x} Zf(x-y,\xi) = M_{(z,0)}\tau_{(y,0)} Zf(x,\xi).$$

If $y = k \in \mathbb{Z}^d$ we have

$$Z(M_z \tau_k f)(x,\xi) = e^{2\pi i z\cdot x} e^{2\pi i k\cdot\xi} Zf(x,\xi). \tag{13}$$

We recall the Plancherel theorem

$$\|\mathcal{F} f\|_{L^2(\mathbb{R}^d)} = \|\hat{f}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}.$$

Similarly for the Zak transform, using Fourier series one has

$$\begin{aligned}\int_{\mathbb{T}^{2d}} \left|Zf(x,\xi)\right|^2 d\xi\, dx &= \int_{\mathbb{T}^d} \sum_{k\in\mathbb{Z}^d} \left|f(x+k)\right|^2 dx \\ &= \int_{\mathbb{R}^d} \left|f(x)\right|^2 dx.\end{aligned}$$

Hence $Z : L^2(\mathbb{R}^d) \to L^2(\mathbb{T}^{2d})$ is an isometry. These operators have inverses given by

$$\mathcal{F}^{-1} f(x) = \check{f}(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi$$

and

$$Z^{-1} f(x) = \int_{\mathbb{T}^d} Zf(x,\xi)\, d\xi.$$

The Fourier and Zak transforms are related in the following manner:

$$Zf(x,\xi) = e^{2\pi i x\cdot\xi} Z\hat{f}(x,-\xi).$$

More information about the Zak transform can be found in [6]. Finally, we will make use of the following lemma, which we present without proof.

Lemma 4.1. *Suppose X and Y are Banach spaces and $T : X \to Y$ is an isomorphism (linear, continuous, one-to-one and onto). Then $\{x_n\}_{n=1}^{\infty}$ is a Schauder basis for X if and only if $\{T(x_n)\}_{n=1}^{\infty}$ is a Schauder basis for Y.*

4.1. Principle shift invariant spaces

Let $\psi \in L^2(\mathbb{R}^d)$, and $\psi_k = \tau_k\psi = \psi(\cdot - k)$, the *principle shift invariant space* generated by ψ is given by

$$\langle\psi\rangle = \overline{\operatorname{span}\{\psi(\cdot - k)\colon k \in \mathbb{Z}^d\}}.$$

Principle shift invariant spaces are related to wavelet expansions and multiresolution analysis (see the book by Hernández and Weiss [9]). Associated to ψ its *periodization function*, p_ψ, is given by

$$p_\psi(\xi) = \sum_{k\in\mathbb{Z}^d} \left|\hat{\psi}(\xi + k)\right|^2.$$

Notice that p_ψ is in $L^1(\mathbb{T}^d)$ with

$$\|p_\psi\|_{L^1(\mathbb{T}^d)} = \|\hat{f}\|^2_{L^2(\mathbb{R}^d)} = \|f\|^2_{L^2(\mathbb{R}^d)}.$$

The main idea is that we may study properties of $\langle\psi\rangle$ by analyzing p_ψ, see [16] and [10]. We have the following theorem.

Theorem 4.2. *Suppose $\psi \in L^2(\mathbb{R}^d)$, then the following are equivalent.*

(1) *For each $\sigma \in \Lambda_{\mathcal{R}}(\mathbb{Z}^d)$, $\{\psi_{\sigma(n)}\}_{n=1}^{\infty}$ is a Schauder basis of $\langle\psi\rangle$ with basis constant C_σ and $\sup_{\sigma\in\Lambda_{\mathcal{R}}(\mathbb{Z}^d)} C_\sigma < \infty$.*
(2) $p_\psi \in A_{2,\mathcal{R}}(\mathbb{T}^d)$.

Proof. The map

$$\begin{aligned} J_\psi : L^2(\mathbb{T}^d, p_\psi) &\to L^2(\mathbb{R}^d) \\ m &\mapsto (m\hat{\psi})^{\vee} \end{aligned}$$

is an isometry onto $\langle\psi\rangle$ and $J_\psi(e_{-k}) = \tau_k\psi = \psi_k$. Notice that if σ is adapted to the rectangle $R_{a,b}$ if and only if $-\sigma$ is adapted to the rectangle $R_{-b,-a}$. Hence $\{\psi_{\sigma(n)}\}_{n=1}^{\infty}$ is a Schauder basis for $\langle\psi\rangle$ if and only if $\{e_{-\sigma(n)}\}_{n=1}^{\infty}$ is a Schauder basis for $L^2(\mathbb{T}^d, p_\psi)$. $\square$

4.2. Gabor systems

Let $g \in L^2(\mathbb{R}^d)$ and let $g_{j,k} = M_j\tau_k g$. The *Gabor system* generated by $\{g_{j,k}\}_{(j,k)\in\mathbb{Z}^{2d}}$ is defined to be

$$\mathcal{G}_g = \overline{\operatorname{span}\{g_{j,k}\colon j,k \in \mathbb{Z}^d\}}.$$

For Gabor systems the Zak transform takes the place of the Fourier transform and we study properties of $\mathcal{G}_g$ by analyzing Zg. We have the following theorem. A version containing less general enumerations can be found in [8] for $d = 1$.

Theorem 4.3. *Suppose $g \in L^2(\mathbb{R}^d)$, then the following are equivalent.*

(1) *For each $\sigma \in \Lambda_{\mathcal{R}}(\mathbb{Z}^{2d})$, $\{g_{\sigma(n)}\}_{n=1}^{\infty}$ forms a Schauder basis of $L^2(\mathbb{R}^d)$ with basis constant C_σ and $\sup_{\sigma\in\Lambda_{\mathcal{R}}(\mathbb{Z}^{2d})} C_\sigma < \infty$.*
(2) $|Zg|^2 \in A_{2,\mathcal{R}}(\mathbb{T}^{2d})$.

Proof. Noticing from (13) it follows that

$$Zg_{j,k}(x,\xi) = e_j(x)e_k(\xi)Zg(x,\xi) = e_{(j,k)}(x,\xi)Zg(x,\xi).$$

It follows that $\operatorname{span}\{g_{j,k}\}_{(j,k)\in\mathbb{Z}^{2d}}$ is dense in $L^2(\mathbb{R}^d)$ if and only if $|Zg(x,\xi)| > 0$ a.e. Since $g \in L^2(\mathbb{R}^d)$, $|Zg|^2 \in L^1(\mathbb{T}^{2d})$ so we may assume $|Zg|^2$ is a weight. Then notice that

$$P_g : L^2(\mathbb{T}^{2d}, |Zg|^2) \to L^2(\mathbb{R}^d)$$
$$\nu \mapsto Z^{-1}(\nu Zg)$$

is an isometry onto $L^2(\mathbb{R}^d)$. This follows from

$$\|P_g \nu\|_{L^2(\mathbb{R}^d)} = \|Z^{-1}(\nu Zg)\|_{L^2(\mathbb{R}^d)} = \|\nu Zg\|_{L^2(\mathbb{T}^{2d})} = \|\nu\|_{L^2(\mathbb{T}^{2d},|Zg|^2)}.$$

Finally, $P_g(e_{(j,k)}) = g_{j,k}$ and if σ is an enumeration of $\mathbb{Z}^{2d}$ then $\{g_{\sigma(n)}\}_{n=1}^{\infty}$ is a Schauder basis of $L^2(\mathbb{R}^d)$ if and only if $\{e_{\sigma(n)}\}_{n=1}^{\infty}$ is a Schauder basis of $L^2(\mathbb{T}^{2d}, |Zg|^2)$. □

5. Conclusions and further questions

Our main result is the characterization of $A_{p,\mathcal{R}}(\mathbb{T}^d)$ in terms of Schauder bases. Theorem 3.6 shows that for $w \in A_{p,\mathcal{R}}(\mathbb{T}^d)$ and any rectangular enumeration σ, $\{e_{\sigma(n)}\}_{n=1}^{\infty}$ forms a Schauder basis of $L^p(\mathbb{T}^d, w)$. When $p = 2$ the weighted theory is applied to principle shift invariant systems and Gabor systems. Given w in a class of weights, it would be interesting to find the largest class of enumerations $\Lambda_{\max}(\mathbb{Z}^d)$ that characterizes when $\{e_{\sigma(n)}\}_{n=1}^{\infty}$ is a Schauder basis of $L^p(\mathbb{T}^d, w)$. For the class of weights $A_{p,\mathcal{R}}(\mathbb{T}^d)$, the set of rectangular enumerations $\Lambda_{\mathcal{R}}(\mathbb{Z}^d)$ is not the largest class of enumerations. Here is one way to expand them.

Definition 5.1. A countable collection of pairwise disjoint rectangles $\{R_j\}_{j=1}^{\infty}$ such that $\bigcup_j R_j = \mathbb{Z}^d$ will be referred as a rectangular tiling of $\mathbb{Z}^d$. We will denote a rectangular tiling of $\mathbb{Z}^d$ by $\mathfrak{R}$. We say that a rectangular tiling $\mathfrak{R} = \{R_j\}$ has depth D if for any $m \in \mathbb{N}$, the union

$$\bigcup_{j=1}^{m} R_j$$

is the disjoint union of at most $D + 1$ rectangles in $\mathbb{Z}^d$. The idea is that as m becomes large the smaller rectangles, R_j combine into large rectangles, never becoming more than $D + 1$ disjoint rectangles.

Definition 5.2. Given a rectangular tiling of $\mathbb{Z}^d$, $\mathfrak{R} = \{R_j\}_{j=1}^{\infty}$, let n_j be the number of elements of R_j. We say an enumeration σ is adapted to $\mathfrak{R}$ if σ is consecutive to R_j on $\{1 + \sum_{m=1}^{j-1} n_m + 1, \ldots, 1 + \sum_{m=1}^{j} n_m\}$ for each $j \in \mathbb{N}$. An enumeration σ belongs to the class of rectangular enumerations of depth D, denoted by $\Lambda_{\mathcal{R}}^{D}(\mathbb{Z}^d)$, if there exists a rectangular tiling $\mathfrak{R}$ with depth D, such that σ is adapted to $\mathfrak{R}$.

Each rectangular enumeration $\sigma \in \Lambda_{\mathcal{R}}(\mathbb{Z}^d)$ is of depth 1, and Theorem 3.6 holds with $\Lambda_{\mathcal{R}}(\mathbb{Z}^d)$ replaced by $\Lambda_{\mathcal{R}}^{D}(\mathbb{Z}^d)$, for a fixed $D < \infty$. The key property of such enumerations is that if $\sigma \in \Lambda_{\mathcal{R}}^{D}(\mathbb{Z}^d)$ and $N \in \mathbb{N}$ then there exists $K \leqslant N$ with $\sigma(\{1, \ldots, K\})$ being the disjoint union of at most D rectangles and $\sigma(\{K + 1, \ldots, N\})$ an incomplete rectangle (hence the disjoint union of at most d rectangles).

The classical definition of $A_p(\mathbb{T}^d)$ is weights that satisfy (3) with the supremum over cubes instead of rectangles, i.e. $w \in A_p(\mathbb{T}^d)$ if

$$\left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\,dx\right)^{p-1} \leqslant C$$

for all cubes Q contained in $\mathbb{T}^d$. It would be interesting to see if there is a class of enumerations that characterize the classic $A_p(\mathbb{T}^d)$ weights. More specifically, does there exist a class of enumerations $\Gamma(\mathbb{Z}^d)$ such that for each $\sigma \in \Gamma(\mathbb{Z}^d)$, $\{e_{\sigma(n)}\}_{n=1}^{\infty}$ forms a Schauder basis for $L^p(\mathbb{T}^d, w)$ if and only if $w \in A_p(\mathbb{T}^d)$?

Acknowledgments

The author would like to thank Guido Weiss and Edward Wilson for suggestions and discussions that improved the quality of this article. The author is also grateful to Hrvoje Šikić for ideas contained in Section 5 concerning more general enumerations.

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