J. Math. Anal. Appl. 382 (2011) 487–502

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the Hamilton–Jacobi equation in the framework of generalized functions

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ARTICLE INFO

Article history:
Received 14 September 2010
Available online 29 April 2011
Submitted by J. Xiao

Keywords:
Generalized functions
Method of characteristics
Hamilton–Jacobi equation

ABSTRACT

In this work we study, in the framework of Colombeau's generalized functions, the Hamilton–Jacobi equation with a given initial condition. We have obtained theorems on existence of solutions and in some cases uniqueness. Our technique is adapted from the classical method of characteristics with a wide use of generalized functions. We were led also to obtain some general results on invertibility and also on ordinary differential equations of such generalized functions.

1. Introduction

In the classical context one has the following result (see [3] or [11]):

Given an open interval I of $\mathbb{R}$ containing 0, an open interval I' of $\mathbb{R}$, open subsets Ω and Ω' of $\mathbb{R}^n$, $H \in \mathcal{C}^\infty(I \times \Omega \times I' \times \Omega')$ and $f \in \mathcal{C}^\infty(\Omega)$, there are an open subset W of $I \times \Omega$ and a function $u \in \mathcal{C}^\infty(W)$ such that $V := \{z \in \Omega \mid (0, z) \in W\}$ is nonempty and u is a solution to the Hamilton–Jacobi equation $\frac{\partial u}{\partial t} + H(t, x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}) = 0$ with the initial condition $u|_{\{0\}\times V} = f|_V$.

The aim of this work is to obtain, under certain conditions, an analogous result for the generalized case, that is, admitting H and f Colombeau's generalized functions. The problem of determining, in this case, the function u is called, in this paper, the HJ-Problem and u is called a solution to the HJ-Problem. Among the existing classical methods, we search to adapt the method of characteristics. This method consists in transforming a Partial Differential Equation with a given initial condition in an Ordinary Differential Equation problem with a certain given initial condition. To adapt the classical method of characteristics, we define (see Definition 3.1) the set $\mathcal{S}(I, \Omega, I', \Omega', H, f, J, W)$ in which the elements (X, U, P) are so that a certain generalized mapping, defined from X, is invertible and the derivative in relation to the first variable of the generalized X, U and P satisfy a certain system of ODE evolving generalized functions (this system, in the classical case, is called Hamiltonian system). This led us to obtain some results on invertible generalized functions (see Section 2) and on local solutions to ODE in the framework of generalized functions (see Section 4). In Section 3 we prove that, under certain conditions, if $\mathcal{S}(I, \Omega, I', \Omega', H, f, J, W)$ is nonempty, then the HJ-Problem has a generalized solution. On uniqueness of solutions to the HJ-Problem, we have obtained partial answers to this question (see Section 5). We finish by presenting, under certain conditions, a theorem of existence and uniqueness of solution to the HJ-Problem.

Results on generalized solutions to nonlinear first-order systems using an algebra of generalized germs $\mathcal{G}(\Omega \times X)$ (see definition in [5]) or other specific techniques such as mollification of derivatives (see presentation in [10]) can be found in [5,7,8].

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.04.069

In [12] one can find, in the framework of tempered generalized functions, results on existence and uniqueness of global solutions to systems of ODE as well as the flow generated by these solutions. Based on these concepts and in some of the arguments of the proofs of the results presented here (see Section 4), E.R. Oliveira has obtained some results on existence of local solutions to systems of ODE of such generalized functions (see [14]).

Recently, S. Konjik, M. Kunzinger and M. Oberguggenberger have introduced a formulation of the calculus of variations in the framework of generalized functions (see [13]).

2. Preliminaries

We briefly introduce the notation that will be used throughout this paper. Our mains references for Colombeau's theory and notation in general are [1,4,6,12]. Moreover we introduce some results and concepts which will be used in this work.

Let Ω be an open subset of $\mathbb{R}^n$ and Ω' an open subset of $\mathbb{R}^m$. The notation $K \Subset \Omega$ means that K is a compact subset of Ω. If L is a subset of $\mathbb{R}^p$ with $L \neq \mathbb{R}^p$, then $\bar{L}$ and $\mathring{L}$ denote the closure of L and the interior of L, respectively.

The algebra $\overline{\mathbb{R}}$ is the quotient $\mathcal{E}_{0M}(\mathbb{R})/\mathcal{N}_0(\mathbb{R})$, where

$$\mathcal{E}_{0M}(\mathbb{R}) := \{u \in \mathbb{R}^{]0,1]} \mid \exists N \in \mathbb{N} \text{ such that } |u(\varepsilon)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \downarrow 0\};$$
$$\mathcal{N}_0(\mathbb{R}) := \{u \in \mathcal{E}_{0M}(\mathbb{R}) \mid \forall q \in \mathbb{N} \text{ one has } |u(\varepsilon)| = O(\varepsilon^{q}) \text{ as } \varepsilon \downarrow 0\}.$$

The Colombeau algebra $\mathcal{G}(\Omega)$ is the quotient $\mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$, where

$$\mathcal{E}_M(\Omega) := \left\{(u_\varepsilon)_\varepsilon \in \left(\mathcal{C}^\infty(\Omega)\right)^{]0,1]} \,\middle|\, \forall K \Subset \Omega,\ \forall \alpha \in \mathbb{N}^n,\ \exists N \in \mathbb{N} \text{ such that } \sup_{x\in K}\left|\partial^\alpha u_\varepsilon(x)\right| = O\left(\varepsilon^{-N}\right) \text{ as } \varepsilon \downarrow 0\right\};$$
$$\mathcal{N}(\Omega) := \left\{(u_\varepsilon)_\varepsilon \in \left(\mathcal{C}^\infty(\Omega)\right)^{]0,1]} \,\middle|\, \forall K \Subset \Omega,\ \forall \alpha \in \mathbb{N}^n \text{ and } \forall q \in \mathbb{N} \text{ one has } \sup_{x\in K}\left|\partial^\alpha u_\varepsilon(x)\right| = O\left(\varepsilon^{q}\right) \text{ as } \varepsilon \downarrow 0\right\}.$$

The elements of $\mathcal{E}_M(\Omega)$ (resp. $\mathcal{N}(\Omega)$) are called moderate (resp. null) nets of smooth functions. $\mathcal{G}(\Omega)$ is a unitary associative, commutative, differential algebra whose elements are equivalence classes $u := [(u_\varepsilon)_\varepsilon]$.

One can prove (see [12, Theorem 1.2.3]) that $u \in \mathcal{E}_M(\Omega)$ is null if and only for all $K \Subset \Omega$ and for all $q \in \mathbb{N}$ one has $\sup_{x\in K}|u_\varepsilon(x)| = O(\varepsilon^q)$ as $\varepsilon \downarrow 0$.

If $u = ([(u_{1\varepsilon})_\varepsilon], \ldots, [(u_{p\varepsilon})_\varepsilon]) \in (\mathcal{G}(\Omega))^p$, then $(u_\varepsilon)_\varepsilon := ((u_{1\varepsilon}, \ldots, u_{p\varepsilon}))_\varepsilon$ will be called a representative of u.

Let J be an open subset of $\mathbb{R}$ and V an open subset of Ω. Then

$$\pi := \left[\left((t, x_1, \ldots, x_n) \in J \times \Omega \mapsto t\right)_\varepsilon\right] \in \mathcal{G}(J \times \Omega);$$
$$\pi_i := \left[\left((t, x_1, \ldots, x_n) \in J \times \Omega \mapsto x_i\right)_\varepsilon\right] \in \mathcal{G}(J \times \Omega), \quad \forall 1 \leqslant i \leqslant n;$$
$$1_\Omega := \left(\left[\left((x_1, \ldots, x_n) \in \Omega \mapsto x_1\right)_\varepsilon\right], \ldots, \left[\left((x_1, \ldots, x_n) \in \Omega \mapsto x_n\right)_\varepsilon\right]\right) \in \left(\mathcal{G}(\Omega)\right)^n;$$
$$u|_V := \left[(u_\varepsilon|_V)_\varepsilon\right] \in \mathcal{G}(V), \quad \text{for } u := \left[(u_\varepsilon)_\varepsilon\right] \in \mathcal{G}(\Omega).$$

Let W be an open subset of $\mathbb{R} \times \mathbb{R}^n$ with $U := \{z \in \mathbb{R}^n \mid (0, z) \in W\} \neq \emptyset$. Then $u|_{\{0\}\times U} := [(u_\varepsilon|_{\{0\}\times U})_\varepsilon] \in \mathcal{G}(U)$, for $u \in \mathcal{G}(W)$.

If $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$, $w := ([(w_{1\varepsilon})_\varepsilon], \ldots, [(w_{n\varepsilon})_\varepsilon]) \in (\mathcal{G}(\Omega))^n$ and $x = (x_1, \ldots, x_n)$ denotes points in $\mathbb{R}^n$, then

$$\nabla u := \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) \in \left(\mathcal{G}(\Omega)\right)^n;$$
$$\mathbf{J}w := \left[(\mathbf{J}w_\varepsilon)_\varepsilon\right] \in \mathcal{G}(\Omega), \quad \text{where } \mathbf{J}w_\varepsilon(x) := \det\left(\frac{\partial w_{i\varepsilon}}{\partial x_j}(x)\right)_{1\leqslant i,j\leqslant n}.$$

The space $\mathcal{G}_*(\Omega; \Omega')$ is the set of all $u \in (\mathcal{G}(\Omega))^m$ for which there is a representative $(u_\varepsilon)_\varepsilon$ such that:

$(*)$ $\forall K \Subset \Omega$, $\exists K' \Subset \Omega'$, $\exists \eta \in]0, 1]$ with $u_\varepsilon(K) \subset K'$, $\forall \varepsilon \in]0, \eta[$.

The elements of $\mathcal{G}_*(\Omega; \Omega')$ are called c-bounded generalized functions from Ω to Ω'. Note that, if $u \in \mathcal{G}_*(\Omega; \Omega')$, then all representatives of u satisfy $(*)$.

We say that $u \in \mathcal{G}_*(\Omega; \Omega')$ is an invertible mapping if, and only if, there is $v \in \mathcal{G}_*(\Omega'; \Omega)$ such that $u \circ v = 1_{\Omega'}$ and $v \circ u = 1_\Omega$. This v is unique and it is called the inverse mapping of u.

First we will present some results on invertibility of a generalized function.

Proposition 2.1. *Let Ω_1 and Ω_2 be open subsets of $\mathbb{R}^n$ and $f \in \mathcal{G}_*(\Omega_1, \Omega_2)$. If f is an invertible mapping, then $\mathbf{J}f$ has multiplicative inverse in $\mathcal{G}(\Omega_1)$.*

Proof. It is enough to note that $\mathbf{J}g \circ f$ is the multiplicative inverse of $\mathbf{J}f$ in $\mathcal{G}(\Omega_1)$, where g is the inverse mapping of f. $\square$

Proposition 2.2. *Let Ω_1 and Ω_2 be open subsets of $\mathbb{R}^n$ and $f \in \mathcal{G}_*(\Omega_1, \Omega_2)$. If there are $(f_\varepsilon)_\varepsilon := ((f_{1\varepsilon}, \dots, f_{n\varepsilon}))_\varepsilon \in (\mathcal{E}_M(\Omega_1))^n$ with $f = ([(f_{1\varepsilon})_\varepsilon], \dots, [(f_{n\varepsilon})_\varepsilon])$ and $\tau \in]0,1]$ satisfying*:

(i) $f_\varepsilon(\Omega_1) = \Omega_2$, $\forall \varepsilon \in]0, \tau[$;
(ii) *f_ε is an invertible mapping with inverse g_ε*, $\forall \varepsilon \in]0, \tau[$;
(iii) $\forall K' \Subset \Omega_2$, $\exists K \Subset \Omega_1$, $\exists \eta \in]0, \tau[$ *such that* $g_\varepsilon(K') \subset K$, $\forall \varepsilon \in]0, \eta[$;
(iv) $\mathbf{J}f_\varepsilon(\Omega_1) \subset \mathbb{R} \setminus \{0\}$, $\forall \varepsilon \in]0, \eta[$;

then the following statements are equivalent:

(a) *$\mathbf{J}f$ has multiplicative inverse in $\mathcal{G}(\Omega_1)$*;
(b) *f is an invertible mapping and its inverse is $g := ([(g_{1\varepsilon})_\varepsilon], \dots, [(g_{n\varepsilon})_\varepsilon])$, where $(g_{1\varepsilon}, \dots, g_{n\varepsilon}) := g_\varepsilon$ if $\varepsilon \in]0, \tau[$ and $(g_{1\varepsilon}, \dots, g_{n\varepsilon}) := g_{\frac{\tau}{2}}$ if $\varepsilon \in [\tau, 1]$.*

Proof. Suppose that (a) is true and let $K' \Subset \Omega_2$ and $\alpha \in \mathbb{N}^n$. We will prove that there is $N \in \mathbb{N}$ such that

$$\sup\{|\partial^\alpha g_{l\varepsilon}(y)| \mid y \in K' \text{ and } 1 \leqslant l \leqslant n\} = O(\varepsilon^{-N}) \quad \text{as } \varepsilon \downarrow 0, \tag{1}$$

and thus $(g_\varepsilon)_\varepsilon \in (\mathcal{E}_M(\Omega_2))^n$. From this, it is easy to check that (b) holds.

Take $K \Subset \Omega_1$ and $\eta \in]0, \tau[$ as in (iii). If $|\alpha| = 0$ then (1) holds for $N = 0$. Assume that $|\alpha| \geqslant 1$. We will prove (1), in this case, using induction on $|\alpha|$.

Denote by $y = (y_1, \dots, y_n)$ points in $\mathbb{R}^n$.

For $(\varepsilon, y) \in]0, \eta[\times \Omega_2$ we know, from (ii), that

$$\frac{\partial g_{k\varepsilon}}{\partial y_j}(y) = \frac{1}{\mathbf{J}f_\varepsilon(g_\varepsilon(y))} a_{jk}, \quad \forall 1 \leqslant j, k \leqslant n,$$

where a_{jk} is a sum of products of elements of the set

$$\left\{ \frac{\partial f_{i\varepsilon}}{\partial x_s}(g_\varepsilon(y)) \;\middle|\; 1 \leqslant i, s \leqslant n \right\} \cup \{1, -1\}.$$

Using that $\mathbf{J}f$ has multiplicative inverse in $\mathcal{G}(\Omega_1)$, (iv) and [2] (see Theorem 5.3), there are $\eta_2 \in]0, \eta[$ and $b \in \mathbb{R}$ such that

$$\varepsilon^b \leqslant \inf\{|\mathbf{J}f_\varepsilon(x)| \mid x \in K\}, \quad \forall \varepsilon \in]0, \eta_2[.$$

Since (iii) holds and $(f_{i\varepsilon})_\varepsilon \in \mathcal{E}_M(\Omega_1)$ for all $1 \leqslant i \leqslant n$, there is $N_1 \in \mathbb{N}$ such that

$$\sup\left\{ \frac{1}{|\mathbf{J}f_\varepsilon(g_\varepsilon(y))|} \;\middle|\; y \in K' \right\} = O(\varepsilon^{-N_1}) \quad \text{as } \varepsilon \downarrow 0;$$
$$\sup_{1 \leqslant i \leqslant n} \{|\partial^\gamma f_{i\varepsilon}(g_\varepsilon(y))| \mid y \in K',\ \gamma \in \mathbb{N}^n \text{ and } |\gamma| \leqslant |\alpha| + 1\} = O(\varepsilon^{-N_1}) \quad \text{as } \varepsilon \downarrow 0.$$

If $|\alpha| = 1$ then (1) is true for $N := (n+1)N_1$.

Let $|\alpha| > 1$ and $\nu \in \mathbb{N}^n$ with $|\nu| = |\alpha| + 1$. By hypothesis of induction, there is $N_2 \in \mathbb{N}$ such that

$$\sup_{1 \leqslant l \leqslant n} \{|\partial^\beta g_{l\varepsilon}(y)| \mid y \in K',\ \beta \in \mathbb{N}^n \text{ and } |\beta| \leqslant |\alpha|\} = O(\varepsilon^{-N_2}) \quad \text{as } \varepsilon \downarrow 0.$$

Noting that given $1 \leqslant k \leqslant n$ there are $1 \leqslant j \leqslant n$ and $\tilde{\gamma} \in \mathbb{N}^n$ with $|\tilde{\gamma}| = |\alpha|$ such that $\partial^\nu g_{k\varepsilon} = \partial^{\tilde{\gamma}} \frac{\partial g_{k\varepsilon}}{\partial y_j}$, it is clear that (1) holds for ν.

By Proposition 2.1 we have that (b) implies (a). $\square$

Next result, apart from supplying invertible mappings, will be used in Section 5.

Proposition 2.3. *Let I be an open interval of $\mathbb{R}$ containing 0, $\tau \in]0,1]$, $(g_\varepsilon)_\varepsilon := ((g_{1\varepsilon}, \dots, g_{n\varepsilon}))_\varepsilon \in (\mathcal{E}_M(I \times \mathbb{R}^n))^n$, $(f_\varepsilon)_\varepsilon := ((f_{0\varepsilon}, f_{1\varepsilon}, \dots, f_{n\varepsilon}))_\varepsilon \in (\mathcal{E}_M(I \times \mathbb{R}^n))^{1+n}$ given by*

$$f_\varepsilon(t, x) := (t, g_\varepsilon(t, x) + x)$$

and $f := ([(f_{0\varepsilon})_\varepsilon], [(f_{1\varepsilon})_\varepsilon], \dots, [(f_{n\varepsilon})_\varepsilon]) \in (\mathcal{G}(I \times \mathbb{R}^n))^{1+n}$. If

(i) $g_{i\varepsilon}(0,x)=0$, $\forall(\varepsilon,x)\in\,]0,\tau[\,\times\mathbb{R}^n$, $\forall 1\leqslant i\leqslant n$;
(ii) *there is* $M>0$ *such that* $\|\partial^\beta g_\varepsilon(t,x)\|\leqslant M$, $\forall(\varepsilon,t,x)\in\,]0,\tau[\,\times I\times\mathbb{R}^n$, $\forall\beta:=(\beta_0,\beta_1,\dots,\beta_n)\in\mathbb{N}^{1+n}$ *with* $1\leqslant\beta_0\leqslant|\beta|\leqslant 2$,

then there is $I_a:=\,]-a,a[$*, for some* $a>0$*, such that* $\overline{I_a}\subset I$*,* $f|_{I_a\times\mathbb{R}^n}\in\mathcal{G}_*(I_a\times\mathbb{R}^n;I_a\times\mathbb{R}^n)$ *and* $f|_{I_a\times\mathbb{R}^n}$ *is an invertible mapping.*

Proof. Denote by $(t,x)=(t,x_1,\dots,x_n)$ points in $I\times\mathbb{R}^n$. Let M be as in (ii), $a^*>0$ with $[-a^*,a^*]\subset I$ and $a>0$ such that

$$a<\min\left\{a^*,\frac{1}{nn!M^n},\frac{n^n}{nn!(n+1)^n}\right\}.$$

Using the Mean Value Theorem, (i) and (ii) we have

$$\left|g_{i\varepsilon}(t,x)\right|\leqslant M|t|,\quad\forall(\varepsilon,t,x)\in\,]0,\tau[\,\times I\times\mathbb{R}^n,\ \forall 1\leqslant i\leqslant n;\tag{1}$$

$$\left|\frac{\partial g_{i\varepsilon}}{\partial x_k}(t,x)\right|\leqslant M|t|,\quad\forall(\varepsilon,t,x)\in\,]0,\tau[\,\times I\times\mathbb{R}^n,\ \forall 1\leqslant i,k\leqslant n.\tag{2}$$

Let $I_a:=\,]-a,a[$. From (1) we obtain $f|_{I_a\times\mathbb{R}^n}\in\mathcal{G}_*(I_a\times\mathbb{R}^n;I_a\times\mathbb{R}^n)$.

Fix $\varepsilon\in\,]0,\tau[$.

Let $(s,x),(t,y)\in I_a\times\mathbb{R}^n$ with $(s,x)\neq(t,y)$. If $s\neq t$ one has $f_\varepsilon(s,x)\neq f_\varepsilon(t,y)$. Suppose $s=t$ and let $1\leqslant j\leqslant n$ such that $|x_i-y_i|\leqslant|x_j-y_j|$ for all $1\leqslant i\leqslant n$. From Mean Value Theorem there is $z\in\mathbb{R}^n$ for which

$$\left|g_{j\varepsilon}(s,x)+x_j-g_{j\varepsilon}(s,y)-y_j\right|=\left|\sum_{i=1}^n\frac{\partial g_{j\varepsilon}}{\partial x_i}(s,z)(x_i-y_i)+x_j-y_j\right|$$

and so, by (2), we obtain

$$\left|g_{j\varepsilon}(s,x)+x_j-g_{j\varepsilon}(s,y)-y_j\right|\geqslant|x_j-y_j|-nMa|x_j-y_j|>0.$$

This implies that $f_\varepsilon(s,x)\neq f_\varepsilon(s,y)$. Hence $f_\varepsilon|_{I_a\times\mathbb{R}^n}$ is one-to-one.

Let $(s,y)\in I_a\times\mathbb{R}^n$, $d:=\|y\|+nM|s|+1$, $B'_d(0):=\{z\in\mathbb{R}^n\mid\|z\|\leqslant d\}$ and $\Psi:x\in B'_d(0)\mapsto y-g_\varepsilon(s,x)$. From (1) we have $\Psi(B'_d(0))\subset B'_d(0)$ and hence, applying Brouwer's Theorem, we conclude that there is $\tilde{x}\in B'_d(0)$ such that $\Psi(\tilde{x})=\tilde{x}$. Thus $f_\varepsilon|_{I_a\times\mathbb{R}^n}$ is onto and if $J\Subset I_a$ and $K'\Subset\mathbb{R}^n$, then $(f_\varepsilon|_{I_a\times\mathbb{R}^n})^{-1}(J\times K')\subset J\times B'_r(0)$, where $r:=\max\{\|z\|+nM|t|\mid(t,z)\in J\times K'\}$.

Let $(\Gamma_\varepsilon)_\varepsilon:=((\Gamma_{0\varepsilon},\Gamma_{1\varepsilon},\dots,\Gamma_{n\varepsilon}))_\varepsilon$ defined by

$$\Gamma_\varepsilon:=(f_\varepsilon|_{I_a\times\mathbb{R}^n})^{-1},\quad\text{for }\varepsilon\in\,]0,\tau[,\quad\text{and}\quad\Gamma_\varepsilon=(f_{\frac{\tau}{2}}|_{I_a\times\mathbb{R}^n})^{-1},\quad\text{for }\varepsilon\in[\tau,1].$$

So $\overline{\Gamma_\varepsilon(J\times K')}\Subset I_a\times\mathbb{R}^n$ for all $J\Subset I_a$, $K'\Subset\mathbb{R}^n$ and $\varepsilon\in\,]0,1]$.

Let $M_1>0$ such that $nn!\max\{M^n,(1+\frac{1}{n})^n\}<M_1<\frac{1}{a}$. From (2), (i), (ii) and the Mean Value Theorem we have, for all $(\varepsilon,t,x)\in\,]0,\tau[\,\times I_a\times\mathbb{R}^n$, that

$$\left|\mathbf{J}f_\varepsilon(t,x)\right|\geqslant\left|\mathbf{J}f_\varepsilon(0,x)\right|-\sup_{s\in I_a}\left|\frac{\partial\mathbf{J}f_\varepsilon}{\partial t}(s,x)\right||t|\geqslant 1-M_1a>0.$$

Thus, by [2] (see Theorem 5.3) (or see [15, Theorem 3.5]), we obtain that $\mathbf{J}f|_{I_a\times\mathbb{R}^n}$ has multiplicative inverse.

Applying Proposition 2.2 we have that $f|_{I_a\times\mathbb{R}^n}$ is an invertible mapping and its inverse has $(\Gamma_\varepsilon)_\varepsilon$ as its representative. □

From the proof of Proposition 2.3 we have:

Remark 2.4. In Proposition 2.3 we can add the following statement:

There are a representative $(\Gamma_\varepsilon)_\varepsilon$ of $(f|_{I_a\times\mathbb{R}^n})^{-1}$, $\tau\in\,]0,1]$ and $M_2>0$ for which one has

$$\Gamma_\varepsilon=(f_\varepsilon|_{I_a\times\mathbb{R}^n})^{-1},\quad\forall\varepsilon\in\,]0,\tau[;$$
$$\left\|\partial^\gamma\Gamma_\varepsilon(t,y)\right\|\leqslant M_2,\quad\forall(\varepsilon,t,y)\in\,]0,\tau[\,\times I_a\times\mathbb{R}^n,\ \forall\alpha\in\mathbb{N}^{1+n}\text{ with }|\alpha|=1.$$

We end this section by introducing two definitions that will be important to achieving our goal.

Definition 2.5. Let Ω be an open subset of $\mathbb{R}^n$ and $(u_\varepsilon)_\varepsilon\in(\mathcal{E}_M(\Omega))^k$. We say that

(i) $(u_\varepsilon)_\varepsilon$ is bounded on $A\subset\Omega$ if there are $M>0$ and $\tau\in\,]0,1]$ such that $\|u_\varepsilon(x)\|\leqslant M$, for all $(\varepsilon,x)\in\,]0,\tau[\,\times A$;
(ii) $(u_\varepsilon)_\varepsilon$ has the property (LLG) (locally logarithmic growth) if it satisfies the following statement:
$\forall K\Subset\Omega$, $\exists N\in\mathbb{N}$, $\exists c>0$, $\exists\eta\in\,]0,1]$ such that $\|u_\varepsilon(x)\|\leqslant\log(c\varepsilon^{-N})$, $\forall(\varepsilon,x)\in\,]0,\eta[\,\times K$.

Clearly the bounded nets have the property (LLG) and if $(u_\varepsilon)_\varepsilon \in (\mathcal{E}_M(\Omega))^k$ has the property (LLG) and $(v_\varepsilon)_\varepsilon \in (\mathcal{E}_M(\Omega))^k$ is such that $(u_\varepsilon - v_\varepsilon)_\varepsilon \in (\mathcal{N}(\Omega))^k$, then $(v_\varepsilon)_\varepsilon$ also has the property (LLG).

Definition 2.6. Let I be an open interval of $\mathbb{R}$, Ω an open subset of $\mathbb{R}^n$ and Ω' an open subset of $\mathbb{R}^m$. We say that:

(i) $(u_\varepsilon)_\varepsilon \in (\mathcal{E}_M(I \times \Omega \times \Omega'))^k$ has the property (LLL) (locally logarithmically Lipschitz) in (I, Ω, Ω') if it satisfies the following statement:
$\forall J \Subset I$, $\forall K \Subset \Omega$, $\forall K' \Subset \Omega'$, $\exists N \in \mathbb{N}$, $\exists c > 0$, $\exists \eta \in]0,1]$ such that
$$\left\| u_\varepsilon(t,x,y) - u_\varepsilon(t,x,z) \right\| \leqslant \log\left(c\varepsilon^{-N}\right) \|y - z\|, \quad \forall \varepsilon \in]0,\eta[,\ \forall t \in J,\ \forall x \in K,\ \forall y,z \in K'.$$

(ii) $(u_\varepsilon)_\varepsilon \in (\mathcal{E}_M(I \times \Omega))^k$ has the property (LLL) (locally logarithmically Lipschitz) in (I, Ω) if it satisfies the following statement:
$\forall J \Subset I$, $\forall K \Subset \Omega$, $\exists N \in \mathbb{N}$, $\exists c > 0$, $\exists \eta \in]0,1]$ such that
$$\left\| u_\varepsilon(t,y) - u_\varepsilon(t,z) \right\| \leqslant \log\left(c\varepsilon^{-N}\right) \|y - z\|, \quad \forall \varepsilon \in]0,\eta[,\ \forall t \in J,\ \forall y,z \in K.$$

3. Method of characteristics

Here Ω and Ω' will be open subsets of $\mathbb{R}^n$, I an open interval of $\mathbb{R}$ containing 0 and I' an open interval of $\mathbb{R}$. Moreover, for a given $t_0 \in \mathbb{R}$ and $a > 0$, we will denote by $I_a(t_0)$ the interval $]t_0 - a, t_0 + a[$. Sometimes we will write I_a instead of $I_a(0)$. We will denote by $x = (x_1, \ldots, x_n)$, $(t,x) = (t, x_1, \ldots, x_n)$ (or $(s,r) = (s, r_1, \ldots, r_n)$) and $(t,x,y,p) = (t, x_1, \ldots, x_n, y, p_1, \ldots, p_n)$ points in Ω, $I \times \Omega$ and $I \times \Omega \times I' \times \Omega'$, respectively.

Now we are ready to present the main object of our study. Consider the following statement (which here is called HJ-Problem):

HJ-Problem. Given $H \in \mathcal{G}(I \times \Omega \times I' \times \Omega')$ and $f \in \mathcal{G}(\Omega)$, there are an open subset W of $I \times \Omega$ and $u \in \mathcal{G}(W)$ such that $V := \{z \in \Omega \mid (0,z) \in W\} \neq \emptyset$ and one has the following statements:
$$\left(u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) \in \mathcal{G}_*\left(W; I' \times \Omega'\right);$$
$$\frac{\partial u}{\partial t} + H \circ \left(\pi, \pi_1, \ldots, \pi_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) = 0 \quad \text{in } \mathcal{G}(W);$$
$$u|_{\{0\}\times V} = f|_V.$$

Our goal is to study under what conditions, given H and f, there are an open subset W and a function u as in HJ-Problem. When there is this function u, we say that *the HJ-Problem has a solution in* $\mathcal{G}(W)$ (or that u *is a solution to the HJ-Problem in* $\mathcal{G}(W)$). To provide such conditions, we have adapted for the generalized case the classical method of characteristics. Before presenting the theorem of existence that we have obtained, we introduce, to simplify the writing, the definition below.

Definition 3.1. Let $H \in \mathcal{G}(I \times \Omega \times I' \times \Omega')$, $f \in \mathcal{G}(\Omega)$, W be an open subset of $I \times \Omega$ with $V := \{z \in \Omega \mid (0,z) \in W\} \neq \emptyset$ and J an open interval of $\mathbb{R}$ with $0 \in J \subset I$. We will denote by $\mathcal{S}(I, \Omega, I', \Omega', H, f, J, W)$ the set of all (X, U, P), where $P = (P_1, \ldots, P_n)$, for which one has:

(i) $(X, U, P) \in \mathcal{G}_*(J \times V; \Omega) \times \mathcal{G}_*(J \times V; I') \times \mathcal{G}_*(J \times V; \Omega')$;
(ii) (X, U, P) is a solution of system:
$$\frac{\partial X}{\partial s} = \frac{\partial H}{\partial p} \circ (\pi, X, U, P);$$
$$\frac{\partial U}{\partial s} = -H \circ (\pi, X, U, P) + \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} \circ (\pi, X, U, P) P_j;$$
$$\frac{\partial P}{\partial s} = -\frac{\partial H}{\partial x} \circ (\pi, X, U, P) - P \frac{\partial H}{\partial y} \circ (\pi, X, U, P);$$
(iii) $(X, U, P)|_{\{0\}\times V} = (1_V, f|_V, \nabla f|_V)$;
(iv) $Y := (\pi, X) \in \mathcal{G}_*(J \times V; W)$ and Y is an invertible mapping.

Unless otherwise stated, H and f belong to $\mathcal{G}(I \times \Omega \times I' \times \Omega')$ and $\mathcal{G}(\Omega)$, respectively.

Theorem 3.2 *(Existence Theorem). Suppose $\mathcal{S}(I,\Omega,I',\Omega',H,f,J,W)\neq\emptyset$. If there is a representative $(H_\varepsilon)_\varepsilon$ of H such that $(\frac{\partial H_\varepsilon}{\partial y})_\varepsilon$ has the property* (LLG)*, then the* HJ-Problem *has a solution in $\mathcal{G}(W)$. More precisely, for all $(X,U,P)\in\mathcal{S}(I,\Omega,I',\Omega',H,f,J,W)$, the function $u:=U\circ(\pi,X)^{-1}$ is a solution to the HJ-Problem in $\mathcal{G}(W)$ and $(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n})=P\circ(\pi,X)^{-1}$.*

Proof. Fix $(X,U,P)\in\mathcal{S}(I,\Omega,I',\Omega',H,f,J,W)$. We will prove that $u:=U\circ(\pi,X)^{-1}$ is a solution to the HJ-Problem in $\mathcal{G}(W)$.

Let $Y:=(\pi,X)$ and $h:=(h_0,h_1,\dots,h_n)=Y^{-1}$. Then

$$\begin{aligned}(\pi|_{\{0\}\times V},\pi_1|_{\{0\}\times V},\dots,\pi_n|_{\{0\}\times V})&=(Y\circ h)|_{\{0\}\times V}\\&=\big(h_0|_{\{0\}\times V},X\circ(h|_{\{0\}\times V})\big)\\&=(h_0|_{\{0\}\times V},h_1|_{\{0\}\times V},\dots,h_n|_{\{0\}\times V}).\end{aligned}$$

Thus $u|_{\{0\}\times V}=U\circ(h|_{\{0\}\times V})=f|_V$.

Since $U=u\circ Y$ and Definition 3.1(ii) holds we have

$$\left(\frac{\partial u}{\partial t}\circ Y\right)+H\circ(\pi,X,U,P)+\sum_{j=1}^{n}\left(\frac{\partial u}{\partial x_j}\circ Y-P_j\right)\frac{\partial X_j}{\partial s}=0; \tag{1}$$

$$\frac{\partial U}{\partial r_i}=\sum_{j=1}^{n}\left(\frac{\partial u}{\partial x_j}\circ Y\right)\frac{\partial X_j}{\partial r_i},\quad\forall 1\leqslant i\leqslant n. \tag{2}$$

Suppose it has been proved that $\frac{\partial U}{\partial r_i}-\sum_{j=1}^{n}P_j\frac{\partial X_j}{\partial r_i}=0$, for all $1\leqslant i\leqslant n$. Then, by (1) and (2), we have the system:

$$\left(\left(\frac{\partial u}{\partial t}\circ Y\right)+H\circ(\pi,X,U,P)\right)+\sum_{j=1}^{n}\left(\frac{\partial u}{\partial x_j}\circ Y-P_j\right)\frac{\partial X_j}{\partial s}=0;$$

$$\sum_{j=1}^{n}\left(\frac{\partial u}{\partial x_j}\circ Y-P_j\right)\frac{\partial X_j}{\partial r_i}=0,\quad\forall 1\leqslant i\leqslant n.$$

Applying that Y is an invertible mapping we conclude that $\mathbf{J}Y$ has multiplicative inverse in $\mathcal{G}(J\times V)$ (see Proposition 2.1). This implies that the system above has only the trivial solution. Hence

$$\left(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right)=P\circ Y^{-1}\in\mathcal{G}_*\big(W,\Omega'\big);$$

$$\frac{\partial u}{\partial t}=-H\circ(\pi,X,U,P)\circ Y^{-1}=-H\circ\left(\pi,\pi_1,\dots,\pi_n,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right).$$

Fix $1\leqslant i\leqslant n$ and define $\varphi:=\frac{\partial U}{\partial r_i}-\sum_{j=1}^{n}P_j\frac{\partial X_j}{\partial r_i}\in\mathcal{G}(J\times V)$.

To finish the proof of this theorem, it remains to verify that $\varphi=0$.

Let $(J_j)_{j\in\mathbb{N}}$ be an exhaustive sequence of compact subsets of J with $0\in\bigcap_{j\in\mathbb{N}}J_j$ and J_j closed interval for all $j\in\mathbb{N}$, $(K'_j)_{j\in\mathbb{N}}$ an exhaustive sequence of compact subsets of V and $(X_\varepsilon)_\varepsilon$, $(U_\varepsilon)_\varepsilon$, $(P_\varepsilon)_\varepsilon$ and $(\varphi_\varepsilon)_\varepsilon$ representatives of X, U, P and φ, respectively.

Fix $j\in\mathbb{N}$, $[a,b]\subset\mathring{J}_j$ containing 0 and $K'\Subset\mathring{K}'_j$. We will prove that

$$\text{given } q\in\mathbb{N} \text{ one has } \sup_{(s,r)\in[a,b]\times K'}\big|\varphi_\varepsilon(s,r)\big|=O\big(\varepsilon^q\big)\text{ as }\varepsilon\downarrow 0, \tag{3}$$

and, as $j\in\mathbb{N}$ is arbitrary, we conclude, by [12] (see Theorem 1.2.3), that $\varphi=0$.

From Definition 3.1 we have that

$$\frac{\partial\varphi}{\partial s}=-\varphi\frac{\partial H}{\partial y}\circ(\pi,X,U,P)\quad\text{and}\quad\varphi|_{\{0\}\times V}=0.$$

Thus there are $(g_\varepsilon)_\varepsilon\in\mathcal{N}(\mathring{J}_j\times\mathring{K}'_j)$, $(h_\varepsilon)_\varepsilon\in\mathcal{N}(\mathring{K}'_j)$ and $\eta_j\in\,]0,1]$ such that

$$\varphi_\varepsilon(s,r)-h_\varepsilon(r)=\int_0^s\big(\psi_\varepsilon\big(w,\varphi_\varepsilon(w,r),r\big)+g_\varepsilon(w,r)\big)\,\mathrm{d}w,$$

where $\psi_\varepsilon(s,t,r)=-t\frac{\partial H_\varepsilon}{\partial y}(s,X_\varepsilon(s,r),U_\varepsilon(s,r),P_\varepsilon(s,r))$, for all $(\varepsilon,s,t,r)\in\,]0,\eta_j[\,\times\mathring{J}_j\times\mathbb{R}\times\mathring{K}'_j$.

Using Definition 3.1(i) and that $(\frac{\partial H_\varepsilon}{\partial y})_\varepsilon$ has the property (LLG) we can find $N \in \mathbb{N}$, $c > 0$ and $\tau \in]0, \eta_j[$ such that

$$\left|\psi_\varepsilon\big(s, \varphi_\varepsilon(s,r), r\big)\right| \leqslant \log\big(c\varepsilon^{-N}\big)\left|\varphi_\varepsilon(s,r)\right|, \quad \forall(\varepsilon, s, r) \in]0, \tau[\times [a,b] \times K'.$$

So, for all $(\varepsilon, s, r) \in]0, \tau[\times [a,b] \times K'$, we obtain that

$$\left|\varphi_\varepsilon(s,r) - h_\varepsilon(r)\right| \leqslant \left|\int_0^s \big(\log\big(c\varepsilon^{-N}\big)\left|\varphi_\varepsilon(w,r) - h_\varepsilon(w)\right| + \log\big(c\varepsilon^{-N}\big)\left|h_\varepsilon(w)\right| + \left|g_\varepsilon(w,r)\right|\big)\,\mathrm{d}w\right|.$$

This, together with $(g_\varepsilon)_\varepsilon \in \mathcal{N}(\mathring{J}_j \times \mathring{K}'_j)$, $(h_\varepsilon)_\varepsilon \in \mathcal{N}(\mathring{K}'_j)$ and Gronwall's Lemma, gives us

$$\sup_{(s,r)\in[a,b]\times K'} \left|\varphi_\varepsilon(s,r) - h_\varepsilon(r)\right| = O\big(\varepsilon^q\big) \quad \text{as } \varepsilon \downarrow 0.$$

Hence (3) holds. $\square$

As an application of the previous theorem we give the following result:

Proposition 3.3. *Let $(h_\varepsilon)_\varepsilon \in \mathcal{E}_M(I \times \mathbb{R}^n)$, $((\mu_{0\varepsilon}, \mu_{1\varepsilon}, \ldots, \mu_{n\varepsilon}))_\varepsilon \in (\mathcal{E}_{0M}(\mathbb{R}))^{1+n}$, $(H_\varepsilon)_\varepsilon \in \mathcal{E}_M(I \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ defined by*

$$H_\varepsilon(t, x_1, \ldots, x_n, y, p_1, \ldots, p_n) := h_\varepsilon(t, p_1, \ldots, p_n) + \mu_{0\varepsilon} y + \sum_{i=1}^n \mu_{i\varepsilon} x_i,$$

$H = [(H_\varepsilon)_\varepsilon] \in \mathcal{G}(I \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ and $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R}^n)$. If

(i) *$\exists K \Subset \mathbb{R}^n$ such that $\mu_\varepsilon := (\mu_{1\varepsilon}, \ldots, \mu_{n\varepsilon}) \in K$, $\forall \varepsilon \in]0,1]$;*
(ii) *$\exists L \Subset I$ such that $0 \notin L$ and $\mu_{0\varepsilon} \in L$, $\forall \varepsilon \in]0,1]$;*
(iii) *$(\partial^\alpha h_\varepsilon)_\varepsilon$ is bounded on $I \times \mathbb{R}^n$, $\forall \alpha := (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{1+n}$ with $0 = \alpha_0 < |\alpha| \leqslant 2$ or $1 = \alpha_0 < |\alpha| = 2$;*
(iv) *$(\partial^\beta f_\varepsilon)_\varepsilon$ is bounded on $\mathbb{R}^n$, $\forall \beta \in \mathbb{N}^n$ with $1 \leqslant |\beta| \leqslant 2$;*
(v) *$f \in \mathcal{G}_*(\mathbb{R}^n; \mathbb{R})$ and $h := [(h_\varepsilon)_\varepsilon] \in \mathcal{G}_*(I \times \mathbb{R}^n; \mathbb{R})$;*

then the HJ-Problem has a solution in $\mathcal{G}(I_a \times \mathbb{R}^n)$, for some $a > 0$.

Proof. Clearly $(\frac{\partial H_\varepsilon}{\partial y})_\varepsilon$ has the property (LLG). So it is enough to prove that $\mathcal{S}(I, \mathbb{R}^n, \mathbb{R}, \mathbb{R}^n, H, f, I_a, I_a \times \mathbb{R}^n) \neq \emptyset$, for some $a > 0$ (see Theorem 3.2).

Denote by $(t, x, y, p) = (t, x_1, \ldots, x_n, y, p_1, \ldots, p_n)$ points in $\mathbb{R}^{1+n+1+n}$ and by $(s, r) = (s, r_1, \ldots, r_n)$ or $(t, p) := (t, p_1, \ldots, p_n)$ points in $\mathbb{R}^{1+n}$.

Let $(P_\varepsilon)_\varepsilon := ((P_{1\varepsilon}, \ldots, P_{n\varepsilon}))_\varepsilon$, $(X_\varepsilon)_\varepsilon := ((X_{1\varepsilon}, \ldots, X_{n\varepsilon}))_\varepsilon$ and $(U_\varepsilon)_\varepsilon$ defined, on $I \times \mathbb{R}^n$, by

$$P_\varepsilon(s,r) := \left(\nabla f_\varepsilon(r) + \frac{\mu_\varepsilon}{\mu_{0\varepsilon}}\right)\exp(-\mu_{0\varepsilon} s) - \frac{\mu_\varepsilon}{\mu_{0\varepsilon}};$$

$$X_\varepsilon(s,r) := r + \int_0^s \frac{\partial h_\varepsilon}{\partial p}\big(t, P_\varepsilon(t,r)\big)\,\mathrm{d}t;$$

$$\begin{aligned} U_\varepsilon(s,r) := {} & \exp(-\mu_{0\varepsilon} s)\left(f_\varepsilon(r) - \int_0^s \exp(\mu_{0\varepsilon} t) h_\varepsilon\big(t, P_\varepsilon(t,r)\big)\,\mathrm{d}t\right) \\ & + \exp(-\mu_{0\varepsilon} s)\sum_{j=1}^n \int_0^s -\exp(\mu_{0\varepsilon} t)\mu_{j\varepsilon} X_{j\varepsilon}(t,r)\,\mathrm{d}t \\ & + \exp(-\mu_{0\varepsilon} s)\sum_{j=1}^n \int_0^s \exp(\mu_{0\varepsilon} t)\frac{\partial h_\varepsilon}{\partial p_j}\big(t, P_\varepsilon(t,r)\big) P_{j\varepsilon}(t,r)\,\mathrm{d}t. \end{aligned}$$

Define $(g_\varepsilon)_\varepsilon := ((g_{1\varepsilon}, \ldots, g_{n\varepsilon}))_\varepsilon$ by $g_\varepsilon : (s,r) \in I \times \mathbb{R}^n \mapsto X_\varepsilon(s,r) - r$.

Note that, for all $(\varepsilon, s, r) \in]0,1] \times I \times \mathbb{R}^n$ and $1 \leqslant i, k \leqslant n$, we have

$$\frac{\partial g_{i\varepsilon}}{\partial s}(s,r) = \frac{\partial h_\varepsilon}{\partial p_i}\big(s, P_\varepsilon(s,r)\big);$$

$$\frac{\partial^2 g_{i\varepsilon}}{\partial s^2}(s,r) = -\sum_{j=1}^{n} \frac{\partial^2 h_\varepsilon}{\partial p_j \partial p_i}\big(s, P_\varepsilon(s,r)\big)\left(\frac{\partial f_\varepsilon}{\partial r_j}(r) + \frac{\mu_{j\varepsilon}}{\mu_{0\varepsilon}}\right)\exp(-\mu_{0\varepsilon} s)\mu_{0\varepsilon} + \frac{\partial^2 h_\varepsilon}{\partial t \partial p_i}\big(s, P_\varepsilon(s,r)\big);$$

$$\frac{\partial^2 g_{i\varepsilon}}{\partial r_k \partial s}(s,r) = \sum_{j=1}^{n} \frac{\partial^2 h_\varepsilon}{\partial p_j \partial p_i}\big(s, P_\varepsilon(s,r)\big)\frac{\partial^2 f_\varepsilon}{\partial r_k \partial r_j}(r)\exp(-\mu_{0\varepsilon} s).$$

Let J be an open interval of $\mathbb{R}$ with $0 \in J \subset \overline{J} \Subset I$. Thus, from (i), (ii), (iii) and (iv), there is $M > 0$ such that for all $1 \leqslant i, k \leqslant n$ one has

$$\max_{(\varepsilon,s,r)\in]0,1]\times J\times\mathbb{R}^n} \left\{ \left|\frac{\partial g_{i\varepsilon}}{\partial s}(s,r)\right|, \left|\frac{\partial^2 g_{i\varepsilon}}{\partial s^2}(s,r)\right|, \left|\frac{\partial^2 g_{i\varepsilon}}{\partial r_k \partial s}(s,r)\right| \right\} \leqslant M. \tag{1}$$

Let $(Y_\varepsilon)_\varepsilon := ((Y_{1\varepsilon}, \ldots, Y_{n\varepsilon}))_\varepsilon$ defined by

$$Y_\varepsilon : (s,r) \in J \times \mathbb{R}^n \mapsto \big(s, X_\varepsilon(s,r)\big)$$

and $Y := ([(Y_{1\varepsilon})_\varepsilon], \ldots, [(Y_{n\varepsilon})_\varepsilon]) \in (\mathcal{G}(J \times \mathbb{R}^n))^n$.

Since $g_{i\varepsilon}(0,r) = 0$, for all $(\varepsilon, r) \in]0,1] \times \mathbb{R}^n$ and $1 \leqslant i \leqslant n$ and (1) holds, we have, by Theorem 2.3, that there is $a > 0$ such that $\overline{I_a} \subset J$, $Y|_{I_a\times\mathbb{R}^n} \in \mathcal{G}_*(I_a \times \mathbb{R}^n; I_a \times \mathbb{R}^n)$ and $Y|_{I_a\times\mathbb{R}^n}$ is an invertible mapping.

It is easy to verify that $(X, U, P) \in \mathcal{S}(I, \mathbb{R}^n, \mathbb{R}, \mathbb{R}^n, H, f, I_a, I_a \times \mathbb{R}^n)$ (note that, from (v), $U \in \mathcal{G}_*(I_a \times \mathbb{R}^n; \mathbb{R})$). $\square$

To find H and f for which the HJ-Problem has a solution we need, according to Theorem 3.2, to solve a system of ordinary differential equations in the framework of generalized functions. This led us to obtain, in this context, some results on ordinary differential equations. These results will be presented in the next section. From these results we have obtained functions H and f for which the HJ-Problem has at least one solution. We have also studied some cases in which one has unique solution (see Section 5).

4. Some results on the existence and uniqueness of solutions to ordinary differential equations

Here Ω will be an open subset of $\mathbb{R}^n$, Ω' an open subset of $\mathbb{R}^m$, I an open interval of $\mathbb{R}$, $t_0 \in I$ and $I_a(t_0) :=]t_0 - a, t_0 + a[$, for $a > 0$.

The purpose of this section is to study under what conditions one has the following statement:

Problem 4.1. Given $f \in (\mathcal{G}(I \times \Omega \times \Omega'))^m$ and $g \in (\mathcal{G}(\Omega))^m$, there are an open subset W of Ω, $a > 0$ and $u \in \mathcal{G}_*(I_a(t_0) \times W; \Omega')$ such that

$$\frac{\partial u}{\partial t}(t, y_1, \ldots, y_n) = f\big(t, y_1, \ldots, y_n, u(t, y_1, \ldots, y_n)\big) \quad \text{in } \big(\mathcal{G}\big(I_a(t_0) \times W\big)\big)^m;$$
$$u|_{\{t_0\}\times W} = g|_W.$$

The function $u \in (\mathcal{G}(I_a(t_0) \times W))^m$ is said to be a *solution to* Problem 4.1 and we say that Problem 4.1 *has a solution in* $(\mathcal{G}(I_a(t_0) \times W))^m$.

The first equation that appears in Problem 4.1 should be read as $\frac{\partial u}{\partial t} = f \circ (\pi, \pi_1, \ldots, \pi_n, u)$.

Theorem 4.2. *Let W be an open subset of Ω, $x_0 := ([(x_{01\varepsilon})_\varepsilon], \ldots, [(x_{0m\varepsilon})_\varepsilon]) \in \overline{\mathbb{R}}^m$ and $f := ([(f_{1\varepsilon})_\varepsilon], \ldots, [(f_{m\varepsilon})_\varepsilon]) \in (\mathcal{G}(I \times \Omega \times \Omega'))^m$ satisfying*

(i) $\overline{W} \Subset \Omega$;
(ii) $\exists K_1 \Subset \Omega'$, $\exists \tau_1 \in]0,1]$ *such that* $(x_{01\varepsilon}, \ldots, x_{0m\varepsilon}) \in K_1$, $\forall \varepsilon \in]0, \tau_1[$ (*i.e.* x_0 *is a compactly supported generalized point in* Ω');
(iii) $f \in \mathcal{G}_*(I \times \Omega \times \Omega'; \mathbb{R}^m)$;
(iv) $(\partial^\alpha f_{i\varepsilon})_\varepsilon$ *has the property* (LLG), $\forall 1 \leqslant i \leqslant m$, $\forall \alpha \in \mathbb{N}^{1+n+m}$ *with* $|\alpha| = 1$ *and* $\alpha = (0, 0, \ldots, 0, \alpha_1, \ldots, \alpha_m)$.

Then Problem 4.1, *for* $g := ([(x \mapsto x_{01\varepsilon})_\varepsilon], \ldots, [(x \mapsto x_{0m\varepsilon})_\varepsilon]) \in (\mathcal{G}(\Omega))^m$, *has a solution in* $(\mathcal{G}(I_a(t_0) \times W))^m$, *for some* $a > 0$.

Proof. Let $a^* > 0$ with $\overline{I_{a^*}(t_0)} \subset I$, K_1 and τ_1 as in (ii), V an open subset of Ω' such that $K_1 \subset V \subset \overline{V} \Subset \Omega'$ and $d > 0$ the distance of K_1 to $\Omega' \setminus V$.

Define $x_{0\varepsilon} := (x_{01\varepsilon}, \dots, x_{0m\varepsilon})$ and $f_\varepsilon := (f_{1\varepsilon}, \dots, f_{m\varepsilon})$, for all $\varepsilon \in]0,1]$.

By (i) and (iii) there are $\tau \in]0, \tau_1[$ and $M > 0$ such that

$$\big\| f_\varepsilon(t,y,z) \big\| \leqslant M, \quad \forall (\varepsilon,t,y,z) \in]0,\tau[\times \overline{I_{a^*}(t_0)} \times \overline{W} \times \overline{V}. \tag{1}$$

Take $a > 0$ with $a < \min\{a^*, d/M\}$. Fixed any $\varepsilon \in]0,\tau[$ there is a unique $u_\varepsilon := (u_{1\varepsilon}, \dots, u_{m\varepsilon}) \in \mathcal{C}(\overline{I_a(t_0)} \times \overline{W}; \overline{V}) \cap \mathcal{C}^\infty(I_a(t_0) \times W; \mathbb{R}^m)$ such that

$$u_\varepsilon(t,y) = x_{0\varepsilon} + \int_{t_0}^{t} f_\varepsilon\big(s,y,u_\varepsilon(s,y)\big)\,\mathrm{d}s, \quad \forall (t,y) \in \overline{I_a(t_0)} \times \overline{W}.$$

Remark that $u_\varepsilon(\overline{I_a(t_0)} \times \overline{W}) \subset \{z \in \mathbb{R}^n \mid \|z - x_{0\varepsilon}\| \leqslant d\} \subset \overline{V}$.

For all $\varepsilon \in [\tau, 1]$, define $u_\varepsilon := u_{\frac{\tau}{2}}$.

By induction on $|\gamma|$, where $\gamma \in \mathbb{N}^{1+n}$, we will prove that there is $N_2 \in \mathbb{N}$ such that

$$\sup_{(t,y)\in I_a(t_0)\times W} \big\| \partial^\gamma u_\varepsilon(t,y) \big\| = O\big(\varepsilon^{-N_2}\big) \quad \text{as } \varepsilon \downarrow 0. \tag{2}$$

Thus $(u_\varepsilon)_\varepsilon \in (\mathcal{E}_M(I_a(t_0) \times W))^m$ and $u := ([(u_{1\varepsilon})_\varepsilon], \dots, [(u_{m\varepsilon})_\varepsilon])$ is a solution to Problem 4.1 in $(\mathcal{G}(I_a(t_0) \times W))^m$.

Note that (2) is true for $|\gamma| = 0$.

To facilitate the writing we will prove (2) for the cases $|\gamma| = 1$ and $|\gamma| = 2$, since the induction argument is similar.

Denote by $(t, y_1, \dots, y_n, z_1, \dots, z_m)$ points in $I \times \Omega \times \Omega'$. Let $\beta \in \mathbb{N}^{1+n}$ and suppose that $\beta = (\beta_0, \beta_1, \dots, \beta_n)$.

Using that $\overline{I_a(t_0)} \times \overline{W} \times \overline{V} \Subset I \times \Omega \times \Omega'$, $(f_\varepsilon)_\varepsilon \in (\mathcal{E}_M[I \times \Omega \times \Omega'])^m$ and (iv), there are $c > 0$, $\eta \in]0,\tau[$ and $N \in \mathbb{N}$ such that

$$\left| \frac{\partial f_{i\varepsilon}}{\partial z_j}\big(s,y,u_\varepsilon(s,y)\big) \right| \leqslant \log\big(c\varepsilon^{-N}\big) \quad \text{and} \quad \big|\partial^\alpha f_{i\varepsilon}\big(s,y,u_\varepsilon(s,y)\big)\big| \leqslant c\varepsilon^{-N}, \tag{3}$$

$\forall (\varepsilon,s,y) \in]0,\eta[\times I_a(t_0) \times W$, $\forall 1 \leqslant i,j \leqslant m$ and $\forall |\alpha| \leqslant |\beta|$.

If $\beta_0 \neq 0$ and $|\beta| = 1$, then $\partial^\beta u_\varepsilon(t,y) = f(t,y,u_\varepsilon(t,u))$, for all $(\varepsilon,t,y) \in]0,\eta[\times I_a(t_0) \times W$. Then, by (1), it is clear that (2) is true for $\gamma = \beta$.

If $\beta_0 = 0$ and $|\beta| = 1$, then

$$\frac{\partial u_{i\varepsilon}}{\partial y_j}(t,y) = \int_{t_0}^{t} \left(\frac{\partial f_{i\varepsilon}}{\partial y_j}\big(s,y,u_\varepsilon(s,y)\big) + \sum_{k=1}^{m} \frac{\partial f_{i\varepsilon}}{\partial z_k}\big(s,y,u_\varepsilon(s,y)\big) \frac{\partial u_{k\varepsilon}}{\partial y_j}(s,y) \right) \mathrm{d}s,$$

$\forall (\varepsilon,t,y) \in]0,\eta[\times I_a(t_0) \times W$, $\forall 1 \leqslant i \leqslant m$ and $\forall 1 \leqslant j \leqslant n$ with $\beta_j = 1$.

From (3), for all $(\varepsilon,t,y) \in]0,\eta[\times I_a(t_0) \times W$, one has

$$\big\| \partial^\beta u_\varepsilon(t,y) \big\| \leqslant \sum_{i=1}^{m} \big| \partial^\beta u_{i\varepsilon}(t,y) \big| \leqslant \left| \int_{t_0}^{t} \big(mc\varepsilon^{-N} + 2m\log\big(c\varepsilon^{-N}\big) \big\| \partial^\beta u_\varepsilon(t,y) \big\| \big)\,\mathrm{d}s \right|.$$

This inequality, together with Gronwall's Lemma, proves (2) for $\gamma = \beta$.

If $\beta_0 \neq 0$ and $|\beta| = 2$, then

$$\frac{\partial^2 u_{i\varepsilon}}{\partial y_j \partial t}(t,y) = \frac{\partial f_{i\varepsilon}}{\partial y_j}\big(t,y,u_\varepsilon(t,y)\big) + \sum_{k=1}^{m} \frac{\partial f_{i\varepsilon}}{\partial z_k}\big(t,y,u_\varepsilon(t,y)\big) \frac{\partial u_{k\varepsilon}}{\partial y_j}(t,y),$$

$\forall (\varepsilon,t,y) \in]0,\eta[\times I_a(t_0) \times W$, $\forall 1 \leqslant i \leqslant m$ and $\forall 1 \leqslant j \leqslant n$ with $\beta_j = 1$.

So, by (2) for $|\gamma| = 1$ and (3), we have that (2) holds for $\gamma = \beta$.

If $\beta_0 = 0$ and $|\beta| = 2$, then

$$\begin{aligned} \partial^\beta u_{i\varepsilon}(t,y) = {} & \int_{t_0}^{t} \frac{\partial^2 f_{i\varepsilon}}{\partial y_\nu \partial y_j}\big(s,y,u_\varepsilon(s,y)\big)\,\mathrm{d}s + \int_{t_0}^{t} \left(\sum_{l=1}^{m} \frac{\partial^2 f_{i\varepsilon}}{\partial z_l \partial y_j}\big(s,y,u_\varepsilon(s,y)\big) \frac{\partial u_{l\varepsilon}}{\partial y_\nu}(s,y) \right) \mathrm{d}s \\ & + \int_{t_0}^{t} \left(\sum_{k=1}^{m} \frac{\partial^2 f_{i\varepsilon}}{\partial y_\nu \partial z_k}\big(s,y,u_\varepsilon(s,y)\big) \frac{\partial u_{k\varepsilon}}{\partial y_j}(s,y) \right) \mathrm{d}s \end{aligned}$$

$$+\int_{t_0}^{t}\left(\sum_{k=1}^{m}\sum_{l=1}^{m}\frac{\partial^2 f_{i\varepsilon}}{\partial z_l\partial z_k}\big(s,y,u_\varepsilon(s,y)\big)\frac{\partial u_{l\varepsilon}}{\partial y_\nu}(s,y)\frac{\partial u_{k\varepsilon}}{\partial y_j}(s,y)\right)\mathrm{d}s$$
$$+\int_{t_0}^{t}\left(\sum_{k=1}^{m}\frac{\partial f_{i\varepsilon}}{\partial z_k}\big(s,y,u_\varepsilon(s,y)\big)\partial^\beta u_{k\varepsilon}(t,y)\right)\mathrm{d}s,$$

$\forall(\varepsilon,t,y)\in\,]0,\eta[\,\times I_a(t_0)\times W$, $\forall 1\leqslant i\leqslant m$ and $\forall 1\leqslant j,\nu\leqslant n$ with $\beta_j=1=\beta_\nu$.

From (2) for $|\gamma|=1$ and (3), there are $c_1>0$, $N_1\in\mathbb{N}$ and $\eta_1\in\,]0,\eta[$ such that, for all $(\varepsilon,t,y)\in\,]0,\eta_1[\,\times I_a(t_0)\times W$, one has

$$\big\|\partial^\beta u_\varepsilon(t,y)\big\|\leqslant\sum_{i=1}^{m}\big|\partial^\beta u_{i\varepsilon}(t,y)\big|\leqslant\left|\int_{t_0}^{t}\big(c_1\varepsilon^{-N_1}+2m\log\big(c\varepsilon^{-N}\big)\big\|\partial^\beta u_\varepsilon(t,y)\big\|\big)\,\mathrm{d}s\right|.$$

This inequality, together with Gronwall's Lemma, proves (2) for $\gamma=\beta$. □

Theorem 4.3 *(First Existence Theorem). Let $f:=([(f_{1\varepsilon})_\varepsilon],\dots,[(f_{m\varepsilon})_\varepsilon])\in(\mathcal{G}(I\times\Omega\times\Omega'))^m$, W be an open subset of Ω and $g:=([(g_{1\varepsilon})_\varepsilon],\dots,[(g_{m\varepsilon})_\varepsilon])\in(\mathcal{G}(W))^m$ such that*

(i) $\overline{W}\Subset\Omega$;
(ii) $\exists$ *an open subset U of Ω', $\exists\tau_1\in\,]0,1]$ such that $\overline{U}\Subset\Omega'$ and*

$$(g_{1\varepsilon},\dots,g_{m\varepsilon})(W)\subset U,\quad\forall\varepsilon\in\,]0,\tau_1[;$$

(iii) $g\in\mathcal{G}_*(W;U)$, *where U is as in* (ii);
(iv) $f\in\mathcal{G}_*(I\times\Omega\times\Omega';\mathbb{R}^m)$;
(v) $(\partial^\alpha f_{i\varepsilon})_\varepsilon$ *has the property* (LLG), $\forall 1\leqslant i\leqslant m$, $\forall\alpha\in\mathbb{N}^{1+n+m}$ *with $|\alpha|=1$ and $\alpha=(0,0,\dots,0,\alpha_1,\dots,\alpha_m)$.*

Then Problem 4.1 *has a solution in $(\mathcal{G}(I_a(t_0)\times W))^m$, for some $a>0$.*

Proof. Let U and τ_1 be as in (ii), take U_1 and U_2 opens subsets of Ω' such that $\overline{U}\subset U_1\subset\overline{U_1}\subset U_2\subset\overline{U_2}\Subset\Omega'$ and let $4d$ the distance of $\overline{U}_1$ to $\Omega'\setminus U_2$.

Note that, if $z\in B_d(0):=\{w\in\mathbb{R}^m\mid\|w\|<d\}$, then $y+z\in\Omega'$ for all $y\in\overline{U_1}$. Thus we can define the moderate net $(h_\varepsilon)_\varepsilon:=((h_{1\varepsilon},\dots,h_{m\varepsilon}))_\varepsilon$ by $h_\varepsilon(s,x,y,z):=f_\varepsilon(s,x,y+z)$, for all $(s,x,y,z)\in I\times\Omega\times U_1\times B_d(0)$.

We will denote by $(s,x,y,z)=(s,x_1,\dots,x_n,y_1,\dots,y_m,z_1,\dots,z_m)$ points in $I\times\Omega\times U_1\times B_d(0)$.

Note that the hypotheses (iii) and (iv) in Theorem 4.2 are satisfied replacing $(f_\varepsilon)_\varepsilon$, Ω and Ω' by $(h_\varepsilon)_\varepsilon$, $\Omega\times U_1$ and $B_d(0)$, respectively. In fact, for $J\Subset I$, $L\Subset\Omega\times U_1$ and $K_2\Subset B_d(0)$, take $K\Subset\Omega$ and $K_1\Subset U_1$ such that $L\subset K\times K_1$. Using that $K_1+K_2\Subset\Omega'$, (iv) and (v), there are $M>0$, $c>0$, $\eta\in\,]0,\tau_1[$ and $N\in\mathbb{N}$ such that, for all $(\varepsilon,s,x,z,y)\in\,]0,\eta[\,\times J\times K\times K_1\times K_2$, one has

$$\big\|h_\varepsilon(s,x,y,z)\big\|\leqslant M\quad\text{and}\quad\left|\frac{\partial h_{i\varepsilon}}{\partial z_j}(s,x,y,z)\right|\leqslant\log\big(c\varepsilon^{-N}\big),\quad\forall 1\leqslant i,j\leqslant m.$$

From Theorem 4.2 and its proof, there are $a>0$ with $\overline{I_a(t_0)}\subset I$, $\tau\in\,]0,\tau_1[$ and $(v_\varepsilon)_\varepsilon\in(\mathcal{E}_M(I_a(t_0)\times W\times U))^m$ such that

$$v_\varepsilon\big(I_a(t_0)\times W\times U\big)\subset B_{\frac{d}{2}}(0),\quad\forall\varepsilon\in\,]0,\tau[;$$
$$v_\varepsilon(t_0,.)=0,\quad\forall\varepsilon\in\,]0,\tau[;$$
$$\frac{\partial v_\varepsilon}{\partial s}(s,y,z)=h_\varepsilon\big(s,y,z,v_\varepsilon(s,y,z)\big),\quad\forall(\varepsilon,s,y,z)\in\,]0,\tau[\,\times I_a(t_0)\times W\times U.$$

From (ii) and (iii) we can define $(u_\varepsilon)_\varepsilon:=((u_{1\varepsilon},\dots,u_{m\varepsilon}))_\varepsilon\in(\mathcal{E}_M(I_a(t_0)\times W))^m$ by

$$u_\varepsilon(t,y):=g_\varepsilon(y)+v_\varepsilon\big(t,y,g_\varepsilon(y)\big),\quad\text{if }\varepsilon\in\,]0,\tau[;$$
$$u_\varepsilon(t,y):=g_{\frac{\tau}{2}}(y)+v_{\frac{\tau}{2}}\big(t,y,g_{\frac{\tau}{2}}(y)\big),\quad\text{if }\varepsilon\in[\tau,1].$$

Let $u:=([(u_{1\varepsilon})_\varepsilon],\dots,[(u_{m\varepsilon})_\varepsilon])\in(\mathcal{G}(I_a(t_0)\times W))^m$.

It is easy to verify that u is a solution to Problem 4.1 in $(\mathcal{G}(I_a(t_0)\times W))^m$ (note that, if $V:=U+B_{\frac{d}{2}}(0)$ then $u_\varepsilon(I_a(t_0)\times W)\subset\overline{V}\Subset\Omega'$, for all $\varepsilon\in\,]0,\tau[$). □

From the proof of previous theorem, we have:

Remark 4.4. In Theorem 4.3 we can add the following statement:

Problem 4.1 has a solution $u \in (\mathcal{G}(I_a(t_0) \times W))^m$, for some $a > 0$, for which there are a representative $((u_{1\varepsilon}, \ldots, u_{m\varepsilon}))_\varepsilon$ and $\tau \in]0, 1]$ such that

$$\begin{aligned} &u_\varepsilon\big(I_a(t_0) \times W\big) \subset K', \quad \forall \varepsilon \in]0, \tau[\text{ and some } K' \Subset \Omega'; \\ &\frac{\partial u_\varepsilon}{\partial t}(t, y) = f_\varepsilon\big(t, y, u_\varepsilon(t, y)\big), \quad \forall (\varepsilon, t, y) \in]0, \tau[\times I_a(t_0) \times W; \\ &u_\varepsilon(t_0, .) = (g_{1\varepsilon}, \ldots, g_{m\varepsilon}), \quad \forall \varepsilon \in]0, \tau[. \end{aligned}$$

Now we will provide conditions for which Problem 4.1 has a solution in $(\mathcal{G}(I_a(t_0) \times \Omega))^m$, for some $a > 0$.

Theorem 4.5. *Let* $f := ([(f_{1\varepsilon})_\varepsilon], \ldots, [(f_{m\varepsilon})_\varepsilon]) \in (\mathcal{G}(I \times \Omega \times \Omega'))^m$ *and* $x_0 := ([(x_{01\varepsilon})_\varepsilon], \ldots, [(x_{0m\varepsilon})_\varepsilon]) \in \overline{\mathbb{R}}^m$ *such that*

(i) *the hypotheses* (ii) *and* (iv) *in* Theorem 4.2 *are true*;
(ii) $((f_{1\varepsilon}, \ldots, f_{m\varepsilon}))_\varepsilon$ *is bounded on* $I \times \Omega \times \Omega'$.

Then Problem 4.1, *for* $g := ([(x \mapsto x_{01\varepsilon})_\varepsilon], \ldots, [(x \mapsto x_{0m\varepsilon})_\varepsilon]) \in (\mathcal{G}(\Omega))^m$, *has a solution in* $(\mathcal{G}(I_a(t_0) \times \Omega))^m$, *for some* $a > 0$.

Proof. Let $a^* > 0$ with $\overline{I_{a^*}(t_0)} \subset I$, K_1 and τ_1 be as in Theorem 4.2(ii), V an open subset of Ω' such that $K_1 \subset V \subset \overline{V} \Subset \Omega'$ and $d > 0$ the distance of K_1 to $\Omega' \setminus V$.

Define $x_{0\varepsilon} := (x_{01\varepsilon}, \ldots, x_{0m\varepsilon})$ and $f_\varepsilon := (f_{1\varepsilon}, \ldots, f_{m\varepsilon})$, for all $\varepsilon \in]0, 1]$.

By (ii), there are $\tau \in]0, \tau_1[$ and $M > 0$ such that

$$\big\| f_\varepsilon(t, x, z)\big\| \leqslant M, \quad \forall (\varepsilon, t, x, z) \in]0, \tau[\times I \times \Omega \times \Omega'.$$

Take $a > 0$ with $a < \min\{a^*, d/M\}$. Using an argument similar to the proof of Theorem 4.2 we have:
$\forall (\varepsilon, y) \in]0, \tau[\times \Omega$, there is a unique $u^\varepsilon_{r_y} \in \mathcal{C}(\overline{I_a(t_0)} \times \overline{B_{r_y}(y)}; \overline{V})$ such that

$$u^\varepsilon_{r_y}(t, z) = x_{0\varepsilon} + \int_{t_0}^{t} f_\varepsilon\big(s, z, u^\varepsilon_{r_y}(s, z)\big)\, ds, \quad \forall (t, z) \in \overline{I_a(t_0)} \times \overline{B_{r_y}(y)},$$

where $r_y > 0$ and $\overline{B_{r_y}(y)} \subset \Omega$ is the closed ball of center y and radius r_y.

Note that, if there is $(\varepsilon, y, z) \in]0, \tau[\times \Omega \times \Omega$ such that $K := \overline{B_{r_y}(y)} \cap \overline{B_{r_z}(z)} \neq \emptyset$, then $u^\varepsilon_{r_y}|_{\overline{I_a(t_0)} \times K} = u^\varepsilon_{r_z}|_{\overline{I_a(t_0)} \times K}$. So we can define the net $(u_\varepsilon)_\varepsilon := ((u_{1\varepsilon}, \ldots, u_{m\varepsilon}))_\varepsilon \in (\mathcal{C}^\infty(I_a(t_0) \times \Omega))^m$ by $u_\varepsilon|_{I_a(t_0) \times B_{r_y}(y)} := u^\varepsilon_{r_y}$ if $\varepsilon \in]0, \tau[$ and u_ε constant in $I_a(t_0) \times \Omega$ if $\varepsilon \in [\tau, 1]$.

By an argument similar to the proof of Theorem 4.2 we conclude that the net $(u_\varepsilon)_\varepsilon$ is moderate and $u := ([(u_{1\varepsilon})_\varepsilon], \ldots, [(u_{m\varepsilon})_\varepsilon])$ is a solution to Problem 4.1 in $(\mathcal{G}(I_a(t_0) \times \Omega))^m$. $\square$

Theorem 4.6 *(Second Existence Theorem). Let* $f := ([(f_{1\varepsilon})_\varepsilon], \ldots, [(f_{m\varepsilon})_\varepsilon]) \in (\mathcal{G}(I \times \Omega \times \mathbb{R}^m))^m$ *and* $g := ([(g_{1\varepsilon})_\varepsilon], \ldots, [(g_{m\varepsilon})_\varepsilon]) \in (\mathcal{G}(\Omega))^m$ *such that*

(i) *the hypothesis* (v) *in* Theorem 4.3 *is true*;
(ii) $g \in \mathcal{G}_*(\Omega, \mathbb{R}^m)$;
(iii) $((f_{1\varepsilon}, \ldots, f_{m\varepsilon}))_\varepsilon$ *is bounded on* $I \times \Omega \times \mathbb{R}^m$.

Then Problem 4.1 *has a solution in* $(\mathcal{G}(I_a(t_0) \times \Omega))^m$, *for some* $a > 0$.

Proof. Define $(f_\varepsilon)_\varepsilon := ((f_{1\varepsilon}, \ldots, f_{m\varepsilon}))_\varepsilon$ and let $(h_\varepsilon)_\varepsilon := ((h_{1\varepsilon}, \ldots, h_{m\varepsilon}))_\varepsilon$ where $h_\varepsilon(s, x, y, z) := f_\varepsilon(s, x, y + z)$, for all $(s, x, y, z) \in I \times \Omega \times \mathbb{R}^m \times \mathbb{R}^m$.

Note that the hypotheses (iii) and (iv) in Theorem 4.2 are satisfied replacing $(f_\varepsilon)_\varepsilon$, Ω and Ω' by $(h_\varepsilon)_\varepsilon$, $\Omega \times \mathbb{R}^m$ and $\mathbb{R}^m$, respectively.

From Theorem 4.5 and its proof, there are $a > 0$ with $\overline{I_a(t_0)} \subset I$, $\tau \in]0, 1]$ and $(v_\varepsilon)_\varepsilon \in (\mathcal{E}_M(I_a(t_0) \times \Omega \times \mathbb{R}^m))^m$ such that

$$\begin{aligned} &v_\varepsilon(t_0, .) = 0, \quad \forall \varepsilon \in]0, \tau[; \\ &\frac{\partial v_\varepsilon}{\partial s}(s, y, z) = h_\varepsilon\big(s, y, z, v_\varepsilon(s, y, z)\big), \quad \forall (\varepsilon, s, y, z) \in]0, \tau[\times I_a(t_0) \times \Omega \times \mathbb{R}^m. \end{aligned}$$

Let $(u_\varepsilon)_\varepsilon$ and u be as in the proof of Theorem 4.3 (replacing W by $\Omega \times \mathbb{R}^m$). This u is a solution to Problem 4.1 in $(\mathcal{G}(I_a(t_0) \times \Omega))^m$. $\square$

It follows from the proof of the last theorem, together with Gronwall's Lemma, the following fact will be important in Section 5 (see Theorem 5.3).

Remark 4.7. If in Theorem 4.6 we add the conditions:

$$\left(\partial^\alpha f_{i\varepsilon}\right)_\varepsilon \text{ is bounded on } I \times \Omega \times \mathbb{R}^m, \forall 1 \leqslant i \leqslant m, \forall \alpha \in \mathbb{N}^{1+n+m} \text{ with } |\alpha| = 1 \text{ and } \alpha = (0, \alpha_1, \ldots, \alpha_{n+m});$$
$$\left(\partial^\gamma g_{i\varepsilon}\right)_\varepsilon \text{ is bounded on } \Omega, \forall 1 \leqslant i \leqslant m, \forall \gamma \in \mathbb{N}^m \text{ with } |\gamma| = 1;$$

then, for some $a > 0$, Problem 4.1 has a solution $u \in (\mathcal{G}(I_a(t_0) \times \Omega))^m$ for which there are a representative $((u_{1\varepsilon}, \ldots, u_{m\varepsilon}))_\varepsilon$ and $\tau \in]0, 1]$ such that

$$\frac{\partial u_\varepsilon}{\partial t}(t, y) = f_\varepsilon\left(t, y, u_\varepsilon(t, y)\right), \quad \forall (\varepsilon, t, y) \in]0, \tau[\times I_a(t_0) \times \Omega;$$
$$u_\varepsilon(t_0, .) = (g_{1\varepsilon}, \ldots, g_{m\varepsilon}), \quad \forall \varepsilon \in]0, \tau[;$$
$$\left(\partial^\beta u_{i\varepsilon}\right)_\varepsilon \text{ is bounded on } J_1 \times \Omega,\ \forall J_1 \Subset I_a(t_0),\ \forall \beta \in \mathbb{N}^{1+n} \text{ with } |\beta| = 1 \text{ and } \beta = (0, \beta_1, \ldots, \beta_n),\ \forall 1 \leqslant i \leqslant m.$$

We finish this section with a theorem on uniqueness of solution to Problem 4.1.

Theorem 4.8 *(Uniqueness Theorem). Let $f := ([(f_{1\varepsilon})_\varepsilon], \ldots, [(f_{m\varepsilon})_\varepsilon]) \in (\mathcal{G}(I \times \Omega \times \Omega'))^m$ such that $((f_{1\varepsilon}, \ldots, f_{m\varepsilon}))_\varepsilon$ has the property* (LLL) *in (I, Ω, Ω'). If u and v are solutions to* Problem 4.1 *in $(\mathcal{G}(I \times \Omega))^m$, then $u = v$.*

Proof. Suppose $u := ([(u_{1\varepsilon})_\varepsilon], \ldots, [(u_{m\varepsilon})_\varepsilon])$ and $v := ([(v_{1\varepsilon})_\varepsilon], \ldots, [(v_{m\varepsilon})_\varepsilon])$.

Let $(u_\varepsilon)_\varepsilon := ((u_{1\varepsilon}, \ldots, u_{m\varepsilon}))_\varepsilon$ and $(v_\varepsilon)_\varepsilon := ((v_{1\varepsilon}, \ldots, v_{m\varepsilon}))_\varepsilon$.

It suffices to prove (see [12, Theorem 1.2.3]) that, if $J \Subset I$ and $K \Subset \Omega$, then one has

$$\sup_{(t,y) \in J \times K} \left\| (v_\varepsilon - u_\varepsilon)(t, y) \right\| = O\left(\varepsilon^q\right) \quad \text{as } \varepsilon \downarrow 0,\ \forall q \in \mathbb{N}. \tag{1}$$

Fix $J \Subset I$, $K \Subset \Omega$ and $q \in \mathbb{N}$. Take $a, b \in \mathbb{R}$ such that $J \cup \{t_0\} \subset [a, b] \Subset I$. Since $u, v \in \mathcal{G}_*(I \times \Omega; \Omega')$, there are $K' \Subset \Omega'$ and $\eta \in]0, 1]$ such that

$$u_\varepsilon\left([a, b] \times K\right) \cup v_\varepsilon\left([a, b] \times K\right) \subset K' \Subset \Omega', \quad \forall \varepsilon \in]0, \eta[.$$

Thus, from $(f_\varepsilon)_\varepsilon := ((f_{1\varepsilon}, \ldots, f_{m\varepsilon}))_\varepsilon$ has the property (LLL) in (I, Ω, Ω'), there are $N \in \mathbb{N}$, $c > 0$ and $\tau \in]0, \eta[$ such that

$$\left\| f_\varepsilon\left(t, y, u_\varepsilon(t, y)\right) - f_\varepsilon\left(t, y, v_\varepsilon(t, y)\right) \right\| \leqslant \log\left(c\varepsilon^{-N}\right) \left\| (u_\varepsilon - v_\varepsilon)(t, y) \right\|, \tag{2}$$

$\forall (\varepsilon, t, y) \in]0, \tau[\times [a, b] \times K$, $\forall 1 \leqslant i, k \leqslant m$.

Let $(g_\varepsilon)_\varepsilon \in (\mathcal{N}(I \times \Omega))^m$ and $(h_\varepsilon)_\varepsilon \in (\mathcal{N}(\Omega))^m$ such that

$$v_\varepsilon(t, y) - u_\varepsilon(t, y) - h_\varepsilon(y) = \int_{t_0}^{t} \left(f_\varepsilon\left(s, y, v_\varepsilon(s, y)\right) - f_\varepsilon\left(s, y, u_\varepsilon(s, y)\right) + g_\varepsilon(s, y)\right) \mathrm{d}s$$

$\forall (\varepsilon, t, y) \in]0, \tau[\times I \times \Omega$.

Let $l_\varepsilon : I \times \Omega \to \mathbb{R}^m$ defined by $l_\varepsilon(t, y) := v_\varepsilon(t, y) - u_\varepsilon(t, y) - h_\varepsilon(y)$, with $\varepsilon \in]0, \tau[$.

Since $(h_\varepsilon)_\varepsilon$ and $(g_\varepsilon)_\varepsilon$ are null there are $c_1 > 0$, $\tau_1 \in]0, \tau[$ and $N_1 \in \mathbb{N}$ with $N_1 > \max\{N, N(b - t_0), N(t_0 - a)\}$ such that

$$\max\left\{\left\|h_\varepsilon(y)\right\|, \left\|g_\varepsilon(t, y)\right\|\right\} \leqslant c_1 \varepsilon^{q+2N_1}, \quad \forall (\varepsilon, t, y) \in]0, \tau_1[\times [a, b] \times K.$$

From this inequality and (2) we have, for all $(\varepsilon, t, y) \in]0, \tau_1[\times [a, b] \times K$, that

$$\left\| l_\varepsilon(t, y) \right\| \leqslant \left| \int_{t_0}^{t} \left(\log\left(c\varepsilon^{-N}\right) \left\| l(\varepsilon, s, y) \right\| + (c + 1) c_1 \varepsilon^{q+N_1} \right) \mathrm{d}s \right|.$$

This, together with Gronwall's Lemma, implies that

$$\sup_{(t,y) \in [a,b] \times K} \left\| (v_\varepsilon - u_\varepsilon)(t, y) - h_\varepsilon(y) \right\| = \sup_{(t,y) \in [a,b] \times K} \left\| l_\varepsilon(t, y) \right\| = O\left(\varepsilon^q\right) \quad \text{as } \varepsilon \downarrow 0.$$

Hence (1) is true. □

For an open interval J of I, we define $\pi_* := [(\pi_{*\varepsilon})_\varepsilon] \in \mathcal{G}(J)$, where $\pi_{*\varepsilon}(t) := t$. Using this definition and with arguments similar to the proof of Theorem 4.2 and Theorem 4.8 can be verified that:

Remark 4.9. Let $x_0 := ([(x_{01\varepsilon})_\varepsilon], \dots, [(x_{0n\varepsilon})_\varepsilon]) \in \overline{\mathbb{R}}^n$ and $f := [(f_{1\varepsilon})_\varepsilon], \dots, [(f_{n\varepsilon})_\varepsilon]) \in \mathcal{G}_*(I \times \Omega; \mathbb{R}^n)$ such that

(i) $\overline{\{(x_{01\varepsilon}, \dots, x_{0n\varepsilon}) \mid \varepsilon \in]0,\tau[\}} \Subset \Omega$, for some $\tau \in]0,1]$;
(ii) $((f_{1\varepsilon}, \dots, f_{n\varepsilon}))_\varepsilon$ has the property (LLL) in (I, Ω).

Then there are $a > 0$ with $\overline{I_a(t_0)} \subset I$ and a unique $u \in \mathcal{G}_*(I_a(t_0); \Omega)$ satisfying: $u' = f \circ (\pi_*, u)$ and $u(t_0) = x_0$. Moreover, from $(\mathcal{G}(.))^n$ be a sheaf of vector spaces on $\mathbb{R}$, one has that there is a unique maximal solution of $\omega' = f \circ (\pi_*, \omega)$ satisfying $w(t_0) = x_0$ (a maximal solution is a solution that cannot be extended).

5. Some results on uniqueness of solution to the HJ-Problem

In this last section we present, using the results of the previous section, some cases in which the HJ-Problem has a unique solution.

Here the sets Ω, Ω', I and I', and the notations are as in Section 3. Moreover, the functions $f := [(f_\varepsilon)_\varepsilon]$ and $H := [(H_\varepsilon)_\varepsilon]$ belong to $\mathcal{G}(\Omega)$ and $\mathcal{G}(I \times \Omega \times I' \times \Omega')$, respectively, and $\mathcal{S}(I, \Omega, I', \Omega', H, f, J, W)$ is as in Definition 3.1.

Under certain conditions, for each element of $\mathcal{S}(I, \Omega, I', \Omega', H, f, J, W)$ is possible to find a solution to the HJ-Problem (see Theorem 3.2). Therefore, it is important to obtain some conditions for which it has at most one element. The result we have obtained is the following:

Proposition 5.1. *If $(H_\varepsilon)_\varepsilon$ is such that $(H_\varepsilon)_\varepsilon$, $(\frac{\partial H_\varepsilon}{\partial x})_\varepsilon$, $(\frac{\partial H_\varepsilon}{\partial y})_\varepsilon$ and $(\frac{\partial H_\varepsilon}{\partial p})_\varepsilon$ have the property* (LLL) *in $(I, \Omega \times I' \times \Omega')$, then $\mathcal{S}(I, \Omega, I', \Omega', H, f, J, W)$ is empty or has a unique element.*

Proof. Let $V := \{z \in \Omega \mid (0, z) \in W\}$.

Define $(\varphi_\varepsilon)_\varepsilon := ((\varphi_{1\varepsilon}, \dots, \varphi_{(2n+1)\varepsilon}))_\varepsilon \in (\mathcal{E}_M(J \times V \times \Omega \times I' \times \Omega'))^{n+1+n}$ and φ by

$$(\varphi_{1\varepsilon}, \dots, \varphi_{n\varepsilon})(t, z, x, y, p) := \frac{\partial H_\varepsilon}{\partial p}(t, x, y, p);$$
$$\varphi_{(n+1)\varepsilon}(t, z, x, y, p) := -H_\varepsilon(t, x, y, p) + \sum_{j=1}^{n} \frac{\partial H_\varepsilon}{\partial p_j}(t, x, y, p) p_j;$$
$$(\varphi_{(n+2)\varepsilon}, \dots, \varphi_{(2n+1)\varepsilon})(t, z, x, y, p) := -\frac{\partial H_\varepsilon}{\partial x}(t, x, y, p) - p\frac{\partial H_\varepsilon}{\partial y}(t, x, y, p);$$
$$\varphi := \big(\big[(\varphi_{1\varepsilon})_\varepsilon\big], \dots, \big[(\varphi_{(2n+1)\varepsilon})_\varepsilon\big]\big) \in \big(\mathcal{G}\big(J \times V \times \big(\Omega \times I' \times \Omega'\big)\big)\big)^{n+1+n}.$$

Then $(\varphi_\varepsilon)_\varepsilon$ has the property (LLL) in $(J, V, \Omega \times I' \times \Omega')$.

Let $(X, U, P), (X^1, U^1, P^1) \in \mathcal{S}(I, \Omega, I', \Omega', H, f, J, W)$. Since the elements of $\mathcal{S}(I, \Omega, I', \Omega', H, f, J, W)$ are solutions of the system

$$\left(\frac{\partial \tilde{X}}{\partial s}, \frac{\partial \tilde{U}}{\partial s}, \frac{\partial \tilde{P}}{\partial s}\right) = \varphi \circ (\pi, \pi_1, \dots, \pi_n, \tilde{X}, \tilde{U}, \tilde{P});$$
$$(\tilde{X}, \tilde{U}, \tilde{P})|_{\{0\} \times V} = (1|_V, f|_V, \nabla f|_V)$$

we obtain $(X, U, P) = (X^1, U^1, P^1)$ (see Theorem 4.8). □

For the case $\Omega = \mathbb{R}^n$ and $I' = \mathbb{R}$ we have the following theorem:

Theorem 5.2 *(Uniqueness Theorem). Let $(H_\varepsilon)_\varepsilon$ such that*

(i) *$(\partial^\alpha(\frac{\partial H_\varepsilon}{\partial p}))_\varepsilon$ has the property* (LLG), *$\forall \alpha \in \mathbb{N}^{1+n+1+n}$ with $|\alpha| = 1$ and $\alpha = (0, \alpha_1, \dots, \alpha_{2n+1})$;*
(ii) *$(H_\varepsilon)_\varepsilon$, $(\frac{\partial H_\varepsilon}{\partial x})_\varepsilon$, $(\frac{\partial H_\varepsilon}{\partial y})_\varepsilon$ and $(\frac{\partial H_\varepsilon}{\partial p})_\varepsilon$ have the property* (LLL) *in $(I, \mathbb{R}^n \times \mathbb{R} \times \Omega')$;*
(iii) *$(\frac{\partial H_\varepsilon}{\partial p})_\varepsilon$ is bounded on $I \times \mathbb{R}^n \times \mathbb{R} \times \Omega'$.*

If $\mathcal{S}(I, \mathbb{R}^n, \mathbb{R}, \Omega', H, f, J, J \times \mathbb{R}^n) \neq \emptyset$, then there is at most one solution u to the HJ-Problem in $\mathcal{G}(J \times \mathbb{R}^n)$ satisfying:

(I) *there are a representative $(u_\varepsilon)_\varepsilon$ of u and $\tau \in]0,1]$ for which*

$$\left(\frac{\partial u_\varepsilon}{\partial x_1}, \dots, \frac{\partial u_\varepsilon}{\partial x_n}\right)\big(J \times \mathbb{R}^n\big) \subset \Omega', \quad \forall \varepsilon \in]0,\tau[;$$

(II) $\partial^\gamma u \in \mathcal{G}_*(J \times \mathbb{R}^n)$, $\forall \gamma \in \mathbb{N}^{1+n}$ *with* $1 \leqslant |\gamma| \leqslant 2$ *and* $\gamma = (0, \gamma_1, \dots, \gamma_n)$.

Moreover, if there is this solution u, then $\mathcal{S}(I, \mathbb{R}^n, \mathbb{R}, \Omega', H, f, J, J \times \mathbb{R}^n) = \{(X, U, P)\}$ *and* $u = U \circ (\pi, X)^{-1}$.

Proof. Note that $\mathcal{S}(I, \mathbb{R}^n, \mathbb{R}, \Omega', H, f, J, J \times \mathbb{R}^n) = \{(X, U, P)\}$ (see Proposition 5.1). Suppose that there is a solution $u := [(u_\varepsilon)_\varepsilon]$ to the HJ-Problem in $\mathcal{G}(J \times \mathbb{R}^n)$ satisfying (I) and (II). Thus we can define $(\psi_\varepsilon)_\varepsilon := ((\psi_{1\varepsilon}, \dots, \psi_{n\varepsilon}))_\varepsilon \in (\mathcal{E}_M(J \times \mathbb{R}^n \times \mathbb{R}^n))^n$ by

$$\psi_\varepsilon(t, z, x) := \frac{\partial H_\varepsilon}{\partial p}\left(t, x, u_\varepsilon(t, x), \frac{\partial u_\varepsilon}{\partial x_1}(t, x), \dots, \frac{\partial u_\varepsilon}{\partial x_n}(t, x)\right)$$

and $\psi := ([(\psi_{1\varepsilon})_\varepsilon], \dots, [(\psi_{n\varepsilon})_\varepsilon] \in (\mathcal{G}(J \times \mathbb{R}^n \times \mathbb{R}^n))^n$.

Applying (i), (iii), (II) and Theorem 4.6, we know that given $t \in J$ there are $a_t > 0$ with $\overline{I_{a_t}(t)} \subset J$ and $\tilde{X}_t \in \mathcal{G}_*(I_{a_t}(t) \times \mathbb{R}^n; \mathbb{R}^n)$ such that

$$\frac{\partial \tilde{X}_t}{\partial s} = \psi \circ (\pi, \pi_1, \dots, \pi_n, \tilde{X}_t) \quad \text{and} \quad \tilde{X}_t|_{\{t\} \times \mathbb{R}^n} = X|_{\{t\} \times \mathbb{R}^n}.$$

Since $J \times \mathbb{R}^n = \bigcup_{t \in J} I_{a_t}(t) \times \mathbb{R}^n$ and $\mathcal{G}_*(., \mathbb{R}^n)$ is a sheaf of vector spaces on $\mathbb{R}$, there is $\tilde{X} \in \mathcal{G}_*(J \times \mathbb{R}^n; \mathbb{R}^n)$ such that $\tilde{X}|_{I_{a_t}(t) \times \mathbb{R}^n} = \tilde{X}_t$ for all $t \in J$. Hence $\tilde{X}|_{\{0\} \times \mathbb{R}^n} = \tilde{X}_0|_{\{0\} \times \mathbb{R}^n} = X|_{\{0\} \times \mathbb{R}^n} = 1_{\mathbb{R}^n}$ and

$$\frac{\partial \tilde{X}}{\partial s} = \frac{\partial H}{\partial p} \circ \left(\pi, \tilde{X}, u \circ (\pi, \tilde{X}), \frac{\partial u}{\partial x_1} \circ (\pi, \tilde{X}), \dots, \frac{\partial u}{\partial x_n} \circ (\pi, \tilde{X})\right).$$

Let $\tilde{U} := u \circ (\pi, \tilde{X})$ and $\tilde{P} = (\frac{\partial u}{\partial x_1} \circ (\pi, \tilde{X}), \dots, \frac{\partial u}{\partial x_n} \circ (\pi, \tilde{X}))$. Define $(\varphi_\varepsilon)_\varepsilon \in (\mathcal{E}_M(J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega'))^{n+1+n}$ and φ as in the proof of Proposition 5.1 (replacing $J \times V \times \Omega \times I' \times \Omega'$ by $J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega'$). Then we have

$$\left(\frac{\partial \tilde{X}}{\partial s}, \frac{\partial \tilde{U}}{\partial s}, \frac{\partial \tilde{P}}{\partial s}\right) = \varphi \circ (\pi, \pi_1, \dots, \pi_n, \tilde{X}, \tilde{U}, \tilde{P});$$
$$(\tilde{X}, \tilde{U}, \tilde{P})|_{\{0\} \times \mathbb{R}^n} = (1_{\mathbb{R}^n}, f|_{\mathbb{R}^n}, \nabla f|_{\mathbb{R}^n}).$$

This together with (ii) and Theorem 4.8 gives us $(\tilde{X}, \tilde{U}, \tilde{P}) = (X, U, P)$. Therefore $u = U \circ (\pi, X)^{-1}$. □

For the case $\Omega = \Omega' = \mathbb{R}^n$ and $I' = \mathbb{R}$ we have the following theorem of existence and uniqueness of solution to the HJ-Problem.

Theorem 5.3 *(Existence and Uniqueness Theorem). Assume that the following statements are true*:

(i) $f \in \mathcal{G}_*(\mathbb{R}^n; \mathbb{R})$;
(ii) $(\partial^\alpha f_\varepsilon)_\varepsilon$ *is bounded on* $\mathbb{R}^n$, $\forall \alpha \in \mathbb{N}^n$ *with* $1 \leqslant |\alpha| \leqslant 2$;
(iii) $(\partial^\gamma H_\varepsilon)_\varepsilon$ *is bounded on* $I \times \mathbb{R}^{n+1+n}$, $\forall \gamma \in \mathbb{N}^{1+n+1+n}$ *with* $|\gamma| \leqslant 2$ *and* $\gamma = (0, \gamma_1, \dots, \gamma_n, \gamma_{n+1}, \gamma_{n+2}, \dots, \gamma_{2n+1})$;
(iv) $(\partial^\gamma H_\varepsilon)_\varepsilon$ *is bounded on* $I \times \mathbb{R}^{n+1+n}$, $\forall \gamma \in \mathbb{N}^{1+n+1+n}$ *with* $|\gamma| = 2$ *and* $\gamma = (1, 0, \dots, 0, 0, \gamma_{n+2}, \dots, \gamma_{2n+1})$.

Then the HJ-Problem *has a unique solution in* $\mathcal{G}(I_a \times \mathbb{R}^n)$, *for some* $a > 0$, *satisfying the assertions* (I) *and* (II) *of* Theorem 5.2.

Proof. Let $(\varphi_\varepsilon)_\varepsilon \in (\mathcal{E}_M(I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n))^{n+1+n}$ and φ be as in the proof of Proposition 5.1 (replacing $J \times V \times \Omega \times I' \times \Omega'$ by $I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$).

Define $(g_\varepsilon)_\varepsilon := ((g_{1\varepsilon}, \dots, g_{(2n+1)\varepsilon}))_\varepsilon \in (\mathcal{E}_M(\mathbb{R}^n))^{n+1+n}$ by

$$g_\varepsilon(z) = \big(z, f_\varepsilon(z), \nabla f_\varepsilon(z)\big)$$

and $g := ([(g_{1\varepsilon})_\varepsilon], \dots, [(g_{(2n+1)\varepsilon})_\varepsilon]) \in (\mathcal{G}(\mathbb{R}^n))^{n+1+n}$.

Applying (i), (ii), (iii) and Remark 4.7 there are $a^* > 0$ with $\overline{I_{a^*}} \subset I$ and $(X, U, P) \in \mathcal{G}_*(I_{a^*} \times \mathbb{R}^n; \mathbb{R}^{n+1+n})$ for which there are $(X_\varepsilon)_\varepsilon := ((X_{1\varepsilon}, \dots, X_{n\varepsilon}))_\varepsilon$, $(U_\varepsilon)_\varepsilon$, $(P_\varepsilon)_\varepsilon := ((P_{1\varepsilon}, \dots, P_{n\varepsilon}))_\varepsilon$ and $\tau \in]0, 1]$ satisfying

$$X = \big(\big[(X_{1\varepsilon})_\varepsilon\big], \dots, \big[(X_{n\varepsilon})_\varepsilon\big]\big); \qquad U = \big[(U_\varepsilon)_\varepsilon\big], \qquad P = \big(\big[(P_{1\varepsilon})_\varepsilon\big], \dots, \big[(P_{n\varepsilon})_\varepsilon\big]\big);$$
$$\left(\frac{\partial X_\varepsilon}{\partial s}, \frac{\partial U_\varepsilon}{\partial s}, \frac{\partial P_\varepsilon}{\partial s}\right) = \varphi_\varepsilon \circ (\pi_\varepsilon, \pi_{1\varepsilon}, \dots, \pi_{n\varepsilon}, X_\varepsilon, U_\varepsilon, P_\varepsilon) \quad \text{on } I_{a^*} \times \mathbb{R}^n,\ \forall \varepsilon \in]0, \tau[;$$
$$\big(X_\varepsilon(0, r), U_\varepsilon(0, r), P_\varepsilon(0, r)\big) = g_\varepsilon(r), \quad \forall (\varepsilon, r) \in]0, \tau[\times \mathbb{R}^n;$$
$$\big(\partial^\alpha X_{i\varepsilon}\big)_\varepsilon, \big(\partial^\alpha U_\varepsilon\big)_\varepsilon, \big(\partial^\alpha P_{i\varepsilon}\big)_\varepsilon \text{ are bounded on } \overline{I_b} \times \mathbb{R}^n,\ \forall b \in]0, a^*[,$$
$$\forall \alpha := (0, \alpha_1, \dots, \alpha_n) \in \mathbb{N}^{1+n} \quad \text{with } |\alpha| = 1 \text{ and } \forall 1 \leqslant i \leqslant n.$$

Fix $b \in]0, a^*[$. Define $(Y_\varepsilon)_\varepsilon := ((Y_{1\varepsilon}, \ldots, Y_{(n+1)\varepsilon}))_\varepsilon \in (\mathcal{E}_M(I_b \times \mathbb{R}^n))^{1+n}$ by $Y_\varepsilon(s,r) := (s, X_\varepsilon(s,r))$, $Y := ([(Y_{1\varepsilon})_\varepsilon], \ldots, [(Y_{(n+1)\varepsilon})_\varepsilon]) \in (\mathcal{G}(I_b \times \mathbb{R}^n))^{1+n}$ and $(l_\varepsilon)_\varepsilon \in (\mathcal{E}_M(I_b \times \mathbb{R}^n))^n$ by $l_\varepsilon(s,r) := X_\varepsilon(s,r) - r$.

Note that, for all $(\varepsilon, s, r) \in]0,1] \times I_b \times \mathbb{R}^n$, one has $Y_\varepsilon(s,r) = (s, l_\varepsilon(s,r) + r)$ and

$$l_\varepsilon(s,r) = \int_0^s \frac{\partial H_\varepsilon}{\partial p}\big(t, X_\varepsilon(t,r), U_\varepsilon(t,r), P_\varepsilon(t,r)\big)\,\mathrm{d}t.$$

Then, by (iii), (iv), Proposition 2.3 and Remark 2.4, there is $a > 0$ with $\overline{I_a} \subset I_b$ such that:

$Y|_{I_a \times \mathbb{R}^n} \in \mathcal{G}_*(I_a \times \mathbb{R}^n; I_a \times \mathbb{R}^n)$ and it is an invertible mapping;
there is a moderate net $(\Gamma_\varepsilon)_\varepsilon := ((\Gamma_{1\varepsilon}, \ldots, \Gamma_{(n+1)\varepsilon}))_\varepsilon$ satisfying

$$(Y|_{I_a \times \mathbb{R}^n})^{-1} = \big(\big[(\Gamma_{1\varepsilon})_\varepsilon\big], \ldots, \big[(\Gamma_{(n+1)\varepsilon})_\varepsilon\big]\big);$$
$$\Gamma_\varepsilon = (Y_\varepsilon|_{I_a \times \mathbb{R}^n})^{-1} \quad \text{for all } \varepsilon \in]0, \tau_1[, \text{ for some } \tau_1 \in]0, \tau[;$$
$$\big(\partial^\alpha \Gamma_\varepsilon\big)_\varepsilon \text{ is bounded on } I_a \times \mathbb{R}^n,\ \forall \alpha \in \mathbb{N}^{1+n} \text{ with } |\alpha| = 1.$$

Hence $(X, U, P)|_{I_a \times \mathbb{R}^n} \in \mathcal{S}(I, \mathbb{R}^n, \mathbb{R}, \mathbb{R}^n, H, f, I_a, I_a \times \mathbb{R}^n)$. Thus, from (iii) and Theorem 3.2, one has that $u = U \circ (Y|_{I_a \times \mathbb{R}^n})^{-1}$ is a solution to the HJ-Problem in $\mathcal{G}(I_a \times \mathbb{R}^n)$ and $(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}) = P \circ (Y|_{I_a \times \mathbb{R}^n})^{-1}$.

Clearly u satisfies the requirements of our proposition. The uniqueness is a consequence of Theorem 5.2 (note that the hypothesis (ii) of Theorem 5.2 is obtained from the Mean Value Theorem and (iii)). □

We conclude our work by providing some $(f_\varepsilon)_\varepsilon$ and $(H_\varepsilon)_\varepsilon$ that satisfy the assumptions of previous theorem.

Example 5.4. Let $\mu, \nu \in {\mathbb{R}^n}^{]0,1]}$ and $\varphi, \Psi \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}^n)$ such that μ, ν, φ, φ', Ψ, Ψ' and Ψ'' are bounded. Suppose $\mu := (\mu_1, \ldots, \mu_n)$, $\nu := (\nu_1, \ldots, \nu_n)$, $\varphi := (\varphi_1, \ldots, \varphi_n)$ and $\Psi := (\Psi_1, \ldots, \Psi_n)$. If $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$ is given by

$$f_\varepsilon(r_1, \ldots, r_n) := \sum_{j=1}^n \int_0^{r_j} \varphi_j\big(\mu_j(\varepsilon)s\big)\,\mathrm{d}s$$

and $(H_\varepsilon)_\varepsilon \in \mathcal{E}_M(I \times \mathbb{R}^{n+1+n})$ is one of the following nets:

$$H_\varepsilon(t, x_1, \ldots, x_n, y, p_1, \ldots, p_n) := \sum_{j=1}^n \Psi_j\big(\nu_j(\varepsilon)(t + x_j + y + p_j)\big);$$
$$H_\varepsilon(t, x_1, \ldots, x_n, y, p_1, \ldots, p_n) := \sum_{j=1}^n \nu_j(\varepsilon)\Psi_j(t + x_j + y + p_j),$$

then $(f_\varepsilon)_\varepsilon$ and $(H_\varepsilon)_\varepsilon$ satisfy the assumptions of Theorem 5.3.

For the next example consider the following embedding of space of Schwartz distributions $\mathcal{D}'(\mathbb{R}^m)$ into $\mathcal{G}(\mathbb{R}^m)$ (see [9]):
Take $\rho \in \mathcal{S}(\mathbb{R}^m)$ even such that

$$\int_{\mathbb{R}^m} \rho(x)\,\mathrm{d}x = 1; \qquad \int_{\mathbb{R}^m} x^p \rho(x)\,\mathrm{d}x = 0, \quad \forall p \in \mathbb{N}^m \setminus \{0\}$$

and $\chi \in \mathcal{C}^\infty(\mathbb{R}^m)$ such that $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on $\overline{B_1(0)}$ and $\chi \equiv 0$ on $\mathbb{R}^m \setminus B_2(0)$. Define

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^m}\rho\left(\frac{x}{\varepsilon}\right), \quad \forall \varepsilon \in]0,1],\ \forall x \in \mathbb{R}^m;$$
$$\theta_\varepsilon(x) := \rho_\varepsilon(x)\chi\big(x\big|\log(\varepsilon)\big|\big), \quad \forall \varepsilon]0,1[; \qquad \theta_1 \equiv 1.$$

Let $i_{\mathcal{D}'} : \mathcal{D}'(\mathbb{R}^m) \to \mathcal{G}(\mathbb{R}^m)$ given by $i_{\mathcal{D}'}(T) = [(T * \theta_\varepsilon)_\varepsilon]$. We recall that $(T * \theta_\varepsilon)(w) = T(x \mapsto \theta_\varepsilon(w - x))$.
For $g \in \mathcal{C}(\mathbb{R}^m)$ denote by T_g the distribution

$$T_g : \varphi \in \mathcal{D}\big(\mathbb{R}^m\big) \mapsto \int_{\mathbb{R}^m} g(x)\varphi(x)\,\mathrm{d}x$$

and by $(T_{g\varepsilon})_\varepsilon$ the representative of $i_{\mathcal{D}'(\mathbb{R}^m)}(T_g)$ defined by

$$T_{g\varepsilon}(w) := \int_{\mathbb{R}^m} g(x)\theta_\varepsilon(w-x)\,\mathrm{d}x, \quad \forall w \in \mathbb{R}^m.$$

Note that, if $g \in \mathcal{C}(\mathbb{R}^m)$ is globally Lipschitz-continuous, then there is $L > 0$ such that

$$\left\| T_{g\varepsilon}(z) - T_{g\varepsilon}(w) \right\| \leqslant L\|z-w\|, \quad \forall z, w \in \mathbb{R}^m,\ \forall \varepsilon \in]0,1].$$

Thus

$$\left(\partial^\alpha T_{g\varepsilon}\right)_\varepsilon \text{ is bounded, } \forall \alpha \in \mathbb{N}^m \text{ with } |\alpha| = 1.$$

From the above definitions and notations we have the following:

Example 5.5. Let $h \in \mathcal{C}(\mathbb{R})$, $g \in \mathcal{C}^1(\mathbb{R}^n)$ and $\Psi \in \mathcal{C}^1(\mathbb{R}^{n+1+n})$ such that

(i) h and Ψ are bounded;
(ii) h, g and Ψ are globally Lipschitz-continuous;
(iii) $\partial^\alpha g$ is globally Lipschitz-continuous, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = 1$;
(iv) $\partial^\gamma \Psi$ is globally Lipschitz-continuous, for all $\gamma \in \mathbb{N}^{n+1+n}$ with $|\gamma| = 1$.

Define $(f_\varepsilon)_\varepsilon := (T_{g\varepsilon})_\varepsilon$ and $(H_\varepsilon)_\varepsilon$ by

$$H_\varepsilon(t, x_1, \ldots, x_n, y, p_1, \ldots, p_n) := T_{h\varepsilon}(t) T_{\Psi\varepsilon}(x_1, \ldots, x_n, y, p_1, \ldots, p_n).$$

Then the HJ-Problem, for $f := [(f_\varepsilon)_\varepsilon]$ and $H := [(H_\varepsilon)_\varepsilon]$, has a unique solution in $\mathcal{G}(I_a \times \mathbb{R}^n)$, for some $a > 0$, satisfying the assertions (I) and (II) of Theorem 5.2 (see Theorem 5.3).

Acknowledgments

We would like to express our gratitude to Jorge Aragona for his encouragement and helpful discussions, as well as a continuous exchange of ideas. We also thank Michael Oberguggenberger for his comments and constructive criticism.

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