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Strong uniqueness sets and t-analytic sets for H^∞ and $H^\infty + C$

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ABSTRACT

We determine the relations of the t-analytic sets for the algebra H^∞ of bounded holomorphic functions in the unit disk with those in the Sarason algebra $H^\infty + C$ and give a description of the strong uniqueness sets for these algebras.

1. Introduction

In this note we introduce the notion of strong uniqueness sets for Banach function algebras A and compare this class of sets with the recently introduced t-analytic sets for A. Recall that a subset E of the spectrum $M(A)$ of A is said to be *t-analytic* (denoted by $E \in \mathscr{A}$), if for every $f \in A$ and every open set U in $M(A)$ with $U \cap E \neq \emptyset$ one has $f \equiv 0$ on E whenever $f \equiv 0$ on $E \cap U$. For example the empty set and every singleton is a t-analytic set. Also, each point in $M(A)$ is contained in a maximal, though not necessary unique, t-analytic set (see [2]).

A non-void set $E \subseteq M(A)$ is called a *uniqueness set* for A, if for every f and g in A, $f = g$ whenever f and g coincide on E. If this property also holds locally, that is, if for every open set U in $M(A)$ with $U \cap E \neq \emptyset$, $f|_{U\cap E} = g|_{U\cap E}$ implies $f = g$, then we say that E is a *strong uniqueness set* for A. The set of strong uniqueness sets for A is denoted by $\mathscr{U}$.

It is clear that any strong uniqueness set is a t-analytic set. These classes are different though, since for example a singleton $\{x\}$, known to be t-analytic, is a strong uniqueness set if and only if $M(A) = \{x\}$ (and so $A = C(\{x\}) \cong \mathbb{C}$). We remark that a t-analytic set E is a strong uniqueness set if and only if the hull-kernel closure $\hat{E}$ of E equals $M(A)$. Recall that $\hat{E}$ is the zero set (or hull) of the ideal

$$I(E, A) = \{f \in A\colon\ f|_E \equiv 0\}.$$

The concept of t-analytic sets, originally considered only for open sets in [1] in connection with local/restricted decomposability of multiplication operators on commutative, semisimple Banach algebras, was first given in this generality in [2]. It turned out that it has a surprising connection to closed prime ideals: if $E \subseteq M(A)$ is a t-analytic set for A, then the ideal $I(E, A)$ is a closed prime ideal. We also unveiled the connection of t-analytic sets with ideals of the form

$$J(x, A) = \bigl\{f \in A\colon\ f \text{ vanishes identically on a neighborhood of } x \text{ in } M(A)\bigr\},$$

that appear in problems on spectral synthesis for Banach function algebras (see for example [5]). In fact, if $E \in \mathscr{A}$ and $x \in E$, then E is contained in the zero set $k_A(x)$ of the ideal $J(x, A)$.

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A description of the t-analytic sets in concrete algebras seems to be a very hard problem. At the moment, such a characterization is only known for the disk-algebra and general regular function algebras (see [2]). In the present paper we will be concerned with the t-analytic sets in the algebra H^∞ of bounded analytic functions in the open unit disk $\mathbb{D}$ and its associated Sarason algebra $H^\infty + C$ of sums of boundary values of functions in H^∞ and (complex-valued) continuous functions on the unit cicle $\mathbb{T}$. We assume that the reader is familiar with the structure of the maximal ideal spaces of these algebras (see [4]).

First results in this direction were given in [2]. For example, it is known [2] that in $H^\infty + C$ the t-analytic sets are very small. In fact, if E is a t-analytic set for $H^\infty + C$ then, due to the fact that $E \subseteq k_{H^\infty+C}(x)$ for some $x \in E$, the set E is nowhere dense and contained in a single fiber.

The situation for H^∞ is quite different. Here $k_{H^\infty}(x)$ equals $M(H^\infty)$ for every $x \in M(H^\infty)$. Moreover, t-analytic sets for H^∞ may be big. For example, the unit disk $\mathbb{D}$ is a t-analytic set for H^∞. Hence, by the corona theorem, the whole spectrum $M(H^\infty)$ is the maximum t-analytic set for H^∞. But also the Shilov boundary, ∂H^∞, of H^∞ is a t-analytic set for H^∞. Or the uniqueness set $[0,1[$. On the other hand, the uniqueness set $[0,1[\cup\{-1/2\}$ is not t-analytic. Neither the corona $M(H^\infty)\setminus\mathbb{D}$ is a t-analytic set for H^∞.

A way for comparing t-analytic sets for H^∞ with those of $H^\infty + C$ comes from the fact that the spectrum $M(H^\infty + C)$ of $H^\infty + C$ can be identified with the corona $M(H^\infty)\setminus\mathbb{D}$ of $\mathbb{D}$ in $M(H^\infty)$. Also, the Shilov boundaries for H^∞ and $H^\infty + C$ coincide and can be identified with $M(L^\infty)$, the maximal ideal space of the algebra of (equivalence classes) of Lebesgue measurable and essentially bounded functions on $\mathbb{T}$.

Natural questions now arise. For instance, which t-analytic sets for $H^\infty + C$ are t-analytic for H^∞? Are there essentially other t-analytic sets for H^∞ besides those mentioned above? Can we describe all the strong uniqueness sets for H^∞, respectively $H^\infty + C$?

In this paper we give answers to these questions.

We conclude the introduction with some additional notations used throughout the paper. For a Banach function algebra A, we always consider A as a set of continuous functions that live on $M(A)$.

If $f \in A$, then $Z(f) = \{x \in M(A)\colon\ f(x) = 0\}$ is the zero set of f. If I is an ideal in A, then $Z(I) = \bigcap_{f\in I} Z(f)$ is the zero set (or hull) of I. The interior of a subset E of a topological space X will be denoted by E°; its closure by $\overline{E}$. If $X \subseteq M(A)$, then $Z_X(f) = Z(f)\cap X$.

2. Some general facts on strong uniqueness sets

In this section we present some general, topological properties of the class $\mathscr{U}$ of strong uniqueness sets. Although the proofs are straightforward, we present them for completeness.

The first question that raises, is whether $\mathscr{U}$ is stable with respect to taking closures; a property enjoyed by the class of t-analytic sets for A (see [2]). The same property now is valid for the class of strong uniqueness sets:

Observation 2.1. *Let A be a Banach function algebra. Then $E \subseteq M(A)$ is a strong uniqueness set for A, that is $E \in \mathscr{U}$, if and only if $\overline{E} \in \mathscr{U}$.*

Proof. Assume that $E \in \mathscr{U}$. Let $U \subseteq M(A)$ be open, $U \cap \overline{E} \neq \emptyset$, and let $f \equiv 0$ on $U \cap \overline{E}$. Then $U \cap E \neq \emptyset$ and $f \equiv 0$ on $U \cap E$. Hence f is the zero function. Thus $\overline{E} \in \mathscr{U}$.

Conversely, let $\overline{E} \in \mathscr{U}$. Let $U \subseteq M(A)$ be open, $U \cap E \neq \emptyset$, and $f \equiv 0$ on $U \cap E$. Then the openness of U and the continuity of f imply that $f \equiv 0$ on $U \cap \overline{E}$. Hence f is the zero function, too. Thus $E \in \mathscr{U}$. □

As an immediate consequence we have

Observation 2.2. *Let $E \subseteq F \subseteq \overline{E}$. Then $F \in \mathscr{U}$ whenever $E \in \mathscr{U}$.*

Observation 2.3. *Let $E \in \mathscr{U}$ and suppose that F is closed. Then $E \setminus F \in \mathscr{U} \cup \{\emptyset\}$. In other words, every non-void relatively open subset of a strong uniqueness set belongs to $\mathscr{U}$, too.*

Whereas the set $\mathscr{A}$ of t-analytic sets always contains the empty set and the singletons, its subset $\mathscr{U}$ of strong uniqueness sets may be void. Indeed, this happens quite frequently, as the following result shows.

Observation 2.4. *If the set of strong uniqueness sets for a Banach function algebra is not empty, then the spectrum of A is connected.*

Proof. We show the contraposition. Suppose that $M(A)$ is disconnected. Then there are two disjoint, non-void open–closed sets S_1 and S_2 such that $S_1 \cup S_2 = M(A)$. Let $E \subseteq M(A)$. Without loss of generality, we may assume that $E \cap S_2 \neq \emptyset$. By Shilov's idempotent theorem (see [3, p. 88]), there is a function $f \in A$ such that $f \equiv 1$ on S_1 and $f \equiv 0$ on S_2. Now we choose $U = S_2$. Then U is open, $f \equiv 0$ on $U \cap E$, but f is not the zero function. Hence $E \notin \mathscr{U}$. □

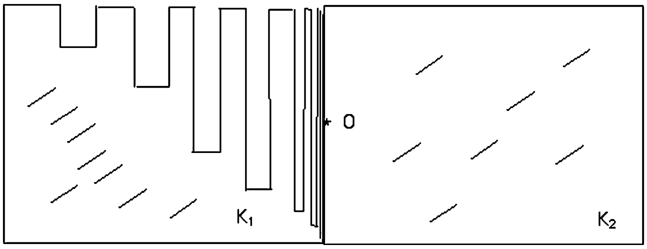

Fig. 1. An instructive example.

Of course the connectivity condition above is far from being sufficient for $\mathscr{U}$ to be non-empty. In fact, $\mathscr{U} = \emptyset$ for any regular algebra strictly containing $\mathbb{C}$. But $\mathscr{U}$ may be empty, too, for other algebras with connected spectrum, as for example $H^\infty + C$ (see Theorem 3.1).

Observation 2.5.

(1) *If $M(A)$ is not a strong uniqueness set, then there exists $x \in M(A)$ such that $k_A(x) \subsetneqq M(A)$.*
(2) *If $\mathscr{U} = \emptyset$, then the set of points x for which $k_A(x) \subsetneqq M(A)$, is dense in $M(A)$.*

Proof. (1) Since $M(A) \notin \mathscr{U}$ there exists a non-void open set $V \subseteq M(A)$ and $f \in A$ such that $f \equiv 0$ on V but $f \not\equiv 0$. Hence, for any $x \in V$, we have $k_A(x) \subsetneqq M(A)$.

(2) Let $\emptyset \neq V$ be open in $M(A)$. Since $\mathscr{U} = \emptyset$, $V \notin \mathscr{U}$. Hence there is a second open set V' such that $V \cap V' \neq \emptyset$ and a non-constant function $f \in A$ such that $f \equiv 0$ on $V \cap V'$. Thus any $x \in V \cap V'$ has the property that $k_A(x) \subsetneqq M(A)$. □

Observation 2.6. *If for every $x \in M(H^\infty)$, $k_A(x)$ is a proper subset of $M(A)$, then $\mathscr{U} = \emptyset$.*

Proof. Let $\emptyset \neq E \subseteq M(A)$. Choose $x \in E$. Since $k_A(x) \neq M(A)$, there exist $y \in M(A) \setminus k_A(x)$ and a function $f \in J(x, A)$ with $f(y) \neq 0$. Hence $E \cap Z(f)^\circ \neq \emptyset$, but $f \not\equiv 0$. Therefore, E is not a strong uniqueness set. □

Whereas the union of two t-analytic sets is, in general, not t-analytic (even if they are non-disjoint and connected) (see [2]), we have the following result concerning the subclass of strong uniqueness sets.

Observation 2.7. *Any union of strong uniqueness sets in a Banach function algebra is a strong uniqueness set again.*

Proof. Let $E_\alpha \in \mathscr{U}$, and set $E = \bigcup E_\alpha$. Note that strong uniqueness sets are never empty. Let U be open and suppose that $f \equiv 0$ on $U \cap E$. We assume that this last set is non-empty. Hence there exists α such that $U \cap E_\alpha \neq \emptyset$. Since $f \equiv 0$ on $U \cap E_\alpha$, our hypothesis implies that $f \equiv 0$. Thus $E \in \mathscr{U}$. □

The class $\mathscr{U}$, though, is not stable with respect to intersections; even if those intersections are non-empty. For example, $[-1, 0]$ and $[0, 1]$ are strong uniqueness sets for the disk-algebra $A(\mathbb{D})$, but their intersection not. As a corollary to Observation 2.7 we obtain:

Observation 2.8. *Let A be a Banach function algebra for which $\mathscr{U} \neq \emptyset$. Then there exists a biggest strong uniqueness set.*

Note that in the class of t-analytic sets for A there always exist maximal elements; but, in general, no maximum t-analytic set (see [2]).

In [2, Example 2.4], an example of a compact set $K = K_1 \cup K_2 \subseteq \mathbb{C}$ is given which shows that for the algebra $A = A(K)$ of all functions continuous on K and holomorphic in the interior K° of K, K_1 and K_2 are (non-disjoint) maximal t-analytic sets. Moreover, $k_A(z) = K = M(A)$ for every $z \in K_1$ and $k_A(z) = K_2 \subsetneqq M(A)$ for any $z \in K_2 \setminus K_1$. Here we can now add that K_1 is the maximum strong uniqueness set for $A(K)$ (see Fig. 1).

Observation 2.9. *Let A be a Banach function algebra for which $\mathscr{U} \neq \emptyset$. Then the biggest strong uniqueness set, $E_{\max}$, is also a maximal t-analytic set.*

Proof. Obviously $E_{\max}$ is t-analytic. Now let $E_{\max} \subseteq E$ for some t-analytic set E. We show that $E \in \mathscr{U}$. Let U be open, $U \cap E \neq \emptyset$, and suppose that $f \equiv 0$ on $U \cap E$. Since E is t-analytic, $f \equiv 0$ on E. In particular $f \equiv 0$ on $E_{\max}$. But $\widehat{E_{\max}} = M(A)$. Hence $f \equiv 0$ and so $E \in \mathscr{U}$. The maximality of $E_{\max}$ now implies that $E = E_{\max}$. □

3. Strong uniqueness sets and t-analytic sets for $H^\infty + C$

In order to be able to compare the different situations for both of the algebras H^∞ and $H^\infty + C$, we start with the following results from [2]; excepted item (4). Recall that a thin point $x \in M(H^\infty) \setminus \mathbb{D}$ is any point lying in the $M(H^\infty)$-closure of a sequence $(z_n) \in \mathbb{D}^{\mathbb{N}}$ satisfying

$$\lim_{j\to\infty} \prod_{n\neq j}^{\infty} \rho(z_n, z_j) = 1,$$

where $\rho(z,w) = |(z-w)/(1-\bar{z}w)|$ is the pseudohyperbolic distance. Moreover, $P(x)$ is the Gleason part associated with a point $x \in M(H^\infty + C)$. The zero sets $k_{H^\infty+C}(x)$ of the ideals $I(x, H^\infty + C)$ are called k-hulls and will be denoted by $k(x)$. See [5–7] for a detailed study of these k-hulls.

Theorem 3.1.

(1) *Let E be a t-analytic set for $H^\infty + C$ and suppose that $x \in E$. Then $E \subseteq k(x)$.*
(2) *If the t-analytic set E meets the Shilov boundary of $H^\infty + C$, then E is a singleton.*
(3) *If E is a maximal t-analytic set containing the thin point x, then $E = \overline{P(x)}$.*
(4) *There are no strong uniqueness sets for $H^\infty + C$.*

Proof. (4) This follows from the Observation 2.6 and the fact that for each $x \in M(H^\infty + C)$, $k(x) \neq M(H^\infty + C)$ (see [5]). □

It is conjectured that in $H^\infty + C$ all maximal t-analytic sets and all hull-kernel closed t-analytic sets with cardinal bigger than two are given by the closures of Gleason parts (see [2]).

4. t-Analytic sets for H^∞

In [2] it was implicitly shown that in the disk-algebra the class of t-analytic sets with cardinal bigger than two and the class of strong uniqueness sets coincide. In H^∞, the class of t-analytic sets containing more than one point is much bigger than $\mathscr{U}$. For instance, the closure of any non-trivial Gleason part in the corona of H^∞ is t-analytic, but obviously not a uniqueness set (see [2]).

However, if the set E meets the Shilov boundary, ∂H^∞, of H^∞, then the result just mentioned for $A(\mathbb{D})$ remains valid.

Proposition 4.1. *A non-void set $E \subseteq M(H^\infty)$ with $E \cap \partial H^\infty \neq \emptyset$ is t-analytic for H^∞ if and only if E is either a singleton or a strong uniqueness set.*

Proof. One direction being obvious, we need only show that every t-analytic set E with $E \cap \partial H^\infty \neq \emptyset$ and containing at least two points is a strong uniqueness set for H^∞. In fact, by [2], the ideal $I(E, H^\infty)$ is a closed prime ideal. By [8, Theorem 3.3], any non-zero closed prime ideal whose hull meets the Shilov boundary, is maximal. Thus $I(E, H^\infty) = \{0\}$ whenever E contains at least two points. Hence E is a strong uniqueness set in that case. □

In what follows, let $\hat{\sigma}$ denote the lifted Lebesgue measure defined on the Borel sets of the extremely disconnected set $M(L^\infty)$ (see [3, p. 17]). Recall that for any $f \in L^\infty$

$$\int_{\mathbb{T}} f\, d\sigma = \int_{M(L^\infty)} \hat{f}\, d\hat{\sigma},$$

and that $\hat{\sigma}(B^\circ) = \hat{\sigma}(B) = \hat{\sigma}(\overline{B})$ for any Borel set $B \subseteq M(L^\infty)$. Here $\hat{f}$ is the Gelfand transform of $f \in L^\infty$. The characteristic function of a set $S \subseteq \mathbb{T}$ is denoted by χ_S. Similarly for sets in $M(L^\infty)$. It is well known that the sets

$$\{\hat{\chi}_S = 1\} := \{x \in M(L^\infty):\ \hat{\chi}_S(x) = 1\},$$

$S \subseteq \mathbb{T}$ Lebesgue-measurable, form a basis of closed–open sets for the topology on $M(L^\infty)$ (see [3, p. 17]).

Let QC be the algebra of quasi-continuous functions on $\mathbb{T}$; that is QC is the biggest C^*-subalgebra of $H^\infty + C$. Moreover, let $QA = QC \cap H^\infty$. See [9,10] for a thorough study of these algebras.

The following lemma has been communicated to me by Keiji Izuchi.

Lemma 4.2. *Let E be a non-void closed subset of $M(L^\infty)$ with $\hat{\sigma}(E) = 0$. Then there exists a non-constant function $f \in H^\infty$ such that $f \equiv 0$ on E.*

Proof. Let K_n be a sequence of closed–open sets in $M(L^\infty)$ satisfying

$$E \subseteq K_{n+1} \subseteq K_n$$

and $\hat{\sigma}(K_n) \to 0$. Let

$$F = \sum_{n=1}^{\infty} (1 - \chi_{K_n})/n^2.$$

Then $F \in C(M(L^\infty))$. Hence there is $q \in L^\infty$ such that $\hat{q} = F$. Moreover,

$$F \equiv 0 \quad \text{on } P := \bigcap_{n=1}^{\infty} K_n.$$

Note that $E \subseteq P$ and that $\hat{\sigma}(P) = 0$. By Wolff [10, Theorem 1], there is a non-zero $f \in QA$ such that $fF \in QC$. Then, with $X = M(L^\infty)$,

$$Z_X(fF) = Z_X(f) \cup Z_X(F) = Z_X(f) \cup P.$$

Since zero-sets of QC-functions have lifted Lebesgue measure 0 (on $M(QC)$), we deduce from [10, Lemma 2.3] that $Z_X(f) \cup P$ is a weak peak interpolation set for QA. Hence there is a non-constant $g \in QA$ such that $g \equiv 0$ on $Z_X(f) \cup P \supseteq E$. $\square$

Theorem 4.3. *Let E be a non-void closed subset of ∂H^∞. The following assertions are equivalent*:

(1) *E is a strong uniqueness set for H^∞*;
(2) *For every Lebesgue measurable set $S \subseteq \mathbb{T}$ with strictly positive Lebesgue measure either $\hat{\sigma}(E \cap \{\hat{\chi}_S = 1\}) > 0$ or $E \cap \{\hat{\chi}_S = 1\} = \emptyset$.*

In particular, $\partial H^\infty \in \mathscr{U}$.

Proof. (2) $\Longrightarrow$ (1): Let $U \subseteq M(H^\infty)$ be any open set with $U \cap E \neq \emptyset$. Let $x \in U \cap E$. Then there is a Lebesgue measurable set $S \subseteq \mathbb{T}$ with $\sigma(S) > 0$ such that

$$x \in \{\hat{\chi}_S = 1\} \subseteq U \cap M\big(L^\infty\big).$$

Hence $\emptyset \neq E \cap \{\hat{\chi}_S = 1\} \subseteq E \cap U$. Suppose that for some $f \in H^\infty$, $f \equiv 0$ on $E \cap U$. Then $f \equiv 0$ on $E \cap \{\hat{\chi}_S = 1\}$, too. Now we use that for any $f \in H^\infty \setminus \{0\}$, $Z(f) \cap \partial H^\infty$ has lifted Lebesgue measure 0. Our hypothesis that $\hat{\sigma}(E \cap \{\chi_S = 1\}) > 0$ now implies that f is the zero function in H^∞. Hence $E \in \mathscr{U}$.

(1) $\Longrightarrow$ (2) will be proven via contraposition. So suppose $E \subseteq \partial H^\infty = M(L^\infty)$ satisfies $E \cap \{\hat{\chi}_S = 1\} \neq \emptyset$, but

$$\hat{\sigma}\big(E \cap \{\hat{\chi}_S = 1\}\big) = 0$$

for some measurable set $S \subseteq \mathbb{T}$ of positive Lebesgue measure. By Lemma 4.2, there is a non-constant $f \in H^\infty$ with $f \equiv 0$ on $E \cap \{\hat{\chi}_S = 1\}$. Hence E cannot be a strong uniqueness set for H^∞. $\square$

Is it possible to give a description of the strong uniqueness sets E in ∂H^∞ using only properties of H^∞ when viewed as a set of functions defined on $\mathbb{T}$?

For example let S be a measurable subset of $\mathbb{T}$. Then S is a 'strong uniqueness set' for $H^\infty|_{\mathbb{T}}$ if and only if $\sigma(S \cap I) > 0$ for every open arc $I \subseteq \mathbb{T}$ with $S \cap I \neq \emptyset$. Which relations can one expect between S and E?

Next we compare the t-analytic sets for H^∞ and $H^\infty + C$.

Lemma 4.4. *Let $x \in M(H^\infty + C)$. Denote the identity function on $\mathbb{T}$ by $\mathbf{z}$. Then the k-hull $k(x)$ of x is contained in a single fiber*

$$M_\lambda = \big\{m \in M\big(H^\infty + C\big)\colon\, m(\mathbf{z}) = \lambda\big\},$$

$|\lambda| = 1$.

Proof. The assertion follows from the facts that $k(x)$ is contained in a unique $C(\mathbb{T})$-level set

$$E_\lambda = \big\{m \in M\big(H^\infty + C\big)\colon\, m(f) = f(\lambda) \text{ for every } f \in C(\mathbb{T})\big\}$$

which coincides with the fibers. $\square$

We shall need several times the following lemma, whose first assertion is a special case of [5, Lemma 2.4].

Lemma 4.5. *Let $x \in M(H^\infty + C) \setminus \partial H^\infty$. Then the ideal $J(x, H^\infty + C)$ is algebraically generated by Blaschke products. Moreover, $k(x)$ is hull-kernel closed in H^∞.*

Proof. Since $J := J(x, H^\infty + C)$ is generated by Blaschke products, we have that $k(x) = \bigcap_{B \in J} Z(B)$. Accordingly, for every $y \in M(H^\infty) \setminus k(x)$ there exists a Blaschke product B with $B \in J$, hence $B \equiv 0$ on $k(x)$, but $B(y) \neq 0$. Thus $k(x)$ is hull-kernel closed in H^∞. □

Lemma 4.6. *Let $E \subseteq M(H^\infty) \setminus \mathbb{D}$ be a t-analytic set for H^∞ and let $x \in E \setminus \partial H^\infty$. Then $E \subseteq k(x)$.*

Proof. By Lemma 4.4, $k(x)$ is contained in a single fiber. In particular, $M(H^\infty + C) \setminus k(x) \neq \emptyset$. So let $y \in M(H^\infty + C) \setminus k(x)$. Since $x \notin \partial H^\infty$, by Lemma 4.5, there is a Blaschke product $B \in J(x, H^\infty + C)$ such that $B(y) \neq 0$. In particular $B \equiv 0$ on an open set U in $M(H^\infty + C)$ with $x \in U$. Note that $E \cap U \neq \emptyset$. Choose an open set V in $M(H^\infty)$ such that $U = V \cap M(H^\infty + C)$. Then $B \equiv 0$ on $E \cap V = E \cap U$. Since E is t-analytic for H^∞, we conclude that $y \notin E$. Hence $E \subseteq k(x)$. □

Corollary 4.7. *Let $E \subseteq M(H^\infty) \setminus \mathbb{D}$ be a t-analytic set for H^∞. Then $\overline{E}$ either is entirely contained in the Shilov boundary or in $M(H^\infty + C) \setminus \partial H^\infty$.*

Proof. Assume that there is $x \in \overline{E} \setminus \partial H^\infty$. By the Observation 2.1, $\overline{E}$ is t-analytic. Hence, by Lemma 4.6, $\overline{E} \subseteq k(x)$. By [5], $k(x) \cap \partial H^\infty = \emptyset$. Thus $\overline{E}$ does not meet ∂H^∞. □

Theorem 4.8. *Let E be a set in $M(H^\infty) \setminus \mathbb{D}$ that does not meet the Shilov boundary of H^∞. Then E is t-analytic for H^∞ if and only if E is t-analytic for $H^\infty + C$.*

Proof. If E is t-analytic for $H^\infty + C$, then it is easily seen that E is t-analytic for H^∞. Indeed, it suffices to observe that any open set U in $M(H^\infty)$ induces the open set $U \cap M(H^\infty + C)$ in $M(H^\infty + C)$.

Conversely, let $E \subseteq M(H^\infty + C)$ be t-analytic for H^∞ with $E \cap \partial H^\infty = \emptyset$. Let $f \in H^\infty + C$ vanish identically on $E \cap \Omega$ for an open set $\Omega \subseteq M(H^\infty + C)$ with $E \cap \Omega \neq \emptyset$. Let $x \in E$. Since $E \cap \partial H^\infty = \emptyset$ we may use Lemma 4.6 to conclude that $E \subseteq k(x)$. Moreover, by Lemma 4.4, $k(x)$ is contained in a single fiber M_λ. Now on fibers, $(H^\infty + C)|_{M_\lambda} = H^\infty|_{M_\lambda}$. Thus we may choose $F \in H^\infty$ such that $F = f$ on M_λ. Now for any open set W in $M(H^\infty)$ with $W \cap M(H^\infty + C) = \Omega$, we have $F \equiv 0$ on $W \cap E$. Since E is t-analytic for H^∞, $F \equiv 0$ on E and so does f. Hence E is t-analytic for $H^\infty + C$. □

Recall that a point $x \in E \subseteq X$, X a topological space, is said to be an isolated point (for E), if there exists an open neighborhood U of x such that $U \cap E = \{x\}$.

Proposition 4.9. *Let E be a subset of $\mathbb{D}$. Suppose that E contains more than one point. Then the following assertions are equivalent*:

(1) *E is a strong uniqueness set for H^∞*;
(2) *E is t-analytic for H^∞*;
(3) *E does not contain any isolated point.*

Proof. (1) $\Longrightarrow$ (2) trivial.

(2) $\Longrightarrow$ (3) Suppose to the contrary that $z_0 \in E$ is an isolated point. The function $z - z_0$ then vanishes in a relative open neighborhood of E, but not at any other point. Thus E is no longer a t-analytic set.

(3) $\Longrightarrow$ (1) This follows immediately from the fact that the zeros of non-constant holomorphic functions are discrete (in $\mathbb{D}$). □

We note that the implication (2) $\Longrightarrow$ (3) holds true in any function algebra; this is a special case of [2, Corollary 4.11].

Theorem 4.10. *Let E be a t-analytic set for H^∞ such that $E \cap \mathbb{D} \neq \emptyset$. Then*

$$E \subseteq \partial H^\infty \cup \overline{E \cap \mathbb{D}}.$$

Proof. Suppose contrariwise that there is some $x \in E \setminus \partial H^\infty$ and $x \notin \overline{E \cap \mathbb{D}}$. Let $z_0 \in E \cap \mathbb{D}$. Choose, as in Lemma 4.6, a Blaschke product B such that B vanishes identically on a neighborhood U^* of x in $M(H^\infty + C)$. Let the open subset U of U^* satisfy $x \in U$ and $\overline{U} \cap \overline{E \cap \mathbb{D}} = \emptyset$. We may also assume that $B(z_0) \neq 0$ (otherwise we just delete the zero z_0). Let $V \subseteq M(H^\infty)$ be open with $V \cap M(H^\infty + C) = U$, $z_0 \notin V$ and $V \cap \overline{E \cap \mathbb{D}} = \emptyset$. Note that $E \setminus \overline{(E \cap \mathbb{D})} \subseteq M(H^\infty + C)$. Then

$$\begin{aligned} V \cap E &= V \cap (E \setminus \overline{E \cap \mathbb{D}}) = V \cap (E \setminus \overline{E \cap \mathbb{D}}) \cap M\big(H^\infty + C\big) \\ &= U \cap (E \setminus \overline{E \cap \mathbb{D}}) = (U \cap E) \setminus \overline{E \cap \mathbb{D}} = U \cap E. \end{aligned}$$

Hence $B \equiv 0$ on $V \cap E$, but $B(z_0) \neq 0$. Accordingly, E is not t-analytic. □

As an immediate corollary we have the following corona-type theorem.

Corollary 4.11. *Let E be a closed t-analytic set for H^∞ with $E \cap \mathbb{D} \neq \emptyset$ and $E \cap \partial H^\infty = \emptyset$. Then $E = \overline{E \cap \mathbb{D}}$.*

Let us note that the set of non-closed t-analytic sets for H^∞ is very huge. For example, in view of Observation 2.2 one has that for all $S \subseteq M(H^\infty + C)$ the set $\mathbb{D} \cup S$ is a strong uniqueness set for H^∞.

Theorem 4.12. *Let $E \subseteq M(H^\infty)$ and suppose that $E \cap \mathbb{D} \neq \emptyset$ or $E \subseteq \partial H^\infty$. Then the following assertions are equivalent*:

(1) *E is a strong uniqueness set for H^∞;*
(2) *E is t-analytic for H^∞ and contains more than one point.*

Proof. (1) $\Longrightarrow$ (2) is trivial. By Proposition 4.1, (2) $\Longrightarrow$ (1) whenever $E \subseteq \partial H^\infty$. Now suppose that $E \cap \mathbb{D} \neq \emptyset$ and that E contains more than one point. As previously mentioned, the t-analyticity of E implies that $E \cap \mathbb{D}$ does not contain any isolated point. Hence, by Proposition 4.9, $E \cap \mathbb{D}$ is a strong uniqueness set. By Observation 2.1, this implies that $\overline{E \cap \mathbb{D}}$ is in $\mathscr{U}$, too. Consider now the set

$$S := E \setminus \overline{E \cap \mathbb{D}}.$$

If $S = \emptyset$, then $E \subseteq \overline{E \cap \mathbb{D}}$. Hence $E \cap \mathbb{D} \subseteq E \subseteq \overline{E \cap \mathbb{D}}$. Since $E \cap \mathbb{D} \in \mathscr{U}$, we have, by Observation 2.2, that $E \in \mathscr{U}$.

Let us now assume that $S \neq \emptyset$. By Theorem 4.10, $S \subseteq \partial H^\infty$. Note that S is not a singleton, since otherwise E would contain an isolated point. This would contradict the fact that E is t-analytic.

We are going to show that S is t-analytic. Let U be an open set in $M(H^\infty)$ with $U \cap S \neq \emptyset$ and let $f \in H^\infty$ be such that $f \equiv 0$ on $U \cap S$. By passing to a subset, we may assume that $U \cap \overline{E \cap \mathbb{D}} = \emptyset$, but still $U \cap S \neq \emptyset$. Hence $U \cap S = U \cap E$. Since E is t-analytic, we get that $f \equiv 0$ on E. In particular, $f \equiv 0$ on S. Thus S is t-analytic. By Proposition 4.1, S is in $\mathscr{U}$. By the Observation 2.7, $S \cup \overline{E \cap \mathbb{D}} \in \mathscr{U}$. Since $E = S \cup \overline{E \cap \mathbb{D}}$, we conclude that $E \in \mathscr{U}$. □

Let $\mathscr{F}$ denote the class of subsets F of $\mathbb{D}$ that do not contain any isolated points, let $\mathscr{U}_c$ denote the class of those strong uniqueness sets for H^∞ that are closed. The following concluding theorems sum up the different situations dealt with above.

Theorem 4.13. *Let $E \subseteq M(H^\infty)$ be closed. Then $E \in \mathscr{U}_c \cup \{\emptyset\}$ if and only if $E = K \cup \overline{F}$, where $F \in \mathscr{F}$ and where $K \subseteq \partial H^\infty$ is a closed set such that for every Lebesgue measurable set $S \subseteq \mathbb{T}$ with strictly positive Lebesgue measure either $\hat{\sigma}(K \cap \{\hat{\chi}_S = 1\}) > 0$ or $K \cap \{\hat{\chi}_S = 1\} = \emptyset$.*

Proof. Let $E = K \cup \overline{F}$, where K and F satisfy the conditions above. By Theorem 4.3, $K \in \mathscr{U}$ whenever $K \neq \emptyset$. By Proposition 4.9, $F \in \mathscr{U}$ whenever $F \neq \emptyset$. By the Observation 2.7, $E \in \mathscr{U}$.

Conversely, let $E \in \mathscr{U}_c$. We discuss two cases:

Case 1. $E \cap \mathbb{D} \neq \emptyset$. Since strong uniqueness sets do not contain isolated points, $E \cap \mathbb{D} \in \mathscr{F}$. Hence, by Proposition 4.9, $E \cap \mathbb{D} \in \mathscr{U}$. Moreover, by Observation 2.1, $\overline{E \cap \mathbb{D}} \in \mathscr{U}$. If $\overline{E \cap \mathbb{D}} = E$, then we are done. So suppose that $\overline{E \cap \mathbb{D}} \subsetneqq E$. By the Observation 2.3, $E \setminus \overline{E \cap \mathbb{D}} \in \mathscr{U}$, and so again,

$$K := \overline{E \setminus \overline{E \cap \mathbb{D}}} \in \mathscr{U}.$$

But by Theorem 4.10, $K \subseteq \partial H^\infty$. Hence we can conclude from Theorem 4.3 that K has the desired property. Since $E = \overline{F} \cup K$, where $F = E \cap \mathbb{D}$, we are done.

Case 2. $E \cap \mathbb{D} = \emptyset$. Corollary 4.7 implies that either $E \subseteq \partial H^\infty$, or $E \cap \partial H^\infty = \emptyset$. We shall see that the hypotheses $E \in \mathscr{U}$ implies that the second case does not occur. So suppose that $E \subseteq M(H^\infty + C) \setminus \partial H^\infty$. Then, by Lemma 4.6, $E \subseteq k(x)$ for $x \in E$. Hence the hull-kernel closure in $M(H^\infty)$ of E is contained in $k(x)$, too (see Lemma 4.5). Since E is t-analytic, we deduce that E cannot be a strong uniqueness set. Hence we must have that $E \subseteq \partial H^\infty$. Using Theorem 4.3 again, we see that $K := E$ has the property we wish. □

Combining Theorems 4.3, 4.12 and 4.13, we get the following result.

Theorem 4.14. *The class $\mathscr{A}_{\mathbb{D}}$ of closed t-analytic sets for H^∞ that meet $\mathbb{D}$ is given by $\mathscr{A}_{\mathbb{D}} = \mathscr{A}_1 \cup \mathscr{A}_2$, where*

$$\mathscr{A}_1 = \big\{\{z_0\}\colon\ z_0 \in \mathbb{D}\big\},$$

and

$$\mathscr{A}_2 = \big\{K \cup \overline{F}\colon\ \emptyset \neq F \subseteq \mathbb{D},\ F \in \mathscr{F},\ K \subseteq \partial H^\infty,\ K \in \mathscr{U}_c \text{ or } K = \emptyset\big\}.$$

Finally, a combination of Theorems 4.7, 4.8 and 4.12 yields:

Theorem 4.15. *The class $\mathscr{A}_{\mathrm{cor}}$ of closed t-analytic sets for H^∞ contained in the corona $M(H^\infty + C)$ of H^∞ is given by $\mathscr{A}_{\mathrm{cor}} = \mathscr{A}_3 \cup \mathscr{A}_4$, where*

$$\mathscr{A}_3 = \big\{E\colon\ E \ t\text{-analytic for } H^\infty + C\big\}$$

and

$$\mathscr{A}_4 = \big\{E\colon\ E \subseteq \partial H^\infty,\ E \in \mathscr{U}_c\big\}.$$

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