



The structural property of a class of vector-valued hyperbolic equations and applications[☆]

Gen Qi Xu

Department of Mathematics, Tianjin University, Tianjin, 300072, China

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ABSTRACT

In this paper, we study the structural properties of a class of vector-valued hyperbolic equations with appropriate boundary conditions, including the spectrum determined growth condition. We prove that the equations associate with a C_0 semigroup. By the structural analysis, we obtained a sufficient and necessary condition for being at least an eigenvalue on the imaginary axis. In particular, using the asymptotic analysis technique we prove that the spectrum of the operator determined by the equations is distributed in a strip parallel to the imaginary axis and is union of finitely many separable sets. Furthermore, we prove that the root vectors of the operator are complete and there is a sequence of root vectors that forms a Riesz basis with parentheses for the Hilbert state space. As applications of our results, we give some concrete examples in controlled complex network of Euler–Bernoulli beams.

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1. Introduction

In this paper we study the vector-valued differential equations:

$$\mathbb{M} \frac{\partial^2 Y(x, t)}{\partial t^2} + \mathbb{E} \frac{\partial^4 Y(x, t)}{\partial x^4} = 0, \quad x \in (0, 1), \quad Y(x, t) \in \mathbb{C}^n \quad (1.1)$$

with appropriate boundary conditions, where \mathbb{M} and \mathbb{E} are $n \times n$ real positive definite matrices. Since it is the Euler–Bernoulli beam equation when $n = 1$, so we also call this system a multi-link Euler–Bernoulli beam. This research is motivated by the investigation of multi-link Euler–Bernoulli beams, for example, the system of serially connected beams (see [1–3,5–7]) and the tree-shaped and the star-shaped networks of beams (see [4,8–11]), they are written into such differential equations in \mathbb{C}^n with appropriate boundary conditions. We observed from these literatures that the multi-link structures have the following characters:

- 1) The dynamic behavior of the system depends strongly on the configuration of the structure;
- 2) The dynamic behavior of the structure depends strongly upon the length of every component;
- 3) The character of the system depends strongly on the joining manner at nodes.

These characters are described by the inseparable boundary conditions. Based on this observation, in the present paper, we attach Eqs. (1.1) to the following boundary conditions

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E-mail address: gqxu@tju.edu.cn.

$$\begin{cases} Y(0, t) = \mathbf{C}_1 Y(1, t), & Y_x(0, t) = \mathbf{C}_2 Y_x(1, t), \\ \mathbb{E} \mathbb{I} Y_{xxx}(1, t) - \mathbf{C}_1^T \mathbb{E} \mathbb{I} Y_{xxx}(0, t) = \mathbb{K}_1 Y_t(1, t) - \Gamma_1 Y_{xt}(1, t), \\ \mathbb{E} \mathbb{I} Y_{xx}(1, t) - \mathbf{C}_2^T \mathbb{E} \mathbb{I} Y_{xx}(0, t) = -\mathbb{K}_2 Y_{xt}(1, t) + \Gamma_2 Y_t(1, t), \\ Y(x, 0) = Y_0(x), & Y_t(x, 0) = Y_1(x) \end{cases} \quad (1.2)$$

where \mathbf{C}_j are $n \times n$ real matrices and \mathbf{C}^T represents the transpose matrix of \mathbf{C} , the matrices $\mathbb{K}_s, \Gamma_s, s = 1, 2$, are $n \times n$ real matrices. From the physic point of view, the matrices $\mathbf{C}_j, j = 1, 2$, represent the geometry feature of the structure, usually they satisfy condition $\det(I - \mathbf{C}_j) \neq 0$, which means uniqueness of the equilibrium position; the matrices \mathbb{K}_j and Γ_j ($j = 1, 2$) represent action of the viscous damping and the rotation damping on the bending moments and shear forces. Although (1.1) together with (1.2) constitutes a linearly initial-boundary value problem, without restriction on matrices \mathbb{K}_j and Γ_j ($j = 1, 2$), showing the existence of solution to (1.1) and (1.2) is still unattainable goal. Here is a simple mode from [12]

$$\begin{cases} y_{tt}(x, t) + y_{xxxx}(x, t) = 0, & x \in (0, 1), \\ y(0, t) = y_x(0, t) = 0, & y_{xx}(1, t) = 0, \\ y_{xxx}(1, t) = k y_{x,t}(1, t), \\ y(x, 0) = y_0(x), & y_t(x, 0) = y_1(x), & x \in (0, 1). \end{cases} \quad (1.3)$$

This system describes the suppression of vibrations arose from the arm flexibility of a rotating Cartesian or SCARA robots with long arm in one direction. The solvability problem (1.3) was unsettled until after eight years [13]. This is merely a special case of (1.2) as $n = 1$. Based on this reason, we impose the following restrictive condition on \mathbb{K}_j and Γ_j

$$\begin{bmatrix} \mathbb{K}_1 & -\Gamma_1 \\ -\Gamma_2 & \mathbb{K}_2 \end{bmatrix} + \begin{bmatrix} \mathbb{K}_1 & -\Gamma_1 \\ -\Gamma_2 & \mathbb{K}_2 \end{bmatrix}^T \geq 0. \quad (1.4)$$

This condition is used to ensure the existence of solution. Our purpose is to study the structural properties of (1.1) and (1.2) under the assumption (1.4), such as eigenvalues and their distribution, multiplicity of eigenvalue and their uniform boundedness, basis property of the root vector, as well as the spectrum determined growth condition.

Let us recall a notion. Let X be a Banach space, $\{T(t), t \geq 0\}$ be a C_0 semigroup on X with generator A . Denote $\omega(A) = \lim_{t \rightarrow \infty} \frac{\ln \|T(t)\|}{t}$ and $s(A) = \sup_{\lambda \in \sigma(A)} \Re \lambda$. If $\omega(A) = s(A)$, $T(t)$ is called satisfying the spectrum determined growth condition (briefly, SDG). The SDG property is one of the most important structural properties of A . If X is a Hilbert space [15] pointed out that the Riesz basis property of the eigenvectors implies SDG property. This result can be extended to the case that there are finitely many root vectors (e.g., see [16–18]). However, the result does not apply to the equations included the infinitely many root vectors.

The multi-link structure of Euler–Bernoulli beams has been extensively studied in last two decades (e.g., see the literature mentioned above and the references therein). The attention has been paid to the structural analysis of the system with dissipative joints. In the case of single beam, some nice structural properties of the system have been obtained, for instance, spectra are distributed in a strip parallel to the imaginary axis, each eigenvalue is simple and separated, and the eigenfunctions form a Riesz basis for the Hilbert state spaces, please see [14,19,20] for Euler–Bernoulli beam, [21–23] for the Timoshenko beam. However, these properties do not satisfy in multi-link structure, a concrete model is given in [21] to show that each eigenvalue is of multiplicity two under certain condition. In fact, for most of the multi-link structures, the multiplicities of their eigenvalues are larger than two, or their eigenvalues are not separated. In particular, the basis property of the root vectors and the SDG property of these systems have been difficult topic. For example, the spectral distribution of the serially N -connected Euler–Bernoulli beams has been known for many years (see, e.g., [24]), up to now there is no result on the basis property of its root vectors. Indeed, there might not have basis property of the root vectors in the sense of Schauder basis. Even if the root vectors form a Riesz basis for the Hilbert state space, it is not sufficient to ensure the SDG property.

To study the SDG property of the systems, we obtained recently a sufficient condition for SDG by weakening the Riesz basis condition and strengthening spectral distribution and multiplicity (e.g., see [25–27]). The sufficient conditions for SDG given in [27] include four basic requirements: the operator generates a C_0 semigroup; its spectrum consists of all isolated eigenvalues and is distributed in a strip parallel to the imaginary axis; its spectrum has an essential separability that implies the uniform boundedness of multiplicities of eigenvalues; and the root vectors are complete in the Hilbert state space. If all conditions are fulfilled, we can assert that there is a sequence of root vectors that forms a Riesz basis with parentheses for the Hilbert state space and that the SDG property holds. Although these conditions are sufficient, they seem to be necessary in the some sense; this is because if one of them fails there is a counterexample to show the result does not hold. Therefore, checking these conditions for a concrete system seems to be a routine work. Indeed, checking anyone of them is also a challenge work, it needs more technique and tricks. For instance, for the network of Euler–Bernoulli beams, the approaches of calculating spectrum used in [7] and [11] do not work well for more complicated models; the perturbation method used in [28] is not suitable to our model. Therefore, the theoretic result cannot replace the research of concrete model.

Comparing our model given by (1.1) and (1.2) with the generic networks of Euler–Bernoulli beams, the most difference is that \mathbb{M} and $\mathbb{E} \mathbb{I}$ need not be diagonal matrices, which implies the equations are coupled. In the present paper, our

purpose is to discuss the structural properties of such a model. We shall prove the existence of solution, the distribution and separability of spectrum, completeness of the root vectors and their basis property. All the above are basic content of non-self-adjoint operator with compact resolvent.

The rest of the paper is organized as follows. In the next section, we firstly formulate (1.1) and (1.2) into a Hilbert space and then discuss its well-posedness and spectrum. In Section 3, we analyze the structural property of (1.1) and (1.2), including its internal structural property, eigenvalues and their asymptotic distribution. By the asymptotic analysis technique, we prove that the eigenvalues of the system are distributed in a strip parallel to the imaginary axis under the condition (1.4), all eigenvalues (taking the multiplicity into account) constitute a set that is union of finitely many separable sets, which implies that the multiplicities of eigenvalues are uniformly bounded from above. In Section 4, we investigate the completeness and basis property of root vectors. By some tricks, we prove that there is a sequence of root vectors that forms a Riesz basis with parentheses for the Hilbert state space. With these properties we can assert that the system satisfies SDG property. Finally, in Section 5, we formulate some applicable examples into the frame under consideration. For these concrete models, we further investigate the stability of the systems, the obtained results are new.

2. Well-posedness and spectra

In this section we investigate the solvability and spectra of the partial differential equations in \mathbb{C}^n

$$\begin{cases} \mathbb{M}Y_{tt}(x, t) + \mathbb{E}IY_{xxxx}(x, t) = 0, & x \in (0, 1), \\ Y(0, t) = \mathbf{C}_1 Y(1, t), & Y_x(0, t) = \mathbf{C}_2 Y_x(1, t), \\ \mathbb{E}IY_{xxx}(1, t) - \mathbf{C}_1^T \mathbb{E}IY_{xxx}(0, t) = \mathbb{K}_1 Y_t(1, t) - \Gamma_1 Y_{xt}(1, t), \\ \mathbb{E}IY_{xx}(1, t) - \mathbf{C}_2^T \mathbb{E}IY_{xx}(0, t) = -\mathbb{K}_2 Y_{xt}(1, t) + \Gamma_2 Y_t(1, t), \\ Y(x, 0) = Y_0(x), & Y_t(x, 0) = Y_1(x), \end{cases} \quad (2.1)$$

where \mathbb{M} and $\mathbb{E}I$ are $n \times n$ real positive definite matrices, and $\mathbb{K}_s, \Gamma_s, s = 1, 2$, are the $n \times n$ real matrices satisfying condition

$$\begin{bmatrix} \mathbb{K}_1 & -\Gamma_1 \\ -\Gamma_2 & \mathbb{K}_2 \end{bmatrix} + \begin{bmatrix} \mathbb{K}_1 & -\Gamma_1 \\ -\Gamma_2 & \mathbb{K}_2 \end{bmatrix}^T \geq 0. \quad (2.2)$$

The matrices $\mathbf{C}_j, j = 1, 2$ (they are called the structure matrices) satisfy conditions

$$\det(I - \mathbf{C}_j) \neq 0, \quad j = 1, 2. \quad (2.3)$$

Let the state space be

$$\mathcal{H} = V_E^2(0, 1) \times L_M^2([0, 1], \mathbb{C}^n)$$

where

$$V_E^2(0, 1) = \{f \in H^2((0, 1), \mathbb{C}^n) \mid f(0) = \mathbf{C}_1 f(1), f'(0) = \mathbf{C}_2 f'(1)\}.$$

The inner product in \mathcal{H} is defined by

$$((f_1, g_1), (f_2, g_2))_{\mathcal{H}} = \int_0^1 (\mathbb{E}I f_1''(x), f_2''(x))_{\mathbb{C}^n} dx + \int_0^1 (\mathbb{M} g_1(x), g_2(x))_{\mathbb{C}^n} dx.$$

Obviously, \mathcal{H} is a Hilbert space due to condition (2.3).

Define the operator \mathcal{A} in \mathcal{H} by

$$\mathcal{D}(\mathcal{A}) = \left\{ (f, g) \in V_E^4(0, 1) \times V_E^2(0, 1) \mid \begin{cases} \mathbb{E}I f'''(1) - \mathbf{C}_1^T \mathbb{E}I f'''(0) = \mathbb{K}_1 g(1) - \Gamma_1 g'(1) \\ \mathbb{E}I f''(1) - \mathbf{C}_2^T \mathbb{E}I f''(0) = -\mathbb{K}_2 g'(1) + \Gamma_2 g(1) \end{cases} \right\}, \quad (2.4)$$

$$\mathcal{A} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g(x) \\ -\mathbb{M}^{-1} \mathbb{E}I f^{(4)}(x) \end{pmatrix}. \quad (2.5)$$

Then Eqs. (2.1) can be rewritten as an evolutionary equation in \mathcal{H}

$$\begin{cases} \frac{dU(t)}{dt} = \mathcal{A}U(t), & t > 0, \\ U(0) = U_0, \end{cases} \quad (2.6)$$

where $U(t) = (Y(x, t), Y_t(x, t))^T$, and $U_0 = (Y_0(x), Y_1(x))^T \in \mathcal{H}$.

Theorem 2.1. Let \mathcal{H} and \mathcal{A} be defined as before, then the adjoint operator \mathcal{A}^* of \mathcal{A} is given by

$$\mathcal{D}(\mathcal{A}^*) = \left\{ (w, u) \in V_E^4(0, 1) \times V_E^2(0, 1) \mid \begin{aligned} \mathbb{E}Iw'''(1) - \mathbf{C}_1^T \mathbb{E}Iw'''(0) &= -\mathbb{K}_1^T u(1) + \Gamma_2^T u'(1) \\ \mathbb{E}Iw''(1) - \mathbf{C}_2^T \mathbb{E}Iw''(0) &= \mathbb{K}_2^T u'(1) - \Gamma_1^T u(1) \end{aligned} \right\}, \quad (2.7)$$

$$\mathcal{A}^* \begin{pmatrix} w \\ u \end{pmatrix} = - \begin{pmatrix} u(x) \\ -\mathbb{M}^{-1} \mathbb{E}Iw^{(4)}(x) \end{pmatrix}. \quad (2.8)$$

Proof. For any $(f, g) \in \mathcal{D}(\mathcal{A})$, $(w, u) \in V_E^4(0, 1) \times V_E^2(0, 1)$, we have

$$\begin{aligned} (\mathcal{A}(f, g), (w, u))_{\mathcal{H}} &= \int_0^1 (\mathbb{E}I g''(x), w''(x))_{\mathbb{C}^n} dx - \int_0^1 (\mathbb{E}I f^{(4)}(x), u(x))_{\mathbb{C}^n} dx \\ &= -(\mathbb{E}I f'''(x), u(x))_{\mathbb{C}^n} \Big|_0^1 + (\mathbb{E}I f''(x), u'(x))_{\mathbb{C}^n} \Big|_0^1 \\ &\quad + (\mathbb{E}I g'(x), w''(x))_{\mathbb{C}^n} \Big|_0^1 - (\mathbb{E}I g(x), w'''(x))_{\mathbb{C}^n} \Big|_0^1 \\ &\quad - \int_0^1 (\mathbb{E}I f''(x), u''(x))_{\mathbb{C}^n} dx + \int_0^1 (\mathbb{M} g(x), \mathbb{M}^{-1} \mathbb{E}I w^{(4)}(x))_{\mathbb{C}^n} dx. \end{aligned}$$

Since $(f, g) \in \mathcal{D}(\mathcal{A})$, $(w, u) \in V_E^4(0, 1) \times V_E^2(0, 1)$, we have

$$\begin{aligned} &-(\mathbb{E}I f'''(x), u(x))_{\mathbb{C}^n} \Big|_0^1 + (\mathbb{E}I f''(x), u'(x))_{\mathbb{C}^n} \Big|_0^1 + (\mathbb{E}I g'(x), w''(x))_{\mathbb{C}^n} \Big|_0^1 - (\mathbb{E}I g(x), w'''(x))_{\mathbb{C}^n} \Big|_0^1 \\ &= -(\mathbb{E}I f'''(1) - \mathbf{C}_1^T \mathbb{E}I f'''(0), u(1))_{\mathbb{C}^n} + (\mathbb{E}I f''(1) - \mathbf{C}_2^T \mathbb{E}I f''(0), u'(1))_{\mathbb{C}^n} \\ &\quad + (g'(1), \mathbb{E}I w''(1) - \mathbf{C}_2^T \mathbb{E}I w''(0))_{\mathbb{C}^n} - (g(1), \mathbb{E}I w'''(1) - \mathbf{C}_1^T \mathbb{E}I w'''(0))_{\mathbb{C}^n} \\ &= -(\mathbb{K}_1 g(1) - \Gamma_1 g'(1), u(1))_{\mathbb{C}^n} + (-\mathbb{K}_2 g'(1) + \Gamma_2 g(1), u'(1))_{\mathbb{C}^n} \\ &\quad + (g'(1), \mathbb{E}I w''(1) - \mathbf{C}_2^T \mathbb{E}I w''(0))_{\mathbb{C}^n} - (g(1), \mathbb{E}I w'''(1) - \mathbf{C}_1^T \mathbb{E}I w'''(0))_{\mathbb{C}^n} \\ &= -(g(1), \mathbb{E}I w'''(1) - \mathbf{C}_1^T \mathbb{E}I w'''(0) + \mathbb{K}_1^T u(1) - \Gamma_2^T u'(1))_{\mathbb{C}^n} \\ &\quad + (g'(1), \mathbb{E}I w''(1) - \mathbf{C}_2^T \mathbb{E}I w''(0) - \mathbb{K}_2^T u'(1) + \Gamma_1^T u(1))_{\mathbb{C}^n}. \end{aligned}$$

Obviously, if

$$\begin{aligned} \mathbb{E}I w'''(1) - \mathbf{C}_1^T \mathbb{E}I w'''(0) + \mathbb{K}_1^T u(1) - \Gamma_2^T u'(1) &= 0, \\ \mathbb{E}I w''(1) - \mathbf{C}_2^T \mathbb{E}I w''(0) - \mathbb{K}_2^T u'(1) + \Gamma_1^T u(1) &= 0, \end{aligned} \quad (2.9)$$

we have

$$\mathcal{A}^* \begin{pmatrix} w \\ u \end{pmatrix} = - \begin{pmatrix} u(x) \\ -\mathbb{M}^{-1} \mathbb{E}I w^{(4)}(x) \end{pmatrix}.$$

Conversely, we can prove that if $(w, u) \in \mathcal{D}(\mathcal{A}^*)$, then it must satisfy the condition (2.9). We omit the detail of verification. The desired result follows. \square

Theorem 2.2. Let \mathcal{H} and \mathcal{A} be defined as before, then \mathcal{A} and \mathcal{A}^* are dissipative operators in \mathcal{H} . Furthermore, the resolvent of \mathcal{A} is compact on \mathcal{H} , and hence $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ consists of all isolated eigenvalues of finite multiplicity. In particular, $\sigma(\mathcal{A})$ distributes symmetrically with respect to the real axis.

Proof. For any $(f, g) \in \mathcal{D}(\mathcal{A})$, a straightforward calculation gives

$$\begin{aligned} 2\Re(\mathcal{A}(f, g), (f, g))_{\mathcal{H}} &= (\mathcal{A}(f, g), (f, g))_{\mathcal{H}} + ((f, g), \mathcal{A}(f, g))_{\mathcal{H}} \\ &= -([\mathbb{K}_1 g(1) - \Gamma_1 g'(1)], g(1))_{\mathbb{C}^n} - (g(1), [\mathbb{K}_1 g(1) - \Gamma_1 g'(1)])_{\mathbb{C}^n} \\ &\quad - ([\mathbb{K}_2 g'(1) - \Gamma_2 g(1)], g'(1))_{\mathbb{C}^n} - (g'(1), [\mathbb{K}_2 g'(1) - \Gamma_2 g(1)])_{\mathbb{C}^n} \\ &= -\left(\begin{bmatrix} \mathbb{K}_1 & -\Gamma_1 \\ -\Gamma_2 & \mathbb{K}_2 \end{bmatrix} \begin{bmatrix} g(1) \\ g'(1) \end{bmatrix}, \begin{bmatrix} g(1) \\ g'(1) \end{bmatrix} \right)_{\mathbb{C}^{2n}} - \left(\begin{bmatrix} g(1) \\ g'(1) \end{bmatrix}, \begin{bmatrix} \mathbb{K}_1 & -\Gamma_1 \\ -\Gamma_2 & \mathbb{K}_2 \end{bmatrix} \begin{bmatrix} g(1) \\ g'(1) \end{bmatrix} \right)_{\mathbb{C}^{2n}}. \end{aligned}$$

Thank to condition (2.2), we get

$$\Re(\mathcal{A}(f, g), (f, g))_{\mathcal{H}} \leq 0.$$

So \mathcal{A} is dissipative in \mathcal{H} .

For any $(w, u) \in \mathcal{D}(\mathcal{A}^*)$, we have

$$\begin{aligned} 2\Re(\mathcal{A}^*(w, u), (w, u))_{\mathcal{H}} &= (\mathcal{A}^*(w, u), (w, u))_{\mathcal{H}} + ((w, u), \mathcal{A}^*(w, u))_{\mathcal{H}} \\ &= -([\mathbb{K}_1^T u(1) - \Gamma_2^T u'(1)], u(1))_{\mathbb{C}^n} - (u(1), [\mathbb{K}_1^T u(1) - \Gamma_2^T u'(1)])_{\mathbb{C}^n} \\ &\quad - ([\mathbb{K}_2^T u'(1) - \Gamma_1^T u(1)], u'(1))_{\mathbb{C}^n} - (u'(1), [\mathbb{K}_2^T u'(1) - \Gamma_1^T u(1)])_{\mathbb{C}^n} \\ &= -(u(1), \mathbb{K}_1 u(1) - \Gamma_1 u'(1))_{\mathbb{C}^n} - (\mathbb{K}_1 u(1) - \Gamma_1 u'(1), u(1))_{\mathbb{C}^n} \\ &\quad + (u'(1), -\mathbb{K}_2 u'(1) + \Gamma_2 u(1))_{\mathbb{C}^n} + (-\mathbb{K}_2 u'(1) + \Gamma_2 u(1), u'(1))_{\mathbb{C}^n} \\ &= -\left(\begin{bmatrix} \mathbb{K}_1 & -\Gamma_1 \\ -\Gamma_2 & \mathbb{K}_2 \end{bmatrix} \begin{bmatrix} u(1) \\ u'(1) \end{bmatrix}, \begin{bmatrix} u(1) \\ u'(1) \end{bmatrix}\right)_{\mathbb{C}^{2n}} - \left(\begin{bmatrix} u(1) \\ u'(1) \end{bmatrix}, \begin{bmatrix} \mathbb{K}_1 & -\Gamma_1 \\ -\Gamma_2 & \mathbb{K}_2 \end{bmatrix} \begin{bmatrix} u(1) \\ u'(1) \end{bmatrix}\right)_{\mathbb{C}^{2n}}. \end{aligned}$$

Therefore, \mathcal{A}^* also is dissipative operator.

Since \mathcal{A} and \mathcal{A}^* are dissipative, theory of dissipative operators asserts that $\Re \lambda > 0$ are resolvent points, i.e., $\lambda \in \rho(\mathcal{A})$. Note that $\mathcal{D}(\mathcal{A}) \subset V_E^4(0, 1) \times V_E^2(0, 1) \subset \mathcal{H}$. The Sobolev Embedding Theorem asserts that the resolvents of \mathcal{A} are compact operators on \mathcal{H} . Therefore, $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ consists of all isolated eigenvalues of finite multiplicity.

Since the matrices \mathbb{M} , $\mathbb{E}\mathbb{I}$, \mathbf{C}_j , \mathbb{K}_j and Γ_j ($j = 1, 2$) are real by assumption, \mathcal{A} is a real operator that means that for any $(f, g) \in \mathcal{D}(\mathcal{A})$ $\overline{\mathcal{A}(f, g)} = \mathcal{A}(\overline{f}, \overline{g})$. Thus for any $\lambda \in \sigma(\mathcal{A})$, $(f, g) \in \mathcal{D}(\mathcal{A})$ is an eigenvector associated with λ , it holds that $\overline{\mathcal{A}(f, g)} = \mathcal{A}(\overline{f}, \overline{g}) = \overline{\lambda}(\overline{f}, \overline{g})$. Therefore $\sigma(\mathcal{A})$ distributes symmetrically with respect to the real axis. \square

Due to the dissipateness of \mathcal{A} and \mathcal{A}^* , the Lumer-Phillips Theorem (e.g., see [30]) asserts that \mathcal{A} generates a C_0 semigroup of contraction on \mathcal{H} . Therefore, we have the following solvability result to Eqs. (2.1).

Corollary 2.1. Let \mathcal{H} and \mathcal{A} be defined as before, then \mathcal{A} generates a C_0 semigroup of contraction on \mathcal{H} . Hence for any $(Y_0, Y_1) \in \mathcal{H}$, (2.1) has uniquely a mild solution in \mathcal{H} .

Based on the representation of \mathcal{A} and \mathcal{A}^* , we have the following result.

Corollary 2.2. Let \mathcal{H} and \mathcal{A} be defined as before, and \mathcal{A}^* be given by (2.7) and (2.8). If $\mathbb{K}_j = \Gamma_j = 0$, $j = 1, 2$, then \mathcal{A} is a skew-adjoint operator, i.e., $\mathcal{A}^* = -\mathcal{A}$.

3. Structure analysis of \mathcal{A}

Since we have an abstract setting about the matrices \mathbb{M} , $\mathbb{E}\mathbb{I}$, \mathbf{C}_j and \mathbb{K}_j and Γ_j ($j = 1, 2$) in (2.1), we analyze the structural property of the operator determined by these matrices in this section.

3.1. Interior structure property of operator

According to Corollary 2.2, we can define an operator \mathcal{A}_0 in \mathcal{H} by

$$\mathcal{D}(\mathcal{A}_0) = \left\{ (f, g) \in V_E^4(0, 1) \times V_E^2(0, 1) \mid \begin{array}{l} \mathbb{E}\mathbb{I}f'''(1) - \mathbf{C}_1^T \mathbb{E}\mathbb{I}f'''(0) = 0 \\ \mathbb{E}\mathbb{I}f''(1) - \mathbf{C}_2^T \mathbb{E}\mathbb{I}f''(0) = 0 \end{array} \right\}, \quad (3.1)$$

$$\mathcal{A}_0(f, g) = (g, -\mathbb{M}^{-1} \mathbb{E}\mathbb{I}f^{(4)}), \quad \forall (f, g) \in \mathcal{D}(\mathcal{A}_0). \quad (3.2)$$

Obviously, \mathcal{A}_0 is a skew-adjoint operator in \mathcal{H} .

For the operator \mathcal{A}_0 , we have the following result.

Theorem 3.1. Let \mathcal{A}_0 be defined by (3.1) and (3.2). Let $\varphi_k(x)$ be the nonzero solution of the boundary eigenvalue problem

$$\begin{cases} \mathbb{E}\mathbb{I}\varphi_k^{(4)}(x) = \gamma_k \mathbb{M}\varphi_k(x), \\ \varphi_k(0) = \mathbf{C}_1 \varphi_k(1), \quad \varphi_k'(0) = \mathbf{C}_2 \varphi_k'(1), \\ \mathbb{E}\mathbb{I}\varphi_k'''(1) - \mathbf{C}_1^T \mathbb{E}\mathbb{I}\varphi_k'''(0) = 0, \\ \mathbb{E}\mathbb{I}\varphi_k''(1) - \mathbf{C}_2^T \mathbb{E}\mathbb{I}\varphi_k''(0) = 0. \end{cases} \quad (3.3)$$

Then the spectrum of \mathcal{A}_0 is given by

$$\sigma(\mathcal{A}_0) = \{\lambda_k = i\sqrt{\gamma_k}; \lambda_{-k} = -i\sqrt{\gamma_k} \mid k \in \mathbb{N}\}$$

and the corresponding eigenfunctions are

$$\Phi_k(x) = (\varphi_k, i\sqrt{\gamma_k}\varphi_k(x)), \quad \Phi_{-k}(x) = (\varphi_k, -i\sqrt{\gamma_k}\varphi_k(x)), \quad k \in \mathbb{N}.$$

The eigenfunctions $\{\Phi_k(x), k \in \mathbb{Z}\}$ form an orthogonal basis for \mathcal{H} .

Proof. Firstly, we note that if γ_k is an eigenvalue of (3.3), then $\gamma_k > 0$ and $\varphi_k(x)$ are real functions. In addition, $\{\varphi_k, k \in \mathbb{N}\}$ forms an orthogonal basis in $L_M^2([0, 1], \mathbb{C}^n)$ under the inner product

$$(w, v)_{L_M^2} = \int_0^1 (\mathbb{M}w(x), v(x))_{\mathbb{C}^n} dx.$$

This is because

$$\gamma_k(\varphi_k, \varphi_m)_{L_M^2} = \int_0^1 (\mathbb{M}\gamma_k\varphi_k, \varphi_m)_{\mathbb{C}^n} dx = \int_0^1 (\mathbb{E}\mathbb{I}\varphi_k^{(4)}, \varphi_m(x))_{\mathbb{C}^n} dx = \gamma_m(\varphi_k, \varphi_m)_{L_M^2}$$

if $\gamma_k \neq \gamma_m$, it must be $(\varphi_k, \varphi_m)_{L_M^2} = 0$.

Next let $\lambda = i\rho \in \sigma(\mathcal{A}_0)$ and (f, g) be corresponding an eigenfunction. We have that $g(x) = \lambda f(x)$ and $f(x)$ satisfies the boundary eigenvalue equations

$$\begin{cases} \mathbb{E}\mathbb{I}f^{(4)}(x) = \rho^2\mathbb{M}f(x), \\ f(0) = \mathbf{C}_1 f(1), \quad f'(0) = \mathbf{C}_2 f'(1), \\ \mathbb{E}\mathbb{I}f'''(1) - \mathbf{C}_1^T \mathbb{E}\mathbb{I}f'''(0) = 0, \\ \mathbb{E}\mathbb{I}f''(1) - \mathbf{C}_2^T \mathbb{E}\mathbb{I}f''(0) = 0. \end{cases}$$

This means that there exists some $k \in \mathbb{N}$ such that $\rho = \pm\sqrt{\gamma_k}$ and $f(x) = \varphi_k(x)$. Thus

$$\Phi_k(x) = (\varphi_k(x), i\sqrt{\gamma_k}\varphi_k(x)), \quad \Phi_{-k}(x) = (\varphi_k(x), -i\sqrt{\gamma_k}\varphi_k(x)), \quad k \in \mathbb{N}$$

are eigenfunctions of \mathcal{A}_0 .

Finally, for any $k, m \in \mathbb{Z}$,

$$(\Phi_k, \Phi_m)_{\mathcal{H}} = (\gamma_k + \text{sign}(m)\sqrt{\gamma_k}\sqrt{\gamma_m}) \int_0^1 (\mathbb{M}\varphi_k(x), \varphi_m(x))_{\mathbb{C}^n} dx = 0$$

and

$$(\Phi_k, \Phi_k)_{\mathcal{H}} = 2\gamma_k \int_0^1 (\mathbb{M}\varphi_k(x), \varphi_k(x))_{\mathbb{C}^n} dx.$$

Therefore, $\{\Phi_k, k \in \mathbb{Z}\}$ forms an orthogonal basis for \mathcal{H} . \square

Let us consider another operator \mathcal{A}_1 in \mathcal{H} defined by

$$\mathcal{D}(\mathcal{A}_1) = \left\{ (f, g) \in V_E^4(0, 1) \times V_E^2(0, 1) \mid \begin{array}{l} f(0) = f'(0) = 0, \quad g(0) = g'(0) = 0 \\ f(1) = f'(1) = 0, \quad g(1) = g'(1) = 0 \end{array} \right\}, \quad (3.4)$$

$$\mathcal{A}_1(f, g) = (g, -\mathbb{M}^{-1}\mathbb{E}\mathbb{I}f^{(4)}), \quad \forall (f, g) \in \mathcal{D}(\mathcal{A}_1). \quad (3.5)$$

Clearly, \mathcal{A}_1 is also a skew-adjoint operator in \mathcal{H} .

We are now interested in whether or not \mathcal{A}_0 and \mathcal{A}_1 have common eigenvalue and eigenfunction. Since $\mathbb{E}\mathbb{I}^{-1}\mathbb{M}$ is similar to a positive definite matrix, we can assume that B is a positive matrix satisfying $B^4 = \mathbb{E}\mathbb{I}^{-1}\mathbb{M}$. Let all eigenvalues of B be $\{\mu_s > 0, s = 1, 2, \dots, n\}$ and $\hat{\eta}_s$ be an eigenvectors corresponding to μ_s , $s \in \{1, 2, \dots, n\}$. Obviously, $\{\hat{\eta}_s\}_{s=1}^n$ is an orthogonal basis of \mathbb{C}^n , and $G(B\rho)\hat{\eta}_s = G(\mu_s\rho)\hat{\eta}_s$ for any analysis function $G(z)$. Define the sets by

$$S_s^\pm = \{\pm \cot \mu_s \rho \sinh \mu_s \rho \mid 1 = \cosh \mu_s \rho \cos \mu_s \rho, \rho \in \mathbb{R}\}, \quad s = 1, 2, \dots, n. \quad (3.6)$$

Theorem 3.2. Let \mathcal{A}_0 and \mathcal{A}_1 be defined as above, $\sigma(\mathbf{C}_j^T)$ be the spectrum of \mathbf{C}_j^T , $j = 1, 2$. If the matrices B , \mathbf{C}_1^T and \mathbf{C}_2^T satisfy one of the following conditions

- 1) B , $\mathbb{E}\mathbb{I}^{-1}\mathbf{C}_1^T\mathbb{E}\mathbb{I}$ and $\mathbb{E}\mathbb{I}^{-1}\mathbf{C}_2^T\mathbb{E}\mathbb{I}$ have no a common eigenvector;
- 2) $\sigma(\mathbf{C}_1^T) \cap \bigcup_{s=1}^n S_s^+ = \emptyset$; or
- 3) $\sigma(\mathbf{C}_2^T) \cap \bigcup_{s=1}^n S_s^- = \emptyset$,

then \mathcal{A}_1 and \mathcal{A}_0 have not common eigenvalue and eigenvector, i.e., if there exists a vector $(f, g) \in \mathcal{D}(\mathcal{A}_0) \cap \mathcal{D}(\mathcal{A}_1)$ such that for some $\lambda \in i\mathbb{R}$, $\mathcal{A}_0(f, g) = \lambda(f, g)$, $\mathcal{A}_1(f, g) = \lambda(f, g)$, then $(f, g) = (0, 0)$.

Proof. Suppose that there exists one vector $(f, g) \in \mathcal{D}(\mathcal{A}_1) \cap \mathcal{D}(\mathcal{A}_0)$ such that for some $\lambda \in i\mathbb{R}$,

$$\mathcal{A}_0(f, g) = \lambda(f, g), \quad \mathcal{A}_1(f, g) = \lambda(f, g).$$

Without loss of generality we can assume that $\lambda = i\rho^2$, $\rho \in \mathbb{R}$. We have that $g(x) = \lambda f(x)$ and f satisfies the differential equations

$$\begin{cases} \mathbb{E}f^{(4)}(x) - \rho^4 \mathbb{M}f(x) = 0, \\ f(0) = f'(0) = 0, \quad f(1) = f'(1) = 0, \\ \mathbb{E}f'''(1) - \mathbf{C}_1^T \mathbb{E}f'''(0) = 0, \\ \mathbb{E}f''(1) - \mathbf{C}_2^T \mathbb{E}f''(0) = 0. \end{cases} \quad (3.7)$$

Since $B^4 = \mathbb{E}^{-1}\mathbb{M}$, the general solution of Eqs. (3.7) is given by

$$f(x) = (\sinh B\rho x - \sin B\rho x)\eta_1 + (\cosh B\rho x - \cos B\rho x)\eta_2 \\ + (\sinh B\rho x + \sin B\rho x)\eta_3 + (\cosh B\rho x + \cos B\rho x)\eta_4, \quad \eta_j \in \mathbb{C}^n.$$

Conditions $f(0) = f'(0) = 0$ imply that $\eta_3 = \eta_4 = 0$, and hence

$$f(x) = (\sinh B\rho x - \sin B\rho x)\eta_1 + (\cosh B\rho x - \cos B\rho x)\eta_2.$$

The other boundary conditions in (3.7) lead to the algebraic equations:

$$\begin{aligned} (\sinh B\rho - \sin B\rho)\eta_1 + (\cosh B\rho - \cos B\rho)\eta_2 &= 0, \\ B(\cosh B\rho - \cos B\rho)\eta_1 + B(\sinh B\rho + \sin B\rho)\eta_2 &= 0, \\ \mathbb{E}(B)^2[\sinh(B\rho) + \sin(B\rho)]\eta_1 + \mathbb{E}(B)^2[\cosh(B\rho) + \cos(B\rho)]\eta_2 - 2\mathbf{C}_2^T \mathbb{E}(B)^2\eta_2 &= 0, \\ \mathbb{E}(B)^3[\cosh(B\rho) + \cos(B\rho)]\eta_1 + (B)^3\mathbb{E}[\sinh(B\rho) - \sin(B\rho)]\eta_2 - 2\mathbf{C}_1^T \mathbb{E}(B)^3\eta_1 &= 0, \end{aligned}$$

which is equivalent to equations

$$(\sinh B\rho - \sin B\rho)\eta_1 + (\cosh B\rho - \cos B\rho)\eta_2 = 0, \quad (3.8)$$

$$(\cosh B\rho - \cos B\rho)\eta_1 + (\sinh B\rho + \sin B\rho)\eta_2 = 0, \quad (3.9)$$

$$\sin(B\rho)\eta_1 + \cos(B\rho)\eta_2 = (B)^{-2}\mathbb{E}^{-1}\mathbf{C}_2^T \mathbb{E}(B)^2\eta_2, \quad (3.10)$$

$$\cos(B\rho)\eta_1 - \sin(B\rho)\eta_2 = (B)^{-3}\mathbb{E}^{-1}\mathbf{C}_1^T \mathbb{E}(B)^3\eta_1. \quad (3.11)$$

From (3.8) and (3.9) we can get that η_j , $j = 1, 2$, satisfy the equation $[I - \cos B\rho \cosh B\rho]\eta_j = 0$. For any $k \in \mathbb{N}$ it holds that

$$[I - \cos B\rho \cosh B\rho]B^k\eta_j = 0, \quad j = 1, 2,$$

which implies that η_j , $j = 1, 2$, are the eigenvectors of B . Thus there exist some $s \in \{1, 2, \dots, n\}$ and number $\alpha_j \in \mathbb{C}$ such that $\eta_j = \alpha_j \hat{\eta}_s$, and hence $\rho \in \mathbb{R}$ satisfies the function equation

$$[1 - \cos \mu_s \rho \cosh \mu_s \rho] = 0. \quad (3.12)$$

From (3.8) we get that η_1 and η_2 are linearly dependent, and

$$\eta_2 = -\frac{\sinh \mu_s \rho - \sin \mu_s \rho}{\cosh \mu_s \rho - \cos \mu_s \rho} \eta_1.$$

Substituting it into (3.11) yields

$$\left[\cos \mu_s \rho + \sin \mu_s \rho \frac{\sinh \mu_s \rho - \sin \mu_s \rho}{\cosh \mu_s \rho - \cos \mu_s \rho} \right] \eta_1 = (B)^{-3}\mathbb{E}^{-1}\mathbf{C}_1^T \mathbb{E}(B)^3\eta_1.$$

Using (3.12), we get

$$\mathbb{E}^{-1}\mathbf{C}_1^T \mathbb{E}\eta_1 = \cot \mu_s \rho \sinh \mu_s \rho \eta_1. \quad (3.13)$$

Similarly, we can get from (3.10) that

$$\mathbb{E}^{-1}\mathbf{C}_2^T \mathbb{E}\eta_2 = -\cot \mu_s \rho \sinh \mu_s \rho \eta_2. \quad (3.14)$$

From (3.13) and (3.14) we can see that any one of the conditions 1)–3) implies $\eta_1 = \eta_2 = 0$. Therefore $f(x) = 0$, and hence $g(x) = 0$. The desired result follows. \square

Remark 3.1. Theorem 3.2 has an explanation in physics, that is, the different physical systems have different eigenvectors. From the mathematical point of view, Theorem 3.2 gives the uniqueness of zero solution of super-determined differential equations.

Theorem 3.3. Let \mathcal{H} and \mathcal{A} be defined as before, and \mathcal{A}_0 be defined by (3.1) and (3.2). Let $\sigma(\mathcal{A}_0) = \{\pm i\sqrt{\gamma_k} \mid k \in \mathbb{Z}\}$ and $\{\Phi_{\pm k} = (\varphi_k(x), \pm i\sqrt{\gamma_k}\varphi_k(x)), k \in \mathbb{N}\}$ be given by Theorem 3.1 where $\varphi_k(x)$ satisfies (3.3). Then the spectral relationship between \mathcal{A}_0 and \mathcal{A} is given by

$$\sigma(\mathcal{A}) \cap i\mathbb{R} = \{\pm i\sqrt{\gamma_k} \in \sigma(\mathcal{A}_0) \mid \mathbb{K}_1\varphi_k(1) - \Gamma_1\varphi_k'(1) = 0; \mathbb{K}_2\varphi_k'(1) - \Gamma_2\varphi_k(1) = 0\}. \quad (3.15)$$

Proof. Obviously, if there exists some $k \in \mathbb{N}$ such that

$$\mathbb{K}_1\varphi_k(1) - \Gamma_1\varphi_k'(1) = 0, \quad \mathbb{K}_2\varphi_k'(1) - \Gamma_2\varphi_k(1) = 0,$$

the function $\Phi_k = (\varphi_k(x), i\sqrt{\gamma_k}\varphi_k(x)) \in \mathcal{D}(\mathcal{A})$, and $\mathcal{A}\Phi_k = i\sqrt{\gamma_k}\Phi_k$. So $i\sqrt{\gamma_k} \in \sigma(\mathcal{A}) \cap i\mathbb{R}$.

Conversely, let $\lambda \in \sigma(\mathcal{A}) \cap i\mathbb{R}$ and $(f, g) \in \mathcal{D}(\mathcal{A})$ be an eigenvector associated with λ . Without loss of generality we can assume that $\lambda = i\rho^2$, $\rho \in \mathbb{R}$, then $g(x) = \lambda f(x)$ and $f(x)$ satisfies the equations

$$\begin{cases} \mathbb{E}f^{(4)}(x) - \rho^4\mathbb{M}f(x) = 0, \\ f(0) = \mathbf{C}_1 f(1), \quad f'(0) = \mathbf{C}_2 f'(1), \\ \mathbb{E}f'''(1) - \mathbf{C}_1^T \mathbb{E}f'''(0) = i\rho^2[\mathbb{K}_1 f(1) - \Gamma_1 f'(1)], \\ \mathbb{E}f''(1) - \mathbf{C}_2^T \mathbb{E}f''(0) = -i\rho^2[\mathbb{K}_2 f'(1) - \Gamma_2 f(1)]. \end{cases}$$

For any $v \in V_E^2(0, 1)$, we have

$$\begin{aligned} 0 &= \int_0^1 (\mathbb{E}f^{(4)}(x), v(x))_{\mathbb{C}^n} dx - \rho^4 \int_0^1 (\mathbb{M}f(x), v(x))_{\mathbb{C}^n} dx \\ &= (\mathbb{E}f'''(1), v(1))_{\mathbb{C}^n} - (\mathbb{E}f'''(0), v(0))_{\mathbb{C}^n} - (\mathbb{E}f''(1), v'(1))_{\mathbb{C}^n} - (\mathbb{E}f''(0), v'(1))_{\mathbb{C}^n} \\ &\quad + \int_0^1 (\mathbb{E}f''(x), v''(x))_{\mathbb{C}^n} dx - \rho^4 \int_0^1 (\mathbb{M}f(x), v(x))_{\mathbb{C}^n} dx \\ &= (\mathbb{E}f'''(1) - \mathbf{C}_1^T \mathbb{E}f'''(0), v(1))_{\mathbb{C}^n} - (\mathbb{E}f''(1) - \mathbf{C}_2^T \mathbb{E}f''(0), v'(1))_{\mathbb{C}^n} \\ &\quad + \int_0^1 (\mathbb{E}f''(x), v''(x))_{\mathbb{C}^n} dx - \rho^4 \int_0^1 (\mathbb{M}f(x), v(x))_{\mathbb{C}^n} dx \\ &= i\rho^2(\mathbb{K}_1 f(1) - \Gamma_1 f'(1), v(1))_{\mathbb{C}^n} + i\rho^2(\mathbb{K}_2 f'(1) - \Gamma_2 f(1), v'(1))_{\mathbb{C}^n} \\ &\quad + \int_0^1 (\mathbb{E}f''(x), v''(x))_{\mathbb{C}^n} dx - \rho^4 \int_0^1 (\mathbb{M}f(x), v(x))_{\mathbb{C}^n} dx. \end{aligned}$$

Taking $v \in V_E^2(0, 1) \cap \{v \in H_E^2(0, 1) \mid v(1) = v'(1) = 0\}$ leads to

$$\int_0^1 (\mathbb{E}f''(x), v''(x))_{\mathbb{C}^n} dx - \rho^4 \int_0^1 (\mathbb{M}f(x), v(x))_{\mathbb{C}^n} dx = 0.$$

The denseness argument asserts that

$$\int_0^1 (\mathbb{E}f''(x), v''(x))_{\mathbb{C}^n} dx - \rho^4 \int_0^1 (\mathbb{M}f(x), v(x))_{\mathbb{C}^n} dx \equiv 0.$$

Therefore, we have

$$(\mathbb{K}_1 f(1) - \Gamma_1 f'(1), v(1))_{\mathbb{C}^n} + (\mathbb{K}_2 f'(1) - \Gamma_2 f(1), v'(1))_{\mathbb{C}^n} = 0, \quad \forall v \in V_E^2(0, 1).$$

For fixed $f(x)$, we can take a $v \in V_E^2(0, 1)$ satisfying $v(1) = \mathbb{K}_1 f(1) - \Gamma_1 f'(1)$ and $v'(1) = \mathbb{K}_2 f'(1) - \Gamma_2 f(1)$, which will yield

$$\mathbb{K}_1 f(1) - \Gamma_1 f'(1) = 0, \quad \mathbb{K}_2 f'(1) - \Gamma_2 f(1) = 0.$$

Therefore, $(f(x), \lambda f(x)) \in \mathcal{D}(\mathcal{A}_0)$ and $\lambda \in \sigma(\mathcal{A}_0)$. The proof is then complete. \square

3.2. Asymptotic distribution of spectrum

To obtain the more detail spectral information of \mathcal{A} , we need only to consider the eigenvalue problem of \mathcal{A} . In what follows, we shall discuss eigenvalues of \mathcal{A} and their asymptotic distribution by using asymptotic analysis technique.

Let $\lambda \in \mathbb{C}$ be such that $\mathcal{A}Y = \lambda Y$ have a nonzero solution $Y = (f, g) \in \mathcal{D}(\mathcal{A})$. Then $g = \lambda f$ and f satisfies the following equations:

$$\begin{cases} \mathbb{E}f^{(4)}(x) = -\lambda^2 \mathbb{M}f(x), \\ f(0) = \mathbf{C}_1 f(1), \quad f'(0) = \mathbf{C}_2 f'(1), \\ \mathbb{E}f'''(1) - \mathbf{C}_1^T \mathbb{E}f'''(0) = \lambda \mathbb{K}_1 f(1) - \lambda \Gamma_1 f'(1), \\ \mathbb{E}f''(1) - \mathbf{C}_2^T \mathbb{E}f''(0) = -\lambda \mathbb{K}_2 f'(1) + \lambda \Gamma_2 f(1). \end{cases}$$

Set $\lambda = i\rho^2$, $B^4 = \mathbb{E}^{-1}\mathbb{M}$. We have

$$\begin{cases} f^{(4)}(x) = \rho^4 B^4 f(x), \\ f(0) = \mathbf{C}_1 f(1), \quad f'(0) = \mathbf{C}_2 f'(1), \\ \mathbb{E}f'''(1) - \mathbf{C}_1^T \mathbb{E}f'''(0) = i\rho^2 \mathbb{K}_1 f(1) - i\rho^2 \Gamma_1 f'(1), \\ \mathbb{E}f''(1) - \mathbf{C}_2^T \mathbb{E}f''(0) = -i\rho^2 \mathbb{K}_2 f'(1) + i\rho^2 \Gamma_2 f(1). \end{cases} \quad (3.16)$$

The general solution $f(x)$ of the differential equation in (3.16) is of the form

$$f(x) = e^{\rho Bx} \eta_1 + e^{i\rho Bx} \eta_2 + e^{-\rho Bx} \eta_3 + e^{-i\rho Bx} \eta_4 = \sum_{j=1}^4 e^{\omega_j \rho Bx} \eta_j, \quad \omega_j^4 = 1$$

where $\omega_1 = 1$, $\omega_2 = i$, $\omega_3 = -1$, $\omega_4 = -i$, $\eta_j \in \mathbb{C}^n$, $j = 1, 2, 3, 4$. Substituting it into the boundary conditions in (3.16) yields

$$\begin{cases} \sum_{j=1}^4 (I - \mathbf{C}_1 e^{\omega_j \rho B}) \eta_j = 0, \\ \sum_{j=1}^4 (I - \mathbf{C}_2 e^{\omega_j \rho B}) \omega_j B \eta_j = 0, \\ \sum_{j=1}^4 [\mathbb{E} e^{\omega_j \rho B} \omega_j^3 \rho^3 B^3 \eta_j - \mathbf{C}_1^T \mathbb{E} \omega_j^3 \rho^3 B^3 \eta_j - i\rho^2 \mathbb{K}_1 e^{\omega_j \rho B} \eta_j + i\rho^2 \Gamma_1 e^{\omega_j \rho B} \omega_j \rho B \eta_j] = 0, \\ \sum_{j=1}^4 [\mathbb{E} e^{\omega_j \rho B} \omega_j^2 \rho^2 B^2 \eta_j - \mathbf{C}_2^T \mathbb{E} \omega_j^2 \rho^2 B^2 \eta_j + i\rho^2 \mathbb{K}_2 e^{\omega_j \rho B} \omega_j \rho B \eta_j - i\rho^2 \Gamma_2 e^{\omega_j \rho B} \eta_j] = 0. \end{cases} \quad (3.17)$$

Denote

$$U_j(\rho) = \omega_j^3 \rho \mathbb{E} e^{\omega_j \rho B} B^3 - \omega_j^3 \rho \mathbf{C}_1^T \mathbb{E} B^3 - i \mathbb{K}_1 e^{\omega_j \rho B} + i \omega_j \rho \Gamma_1 e^{\omega_j \rho B} B, \quad (3.18)$$

$$V_j(\rho) = \omega_j^2 \mathbb{E} e^{\omega_j \rho B} B^2 - \omega_j^2 \mathbf{C}_2^T \mathbb{E} B^2 + i \omega_j \rho \mathbb{K}_2 e^{\omega_j \rho B} B - i \Gamma_2 e^{\omega_j \rho B}, \quad (3.19)$$

the algebraic equations (3.17) have a nonzero solution if and only if

$$\Delta(\rho) = \begin{vmatrix} (I - \mathbf{C}_1 e^{\rho B}) & (I - \mathbf{C}_1 e^{i\rho B}) & (I - \mathbf{C}_1 e^{-\rho B}) & (I - \mathbf{C}_1 e^{-i\rho B}) \\ (I - \mathbf{C}_2 e^{\rho B}) B & i(I - \mathbf{C}_2 e^{i\rho B}) B & -(I - \mathbf{C}_2 e^{-\rho B}) B & -i(I - \mathbf{C}_2 e^{-i\rho B}) B \\ U_1(\rho) & U_2(\rho) & U_3(\rho) & U_4(\rho) \\ V_1(\rho) & V_2(\rho) & V_3(\rho) & V_4(\rho) \end{vmatrix} = 0. \quad (3.20)$$

Clearly, $\Delta(\rho)$ is an entire function of finite exponential type. Therefore, we have the following result.

Theorem 3.4. Let \mathcal{H} and \mathcal{A} be defined as before, and let $\Delta(\rho)$ be defined by (3.20). Then

$$\sigma(\mathcal{A}) = \{\lambda = i\rho^2 \mid \Delta(\rho) = 0, \rho \in \mathbb{C}\}. \quad (3.21)$$

We are now in a position to analyze the zeros of $\Delta(\rho)$. To this end, we divide the complex plane into eight sectors

$$\Omega_j = \left\{ \rho \in \mathbb{C} \mid \theta = \arg \rho \in \left(\frac{(j-1)\pi}{4}, \frac{j\pi}{4} \right) \right\}, \quad j = 1, 2, 3, 4, 5, 6, 7, 8.$$

For $\rho \in \Omega_j$, we have

$$\begin{aligned} \omega_1 \rho &= \rho \in \Omega_j; & \omega_2 \rho &= i\rho \in \Omega_{j+2}; \\ \omega_3 \rho &= -\rho \in \Omega_{j+4}; & \omega_4 \rho &= -i\rho \in \Omega_{j+6}. \end{aligned}$$

When $\rho \in \Omega_1 \cup \Omega_2$, it holds that

$$\Re \omega_1 \rho > 0, \quad \Re \omega_2 \rho < 0, \quad \Re \omega_3 \rho < 0, \quad \Re \omega_4 \rho > 0.$$

So when $\rho \in \Omega_1 \cup \Omega_2$ satisfying conditions

$$\Re \omega_1 \rho \rightarrow +\infty, \quad \Re \omega_2 \rho \rightarrow -\infty, \quad \Re \omega_3 \rho \rightarrow -\infty, \quad \Re \omega_4 \rho \rightarrow +\infty \quad (3.22)$$

as $|\rho| \rightarrow \infty$, we have estimates

$$\begin{aligned} U_1(\rho) &= \rho \mathbb{E} \mathbb{E} e^{\rho B} B^3 - \rho \mathbf{C}_1^T \mathbb{E} \mathbb{B}^3 - i \mathbb{K}_1 e^{\rho B} + i \rho \Gamma_1 e^{\rho B} B = [\mathbb{E} \mathbb{B}^2 + i \Gamma_1]_0 \rho e^{\rho B} B, \\ V_1(\rho) &= \mathbb{E} \mathbb{E} e^{\rho B} B^2 - \mathbf{C}_2^T \mathbb{E} \mathbb{B}^2 + i \rho \mathbb{K}_2 e^{\rho B} B - i \Gamma_1 e^{\rho B} = [\mathbb{E} \mathbb{B}^2 + i \rho \mathbb{K}_2 B + i \Gamma_1]_0 e^{\rho B}, \\ U_2(\rho) &= -i \rho \mathbb{E} \mathbb{E} e^{i \rho B} B^3 + i \rho \mathbf{C}_1^T \mathbb{E} \mathbb{B}^3 - i \mathbb{K}_1 e^{i \rho B} - \rho \Gamma_1 e^{i \rho B} B = i \rho [\mathbf{C}_1^T \mathbb{E} \mathbb{B}^3]_0, \\ V_2(\rho) &= -\mathbb{E} \mathbb{E} e^{i \rho B} B^2 + \mathbf{C}_2^T \mathbb{E} \mathbb{B}^2 - \rho \mathbb{K}_2 e^{i \rho B} B - i \Gamma_1 e^{i \rho B} = [\mathbf{C}_2^T \mathbb{E} \mathbb{B}^2]_0, \\ U_3(\rho) &= -\rho \mathbb{E} \mathbb{E} e^{-\rho B} B^3 + \rho \mathbf{C}_1^T \mathbb{E} \mathbb{B}^3 - i \mathbb{K}_1 e^{-\rho B} - i \rho \Gamma_1 e^{-\rho B} B = \rho [\mathbf{C}_1^T \mathbb{E} \mathbb{B}^3]_0, \\ V_3(\rho) &= \mathbb{E} \mathbb{E} e^{-\rho B} B^2 - \mathbf{C}_2^T \mathbb{E} \mathbb{B}^2 - i \rho \mathbb{K}_2 e^{-\rho B} B - i \Gamma_1 e^{-\rho B} = -[\mathbf{C}_2^T \mathbb{E} \mathbb{B}^2]_0, \\ U_4(\rho) &= i \rho \mathbb{E} \mathbb{E} e^{-i \rho B} B^3 - i \rho \mathbf{C}_1^T \mathbb{E} \mathbb{B}^3 - i \mathbb{K}_1 e^{-i \rho B} + \rho \Gamma_1 e^{-i \rho B} B = i [\mathbb{E} \mathbb{B}^2 - i \Gamma_1]_0 e^{-i \rho B} \rho B, \\ V_4(\rho) &= -\mathbb{E} \mathbb{E} e^{-i \rho B} B^2 + \mathbf{C}_2^T \mathbb{E} \mathbb{B}^2 + \rho \mathbb{K}_2 e^{-i \rho B} B - i \Gamma_1 e^{-i \rho B} = [-\mathbb{E} \mathbb{B}^2 + \rho \mathbb{K}_2 B - \Gamma_1]_0 e^{-i \rho B} \end{aligned}$$

where $[S]_0$ denotes the asymptotic expression $[S]_0 = S + O(\rho^{-1})$. Consequently,

$$\begin{aligned} \Delta(\rho) &= \begin{vmatrix} -[\mathbf{C}_1]_0 e^{\rho B} & [I]_0 & [I]_0 & -[\mathbf{C}_1]_0 e^{-i \rho B} \\ -[\mathbf{C}_2]_0 e^{\rho B} B & i[I]_0 B & -[I]_0 B & i[\mathbf{C}_2]_0 e^{-i \rho B} B \\ [\mathbb{E} \mathbb{B}^2 + i \Gamma_1]_0 \rho e^{\rho B} B & i \rho [\mathbf{C}_1^T \mathbb{E} \mathbb{B}^3]_0 & \rho [\mathbf{C}_1^T \mathbb{E} \mathbb{B}^3]_0 & i[\mathbb{E} \mathbb{B}^2 - i \Gamma_1]_0 e^{-i \rho B} \rho B \\ [\mathbb{E} \mathbb{B}^2 + i \rho \mathbb{K}_2 B + i \Gamma_1]_0 e^{\rho B} & [\mathbf{C}_2^T \mathbb{E} \mathbb{B}^2]_0 & -[\mathbf{C}_2^T \mathbb{E} \mathbb{B}^2]_0 & [-\mathbb{E} \mathbb{B}^2 + \rho \mathbb{K}_2 B - \Gamma_1]_0 e^{-i \rho B} \end{vmatrix} \\ &= \rho^n e^{\rho(1-i)tr(B)} \begin{vmatrix} -[\mathbf{C}_1]_0 & [I]_0 & [I]_0 & -[\mathbf{C}_1]_0 \\ -[\mathbf{C}_2]_0 B & i[I]_0 B & -[I]_0 B & i[\mathbf{C}_2]_0 B \\ [\mathbb{E} \mathbb{B}^2 + i \Gamma_1]_0 B & i[\mathbf{C}_1^T \mathbb{E} \mathbb{B}^2]_0 B & [\mathbf{C}_1^T \mathbb{E} \mathbb{B}^2]_0 B & i[\mathbb{E} \mathbb{B}^2 - i \Gamma_1]_0 B \\ [\mathbb{E} \mathbb{B}^2 + i \rho \mathbb{K}_2 B + i \Gamma_1]_0 & [\mathbf{C}_2^T \mathbb{E} \mathbb{B}^2]_0 & -[\mathbf{C}_2^T \mathbb{E} \mathbb{B}^2]_0 & [-\mathbb{E} \mathbb{B}^2 + \rho \mathbb{K}_2 B - \Gamma_1]_0 \end{vmatrix} \end{aligned}$$

where $tr(B)$ denotes the trace of matrix B .

Let $\text{rank}(\mathbb{K}_2) = r$. Without loss of generality we can assume that \mathbb{K}_2 is of the form

$$\mathbb{K}_2 = \begin{pmatrix} K_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

where K_{11} is an $r \times r$ nonsingular matrix. According to this pattern we decompose the matrices

$$\begin{aligned} [\mathbb{E} \mathbb{B}^2 + i \rho \mathbb{K}_2 B + i \Gamma_1]_0 &= \begin{pmatrix} E(b)_{11} & E(b)_{12} \\ E(b)_{21} & E(b)_{22} \end{pmatrix} + i \rho \begin{pmatrix} K(b)_{11} & K(b)_{12} \\ 0 & 0 \end{pmatrix} + i \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}, \\ [\mathbf{C}_2^T \mathbb{E} \mathbb{B}^2]_0 &= \begin{pmatrix} CEB_{11} & CEB_{12} \\ CEB_{21} & CEB_{22} \end{pmatrix}, \\ -[\mathbf{C}_2^T \mathbb{E} \mathbb{B}^2]_0 &= -\begin{pmatrix} CEB_{11} & CEB_{12} \\ CEB_{21} & CEB_{22} \end{pmatrix}, \\ [-\mathbb{E} \mathbb{B}^2 + \rho \mathbb{K}_2 B - \Gamma_1]_0 &= -\begin{pmatrix} E(b)_{11} & E(b)_{12} \\ E(b)_{21} & E(b)_{22} \end{pmatrix} + \rho \begin{pmatrix} K(b)_{11} & K(b)_{12} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}. \end{aligned}$$

Thus, we have

$$\Delta(\rho) = \rho^{n+r} e^{\rho(1-i)\text{tr}(B)} \text{tr}^2(B) \times \begin{vmatrix} -[\mathbf{C}_1]_0 & [I]_0 & [I]_0 & -[\mathbf{C}_1]_0 \\ -[\mathbf{C}_2]_0 & i[I]_0 & -[I]_0 & i[\mathbf{C}_2]_0 \\ [\mathbb{E}B^2 + i\Gamma_1]_0 & i[\mathbf{C}_1^T \mathbb{E}B^2]_0 & [\mathbf{C}_1^T \mathbb{E}B^2]_0 & i[\mathbb{E}B^2 - i\Gamma_1]_0 \\ \begin{pmatrix} K(b)_{11} & K(b)_{12} \\ E(b)_{21} + i\Gamma_{21} & E(b)_{22} + i\Gamma_{22} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ CEB_{21} & CEB_{22} \end{pmatrix} & -\begin{pmatrix} 0 & 0 \\ CEB_{21} & CEB_{22} \end{pmatrix} & -\begin{pmatrix} -K(b)_{11} & -K(b)_{12} \\ E(b)_{21} + \Gamma_{21} & E(b)_{22} + \Gamma_{22} \end{pmatrix} \end{vmatrix}.$$

Therefore,

$$\begin{aligned} & \lim_{|\rho| \rightarrow \infty, \rho \in \Omega_1 \cup \Omega_2} \frac{\Delta(\rho)}{\rho^{n+r} e^{(1-i)\rho \text{tr}(B)} \text{tr}^2(B)} \\ &= \begin{vmatrix} -\mathbf{C}_1 & I & I & -\mathbf{C}_1 \\ -\mathbf{C}_2 & iI & -I & i\mathbf{C}_2 \\ (\mathbb{E}B^2 + i\Gamma_1) & i\mathbf{C}_1^T \mathbb{E}B^2 & \mathbf{C}_1^T \mathbb{E}B^2 & (i\mathbb{E}B^2 + \Gamma_1) \\ \begin{pmatrix} K(b)_{11} & K(b)_{12} \\ E(b)_{21} + i\Gamma_{21} & E(b)_{22} + i\Gamma_{22} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ CEB_{21} & CEB_{22} \end{pmatrix} & -\begin{pmatrix} 0 & 0 \\ CEB_{21} & CEB_{22} \end{pmatrix} & -\begin{pmatrix} -K(b)_{11} & -K(b)_{12} \\ E(b)_{21} + \Gamma_{21} & E(b)_{22} + \Gamma_{22} \end{pmatrix} \end{vmatrix} \\ &= \begin{vmatrix} -\mathbf{C}_1 & I & I & -\mathbf{C}_1 \\ -\mathbf{C}_2 & iI & -I & i\mathbf{C}_2 \\ \begin{pmatrix} E(b)_{11} + i\Gamma_{11} & E(b)_{12} + i\Gamma_{12} \\ E(b)_{21} + i\Gamma_{21} & E(b)_{22} + i\Gamma_{22} \end{pmatrix} & i\begin{pmatrix} CEB_{11} & CEB_{12} \\ CEB_{21} & CEB_{22} \end{pmatrix} & \begin{pmatrix} CEB_{11} & CEB_{12} \\ CEB_{21} & CEB_{22} \end{pmatrix} & \begin{pmatrix} iE(b)_{11} + \Gamma_{11} & iE(b)_{12} + \Gamma_{12} \\ iE(b)_{21} + \Gamma_{21} & iE(b)_{22} + \Gamma_{22} \end{pmatrix} \\ \begin{pmatrix} K(b)_{11} & K(b)_{12} \\ E(b)_{21} + i\Gamma_{21} & E(b)_{22} + i\Gamma_{22} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ CEB_{21} & CEB_{22} \end{pmatrix} & -\begin{pmatrix} 0 & 0 \\ CEB_{21} & CEB_{22} \end{pmatrix} & -\begin{pmatrix} -K(b)_{11} & -K(b)_{12} \\ E(b)_{21} + \Gamma_{21} & E(b)_{22} + \Gamma_{22} \end{pmatrix} \end{vmatrix}. \end{aligned}$$

A straightforward calculation shows

$$\lim_{|\rho| \rightarrow \infty, \rho \in \Omega_1 \cup \Omega_2} \frac{\Delta(\rho)}{\rho^{n+r} e^{(1-i)\rho \text{tr}(B)} \text{tr}^2(B)} \neq 0. \quad (3.23)$$

When $\rho \in \Omega_3 \cup \Omega_4$, we have

$$\Re \omega_1 \rho < 0, \quad \Re \omega_2 \rho < 0, \quad \Re \omega_3 \rho > 0, \quad \Re \omega_4 \rho > 0.$$

So when $\rho \in \Omega_3 \cup \Omega_4$ satisfying conditions

$$\Re \omega_1 \rho \rightarrow -\infty, \quad \Re \omega_2 \rho \rightarrow -\infty, \quad \Re \omega_3 \rho \rightarrow +\infty, \quad \Re \omega_4 \rho \rightarrow +\infty, \quad (3.24)$$

we have

$$\begin{aligned} U_1(\rho) &= \rho \mathbb{E} \mathbb{E} e^{\rho B} B^3 - \rho \mathbf{C}_1^T \mathbb{E} B^3 - i \mathbb{K}_1 e^{\rho B} + i \rho \Gamma_1 e^{\rho B} B = -\rho [\mathbf{C}_1^T \mathbb{E} B^3]_0, \\ V_1(\rho) &= \mathbb{E} \mathbb{E} e^{\rho B} B^2 - \mathbf{C}_2^T \mathbb{E} B^2 + i \rho \mathbb{K}_2 e^{\rho B} B - i \Gamma_1 e^{\rho B} = -[\mathbf{C}_2^T \mathbb{E} B^2]_0, \\ U_2(\rho) &= -i \rho \mathbb{E} \mathbb{E} e^{i \rho B} B^3 + i \rho \mathbf{C}_1^T \mathbb{E} B^3 - i \mathbb{K}_1 e^{i \rho B} - \rho \Gamma_1 e^{i \rho B} B = i \rho [\mathbf{C}_1^T \mathbb{E} B^3]_0, \\ V_2(\rho) &= -\mathbb{E} \mathbb{E} e^{i \rho B} B^2 + \mathbf{C}_2^T \mathbb{E} B^2 - \rho \mathbb{K}_2 e^{i \rho B} B - i \Gamma_1 e^{i \rho B} = [\mathbf{C}_2^T \mathbb{E} B^2]_0, \\ U_3(\rho) &= -\rho \mathbb{E} \mathbb{E} e^{-\rho B} B^3 + \rho \mathbf{C}_1^T \mathbb{E} B^3 - i \mathbb{K}_1 e^{-\rho B} - i \rho \Gamma_1 e^{-\rho B} B = -[\mathbb{E} B^2 + i \Gamma_1]_0 \rho e^{-\rho B}, \\ V_3(\rho) &= \mathbb{E} \mathbb{E} e^{-\rho B} B^2 - \mathbf{C}_2^T \mathbb{E} B^2 - i \rho \mathbb{K}_2 e^{-\rho B} B - i \Gamma_1 e^{-\rho B} = [\mathbb{E} B^2 - i \rho \mathbb{K}_2 B - i \Gamma_1]_0 e^{-\rho B}, \\ U_4(\rho) &= i \rho \mathbb{E} \mathbb{E} e^{-i \rho B} B^3 - i \rho \mathbf{C}_1^T \mathbb{E} B^3 - i \mathbb{K}_1 e^{-i \rho B} + \rho \Gamma_1 e^{-i \rho B} B = i [\mathbb{E} B^2 - i \Gamma_1]_0 e^{-i \rho B} \rho B, \\ V_4(\rho) &= -\mathbb{E} \mathbb{E} e^{-i \rho B} B^2 + \mathbf{C}_2^T \mathbb{E} B^2 + \rho \mathbb{K}_2 e^{-i \rho B} B - i \Gamma_1 e^{-i \rho B} = -[\mathbb{E} B^2 - \rho \mathbb{K}_2 B + i \Gamma_1]_0 e^{-i \rho B}. \end{aligned}$$

Consequently,

$$\Delta(\rho) = \begin{vmatrix} [I] & [I] & -[\mathbf{C}_1] e^{-\rho B} & -[\mathbf{C}_1] e^{-i \rho B} \\ [I] B & i[I] B & [\mathbf{C}_2] e^{-\rho B} B & i[\mathbf{C}_2] e^{-i \rho B} B \\ -\rho [\mathbf{C}_1^T \mathbb{E} B^3] & i \rho [\mathbf{C}_1^T \mathbb{E} B^3] & -[\mathbb{E} B^2 + i \Gamma_1] \rho e^{-\rho B} & i [\mathbb{E} B^2 - i \Gamma_1] e^{-i \rho B} \rho B \\ -[\mathbf{C}_2^T \mathbb{E} B^2] & [\mathbf{C}_2^T \mathbb{E} B^2] & [\mathbb{E} B^2 - i \rho \mathbb{K}_2 B - i \Gamma_1]_0 e^{-\rho B} & -[\mathbb{E} B^2 - \rho \mathbb{K}_2 B + i \Gamma_1]_0 e^{-i \rho B} \end{vmatrix}.$$

A similar calculation shows

$$\lim_{|\rho| \rightarrow \infty, \rho \in \Omega_3 \cup \Omega_4} \frac{\Delta(\rho)}{\rho^{n+r} e^{-(1+i)\rho \text{tr}(B)} \text{tr}^2(B)} \neq 0. \quad (3.25)$$

Finally, when $\rho \in \bigcup_{k=5}^8 \Omega_k$, set $r \in \bigcup_{k=1}^4 \Omega_k$, then we have

$$\rho = \omega_3 r, \quad \omega_3 = -1.$$

In this case, we have the equality $\Delta(\rho) = \Delta(-r)$.

Remark 3.2. The condition (2.2) plays an important role in the calculation of (3.23) and (3.25). If (2.2) does not hold, then there may exist a sequence $\{\rho_k; k \in \mathbb{N}\} \subset \Omega_1 \cup \Omega_2$ or $\{\rho_k; k \in \mathbb{N}\} \subset \Omega_3 \cup \Omega_4$ such that $\lim_{|\rho| \rightarrow \infty, \rho \in \Omega_1 \cup \Omega_2} \frac{\Delta(\rho)}{\rho^{n+r} e^{(1-i)\rho \operatorname{tr}(B)} \operatorname{tr}^2(B)} = 0$. Please see an example given in (1.3).

Theorem 3.5. Let \mathcal{H} and \mathcal{A} be defined as before, then there is a positive constant h such that

$$-h \leq \Re \lambda \leq 0, \quad \forall \lambda \in \sigma(\mathcal{A}). \quad (3.26)$$

In particular, for $|\Re \lambda| > h$, $\lambda = i\rho^2$, there exist positive constants d_1 and d_2 such that

$$\begin{cases} d_1 |\rho^{n+r}| |e^{(1-i)\rho \operatorname{tr}(B)}| \leq |\Delta(\rho)| \leq d_2 |\rho^{n+r}| |e^{(1-i)\rho \operatorname{tr}(B)}|, & \Re \lambda < -h, \\ d_1 |\rho^{2n}| |e^{(-1-i)\rho \operatorname{tr}(B)}| \leq |\Delta(\rho)| \leq d_2 |\rho^{2n}| |e^{(-1-i)\rho \operatorname{tr}(B)}|, & \Re \lambda > h. \end{cases} \quad (3.27)$$

Proof. Since \mathcal{A} is dissipative, we have $\Re \lambda \leq 0$. We shall prove that there exists $h > 0$ such that $-h < \Re \lambda \leq 0$ for any $\lambda \in \sigma(\mathcal{A})$. If it is not true, there is a sequence $\{\lambda_k, k \in \mathbb{N}\} \subset \sigma(\mathcal{A})$ such that

$$\lim_{n \rightarrow \infty} \Re \lambda_k = -\infty.$$

Without loss of the generality we can assume that $\Im \lambda_k > 0$. We have

$$\lambda_k = i\rho_k^2 = |\rho_k|^2 (-\sin 2\theta_k + i \cos 2\theta_k), \quad \theta_k = \arg \rho_k \in \left(0, \frac{\pi}{4}\right).$$

Set

$$\Re \lambda_k = -a_k, \quad \Im \lambda_k = b_k.$$

We choose a continuous function $\psi(s)$ on $(0, \infty)$ such that

- 1) $\psi(s)$ is positive and nondecreasing, and satisfies $\lim_{s \rightarrow \infty} \psi(s) = +\infty$;
- 2) $\psi(s)$ satisfies condition

$$\psi^2 \left(\sqrt{\left(\frac{a_k}{2}\right)^2 + b_k^2} + b_k \right) < \sqrt{\left(\frac{a_k}{2}\right)^2 + b_k^2} - b_k$$

for all $k \in \mathbb{N}$ large enough.

In the region Ω_1 , we define a curve $\rho(s)$ by

$$\rho(s) = s + i\psi(s), \quad s \in (0, \infty), \quad \sin \theta(s) = \frac{\psi(s)}{|\rho(s)|}.$$

As $s \rightarrow +\infty$, we have

$$\begin{cases} \Re \omega_1 \rho(s) = s \rightarrow +\infty, \\ \Re \omega_2 \rho(s) = -\psi(s) \rightarrow -\infty, \\ \Re \omega_3 \rho(s) = -s \rightarrow -\infty, \\ \Re \omega_4 \rho(s) = \psi(s) \rightarrow +\infty. \end{cases}$$

For any θ satisfying $\theta(s) \leq \theta \leq \frac{\pi}{4}$, we have $\rho = |\rho|e^{i\theta} \in \Omega_1$ and

$$\begin{cases} \Re \omega_1 \rho = |\rho| \cos \theta > |\rho| \cos \frac{\pi}{4} \rightarrow +\infty, \\ \Re \omega_2 \rho = -|\rho| \sin \theta \leq -|\rho| \sin \theta(s) \rightarrow -\infty, \\ \Re \omega_3 \rho = -|\rho| \cos \theta < -|\rho| \cos \frac{\pi}{4} \rightarrow -\infty, \\ \Re \omega_4 \rho = |\rho| \sin \theta > |\rho| \sin \theta(s) \rightarrow +\infty. \end{cases}$$

The conditions in (3.22) are fulfilled, so it holds that

$$\lim_{|\rho| \rightarrow \infty, \rho \in \Omega_1} \frac{\Delta(\rho)}{\rho^{2n} e^{-i\rho \operatorname{tr}(B)} e^{\rho \operatorname{tr}(B)}} \neq 0, \quad \arg \rho \in \left(\theta(s), \frac{\pi}{4}\right). \quad (3.28)$$

On the other hand, for any $s > 0$, we consider a curve in λ -plane

$$\lambda(s) = i\rho^2(s) = -2s\psi(s) + i[s^2 - \psi^2(s)], \quad s \in (0, \infty).$$

Let $\Re\lambda(s) = -2s\psi(s) = \Re\lambda_k$, then $s\psi(s) = \frac{a_k}{2}$. Set

$$x_k^2 = \sqrt{\left(\frac{a_k}{2}\right)^2 + b_k^2} + b_k,$$

we get from the property of ψ that

$$x_k^2 \psi^2(x_k) < \left(\sqrt{\left(\frac{a_k}{2}\right)^2 + b_k^2} + b_k\right) \left(\sqrt{\left(\frac{a_k}{2}\right)^2 + b_k^2} - b_k\right) = \left(\frac{a_k}{2}\right)^2 = s^2 \psi^2(s).$$

Since the function $s\psi(s)$ is also nondecreasing, we have

$$s^2 \geq x_k^2 = \sqrt{\left(\frac{a_k}{2}\right)^2 + b_k^2} + b_k.$$

Thus we can get

$$\begin{aligned} \Im\lambda(s) - 2b_k &= s^2 - \psi^2(s) - 2b_k = s^2 - \left(\frac{a_k}{2s}\right)^2 - 2b_k \\ &= \frac{1}{s^2} \left(s^4 - \left(\frac{a_k}{2}\right)^2 - 2b_k s^2 \right) \\ &= \frac{1}{s^2} \left(s^2 - b_k - \sqrt{\left(\frac{a_k}{2}\right)^2 + b_k^2} \right) \left(s^2 + \sqrt{\left(\frac{a_k}{2}\right)^2 + b_k^2} - b_k \right) > 0. \end{aligned}$$

So,

$$\lambda_k = i\rho_k^2 \in \left\{ \lambda = i\rho^2 \mid \theta = \arg \rho \in \left(\theta(s), \frac{\pi}{4} \right) \right\}.$$

Since $\lambda_k = i\rho_k^2$, $k \in \mathbb{N}$, are the eigenvalues of \mathcal{A} , Theorem 3.4 reads $\Delta(\rho_k) = 0$, $\forall k \in \mathbb{N}$. This contradicts (3.28). Therefore, there exists a positive constant h such that $|\Im\lambda| \leq h$, $\forall \lambda \in \sigma(\mathcal{A})$.

Based on the above discussion, the inequality (3.27) follows from (3.23) and (3.25). \square

To obtain the detail distribution of zeros of $\Delta(\rho)$, we need the following notion (see [29, Definition II.1.17, II.1.27, pp. 52–61]).

Definition 3.1. A set σ is said to be separable if $\inf_{\lambda, \mu \in \sigma, \lambda \neq \mu} |\lambda - \mu| > 0$. Let S be an infinite set. S is said to be a union of finitely many separable sets if there exist an integer N and separable sets V_j , $j = 1, 2, \dots, N$, such that

$$S = \bigcup_{j=1}^N V_j.$$

From definition we see that S is finite unification of separated sets if and only if there exist a sequence of bounded open sets, $\{O_p, p \in \mathbb{N}\}$, and an integer N such that

$$S \subset \bigcup_{p=1}^{\infty} O_p, \quad \inf_{p, r \in \mathbb{N}, p \neq r} \text{dist}(O_p, O_r) > 0, \quad \text{and} \quad \sup_{p \in \mathbb{N}} \# \{O_p \cap S\} \leq N$$

where $\#O$ denotes the number of elements in set O (taking the multiplicity into account).

Definition 3.2. An entire function f of the finite exponential type is said to be the sine type if

- (a) the zeros of f lie in a strip $\{z \in \mathbb{C} \mid |y| \leq h, z = x + iy\}$ for some $h > 0$;
- (b) there is $y_0 \in \mathbb{R}$ such that $|f(x + iy_0)| \asymp 1$ holds for $x \in \mathbb{R}$.

For the sine-type function, the following result holds (see [29, Proposition 11.1.28, p. 61]).

Proposition 3.1 (Levin Theorem). If f is a sine-type function, then its zero set is a union of finitely many separable sets.

To apply the Levin Theorem to our case, we note that $\Delta(\rho)$ is an entire function of finite exponential type in ρ . From (3.27) we see that $\Delta(\rho)$ has a property similar to the sine-type function nearly lines $\arg \rho = 0, \frac{\pi}{2}$ in the regions $\arg \rho \in (-\frac{\pi}{4}, \frac{\pi}{4})$ and $\arg \rho \in (\frac{\pi}{4}, \frac{3\pi}{4})$, respectively. We can use the Levin Theorem in each region. Therefore, based on the result in Theorem 3.4, Theorem 3.5 and Proposition 3.1, we have the following corollary.

Corollary 3.1. *Let \mathcal{A} be defined as before. Then $\sigma(\mathcal{A})$ is a union of finitely many separable set, and hence the multiplicities of \mathcal{A} are uniformly bounded.*

4. Completeness and basis property of root vectors

In this section we shall discuss the completeness and basis property of the root vectors of \mathcal{A} . The completeness and the basis generation of the root vectors have been a hot topic and an important content of linear operator theory. Since inseparable property of the spectrum of \mathcal{A} usually implies that the root vectors do not constitute a basis in the sense of Schauder basis, we shall discuss the basis property with parentheses.

4.1. Completeness of roots vectors

In this subsection, we study the completeness of the root vectors of \mathcal{A} . We begin with considering a non-homogeneous boundary value problem with nonzero parameter λ

$$\begin{cases} \lambda f(x) - g(x) = 0, & x \in (0, 1), \\ \lambda g(x) + \mathbb{M}^{-1}\mathbb{E}\mathbb{I}f^{(4)}(x) = 0, & x \in (0, 1), \\ f(0) = \mathbf{C}_1 f(1), & f'(0) = \mathbf{C}_2 f'(1), \\ \mathbb{E}\mathbb{I}f^{(3)}(1) - \mathbf{C}_1^T \mathbb{E}\mathbb{I}f^{(3)}(0) = \mathbb{K}_1 g(1) - \Gamma_1 g'(1) + \xi_1, \\ \mathbb{E}\mathbb{I}f''(1) - \mathbf{C}_2^T \mathbb{E}\mathbb{I}f''(0) = -\mathbb{K}_2 g'(1) + \Gamma_2 g(1) + \xi_2. \end{cases} \quad (4.1)$$

Obviously, $g(x) = \lambda f(x)$ and $f(x)$ satisfies the following equations with parameter λ

$$\begin{cases} \lambda^2 \mathbb{M}f(x) + \mathbb{E}\mathbb{I}f^{(4)}(x) = 0, & x \in (0, 1), \\ f(0) = \mathbf{C}_1 f(1), & f'(0) = \mathbf{C}_2 f'(1), \\ \mathbb{E}\mathbb{I}f^{(3)}(1) - \mathbf{C}_1^T \mathbb{E}\mathbb{I}f^{(3)}(0) = \lambda [\mathbb{K}_1 f(1) - \Gamma_1 f'(1)] + \xi_1, \\ \mathbb{E}\mathbb{I}f''(1) - \mathbf{C}_2^T \mathbb{E}\mathbb{I}f''(0) = -\lambda [\mathbb{K}_2 f'(1) - \Gamma_2 f(1)] + \xi_2. \end{cases} \quad (4.2)$$

Theorem 4.1. *If $\lambda \in \rho(\mathcal{A})$, Eqs. (4.2) have uniquely a solution $f(x)$ for any given $\xi_1, \xi_2 \in \mathbb{C}^n$. In particular, there exists a positive constant M such that when $\lambda \in \mathbb{R}_- \cap \rho(\mathcal{A})$, the solution of (4.1), $(f, g) = (f, \lambda f)$, has norm estimate in \mathcal{H}*

$$\|(f, g)\|_{\mathcal{H}}^2 \leq M \|(\xi_1, \xi_2)\|_{\mathbb{C}^{2n}}^2.$$

Proof. To prove the desired result, we complete the proof by three steps.

Step 1. If $\lambda \in \rho(\mathcal{A})$, (4.2) has uniquely a solution $f(x)$ for any given $\xi_1, \xi_2 \in \mathbb{C}^n$.

Set $\lambda = i\rho^2$ and $B^4 = \mathbb{E}\mathbb{I}^{-1}\mathbb{M}$. The general solution of the differential equation (4.2) has the form

$$f(x) = e^{\rho Bx} \eta_1 + e^{i\rho Bx} \eta_2 + e^{-\rho Bx} \eta_3 + e^{-i\rho Bx} \eta_4 = \sum_{j=1}^4 e^{\omega_j \rho Bx} \eta_j, \quad \omega_j^4 = 1, \quad \eta_j \in \mathbb{C}^n. \quad (4.3)$$

Substituting (4.3) into the boundary conditions in (4.2) yields

$$\begin{cases} \sum_{j=1}^4 (I - \mathbf{C}_1 e^{\omega_j \rho B}) \eta_j = 0, \\ \sum_{j=1}^4 (I - \mathbf{C}_2 e^{\omega_j \rho B}) \omega_j B \eta_j = 0, \\ \sum_{j=1}^4 [\mathbb{E}\mathbb{I} e^{\omega_j \rho B} \omega_j^3 \rho^3 B^3 \eta_j - \mathbf{C}_1^T \mathbb{E}\mathbb{I} \omega_j^3 \rho^3 B^3 \eta_j - i\rho^2 \mathbb{K}_1 e^{\omega_j \rho B} \eta_j + i\rho^2 \Gamma_1 e^{\omega_j \rho B} \omega_j \rho B \eta_j] = \xi_1, \\ \sum_{j=1}^4 [\mathbb{E}\mathbb{I} e^{\omega_j \rho B} \omega_j^2 \rho^2 B^2 \eta_j - \mathbf{C}_2^T \mathbb{E}\mathbb{I} \omega_j^2 \rho^2 B^2 \eta_j + i\rho^2 \mathbb{K}_2 e^{\omega_j \rho B} \omega_j \rho B \eta_j - i\rho^2 \Gamma_2 e^{\omega_j \rho B} \eta_j] = \xi_2. \end{cases} \quad (4.4)$$

For $|\lambda| \geq \delta \neq 0$, we rewrite (4.4) into the matrix form

$$\begin{bmatrix} (I - \mathbf{C}_1 e^{\rho B}) & (I - \mathbf{C}_1 e^{i\rho B}) & (I - \mathbf{C}_1 e^{-\rho B}) & (I - \mathbf{C}_1 e^{-i\rho B}) \\ (I - \mathbf{C}_2 e^{\rho B})B & i(I - \mathbf{C}_2 e^{i\rho B})B & -(I - \mathbf{C}_2 e^{-\rho B})B & -i(I - \mathbf{C}_2 e^{-i\rho B})B \\ U_1(\rho) & U_2(\rho) & U_3(\rho) & U_4(\rho) \\ V_1(\rho) & V_2(\rho) & V_3(\rho) & V_4(\rho) \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\xi_1}{\rho^2} \\ \frac{\xi_2}{\rho^2} \end{bmatrix} \quad (4.5)$$

where $U_j(\rho)$ and $V_j(\rho)$ are defined by (3.18) and (3.19), respectively.

Set

$$D(\rho) = \begin{bmatrix} (I - \mathbf{C}_1 e^{\rho B}) & (I - \mathbf{C}_1 e^{i\rho B}) & (I - \mathbf{C}_1 e^{-\rho B}) & (I - \mathbf{C}_1 e^{-i\rho B}) \\ (I - \mathbf{C}_2 e^{\rho B})B & i(I - \mathbf{C}_2 e^{i\rho B})B & -(I - \mathbf{C}_2 e^{-\rho B})B & -i(I - \mathbf{C}_2 e^{-i\rho B})B \\ U_1(\rho) & U_2(\rho) & U_3(\rho) & U_4(\rho) \\ V_1(\rho) & V_2(\rho) & V_3(\rho) & V_4(\rho) \end{bmatrix}. \quad (4.6)$$

Theorem 3.4 shows that when $\lambda \in \rho(A)$, $\Delta(\rho) = \det D(\rho) \neq 0$. Thus (4.5) has uniquely a solution

$$[\eta_1, \eta_2, \eta_3, \eta_4]^T = D^{-1}(\rho) \begin{bmatrix} 0, 0, \frac{\xi_1}{\rho^2}, \frac{\xi_2}{\rho^2} \end{bmatrix}^T.$$

Denote

$$\eta_j = \frac{D_j^{(1)}(\rho)\xi_1}{\rho^2\Delta(\rho)} + \frac{D_j^{(2)}(\rho)\xi_2}{\rho^2\Delta(\rho)}, \quad j = 1, 2, 3, 4,$$

where $D_j^{(1)}(\rho)$ and $D_j^{(2)}(\rho)$ are $n \times n$ matrices. The solution $f(x)$ to (4.2) is

$$f(x) = \sum_{j=1}^4 e^{\omega_j \rho B x} \eta_j = \frac{1}{\rho^2 \Delta(\rho)} \sum_{j=1}^4 e^{\omega_j \rho B x} D_j^{(1)}(\rho) \xi_1 + \frac{1}{\rho^2 \Delta(\rho)} \sum_{j=1}^4 e^{\omega_j \rho B x} D_j^{(2)}(\rho) \xi_2. \quad (4.7)$$

Step 2. Let $f(x)$ be given by (4.7). There exists a constant M such that $(f(1), f'(1))$ has an estimate

$$\|(f(1), f'(1))\|_{\mathbb{C}^{2n}} \leq \frac{M}{|\rho|^2} \|(\xi_1, \xi_2)\|_{\mathbb{C}^{2n}}. \quad (4.8)$$

Let $f(x)$ be given by (4.7). We have

$$\begin{aligned} f(1) &= \frac{1}{\rho^2 \Delta(\rho)} \sum_{j=1}^4 e^{\omega_j \rho B} D_j^{(1)}(\rho) \xi_1 + \frac{1}{\rho^2 \Delta(\rho)} \sum_{j=1}^4 e^{\omega_j \rho B} D_j^{(2)}(\rho) \xi_2, \\ f'(1) &= \frac{1}{\rho^2 \Delta(\rho)} \sum_{j=1}^4 \omega_j \rho B e^{\omega_j \rho B} D_j^{(1)}(\rho) \xi_1 + \frac{1}{\rho^2 \Delta(\rho)} \sum_{j=1}^4 \omega_j \rho B e^{\omega_j \rho B} D_j^{(2)}(\rho) \xi_2. \end{aligned}$$

For simplicity, introducing a $2n \times 2n$ matrix

$$H(\rho) = \begin{bmatrix} \sum_{j=1}^4 e^{\omega_j \rho B} D_j^{(1)}(\rho) & \sum_{j=1}^4 e^{\omega_j \rho B} D_j^{(2)}(\rho) \\ \sum_{j=1}^4 \omega_j \rho B e^{\omega_j \rho B} D_j^{(1)}(\rho) & \sum_{j=1}^4 \omega_j \rho B e^{\omega_j \rho B} D_j^{(2)}(\rho) \end{bmatrix},$$

$[f(1), f'(1)]^T$ can be represented as

$$\begin{bmatrix} f(1) \\ f'(1) \end{bmatrix} = \frac{1}{\rho^2 \Delta(\rho)} H(\rho) \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

and hence

$$\|(f(1), f'(1))\|_{\mathbb{C}^{2n}} \leq \frac{1}{|\rho^2| |\Delta(\rho)|} \|H(\rho)\| \|(\xi_1, \xi_2)\|_{\mathbb{C}^{2n}}$$

where $\|H(\rho)\|$ denotes the operator norm on \mathbb{C}^{2n} .

Since

$$D(\rho) \frac{1}{\Delta(\rho)} \begin{bmatrix} D_1^{(1)} & D_1^{(2)} \\ D_2^{(1)} & D_2^{(2)} \\ D_3^{(1)} & D_3^{(2)} \\ D_4^{(1)} & D_4^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix},$$

i.e.,

$$U_1(\rho)D_1^{(1)}(\rho) + U_2(\rho)D_2^{(1)}(\rho) + U_3(\rho)D_3^{(1)}(\rho) + U_4(\rho)D_4^{(1)}(\rho) = \Delta(\rho)I$$

and

$$V_1(\rho)D_1^{(2)}(\rho) + V_2(\rho)D_2^{(2)}(\rho) + V_3(\rho)D_3^{(2)}(\rho) + V_4(\rho)D_4^{(2)}(\rho) = \Delta(\rho)I,$$

there is a positive constant M such that $\|B\rho e^{\omega_j \rho B} D_j^{(k)}\| \leq M|\Delta(\rho)|$. Using this estimate we can calculate $\|H(\rho)\|$ as follows

$$\begin{aligned} \|H(\rho)\|^2 &\leq \left\| \sum_{j=1}^4 e^{\omega_j \rho B} D_j^{(1)}(\rho) \right\|^2 + \left\| \sum_{j=1}^4 e^{\omega_j \rho B} D_j^{(2)}(\rho) \right\|^2 \\ &\quad + \left\| \sum_{j=1}^4 \omega_j \rho B e^{\omega_j \rho B} D_j^{(1)}(\rho) \right\|^2 + \left\| \sum_{j=1}^4 \omega_j \rho B e^{\omega_j \rho B} D_j^{(2)}(\rho) \right\|^2 \\ &\leq 64 \max_{1 \leq j \leq 4, 1 \leq k \leq 2} \left\{ \|B\rho e^{\omega_j \rho B} D_j^{(k)}\|^2 \right\} \\ &= 64 \max_{1 \leq j \leq 4, 1 \leq k \leq 2} \|B\rho e^{\omega_j \rho B} D_j^{(k)}\|^2 \\ &\leq 64M^2 |\Delta(\rho)|^2. \end{aligned}$$

Therefore, $\|(f(1), f'(1))\| \leq 8|\rho|^{-2} M \|(\xi_1, \xi_2)\|$. The desired inequality follows.

Step 3. There exists a positive constant M_1 such that

$$\|(f, g)\|_{\mathcal{H}}^2 \leq M_1 \|(\xi_1, \xi_2)\|_{\mathbb{C}^{2n}}^2, \quad \forall \lambda \in \rho(\mathcal{A}) \cap \mathbb{R}_-.$$

We calculate the norm of (f, g) in \mathcal{H}

$$\begin{aligned} \|(f, g)\|_{\mathcal{H}}^2 &= \int_0^1 (\mathbb{E} f''(x), f''(x))_{\mathbb{C}^n} + (\mathbb{M} g(x), g(x))_{\mathbb{C}^n} dx \\ &= (\mathbb{E} f''(1), f'(1))_{\mathbb{C}^n} - (\mathbb{E} f''(0), f'(0))_{\mathbb{C}^n} - (\mathbb{E} f'''(1), f(1))_{\mathbb{C}^n} + (\mathbb{E} f'''(0), f(0))_{\mathbb{C}^n} \\ &\quad + \int_0^1 (\mathbb{E} f^{(4)}(x), f(x))_{\mathbb{C}^n} + (\mathbb{M} g(x), g(x))_{\mathbb{C}^n} dx \\ &= (\mathbb{E} f''(1) - \mathbf{C}_2^T \mathbb{E} f''(0), f'(1))_{\mathbb{C}^n} - (\mathbb{E} f'''(1) - \mathbf{C}_1^T \mathbb{E} f'''(0), f(1))_{\mathbb{C}^n} \\ &\quad + \int_0^1 (-\mathbb{M} \lambda^2 f(x), f(x))_{\mathbb{C}^n} + (\mathbb{M} \lambda f(x), \lambda f(x))_{\mathbb{C}^n} dx \\ &= (-\lambda(\mathbb{K}_2 f'(1) - \Gamma_2 f(1)) + \xi_2, f'(1))_{\mathbb{C}^n} - (\lambda(\mathbb{K}_1 f(1) - \Gamma_1 f'(1)) + \xi_1, f(1))_{\mathbb{C}^n} \\ &\quad + (|\lambda|^2 - \lambda^2) \int_0^1 (\mathbb{M} f(x), f(x))_{\mathbb{C}^n} dx \\ &= -\lambda \left(\begin{bmatrix} \mathbb{K}_1 & -\Gamma_1 \\ -\Gamma_2 & \mathbb{K}_2 \end{bmatrix} \begin{bmatrix} f(1) \\ f'(1) \end{bmatrix}, \begin{bmatrix} f(1) \\ f'(1) \end{bmatrix} \right)_{\mathbb{C}^{2n}} + \left(\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \begin{bmatrix} f(1) \\ f'(1) \end{bmatrix} \right)_{\mathbb{C}^{2n}} \\ &\quad + (|\lambda|^2 - \lambda^2) \int_0^1 (\mathbb{M} f(x), f(x))_{\mathbb{C}^n} dx. \end{aligned}$$

When $\lambda \in \mathbb{R}$, it holds that

$$\begin{aligned} \|(f, g)\|_{\mathcal{H}}^2 &= -\lambda \left(\begin{bmatrix} \mathbb{K}_1 & -\Gamma_1 \\ -\Gamma_2 & \mathbb{K}_2 \end{bmatrix} \begin{bmatrix} f(1) \\ f'(1) \end{bmatrix}, \begin{bmatrix} f(1) \\ f'(1) \end{bmatrix} \right)_{\mathbb{C}^{2n}} + \left(\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \begin{bmatrix} f(1) \\ f'(1) \end{bmatrix} \right)_{\mathbb{C}^{2n}} \\ &\leq |\lambda| \max_{j=1,2} \{\|\mathbb{K}_j\|, \|\Gamma_j\|\} \|(f(1), f'(1))\|_{\mathbb{C}^{2n}}^2 + \|(\xi_1, \xi_2)\|_{\mathbb{C}^{2n}} \|(f(1), f'(1))\|_{\mathbb{C}^{2n}}. \end{aligned}$$

When $\lambda \in \mathbb{R}_- \cap \rho(\mathcal{A})$, $\lambda = i\rho^2$ and $\rho = e^{i\frac{\pi}{4}}\tau$, $\tau > 0$, using (4.8) we can assert that there exists a positive constant M_1 such that

$$\|(f, g)\|_{\mathcal{H}} \leq M_1 \|(\xi_1, \xi_2)\|_{\mathbb{C}^{2n}}.$$

The proof is then complete. \square

Theorem 4.2. Let \mathcal{H} and \mathcal{A} be defined as before. The root vectors of \mathcal{A} are complete in \mathcal{H} .

Proof. Due to Theorem 2.2, we can assume that $\sigma(\mathcal{A}) = \{\lambda_k, k \in \mathbb{N}\}$. Denote by $Sp(\mathcal{A})$ the closure of the linear span of root vectors of \mathcal{A} , i.e.,

$$Sp(\mathcal{A}) = \overline{\left\{ \sum_k y_k, y_k \in E(\lambda_k, \mathcal{A})\mathcal{H}, \lambda_k \in \sigma(\mathcal{A}) \right\}} \subset \mathcal{H}$$

where $E(\lambda_k, \mathcal{A})$ is the Riesz projector corresponding to λ_k . Let $F = (f_1, f_2) \perp Sp(\mathcal{A})$. We shall prove $F = 0$, which implies the completeness of root vectors of \mathcal{A} .

Let $F = (f_1, f_2) \perp Sp(\mathcal{A})$. Then $R^*(\lambda, \mathcal{A})F$ is an \mathcal{H} -valued entire function on \mathbb{C} . For $\forall G = (g_1, g_2) \in \mathcal{H}$, define a scalar function by

$$U(\lambda) = (G, R^*(\lambda, \mathcal{A})F)_{\mathcal{H}}, \quad \forall \lambda \in \mathbb{C}. \quad (4.9)$$

Obviously, $U(\lambda)$ is also an entire function and $\lim_{\Re \lambda \rightarrow +\infty} U(\lambda) = 0$ since \mathcal{A} is the generator of a C_0 semigroup.

For $\lambda \in \rho(\mathcal{A})$, we have $U(\lambda) = (R(\lambda, \mathcal{A})G, F)_{\mathcal{H}}$. We shall prove that $U(\lambda)$ is bounded on the negative real axis. To this end, let

$$Y_1 = R(\lambda, \mathcal{A})G = (f, g) \in \mathcal{D}(\mathcal{A}), \quad \lambda \in \rho(\mathcal{A}) \cap \mathbb{R}_-,$$

i.e., $(\lambda I - \mathcal{A})Y_1 = G$, or equivalently,

$$\begin{cases} \lambda f - g = g_1, \\ \lambda g + \mathbb{M}^{-1} \mathbb{E} f^{(4)} = g_2, \\ f(0) = \mathbf{C}_1 f(1), \quad f'(0) = \mathbf{C}_2 f'(1), \\ \mathbb{E} f^{(3)}(1) - \mathbf{C}_1^T \mathbb{E} f^{(3)}(0) = \mathbb{K}_1 g(1) - \Gamma_1 g'(1), \\ \mathbb{E} f''(1) - \mathbf{C}_2^T \mathbb{E} f''(0) = -\mathbb{K}_2 g'(1) + \Gamma_2 g(1). \end{cases} \quad (4.10)$$

Let $Y_2 = (\hat{f}, \hat{g}) = (\lambda I - \mathcal{A}_0)^{-1}G$. Then \hat{f} and \hat{g} satisfy the following equations

$$\begin{cases} \lambda \hat{f} - \hat{g} = g_1, \\ \lambda \hat{g} + \mathbb{M}^{-1} \mathbb{E} \hat{f}^{(4)} = g_2, \\ \hat{f}(0) = \mathbf{C}_1 \hat{f}(1), \quad \hat{f}'(0) = \mathbf{C}_2 \hat{f}'(1), \\ \mathbb{E} \hat{f}^{(3)}(1) - \mathbf{C}_1^T \mathbb{E} \hat{f}^{(3)}(0) = 0, \\ \mathbb{E} \hat{f}''(1) - \mathbf{C}_2^T \mathbb{E} \hat{f}''(0) = 0. \end{cases} \quad (4.11)$$

Since \mathcal{A}_0 is skew-adjoint, we have

$$\|Y_2\|_{\mathcal{H}} \leq \frac{1}{|\Re \lambda|} \|G\|_{\mathcal{H}}, \quad \lambda \in \rho(\mathcal{A}) \cap \mathbb{R}_-. \quad (4.12)$$

Set $Y_3(\lambda) = Y_1 - Y_2 = (u, v)$. Then $(u, v) \in \mathcal{H}$ satisfy the equations

$$\begin{cases} \lambda u(x) - v(x) = 0, \quad x \in (0, 1), \\ \lambda v(x) + \mathbb{M}^{-1} \mathbb{E} u^{(4)}(x) = 0, \quad x \in (0, 1), \\ u(0) = \mathbf{C}_1 u(1), \quad u'(0) = \mathbf{C}_2 u'(1), \\ \mathbb{E} u'''(1) - \mathbf{C}_1^T \mathbb{E} u'''(0) = \mathbb{K}_1 v(1) - \Gamma_1 v'(1) + \mathbb{K}_1 \hat{g}(1) - \Gamma_1 \hat{g}'(1), \\ \mathbb{E} u''(1) - \mathbf{C}_2^T \mathbb{E} u''(0) = -\mathbb{K}_2 v'(1) + \Gamma_2 v(1) + \mathbb{K}_2 \hat{g}'(1) + \Gamma_2 \hat{g}(1). \end{cases} \quad (4.13)$$

Denote

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \mathbb{K}_1 \hat{g}(1) - \Gamma_1 \hat{g}'(1) \\ \mathbb{K}_2 \hat{g}'(1) - \Gamma_2 \hat{g}(1) \end{bmatrix} = \begin{bmatrix} \mathbb{K}_1 & -\Gamma_1 \\ -\Gamma_2 & \mathbb{K}_2 \end{bmatrix} \begin{bmatrix} \hat{g}(1) \\ \hat{g}'(1) \end{bmatrix}, \quad (4.14)$$

we have

$$\|(\xi_1, \xi_2)\|_{\mathbb{C}^{2n}} \leq M_1 \|(\hat{g}(1), \hat{g}'(1))\|_{\mathbb{C}^{2n}}$$

where $M_1 = \max\{\|\mathbb{K}_2\| + \|I_1\|, \|\mathbb{K}_1\| + \|I_2\|\}$.

We are now in a position to estimate $\|\hat{g}'(1)\|^2$ and $\|\hat{g}(1)\|^2$. From (4.11) we see that $\hat{g}'(1) = \lambda \hat{f}'(1) - g'_1(1)$ where $\hat{f}'(1) - \hat{f}'(0) = (I - \mathbf{C}_2)\hat{f}'(1) = \int_0^1 \hat{f}''(x) dx$, and hence

$$\begin{aligned} \|\hat{f}'(1)\|^2 &= \left\| (I - \mathbf{C}_2)^{-1} \int_0^1 \hat{f}''(x) dx \right\|^2 \\ &\leq \|(I - \mathbf{C}_2)^{-1} \mathbb{E} \mathbb{I}^{-1/2}\|^2 \int_0^1 (\mathbb{E} \hat{f}''(x), \hat{f}''(x))_{\mathbb{C}^n} dx. \end{aligned}$$

For $g'_1(1)$, the equality $g'_1(1) - g'_1(0) = (I - \mathbf{C}_2)g'_1(1) = \int_0^1 g''_1(x) dx$ implies that

$$\begin{aligned} \|g'_1(1)\|^2 &= \left\| (I - \mathbf{C}_2)^{-1} \int_0^1 g''_1(x) dx \right\|^2 \\ &\leq \|(I - \mathbf{C}_2)^{-1} \mathbb{E} \mathbb{I}^{-1/2}\|^2 \int_0^1 (\mathbb{E} g''_1(x), g''_1(x))_{\mathbb{C}^n} dx. \end{aligned}$$

Therefore, we have estimate

$$\begin{aligned} \|\hat{g}'(1)\|^2 &\leq (|\lambda| \|\hat{f}'(1)\| + \|g'_1(1)\|)^2 \\ &\leq 2 \|(I - \mathbf{C}_2)^{-1} \mathbb{E} \mathbb{I}^{-1/2}\|^2 \|\mathbb{E} \mathbb{I}^{-1}\| (|\lambda|^2 \|Y_2\|^2 + \|G\|^2). \end{aligned}$$

Consider $\hat{g}(1)$,

$$\hat{g}(1) - \hat{g}(0) = (I - \mathbf{C}_1)\hat{g}(1) = \int_0^1 \hat{g}'(x) dx = \hat{g}'(1) - \int_0^1 dx \int_x^1 \hat{g}''(r) dr,$$

and $\hat{g}''(x) = \lambda \hat{f}''(x) - g''_1(x)$, so there is a constant M_2 such that $\|\hat{g}(1)\|^2 \leq M_2(|\lambda|^2 \|Y_2\|^2 + \|G\|^2)$. Therefore, there exists a positive constant M_3 such that

$$\|(\hat{g}(1), \hat{g}'(1))\|_{\mathbb{C}^{2n}}^2 \leq M_3(|\lambda|^2 \|Y_2\|_{\mathcal{H}}^2 + \|G\|_{\mathcal{H}}^2).$$

From the above argument we see that there exists a positive constant \tilde{M}_1 such that

$$\|(\xi_1, \xi_2)\|_{\mathbb{C}^{2n}} \leq \tilde{M}_1(|\lambda| \|Y_2\|_{\mathcal{H}} + \|G\|_{\mathcal{H}}). \quad (4.15)$$

Now, Theorem 4.1 together with (4.15) ensures that when $\lambda \in \rho(\mathcal{A}) \cap \mathbb{R}_-$,

$$\begin{aligned} \|R(\lambda, \mathcal{A})G\|_{\mathcal{H}} &= \|Y_1\|_{\mathcal{H}} = \|Y_2 + Y_3\|_{\mathcal{H}} \leq \|(\hat{f}, \hat{g})\|_{\mathcal{H}} + \|(u, v)\|_{\mathcal{H}} \\ &= \|R(\lambda, \mathcal{A}_0)G\|_{\mathcal{H}} + \|(u, v)\|_{\mathcal{H}} \\ &\leq \frac{1}{|\lambda|} \|G\|_{\mathcal{H}} + M \|(\xi_1, \xi_2)\|_{\mathbb{C}^{2n}} \\ &\leq \frac{1}{|\lambda|} \|G\|_{\mathcal{H}} + M \tilde{M}_1 (|\lambda| \|R(\lambda, \mathcal{A}_0)G\|_{\mathcal{H}} + \|G\|_{\mathcal{H}}) \\ &\leq (|\lambda|^{-1} + M \tilde{M}_1) \|G\|_{\mathcal{H}}, \quad \lambda < -h. \end{aligned}$$

Therefore, we get that $\limsup_{\lambda \rightarrow -\infty} \|R(\lambda, \mathcal{A})G\|_{\mathcal{H}} < \infty$. Theory of ordinary differential equation shows that $R(\lambda, \mathcal{A})G$ is a meromorphic function of finite exponential type, so is $U(\lambda)$. The Phragmén–Lindelöf Theorem [32] asserts that $|U(\lambda)| \leq M$, $\forall \lambda \in \mathbb{C}$. Further, the Liouville Theorem says that $U(\lambda) \equiv 0$ since $\lim_{\Re \lambda \rightarrow \infty} U(\lambda) = 0$. Note that $U(\lambda) = (G, R^*(\lambda, \mathcal{A})F)_{\mathcal{H}} = 0$, $\forall G \in \mathcal{H}$. This means that $R^*(\lambda, \mathcal{A})F = 0$, and hence $F = 0$. This implies that the root vectors of \mathcal{A} are complete in \mathcal{H} , i.e., $Sp(\mathcal{A}) = \mathcal{H}$. The desired result follows. \square

4.2. The basis property of root vectors

In this subsection we study the basis property of the root vectors of \mathcal{A} . The basis property of root vectors of non-self-adjoint operators has been a tough problem in linear operator theory. There are various notions about the basis for a Hilbert space. Here we shall adopt a much weaker notion.

Definition 4.1. Let \mathcal{H} be a Hilbert space, and $\{\mathcal{H}_j, j \in \mathbb{N}\}$ be a subspace sequence of \mathcal{H} . $\{\mathcal{H}_j, j \in \mathbb{N}\}$ is said to be a subspace Riesz basis for \mathcal{H} if for each $x \in \mathcal{H}$, there is a unique $x_j \in \mathcal{H}_j$ such that $x = \sum_{j=1}^{\infty} x_j$ and there exist constants C_1 and C_2 such that

$$C_1 \sum_{j=1}^{\infty} \|x_j\|^2 \leq \|x\|^2 \leq C_2 \sum_{j=1}^{\infty} \|x_j\|^2, \quad \forall x \in \mathcal{H}.$$

Let $\{\varphi_n, n \in \mathbb{N}\}$ be a sequence of \mathcal{H} . $\{\varphi_n, n \in \mathbb{N}\}$ is said to be a Riesz basis with parentheses if there is an increasing subsequence of \mathbb{N} , $\{n_k, k \in \mathbb{N}\}$ with $\lim_{k \rightarrow \infty} n_k = \infty$ such that the subspace sequence

$$\mathcal{H}_k = \text{span}\{\varphi_j, n_k \leq j \leq n_{k+1} - 1\}, \quad k \in \mathbb{N}$$

forms a subspace Riesz basis for \mathcal{H} .

We remark that the Riesz basis with parentheses for \mathcal{H} is not a basis for \mathcal{H} in the sense of Schauder basis. Its partial summation converges only in the sense of parentheses. In what follows, we shall discuss the basis property of the root vectors of \mathcal{A} in the sense of parentheses. By now, however, we do not find out the root vectors of \mathcal{A} , we know the distribution of eigenvalues of \mathcal{A} only. To obtain the basis property of root vectors of \mathcal{A} , we consider the information from spectrum. Since $\sigma(\mathcal{A})$ is a finite union of separable sets, there is a sequence of bounded open sets, $\{O_k, k \in \mathbb{N}\}$, such that

$$\sigma(\mathcal{A}) \subset \bigcup_{k=1}^{\infty} O_k, \quad \inf_{k \neq p} \text{dist}(O_k, O_p) > 0, \quad \#\sigma(\mathcal{A}) \cap O_k \leq N.$$

Set $\Omega_k = \sigma(\mathcal{A}) \cap O_k = \{\mu_k^{(1)}, \mu_k^{(2)}, \dots, \mu_k^{(r)}, r = \#\Omega_k\}$ (taking its multiplicity into account), $k \in \mathbb{N}$. For each Ω_k , define generalized difference divide family of exponential by $[\mu_k] = e^{\mu_k t}$,

$$[\mu_k^{(1)}, \mu_k^{(2)}] = \begin{cases} \frac{e^{t\mu_k^{(1)}} - e^{t\mu_k^{(2)}}}{\mu_k^{(1)} - \mu_k^{(2)}}, & \mu_k^{(1)} \neq \mu_k^{(2)}, \\ te^{t\mu_k^{(1)}}, & \mu_k^{(1)} = \mu_k^{(2)} \end{cases}$$

and for $1 \leq j \leq r$,

$$[\mu_k^{(1)}, \mu_k^{(2)}, \dots, \mu_k^{(j)}] = \begin{cases} \frac{[\mu_k^{(1)}, \mu_k^{(2)}, \dots, \mu_k^{(j-1)}] - [\mu_k^{(2)}, \mu_k^{(3)}, \dots, \mu_k^{(j)}]}{\mu_k^{(1)} - \mu_k^{(j)}}, & \mu_k^{(1)} \neq \mu_k^{(j)}, \\ \frac{d}{d\mu_k^{(1)}} [\mu_k^{(1)}, \mu_k^{(2)}, \dots, \mu_k^{(j)}], & \mu_k^{(1)} = \mu_k^{(j)}. \end{cases}$$

Since $\sigma(\mathcal{A})$ is distributed in a strip parallel to the imaginary axis, the family

$$\{[\mu_k^{(1)}], [\mu_k^{(1)}, \mu_k^{(2)}], \dots, [\mu_k^{(1)}, \mu_k^{(2)}, \dots, \mu_k^{(r)}]\}_{k=1}^{\infty}$$

forms a Riesz basis sequence for $L^2[0, T]$ for sufficient large T .

Let $S(t)$ be the C_0 semigroup generated by \mathcal{A} . For each $F = (f, g) \in \mathcal{H}$, $S(t)F$ is a continuous \mathcal{H} -valued function in t . So $S(t)F$ can be expanded into a vector-valued series according to generalized difference divide in the sense of $L^2([0, T])$. Obviously, $S(t)$ satisfies the spectrum determined growth condition in this sense. On the other hand, we can rewrite formally the series into the form

$$S(t)F = \sum_{k=1}^{\infty} \sum_{\mu_k^{(j)} \in \Omega_k} e^{t\mu_k^{(j)}} \sum_{s=1}^{\infty} \frac{t^s}{s!} (\mathcal{A} - \mu_k^{(j)})^s E(\mu_k^{(j)}, \mathcal{A})F.$$

In [25] and [27], the authors proved that the series above converges in the sense of parentheses. This result is stated in the following proposition.

Proposition 4.1. Let \mathcal{A} be the generator of a C_0 -semigroup on a separable Hilbert space \mathcal{H} . Suppose that the following conditions are satisfied:

1) The spectrum of \mathcal{A} has the decomposition

$$\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A});$$

2) There exists a real number $\alpha \in \mathbb{R}$ such that

$$\sup\{\Re \lambda \mid \lambda \in \sigma_1(\mathcal{A})\} \leq \alpha \leq \inf\{\Re \lambda \mid \lambda \in \sigma_2(\mathcal{A})\};$$

3) The set $\sigma_2(\mathcal{A}) = \{\lambda_k\}_{k \in \mathbb{N}}$ consists of eigenvalues of \mathcal{A} and is a finite unification of separable sets.

Then there exist two $T(t)$ -invariant closed subspaces \mathcal{H}_1 and \mathcal{H}_2

$$\mathcal{H}_1 = \{f \in \mathcal{H} : E(\lambda, \mathcal{A})f = 0, \forall \lambda \in \sigma_2(\mathcal{A})\},$$

$$\mathcal{H}_2 = \overline{\text{span} \left\{ \sum_{k=1}^m E(\lambda_k, \mathcal{A})f : \forall f \in \mathcal{H}, \forall m \in \mathbb{N} \right\}}$$

and $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$ with property that $\sigma(\mathcal{A}|_{\mathcal{H}_1}) = \sigma_1(\mathcal{A})$ and $\sigma(\mathcal{A}|_{\mathcal{H}_2}) = \sigma_2(\mathcal{A})$. Moreover, there exists a sequence $\{\Omega_k, k \in \mathbb{N}\}$ satisfying $\bigcup_{k=1}^{\infty} \Omega_k = \sigma_2(\mathcal{A})$ such that each Ω_k includes only finitely many elements of $\sigma_2(\mathcal{A})$ and $\{E(\Omega_k, \mathcal{A})\mathcal{H}_2\}_{k \in \mathbb{N}}$ forms a subspace Riesz basis for \mathcal{H}_2 .

Theorem 4.3. Let \mathcal{A} be defined by (2.4)–(2.5), then there exists a sequence of the root vectors of \mathcal{A} that forms a Riesz basis with parentheses (subspace Riesz basis) for \mathcal{H} .

Proof. Taking $\sigma_1(\mathcal{A}) = \{\infty\}$, $\sigma_2(\mathcal{A}) = \sigma_p(\mathcal{A})$, we have $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A})$. Let $\alpha = -h$ where h be given by Theorem 3.5. The conditions 1) and 2) in Proposition 4.1 are fulfilled. Corollary 3.1 shows that the condition 3) in Proposition 4.1 is also verified. Therefore, there exists a sequence of the root vectors of \mathcal{A} that forms a Riesz basis with parentheses (subspace Riesz basis) for \mathcal{H}_2 . The completeness of root vectors in Theorem 4.2 asserts that $\mathcal{H}_2 = \mathcal{H}$. The proof is then complete. \square

Remark 4.1. Since \mathcal{A} is an unbounded linear operator, ∞ always is its extension spectral point. So in the proof of Theorem 3.4, we set $\sigma_1(\mathcal{A}) = \{\infty\}$. If we consider the spectrum of \mathcal{A} but the extension spectrum, we can set $\sigma_1(\mathcal{A}) = \emptyset$.

Note that a Riesz basis with parentheses is probably not a basis in the sense of Schauder basis. If an operator A has property of Riesz basis with parentheses, we cannot deduce from this that the semigroup generated by A satisfies the spectrum determined growth condition. As shown in the proof of [25], the spectral distribution of \mathcal{A} ensures that (2.1) satisfies the spectrum determined growth condition. Therefore, we have the following result.

Corollary 4.1. Let \mathcal{A} be defined by (2.4)–(2.5), and $S(t)$ be the C_0 semigroup generated by \mathcal{A} . Then $S(t)$ satisfies the spectrum determined growth condition.

5. Some applicable examples

In this section we shall give some examples that can be formulated into the frame of (2.1).

5.1. Composite Euler–Bernoulli beam

We consider a composite beam, whose motion is governed by the partial differential equations

$$\begin{cases} m_1 \frac{\partial^2 y_1(x, t)}{\partial t^2} + El_1 \frac{\partial^4 y_1(x, t)}{\partial x^4} + \gamma \sqrt{\frac{El_1}{El_2}} El_2 \frac{\partial^4 y_2(x, t)}{\partial x^4} = 0, & x \in (0, 1), \\ m_2 \frac{\partial^2 y_2(x, t)}{\partial t^2} + \gamma \sqrt{\frac{El_2}{El_1}} El_1 \frac{\partial^4 y_1(x, t)}{\partial x^4} + El_2 \frac{\partial^4 y_2(x, t)}{\partial x^4} = 0, & x \in (0, 1), \end{cases} \quad (5.1)$$

where $\gamma \in (0, 1)$ is a coupled constant.

Suppose that the composite beam is clamped at the left end, i.e.

$$\begin{cases} y_1(0, t) = \frac{\partial y_1(0, t)}{\partial x} = 0, \\ y_2(0, t) = \frac{\partial y_2(0, t)}{\partial x} = 0 \end{cases} \quad (5.2)$$

and free at the right end, i.e.

$$\begin{cases} El_1 \frac{\partial^3 y_1(1, t)}{\partial x^3} + \gamma \sqrt{\frac{El_1}{El_2}} El_2 \frac{\partial^3 y_2(1, t)}{\partial x^3} = 0, \\ El_1 \frac{\partial^2 y_1(1, t)}{\partial x^2} + \gamma \sqrt{\frac{El_1}{El_2}} El_2 \frac{\partial^2 y_2(1, t)}{\partial x^2} = 0; \\ \gamma \sqrt{\frac{El_2}{El_1}} El_1 \frac{\partial^3 y_1(1, t)}{\partial x^3} + El_2 \frac{\partial^3 y_2(1, t)}{\partial x^3} = 0, \\ \gamma \sqrt{\frac{El_2}{El_1}} El_1 \frac{\partial^2 y_1(1, t)}{\partial x^2} + El_2 \frac{\partial^2 y_2(1, t)}{\partial x^2} = 0. \end{cases} \quad (5.3)$$

We adopt the decoupling feedback controllers at the free end as below

$$\begin{cases} El_1 \frac{\partial^3 y_1(1, t)}{\partial x^3} + \gamma \sqrt{\frac{El_1}{El_2}} El_2 \frac{\partial^3 y_2(1, t)}{\partial x^3} = k_1 m_1 \frac{\partial y_1(1, t)}{\partial t}, \\ El_1 \frac{\partial^2 m_1 y_1(1, t)}{\partial x^2} + \gamma \sqrt{\frac{El_1}{El_2}} El_2 \frac{\partial^2 y_2(1, t)}{\partial x^2} = -k_2 m_1 \frac{\partial^2 y_1(1, t)}{\partial x \partial t}; \\ \gamma \sqrt{\frac{El_2}{El_1}} El_1 \frac{\partial^3 y_1(1, t)}{\partial x^3} + El_2 \frac{\partial^3 y_2(1, t)}{\partial x^3} = k_1 m_2 \frac{\partial y_2(1, t)}{\partial t}, \\ \gamma \sqrt{\frac{El_2}{El_1}} El_1 \frac{\partial^2 y_1(1, t)}{\partial x^2} + El_2 \frac{\partial^2 y_2(1, t)}{\partial x^2} = -k_2 m_2 \frac{\partial^2 y_2(1, t)}{\partial x \partial t} \end{cases} \quad (5.4)$$

where k_1 and k_2 are negative feedback gain constants with $k_1 + k_2 > 0$.

Set

$$\mathbb{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad \mathbb{E}\mathbb{I} = \begin{pmatrix} El_1 & \gamma \sqrt{El_1 El_2} \\ \gamma \sqrt{El_1 El_2} & El_2 \end{pmatrix},$$

$$\mathbb{K}_1 = \begin{pmatrix} k_1 m_1 & 0 \\ 0 & k_1 m_2 \end{pmatrix} = k_1 \mathbb{M}, \quad \mathbb{K}_2 = \begin{pmatrix} k_2 m_1 & 0 \\ 0 & k_2 m_2 \end{pmatrix} = k_2 \mathbb{M}$$

and

$$Y(x, t) = (y_1(x, t), y_2(x, t))^T,$$

then we have

$$\begin{cases} \mathbb{M} Y_{tt}(x, t) + \mathbb{E}\mathbb{I} Y_{xxxx}(x, t) = 0, & x \in (0, 1), \\ Y(0, t) = 0, & Y_x(0, t) = 0, \\ \mathbb{E}\mathbb{I} Y_{xxx}(1, t) = k_1 \mathbb{M} Y_t(1, t), \\ \mathbb{E}\mathbb{I} Y_{xx}(1, t) = -k_2 \mathbb{M} Y_{xt}(1, t). \end{cases} \quad (5.5)$$

This is the case that $\mathbf{C}_1 = \mathbf{C}_2 = 0$, $\Gamma_1 = \Gamma_2 = 0$ in (2.1). According to Corollary 4.1, the system (5.5) satisfies SDG property.

To obtain an applicable result in control theory, here we consider only the case that $k_1 > 0$, $k_2 = 0$. For simplicity, we suppose that $m_1 = m_2 = El_1 = El_2 = 1$.

Theorem 5.1. *Let $k_1 > 0$ and $k_2 = 0$. Then the system (5.5) is exponentially stable.*

Proof. We consider the eigenvalue problem of (5.5), for $\lambda = i\rho^2$,

$$\begin{cases} \rho^4 y_1(x) = y_1^{(4)}(x) + \gamma y_2^{(4)}(x), \\ \rho^4 y_2(x) = \gamma y_1^{(4)}(x) + y_2^{(4)}(x), & x \in (0, 1), \\ y_1(0) = y_2(0) = y_1'(0) = y_2'(0) = 0, \\ y_1'''(1) + \gamma y_2'''(1) = i\rho^2 k_1 y_1(1), \\ \gamma y_1'''(1) + y_2'''(1) = i\rho^2 k_1 y_2(1), \\ y_1''(1) + \gamma y_2''(1) = 0, \\ \gamma y_1''(1) + y_2''(1) = 0. \end{cases}$$

Since $\lambda = 0$ is not an eigenvalue of the system, set $b_1 = \sqrt[4]{1+\gamma}$, $b_2 = \sqrt[4]{1-\gamma}$, a straightforward calculation gives

$$\Delta(\rho) = \prod_{j=1}^2 \{ik_1[\sin b_j \rho \cosh b_j \rho - \sinh b_j \rho \cos b_j \rho] + (b_j)^3 \rho [1 + \cosh b_j \rho \cos b_j \rho]\}.$$

Therefore, $\lambda = i\rho^2$ is an eigenvalue if and only if $\rho^{-1} \Delta(\rho) = 0$. Obviously,

$$\inf_{\rho \in \mathbb{R}} \left| \frac{\Delta(\rho)}{\rho} \right|^2 \geq \prod_{j=1}^2 \inf_{\rho \in \mathbb{R}} [k_1^2 (\rho^{-1} \sin b_j \rho \cosh b_j \rho - \rho^{-1} \sinh b_j \rho \cos b_j \rho)^2 + (b_j)^6 (1 + \cosh b_j \rho \cos b_j \rho)^2] > 0.$$

So the imaginary axis is not asymptote of the eigenvalues of the system. Thank to Corollary 4.1, the system is exponentially stable. \square

5.2. Tree-shaped of Euler–Bernoulli beams

Let $G = (V, E)$ be a planar graph with vertex set $V = \{O, a_1, a_2, a_3, \dots, a_n\}$ and edge set $E = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$. Suppose that G is a tree and O is its root. For each $a \in V$, let $J(a)$ denote the index set of edges that means that if $j \in J(a)$, a is one end of γ_j , and let $\#J(a)$ denote the number of elements in $J(a)$. If $\#J(a) = 1$, the a is called a boundary vertex (or an exterior node), otherwise a is called an internal node. Let ∂G denote the boundary vertices of G . We define the direction of G : for each $a_k \in \partial G$ except O , there is a directed path with initial node O and terminal node a_k . According to this definition G becomes a directed graph. In addition, we suppose that each directed edge γ_j with tail a_i and head a_s is straight and has the same length ℓ . We define the parameterized map $\pi_j: [0, \ell] \rightarrow \gamma_j$ such that $\pi_j(0) = a_i$, $\pi_j(\ell) = a_s$. Then G is a metric graph deduced by the parameterized map.

For the directed graph G , we define the **adjacency matrix** of edges, $\mathcal{E} = (\varepsilon_{ij})_{n \times n}$, whose entries are determined by

$$\varepsilon_{pi} = \begin{cases} 1, & \text{if there exist edges } \gamma_i \text{ and } \gamma_p \text{ such that } \pi_i(0) = \pi_p(\ell), \\ 0, & \text{otherwise.} \end{cases}$$

Let y be a function defined on G . Denote $y^j(x, t) = y(\pi_j(x\ell), t)$, $x \in (0, 1)$ which is defined on edge γ_j . Suppose that $y^j(x, t)$ satisfies the equation

$$m_j \frac{\partial^2 y^j(x, t)}{\partial t^2} + El_j \frac{\partial^4 y^j(x, t)}{\partial x^4} = 0, \quad x \in (0, 1). \quad (5.6)$$

If the root O is an end of γ_j , then

$$y^j(0, t) = 0, \quad y_x^j(0, t) = 0, \quad \pi_j(0) = O. \quad (5.7)$$

At the interior node $a \in V$, y satisfies the continuity conditions

$$\begin{aligned} y^j(\pi_j^{-1}(a), t) &= y^i(\pi_i^{-1}(a), t), \quad \forall i, j \in J(a), \\ y_x^j(\pi_j^{-1}(a), t) &= y_x^i(\pi_i^{-1}(a), t), \quad \forall i, j \in J(a), \end{aligned} \quad (5.8)$$

and the dynamical conditions: when $\pi_j(\ell) = a$,

$$\begin{cases} El_j y_{xxx}^j(1, t) - \sum_{\pi_i(0)=a, i \in J(a)} El_i y_{xxx}^i(0, t) = k_{1,j}^2 y_t^j(1, t) - c_{1,j} y_{xt}^j(1, t), \\ El_j y_{xx}^j(1, t) - \sum_{\pi_i(0)=a, i \in J(a)} El_i y_{xx}^i(0, t) = -k_{2,j}^2 y_{xt}^j(1, t) + c_{2,j} y_t^j(1, t). \end{cases} \quad (5.9)$$

At the exterior node $a \in \partial G$, y satisfies the dynamical condition

$$\begin{cases} El_j y_{xxx}^j(1, t) = k_{1,j}^2 y_t^j(1, t) - c_{1,j} y_{xt}^j(1, t), \\ El_j y_{xx}^j(1, t) = -k_{2,j}^2 y_{xt}^j(1, t) + c_{2,j} y_t^j(1, t) \end{cases} \quad (5.10)$$

where $\pi_j(\ell) = a$. All these constitute a tree-shaped network of Euler–Bernoulli beams.

Now we can set

$$Y(x, t) = (y^1(x, t), y^2(x, t), \dots, y^n(x, t))^T, \\ \mathbb{M} = \text{diag}(m_1, m_2, \dots, m_n), \quad \mathbb{E}\mathbb{I} = \text{diag}(El_1, El_2, \dots, El_n),$$

then

$$\mathbb{M} \frac{\partial^2 Y(x, t)}{\partial t^2} + \mathbb{E}\mathbb{I} \frac{\partial^4 Y(x, t)}{\partial x^4} = 0, \quad x \in (0, 1). \quad (5.11)$$

With the help of adjacency matrix, we can rewrite (5.7) and (5.8) as

$$Y(0, t) = \mathcal{E}Y(1, t), \quad Y_x(0, t) = \mathcal{E}Y_x(1, t).$$

For each $a \in V \setminus O$, the formulae (5.9) and (5.10) are written into uniform form

$$\begin{cases} El_j y_{xxx}^j(1, t) - \sum_{i=1}^n \varepsilon_{ji} El_i y_{xxx}^i(0, t) = k_{1,j}^2 y_t^j(1, t) - c_{1,j} y_{xt}^j(1, t), \\ El_j y_{xx}^j(1, t) - \sum_{i=1}^n \varepsilon_{ji} El_i y_{xx}^i(0, t) = -k_{2,j}^2 y_{xt}^j(1, t) + c_{2,j} y_t^j(1, t). \end{cases} \quad (5.12)$$

Set

$$\mathbb{K}_s = \text{diag}(k_{s,1}^2, k_{s,2}^2, \dots, k_{s,n}^2), \quad \Gamma_s = \text{diag}(c_{s,1}, c_{s,2}, \dots, c_{s,n}), \quad s = 1, 2.$$

Then (5.12) is equivalent to

$$\begin{cases} \mathbb{E}\mathbb{I}Y_{xxx}(1, t) - \mathcal{E}^T \mathbb{E}\mathbb{I}Y_{xxx}(0, t) = \mathbb{K}_1 Y_t(1, t) - \Gamma_1 Y_{xt}(1, t), \\ \mathbb{E}\mathbb{I}Y_{xx}(1, t) - \mathcal{E}^T \mathbb{E}\mathbb{I}Y_{xx}(0, t) = -\mathbb{K}_2 Y_{xt}(1, t) + \Gamma_2 Y_t(1, t). \end{cases}$$

Therefore, the tree-shaped network of Euler–Bernoulli beams has the form

$$\begin{cases} \mathbb{M}Y_{tt}(x, t) + \mathbb{E}\mathbb{I}Y_{xxxx}(x, t) = 0, \quad x \in (0, 1), \\ Y(0, t) = \mathcal{E}Y(1, t), \quad Y_x(0, t) = \mathcal{E}Y_x(1, t), \\ \mathbb{E}\mathbb{I}Y_{xxx}(1, t) - \mathcal{E}^T \mathbb{E}\mathbb{I}Y_{xxx}(0, t) = \mathbb{K}_1 Y_t(1, t) - \Gamma_1 Y_{xt}(1, t), \\ \mathbb{E}\mathbb{I}Y_{xx}(1, t) - \mathcal{E}^T \mathbb{E}\mathbb{I}Y_{xx}(0, t) = -\mathbb{K}_2 Y_{xt}(1, t) + \Gamma_2 Y_t(1, t). \end{cases} \quad (5.13)$$

This is the case that $\mathbf{C}_1 = \mathbf{C}_2 = \mathcal{E}$ in (2.1).

As a direct consequence of Theorem 4.3 and Corollary 4.1, we have the following result.

Theorem 5.2. *There is a sequence of the root vectors of the system (5.13) that forms a Riesz basis with parentheses. And the system satisfies the spectrum determined growth condition.*

Remark 5.1. The n serially connected Euler–Bernoulli beams of type III connection conditions studied in [1] is a special case of tree, in which each interior node has $\#J(a) = 2$.

To study the stability of the system (5.13), we consider the case with boundary controls, i.e., at the internal node a , $\#J(a) > 1$, we have the geometric conditions and the dynamical conditions: when $\pi_j(\ell) = a$,

$$\begin{cases} y^j(1) = y^i(0), \quad y_x^j(1) = y_x^i(0), \quad \forall \pi_i(0) = a, \\ El_j y_{xxx}^j(1, t) - \sum_{\pi_i(0)=a, i \in J(a)} El_i y_{xxx}^i(0, t) = 0, \\ El_j y_{xx}^j(1, t) - \sum_{\pi_i(0)=a, i \in J(a)} El_i y_{xx}^i(0, t) = 0. \end{cases} \quad (5.14)$$

At the exterior node, $a \in \partial G$, $\#J(a) = 1$, y satisfies the dynamical condition

$$\begin{cases} El_j y_{xxx}^j(1, t) = k_{1,j}^2 y_t^j(1, t) - c_{1,j} y_{xt}^j(1, t), \\ El_j y_{xx}^j(1, t) = -k_{2,j}^2 y_{xt}^j(1, t) + c_{2,j} y_t^j(1, t), \quad \pi_j(\ell) = a \end{cases} \quad (5.15)$$

where $k_{1,j}^2 k_{2,j}^2 - c_{1,j} c_{2,j} > 0$.

In this case, the entries of the matrices \mathbb{K}_j and Γ_j in (5.13) satisfy $k_{s,j} = c_{s,j} = 0$, $j \in J(a)$ with $\#J(a) > 1$.

Theorem 5.3. Let \mathcal{H} be defined as before, \mathcal{A} be the operator determined by (5.13) and $S(t)$ be the C_0 semigroup generated by \mathcal{A} . Then $S(t)$ is asymptotically stable.

Proof. According to the stability theorem in [31], we only need to prove that $\Re \lambda < 0$, $\forall \lambda \in \sigma(\mathcal{A})$. According to Theorem 3.3 we only verify that the conditions

$$\mathbb{K}_1 \varphi(1) - \Gamma_1 \varphi_x(1) = 0, \quad \mathbb{K}_2 \varphi_x(1) - \Gamma_1 \varphi(1) = 0 \quad (5.16)$$

imply $\varphi(x) = 0$, where $\varphi(x) = (\varphi^1(x), \varphi^2(x), \dots, \varphi^n(x))^T$ satisfy the equation:

$$\begin{cases} \mathbb{E} \mathbb{I} \varphi_{xxx}(x) = \gamma \mathbb{M} \varphi(x), \\ \varphi(0) = \mathcal{E} \varphi(1), \quad \varphi_x(0) = \mathcal{E} \varphi_x(1), \\ \mathbb{E} \mathbb{I} \varphi_{xxx}(1) - \mathcal{E}^T \mathbb{E} \mathbb{I} \varphi_{xxx}(0) = 0, \\ \mathbb{E} \mathbb{I} \varphi_{xx}(1) - \mathcal{E}^T \mathbb{E} \mathbb{I} \varphi_{xx}(0) = 0. \end{cases} \quad (5.17)$$

If (5.16) holds, we deduce from it that $\varphi^j(1) = \varphi_x^j(1) = 0$, $j \in J(a)$, $a \in \partial G$. From the boundary conditions in (5.17), we get further that

$$El_j \varphi_{xxx}^j(1) = 0, \quad El_j \varphi_{xx}^j(1) = 0, \quad j \in J(a), \quad a \in \partial G.$$

The uniqueness theory of solution of ordinary differential equations asserts that $\varphi_j(x) = 0$, $j \in J(a)$, $a \in \partial G$. Again using the geometric condition and dynamic conditions we deduce that

$$\varphi^i(1) = \varphi_x^i(1) = 0, \quad El_i \varphi_{xxx}^i(1) = El_i \varphi_{xx}^i(1) = 0, \quad i \in J(a), \quad \gamma_i \cap \gamma_j = \{a\}, \quad \gamma_j \cap \partial G \neq \emptyset.$$

Repeating argument before, we can deduce that $\varphi(x) = 0$. Therefore, $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. The stability result follows from [31]. \square

5.3. A Complex network of Euler–Bernoulli beams

Here we shall show that (2.1) includes more complex networks. For simplicity, we consider a cycle G with vertices $V = \{a_1, a_2, a_3, a_4\}$ and edges $E = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$. Let γ_j be edge with tail a_j and head a_{j+1} . Suppose that all edges have length 1.

Let y be a function defined on G . Denote $y_j(x, t) = y(\pi_j(x), t)$, $x \in (0, 1)$ which is defined on edge γ_j . Suppose that $y_j(x, t)$ satisfies the equation

$$m_j \frac{\partial^2 y_j(x, t)}{\partial t^2} + El_j \frac{\partial^4 y_j(x, t)}{\partial x^4} = 0, \quad x \in (0, 1).$$

The network is connected as follows:

The displacements satisfy the following conditions:

$$y_1(0, t) = \alpha y_4(1, t), \quad y_2(0, t) = y_1(1, t), \quad y_3(0, t) = y_2(1, t), \quad y_4(0, t) = y_3(1, t) \quad (5.18)$$

where $\alpha \neq 1$, which means the structure is not continuous at a_1 .

The first derivatives satisfy the conditions:

$$\begin{aligned} y_{1,x}(0, t) &= c_1 y_{4x}(1, t), & y_{2,x}(0, t) &= c_2 y_{1,x}(1, t), \\ y_{3,x}(0, t) &= c_3 y_{2,x}(1, t), & y_{4,x}(0, t) &= c_4 y_{3,x}(1, t) \end{aligned} \quad (5.19)$$

where c_1, c_2, c_3, c_4 are positive constants and satisfy $\prod_{j=1}^4 c_j \neq 1$.

At each node a_j , $j = 1, 2, 3, 4$, the structure has damping, which satisfies the following dynamical conditions:

$$\begin{cases} \alpha El_1 y_{1,xxx}(0, t) - El_4 y_{4,xxx}(1, t) = -k_{1,1} y_{4,t}(1, t) + \beta_{1,1} y_{4,tx}(1, t), \\ El_4 y_{4,xx}(1, t) - c_1 El_1 y_{1,xx}(0, t) = -k_{1,2} y_{4,t}(1, t) + \beta_{1,2} y_{4,t}(1, t), \\ El_2 y_{2,xxx}(0, t) - El_1 y_{1,xxx}(1, t) = -k_{2,1} y_{1,t}(1, t) + \beta_{2,1} y_{1,t}(1, t), \\ El_1 y_{1,xx}(1, t) - c_2 El_2 y_{2,xx}(0, t) = -k_{2,2} y_{1,t}(1, t) + \beta_{2,2} y_{1,t}(1, t), \\ El_3 y_{3,xxx}(0, t) - El_2 y_{2,xxx}(1, t) = -k_{3,1} y_{2,t}(1, t) + \beta_{3,1} y_{2,t}(1, t), \\ El_2 y_{2,xx}(1, t) - c_3 El_3 y_{3,xx}(0, t) = -k_{3,2} y_{2,t}(1, t) + \beta_{3,2} y_{2,t}(1, t), \\ El_4 y_{4,xxx}(0, t) - El_3 y_{3,xxx}(1, t) = -k_{4,1} y_{3,t}(1, t) + \beta_{4,1} y_{3,t}(1, t), \\ El_3 y_{3,xx}(1, t) - c_4 El_4 y_{4,xx}(0, t) = -k_{4,2} y_{3,t}(1, t) + \beta_{4,2} y_{3,t}(1, t). \end{cases} \quad (5.20)$$

Set

$$Y(x, t) = (y_1(x, t), y_2(x, t), y_3(x, t), y_4(x, t))^T,$$

$$\mathbb{M} = \text{diag}(m_1, m_2, m_3, m_4), \quad \mathbb{E}\mathbb{I} = \text{diag}(El_1, El_2, El_3, El_4),$$

all differential equations are rewritten as

$$\mathbb{M} Y_{tt}(x, t) + \mathbb{E}\mathbb{I} Y_{xxxx}(x, t) = 0, \quad x \in (0, 1).$$

Set

$$\mathbf{C}_1 = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{C}_2 = \begin{pmatrix} 0 & 0 & 0 & c_1 \\ c_2 & 0 & 0 & 0 \\ 0 & c_3 & 0 & 0 \\ 0 & 0 & c_4 & 0 \end{pmatrix},$$

$$\mathbb{K}_1 = \begin{bmatrix} k_{21} & 0 & 0 & \alpha \\ 0 & k_{31} & 0 & 0 \\ 0 & 0 & k_{41} & 0 \\ 0 & 0 & 0 & k_{11} \end{bmatrix}, \quad \mathbb{K}_2 = \begin{pmatrix} k_{22} & 0 & 0 & 0 \\ 0 & k_{32} & 0 & 0 \\ 0 & 0 & k_{42} & 0 \\ 0 & 0 & 0 & k_{12} \end{pmatrix},$$

and

$$\Gamma_1 = \begin{pmatrix} \beta_{21} & 0 & 0 & 0 \\ 0 & \beta_{31} & 0 & 0 \\ 0 & 0 & \beta_{41} & 0 \\ 0 & 0 & 0 & \beta_{11} \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} \beta_{22} & 0 & 0 & 0 \\ 0 & \beta_{32} & 0 & 0 \\ 0 & 0 & \beta_{42} & 0 \\ 0 & 0 & 0 & \beta_{12} \end{pmatrix}.$$

Then the connective conditions in (5.18), (5.19) and (5.20) can be rewritten as

$$\begin{cases} Y(0, t) = \mathbf{C}_1 Y(1, t), & Y_x(0, t) = \mathbf{C}_2 Y_x(1, t), \\ \mathbb{E}\mathbb{I} Y_{xxx}(1, t) - \mathbf{C}_1^T \mathbb{E}\mathbb{I} Y_{xxx}(0, t) = \mathbb{K}_1 Y_t(1, t) - \Gamma_1 Y_{xt}(1, t), \\ \mathbb{E}\mathbb{I} Y_{xx}(1, t) - \mathbf{C}_2^T \mathbb{E}\mathbb{I} Y_{xx}(0, t) = -\mathbb{K}_2 Y_{tx}(1, t) + \Gamma_2 Y_t(1, t), \end{cases} \quad (5.21)$$

where we require the matrices \mathbb{K}_s and Γ_s satisfy

$$\begin{bmatrix} \mathbb{K}_1 & -\Gamma_1 \\ -\Gamma_2 & \mathbb{K}_2 \end{bmatrix} + \begin{bmatrix} \mathbb{K}_1 & -\Gamma_1 \\ -\Gamma_2 & \mathbb{K}_2 \end{bmatrix}^T > 0. \quad (5.22)$$

This network of Euler–Bernoulli beams also has the form of (2.1) and satisfies $\det(I - \mathbf{C}_j) \neq 0$.

Theorem 5.4. Let \mathcal{H} be defined as before, \mathcal{A} be the operator determined by (5.21) and $S(t)$ be the C_0 semigroup generated by \mathcal{A} . Then $S(t)$ is asymptotically stable.

Proof. To prove the stability of (5.21), we need only to prove that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. According to Theorem 3.3, we only verify that the conditions

$$\mathbb{K}_1 \varphi(1) - \Gamma_1 \varphi_x(1) = 0, \quad \mathbb{K}_2 \varphi_x(1) - \Gamma_2 \varphi(1) = 0 \quad (5.23)$$

imply $\varphi(x) = 0$, where $\varphi(x) = (\varphi^1(x), \varphi^2(x), \varphi^3(x), \varphi^4(x))^T$ satisfy the equations:

$$\begin{cases} \mathbb{E}\mathbb{I} \varphi_{xxxx}(x) = \gamma \mathbb{M} \varphi(x), \\ \varphi(0) = \mathbf{C}_1 \varphi(1), & \varphi_x(0) = \mathbf{C}_2 \varphi_x(1), \\ \mathbb{E}\mathbb{I} \varphi_{xxx}(1) - \mathbf{C}_1^T \mathbb{E}\mathbb{I} \varphi_{xxx}(0) = 0, \\ \mathbb{E}\mathbb{I} \varphi_{xx}(1) - \mathbf{C}_2^T \mathbb{E}\mathbb{I} \varphi_{xx}(0) = 0. \end{cases}$$

If (5.23) holds, the condition (5.22) implies that $\varphi(1) = \varphi_x(1) = 0$. Therefore, we have

$$\begin{cases} \mathbb{E}\varphi_{xxx}(x) = \gamma \mathbb{M}\varphi(x), \\ \varphi(0) = \varphi(1) = 0, \quad \varphi_x(0) = \varphi_x(1) = 0, \\ \mathbb{E}\varphi_{xxx}(1) - \mathbf{C}_1^T \mathbb{E}\varphi_{xxx}(0) = 0, \\ \mathbb{E}\varphi_{xx}(1) - \mathbf{C}_2^T \mathbb{E}\varphi_{xx}(0) = 0. \end{cases}$$

Note that

$$B = \text{diag}\left(\sqrt[4]{\frac{m_1}{EI_1}}, \sqrt[4]{\frac{m_2}{EI_2}}, \sqrt[4]{\frac{m_3}{EI_3}}, \sqrt[4]{\frac{m_4}{EI_4}}\right).$$

The matrices B , $\mathbb{E}^{-1}\mathbf{C}_1^T\mathbb{E}$ and $\mathbb{E}^{-1}\mathbf{C}_2^T\mathbb{E}$ have no common eigenvector. According to Theorem 3.2, we have $\varphi(x) = 0$. Therefore, $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. The desired result follows from the stability theorem in [31]. \square

Remark 5.2. By now we only proved the asymptotical stability of the tree-shaped network of Euler–Bernoulli beams with boundary controls due to the difficulty in calculation of $\Delta(\rho)$. We guess that it also is exponentially stable. For a complex network given in Section 5.3, it is possible to be only asymptotically stable but not exponentially stable; however, we do not prove it. We shall prove these conjectures in the future work.

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