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Asymptotics on Laguerre or Hermite polynomial expansions and their applications in Gauss quadrature

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ABSTRACT

In this paper, we present asymptotic analysis on the coefficients of functions expanded in forms of Laguerre or Hermite polynomial series, which shows the decay of the coefficients and derives new error bounds on the truncated series. Moreover, by applying the asymptotics, new estimates on the errors for Gauss–Laguerre, Radau–Laguerre and Gauss–Hermite quadrature are deduced. These results show that Gauss–Laguerre-type and Gauss-Hermite-type quadratures are nearly of same convergence rates.

1. Introduction

Laguerre polynomials $L_n^{(\alpha)}(x)$ and Hermite polynomials $H_n(x)$ are well-known in Gaussian quadrature to numerically compute integrals of the forms

$$\int_0^{+\infty} x^\alpha e^{-x} f(x)dx \quad (\alpha > -1), \qquad \int_{-\infty}^{+\infty} e^{-x^2} f(x)dx.$$

Laguerre or Hermite expansions have many uses in the Mathieu equation, prolate spheroidal wave equation, Laplace's tidal equation, Vlasov–Maxwell equation, quantum mechanics etc. The expressions of the derivatives of these polynomials are quite simple and thus it is easy to use them to solve differential equations [1–8].

The decay of the coefficients of $f(x)$ expanded in an orthogonal polynomial series in a finite interval has been extensively studied [9,1,10–16]. Unlike most other sets of orthogonal polynomials in a finite interval, the Laguerre and Hermite polynomials increase exponentially with the degree n, so it is difficult to work with unnormalized functions without encountering overflow [1,17].

Suppose $f(x)$ can be expanded in the form of series of $\{L_j^{(\alpha)}(x)\}_{j=0}^{\infty}$ or $\{H_j(x)\}_{j=0}^{\infty}$ [1,10,18–21]

$$f(x) = \sum_{j=0}^{\infty} a_j L_j^{(\alpha)}(x), \quad a_j = \frac{1}{\sigma_j^\alpha} \int_0^{+\infty} e^{-x} x^\alpha f(x) L_j^{(\alpha)}(x)dx \tag{1.1a}$$

$$f(x) = \sum_{j=0}^{\infty} h_j H_j(x), \quad h_j = \frac{1}{\gamma_n} \int_{-\infty}^{+\infty} e^{-x^2} f(x) H_j(x)dx. \tag{1.1b}$$

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A natural approximation to $f(x)$ is the truncated polynomial

$$\mathcal{P}_N^f(x)=\sum_{j=0}^{N}a_jL_j^{(\alpha)}(x)\quad\text{or}\quad \mathcal{P}_N^f(x)=\sum_{j=0}^{N}h_jH_j(x).$$

The Parseval identity leads to a truncated error

$$\|f(x)-\mathcal{P}_N^f(x)\|^2_{L^2_{w(x)}[0,+\infty)}=\sum_{j=N+1}^{\infty}a_j^2\sigma_j\quad\text{or}\quad \|f(x)-\mathcal{P}_N^f(x)\|^2_{L^2_{w(x)}(-\infty,+\infty)}=\sum_{j=N+1}^{\infty}a_j^2\gamma_j,$$

which implies that the convergence of the truncated error solely depends on the decay of the expansion coefficients [20].

Let $\{x_j\}_{j=1}^N$ be zeros of $L_N^{(\alpha)}(x)$ or $H_N(x)$, and w_i be the weights in the Gauss–Laguerre quadrature $Q_N^{GL}[f]$ or Gauss–Hermite quadrature $Q_N^{GH}[f]$. Here x_i and w_i can be computed quickly by Golub and Welsch [22] with $O(N^2)$ operations and Glaser et al. [17] with $O(N)$ operations, respectively (the efficient algorithms can be found in [23]).

Using the orthogonality of the polynomials, from $I[L_n^{(\alpha)}(x)]=0$ and $I[H_n(x)]=0$ for $n\ge 1$, and $Q_N^{GL}[L_n^{(\alpha)}(x)]=I[L_n^{(\alpha)}(x)]$ and $Q_N^{GH}[H_n(x)]=I[H_n(x)]$ for $0\le n\le 2N-1$, we see that

$$I[f]-Q_N^{GL}[f]=\sum_{n=2N}^{\infty}a_nQ_N^{GL}[L_n^{(\alpha)}(x)]$$

and

$$I[f]-Q_N^{GH}[f]=\sum_{n=2N}^{\infty}h_nQ_N^{GH}[H_n(x)],$$

which implies that the error bounds for Gauss–Laguerre and Gauss–Hermite quadrature can be estimated by the asymptotics of the coefficents of the expansions.

The following error estimates are widely cited [18, p. 223]

$$\int_0^{+\infty}x^\alpha e^{-x}f(x)dx=\sum_{n=1}^{N}w_nf(x_n)+\frac{(N!)^2}{(2N)!}f^{(2N)}(\xi),\quad 0<\xi<+\infty,\tag{1.2a}$$

$$\int_{-\infty}^{+\infty}e^{-x^2}f(x)dx=\sum_{n=1}^{N}w_nf(x_n)+\frac{N!\sqrt{\pi}}{2^N(2N)!}f^{(2N)}(\xi),\quad -\infty<\xi<+\infty.\tag{1.2b}$$

However, in (1.2a)–(1.2b), ξ is difficult to determine. In particular, for some special functions such as $f(x)=\sin(x)e^{x/2}$, the estimate on $f^{(2N)}(\xi)$ can be very large if ξ is not specified.

Considering the convergence of formulas of the Gauss–Laguerre and Gauss–Hermite quadrature, Uspensky [24] showed that if the function $f(x)$ satisfies the inequality for all sufficiently large values of x

$$|f(x)|\le\frac{e^x}{x^{\alpha+1+\rho}},\quad\text{for some }\rho>0,$$

or

$$|f(x)|\le\frac{e^{x^2}}{|x|^{1+\rho}},\quad\text{for some }\rho>0,$$

then

$$\lim_{N\to\infty}Q_N^{GL}[f]=\int_0^{+\infty}x^\alpha e^{-x}f(x)dx,\qquad \lim_{N\to\infty}Q_N^{GH}[f]=\int_{-\infty}^{+\infty}e^{-x^2}f(x)dx,$$

respectively. Particularly, for entire functions represented by $f(z)=\sum_{n=0}^{\infty}b_nz^n$, Lubinsky [25] proved geometric convergence of $Q_N^{GL}[f]$ and $Q_N^{GH}[f]$: Let

$$A=\limsup_{n\to\infty}\frac{n\sqrt[n]{|b_n|}}{2},\qquad B=\limsup_{n\to\infty}\sqrt[n]{|b_n|}\sqrt{n/2}.\tag{1.3}$$

If $A<1$ and $B<1$ then, for sufficiently large N,

$$\left|\int_0^{+\infty}x^\alpha e^{-x}f(x)dx-Q_N^{GL}[f]\right|\le A^{2N}\tag{1.4a}$$

$$\left|\int_{-\infty}^{+\infty}e^{-x^2}f(x)dx-Q_N^{GH}[f]\right|\le B^{2N}.\tag{1.4b}$$

In this paper, we will present new asymptotics on the coefficients a_n and h_n for the Laguerre and Hermite expansions. Applying these asymptotics, we will derive new error bounds on the truncated series, Gauss–Laguerre and Gauss–Hermite type quadrature.

2. Laguerre expansions and Gauss–Laguerre quadrature

Assume $f(x)$ is a suitably smooth function in $[0, +\infty)$ of finite regularity and $\int_0^{+\infty} e^{-x}x^\alpha f(x)dx < \infty$ for $\alpha > -1$. Then $f(x)$ can be expanded with respect to $w(x) = e^{-x}x^\alpha$ into

$$f(x) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) \tag{2.1}$$

[26, p. 110] with the expansion coefficient

$$a_n = \frac{1}{\sigma_n^\alpha} \int_0^{+\infty} e^{-x}x^\alpha f(x) L_n^{(\alpha)}(x)dx,$$

where $L_n^{(\alpha)}(x)$ is the Laguerre polynomial of degree n and

$$\sigma_n^\alpha = \frac{\Gamma(n+\alpha+1)}{n!}$$

[27, p. 774].

Theorem 2.1. *Suppose $f, f', \ldots, f^{(k-1)}$ are absolutely continuous in $[0, +\infty)$ and satisfies for $j = 0, 1, \ldots, k$ for some $k \geq 1$ that*

$$\lim_{x\to+\infty} e^{-x/2}x^{1+j+\alpha} f^{(j)}(x) = 0, \qquad V = \sqrt{\int_0^{+\infty} x^{1+k+\alpha} e^{-x}[f^{(k+1)}(x)]^2 dx} < \infty, \tag{2.2}$$

then for the Laguerre expansion it follows that

$$|a_n| \leq \frac{V}{\sqrt{n(n-1)\cdots(n-k)}} \sqrt{\frac{n!}{\Gamma(1+n+\alpha)}}, \quad k \geq 1 \tag{2.3a}$$

$$\|f(x) - \mathcal{P}_N^f(x)\|_{L^2_w[0,+\infty)} \leq \frac{2V\sqrt{N}}{(k-1)\sqrt{(N-1)\cdots(N-k)}}, \quad k \geq 2. \tag{2.3b}$$

Proof. From Rodrigues's formulas [26, p. 101]

$$e^{-x}x^\alpha L_n^{(\alpha)}(x) = \frac{1}{n!} \frac{d^n\left[e^{-x}x^{n+\alpha}\right]}{dx^n} = \frac{1}{n} \cdot \frac{1}{(n-1)!} \frac{d}{dx}\left[\frac{d^{n-1}(e^{-x}x^{n+\alpha})}{dx^{n-1}}\right]$$

we see that

$$ne^{-x}x^\alpha L_n^{(\alpha)}(x) = \frac{de^{-x}x^{1+\alpha}L_{n-1}^{(1+\alpha)}(x)}{dx}$$

and

$$\begin{aligned} a_n &= \frac{1}{n\sigma_n^\alpha} \int_0^{+\infty} f(x) de^{-x}x^{1+\alpha}L_{n-1}^{(1+\alpha)}(x) \\ &= -\frac{1}{n\sigma_n^\alpha} \int_0^{+\infty} e^{-x}x^{1+\alpha}L_{n-1}^{(1+\alpha)}(x)f'(x)dx \\ &= \cdots \\ &= \frac{(-1)^{k+1}}{\sigma_n^\alpha n(n-1)\cdots(n-k)} \int_0^{+\infty} e^{-x}x^{1+k+\alpha}L_{n-k-1}^{(1+k+\alpha)}(x)f^{(k+1)}(x)dx, \end{aligned}$$

where we used the following inequalities [27, p. 786], [19, p. 31]

$$|e^{-x/2}L_n^{(\alpha)}(x)| \leq \begin{cases} \left(2 - \dfrac{\Gamma(1+\alpha+n)}{n!\Gamma(1+\alpha)}\right), & -1 < \alpha \leq 0 \\ \dfrac{\Gamma(1+\alpha+n)}{n!\Gamma(1+\alpha)}, & \alpha > 0 \end{cases} \quad x \geq 0,\ n = 0, 1, \ldots, \tag{2.4}$$

and identities for $j = 0, 1, \ldots, k$

$$e^{-x}x^{1+j+\alpha}f^{(j)}(x)L_{n-j-1}^{(1+j+\alpha)}(x)\,|_0^{+\infty} = e^{-x/2}x^{1+j+\alpha}f^{(j)}(x)e^{-x/2}L_{n-j-1}^{(1+j+\alpha)}(x)\,|_0^{+\infty} = 0.$$

By using the Cauchy–Schwarz inequality we deduce

$$
\begin{aligned}
|a_n| &= \left|\frac{(-1)^{k+1}}{\sigma_n^\alpha n(n-1)\cdots(n-k)}\int_0^{+\infty} e^{-x}x^{1+k+\alpha}L_{n-k-1}^{(1+k+\alpha)}(x)f^{(k+1)}(x)dx\right| \\
&= \frac{\left|\int_0^{+\infty}\left[e^{-x/2}x^{(1+k+\alpha)/2}L_{n-k-1}^{(1+k+\alpha)}(x)\right]\left[e^{-x/2}x^{(1+k+\alpha)/2}f^{(k+1)}(x)\right]dx\right|}{\sigma_n^\alpha n(n-1)\cdots(n-k)} \\
&\le \frac{V\sqrt{\sigma_{n-k-1}^{1+k+\alpha}}}{\sigma_n^\alpha n(n-1)\cdots(n-k)},
\end{aligned} \tag{2.5}
$$

which together with

$$
\frac{\sqrt{\sigma_{n-k-1}^{1+k+\alpha}}}{\sigma_n^\alpha} = \sqrt{n(n-1)\cdots(n-k)\frac{n!}{\Gamma(1+n+\alpha)}}
$$

yields (2.3a).

Expression (2.3b) follows from

$$
\begin{aligned}
\|f(x)-\mathcal{P}_N^f(x)\|_{L_w^2[0,+\infty)} &= \left[\sum_{n=N+1}^{\infty}|a_n|^2\sigma_n^\alpha\right]^{\frac{1}{2}} \\
&\le \sum_{n=N+1}^{\infty}|a_n|\sqrt{\sigma_n^\alpha} \\
&\le \sum_{n=N+1}^{\infty}\frac{V}{n(n-1)\cdots(n-k)}\sqrt{\frac{\sigma_{n-k-1}^{1+k+\alpha}}{\sigma_n^\alpha}} \quad (\text{by } (2.5)) \\
&= \sum_{n=N+1}^{\infty}\frac{V}{\sqrt{n(n-1)\cdots(n-k)}} \\
&\le \frac{V}{\sqrt{\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{k}{N}\right)}}\sum_{n=N+1}^{\infty}\frac{1}{n^{\frac{k+1}{2}}} \\
&\le \frac{V}{\sqrt{\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{k}{N}\right)}}\int_N^{+\infty}\frac{1}{x^{\frac{k+1}{2}}}dx \\
&\le \frac{2V\sqrt{N}}{(k-1)\sqrt{(N-1)\cdots(N-k)}}. \quad \square
\end{aligned}
$$

Remark 1. From Theorem 2.1, we see that, for $\alpha = 0$,

$$
\left\|e^{-x/2}f(x)-\sum_{n=0}^{N}a_n\widetilde{L}_n(x)\right\|_\infty \le \sum_{n=N+1}^{\infty}|a_n| \le \frac{2V\sqrt{N}}{(k-1)\sqrt{(N-1)\cdots(N-k)}},
$$

where $\widetilde{L}_n(x) = e^{-x/2}L_n(x)$.

The asymptotics can be applied to establish the computational error bounds for Gauss–Laguerre quadrature for functions of finite regularity.

Theorem 2.2 (*Error Bounds for Gauss–Laguerre Quadrature*)**.** *Suppose $f(x)$ satisfies (2.2) for some $k \ge 3$, then for each $N \ge (k+1)/2+1$,*

$$
|I[f]-Q_N^{GL}[f]| \le \begin{cases} \dfrac{2^{3+\alpha}V(2N-1)}{(k-2)\sqrt{(2N-2)(2N-3)\cdots(2N-k-1)}}, & -1<\alpha<0 \\ \dfrac{4V\sqrt{2N-1}}{(k-1)\sqrt{(2N-2)(2N-3)\cdots(2N-k-1)}}, & \alpha = 0 \\ \dfrac{4V(2N-1)}{(k-2)\sqrt{(2N-2)(2N-3)\cdots(2N-k-1)}}, & 0<\alpha\le 1. \end{cases} \tag{2.6}
$$

Proof. From expression (2.1) and by $I[L_n^{(\alpha)}(x)] = 0$ for $n \geq 1$, we have

$$|I[f] - Q_N^{GL}[f]| = \left|\sum_{n=2N}^{\infty} a_n Q_N^{GL}[L_n^{(\alpha)}(x)]\right| \leq \sum_{n=2N}^{\infty} |a_n|\, |Q_N^{GL}[L_n^{(\alpha)}(x)]|.$$

Applying [18, p. 223]

$$w_i = \frac{\Gamma(1+\alpha+N)}{N!} \cdot \frac{x_i}{[L_{N+1}^{(\alpha)}(x_i)]^2} \quad (x_i \text{ are the zeros of } L_N^{(\alpha)}(x))$$

yields

$$\begin{aligned}
|Q_N^{GL}[L_n^{(\alpha)}(x)]| &= \left|\sum_{i=1}^{N} w_i L_n^{(\alpha)}(x_i)\right| \leq \sum_{i=1}^{N} \frac{\Gamma(1+\alpha+N)}{N!} \cdot \frac{x_i e^{x_i/2}}{[L_{N+1}^{(\alpha)}(x_i)]^2} |e^{-x_i/2} L_n^{(\alpha)}(x_i)| \\
&\leq \|e^{-x/2} L_n^{(\alpha)}(x)\|_\infty Q_N^{GL}[e^{x/2}] \\
&\leq \begin{cases} 2^{1+\alpha}\Gamma(1+\alpha)\left(2 - \dfrac{\Gamma(1+\alpha+n)}{n!\,\Gamma(1+\alpha)}\right), & -1 < \alpha < 0 \\ 2, & \alpha = 0 \\ 2^{1+\alpha}\dfrac{\Gamma(1+\alpha+n)}{n!}, & 0 < \alpha \end{cases} \\
&\leq \begin{cases} 2^{2+\alpha}\Gamma(1+\alpha), & -1 < \alpha < 0 \\ 2, & \alpha = 0 \\ 2^{1+\alpha}\dfrac{\Gamma(1+\alpha+n)}{n!}, & 0 < \alpha, \end{cases}
\end{aligned}$$

where in the proof of the above third inequality we use inequality (2.4) and the estimate on $Q_N^{GL}[e^{x/2}]$ by (1.2a)

$$\begin{aligned}
0 \leq Q_N^{GL}[e^{x/2}] &= \sum_{i=1}^{N} \frac{\Gamma(1+\alpha+N)}{N!} \cdot \frac{x_i e^{x_i/2}}{[L_{N+1}^{(\alpha)}(x_i)]^2} \\
&= \int_0^{+\infty} e^{-x} x^\alpha e^{x/2} dx - \frac{(N!)^2}{(2N)!} (e^{x/2})^{(2N)}(\xi) \\
&\leq \int_0^{+\infty} e^{-x} x^\alpha e^{x/2} dx \\
&= 2^{1+\alpha}\Gamma(1+\alpha).
\end{aligned}$$

These together with (2.3a) yield

$$\begin{aligned}
|I[f] - Q_N^{GL}[f]| &\leq \sum_{n=2N}^{\infty} \frac{V|Q_N^{GL}[L_n^{(\alpha)}(x)]|}{\sqrt{n(n-1)\cdots(n-k)}} \sqrt{\frac{n!}{\Gamma(1+n+\alpha)}} \\
&\leq \begin{cases} \displaystyle\sum_{n=2N}^{\infty} \frac{2^{2+\alpha}\Gamma(1+\alpha)V}{\sqrt{n(n-1)\cdots(n-k)}} \sqrt{\frac{n!}{\Gamma(1+n+\alpha)}}, & -1 < \alpha < 0 \\ \displaystyle\sum_{n=2N}^{\infty} \frac{2V}{\sqrt{n(n-1)\cdots(n-k)}}, & \alpha = 0 \\ \displaystyle\sum_{n=2N}^{\infty} \frac{2^{1+\alpha}V}{\sqrt{n(n-1)\cdots(n-k)}} \sqrt{\frac{\Gamma(1+n+\alpha)}{n!}}, & 0 < \alpha \leq 1, \end{cases} \\
&\leq \begin{cases} \displaystyle\sum_{n=2N}^{\infty} \frac{2^{2+\alpha}\Gamma(1+\alpha)V}{\sqrt{(n-1)\cdots(n-k)}}, & -1 < \alpha < 0 \\ \displaystyle\sum_{n=2N}^{\infty} \frac{2V}{\sqrt{n(n-1)\cdots(n-k)}}, & \alpha = 0 \\ \displaystyle\sum_{n=2N}^{\infty} \frac{2^{1+\alpha}V\sqrt{1+\frac{1}{2N}}}{\sqrt{(n-1)\cdots(n-k)}}, & 0 < \alpha \leq 1, \end{cases}
\end{aligned}$$

where in the proof of the last inequality we use

$$\frac{n!}{\Gamma(1+n+\alpha)} \le n \quad \text{for } -1<\alpha<0, \qquad \frac{\Gamma(1+n+\alpha)}{n!} \le n+1 \quad \text{for } 0<\alpha\le 1.$$

By a similar proof to (2.3b) on $\sum_{n=2N}^{\infty} \frac{1}{\sqrt{(n-1)\cdots(n-k)}}$ it leads to the desired result. □

For entire functions the geometric convergence of $Q_N^{GL}[f]$ can be improved as

Theorem 2.3. *Suppose $f(x)=\sum_{n=0}^{\infty} b_n x^n$ and*

$$A_1 = e^{-1} \limsup_{n\to\infty} n\sqrt[n]{|b_n|} < 1$$

then for each $\delta>0$ with $A_1+\delta<1$, there exists $N_0>0$ such that for $N>N_0$

$$\left|\int_0^{+\infty} x^{\alpha} e^{-x} f(x)dx - Q_N^{GL}[f]\right| \le \frac{(A_1+\delta)^{2N}}{1-A_1-\delta}, \quad -1<\alpha. \tag{2.7}$$

Proof. From (1.2a) it follows that

$$0 \le Q_N^{GL}[x^n] \le \int_0^{+\infty} x^{n+\alpha} e^{-x} dx = \Gamma(1+\alpha+n)$$

and then

$$\begin{aligned}\left|\int_0^{+\infty} x^{\alpha} e^{-x} f(x)dx - Q_N^{GL}[f]\right| &\le \sum_{n=2N}^{\infty} |b_n|\,|I[x^n]-Q_N^{GL}I[x^n]| \\ &\le \sum_{n=2N}^{\infty} |b_n| I[x^n] \\ &= \sum_{n=2N}^{\infty} |b_n| \Gamma(1+\alpha+n).\end{aligned}$$

Applying $\Gamma(n+\eta) \sim \sqrt{2\pi} e^n n^{n+\eta-\frac{1}{2}}$ [27, Eq. (6.1.39)] yields

$$\lim_{n\to\infty} \sqrt[n]{\frac{\Gamma(1+\alpha+n)}{n!}} = 1, \qquad \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = e^{-1}, \qquad \limsup_{n\to\infty} \sqrt[n]{|b_n|\Gamma(1+\alpha+n)} = A_1.$$

Thus, for each $\delta>0$ with $A_1+\delta<1$, there exists $N_0>0$ such that for $n>N_0$

$$|b_n|\Gamma(1+\alpha+n) \le (A_1+\delta)^n.$$

These together prove (2.7). □

Remark 2. Comparing A_1 with A in (1.3), we find that $A_1 = 2e^{-1}A$, which shows that the upper bound in Theorem 2.3 is sharper than that given by Lubinsky [25].

Corollary 2.1 (*Error Bounds for Radau–Laguerre Quadrature*)**.** *Suppose $f(x)$ satisfies (2.2) for some $k\ge 3$, then for each $N\ge k/2+1$,*

$$|I[f]-Q_N^{RL}[f]| \le \begin{cases} \dfrac{2^{4+\alpha}VN}{(k-2)\sqrt{(2N-1)(2N-2)\cdots(2N-k)}}, & -1<\alpha<0 \\ \dfrac{4V\sqrt{2N}}{(k-1)\sqrt{(2N-1)(2N-2)\cdots(2N-k)}}, & \alpha=0 \\ \dfrac{8VN}{(k-2)\sqrt{(2N-1)(2N-2)\cdots(2N-k)}}, & 0<\alpha\le 1. \end{cases} \tag{2.8}$$

Proof. Corresponding to the Radau rule with a preassigned abscissa at 0

$$Q_N^{RL}[f] = \frac{N!\Gamma(1+\alpha)\Gamma(2+\alpha)}{\Gamma(2+\alpha+N)} f(0) + \sum_{n=1}^{N} \hat{w}_n f(\hat{x}_n)$$

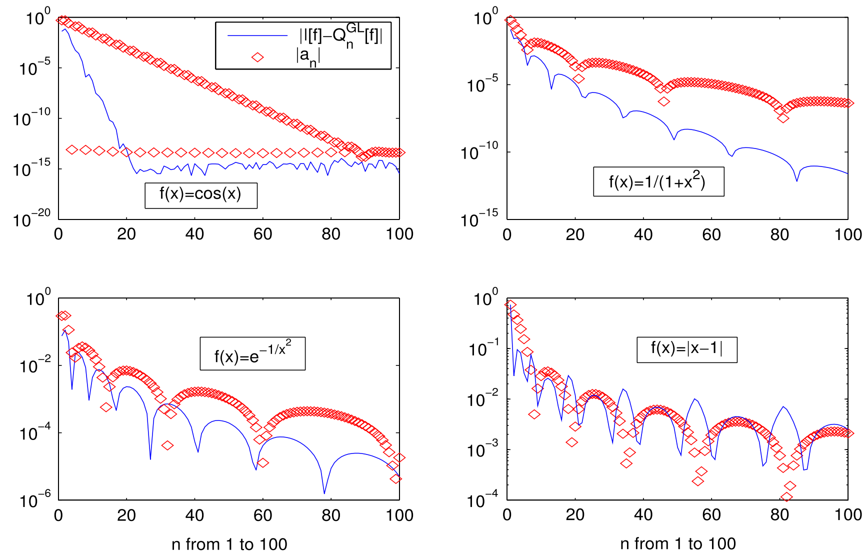

Fig. 1. Absolute errors $|I[f]-Q_n^{GL}[f]|$ for Gauss–Laguerre quadrature and a_n: $n=1:100$.

(see [18, p. 223]), it follows that

$$\int_0^{+\infty} x^\alpha e^{-x} f(x)dx = \frac{N!\Gamma(1+\alpha)\Gamma(2+\alpha)}{\Gamma(2+\alpha+N)} f(0) + \sum_{n=1}^{N} \hat{w}_n f(\hat{x}_n) + \frac{N!\Gamma(2+\alpha+N)}{(2N+1)!} f^{(2N+1)}(\zeta)$$

for some $0<\zeta<+\infty$ (see [18, p 224]), where $\hat{x}_i$ are the zeros of $L_N^{(1+\alpha)}(x)$ and

$$\hat{w}_i = \frac{\Gamma(1+\alpha+N)}{N!(1+\alpha+N)[L_N^{(\alpha)}(\hat{x}_i)]^2}.$$

Applying a similar proof to Theorem 2.2 yields (2.8). □

Remark 3. From the proof of Theorem 2.1 and by using $\frac{\Gamma(1+\alpha+n)}{n!} = O(n^\alpha)$ [27,26], we see that

$$|a_N| = O(N^{-(k+1+\alpha)/2}), \quad \alpha > -1$$

and

$$I[f]-Q_N^{GL}[f] = O(N^{-(k-1-|\alpha|)/2}), \qquad I[f]-Q_N^{RL}[f] = O(N^{-(k-1-|\alpha|)/2}), \quad -1<\alpha\le 1,$$

which shows that the smoother $f(x)$ is, the faster the decay of the coefficients and the errors of Gauss-type quadrature are as N increases.

Remark 4. Comparing the error bounds of Gauss–Laguerre quadrature with Radau–Laguerre quadrature, we see that these two quadratures have almost the same convergence.

In the following, we illustrate the Gauss–Laguerre quadrature $Q_N^{GL}[f]$ ($\alpha=0$) and the asympotics of the coefficients a_n for $f(x)$ being an entire function $\cos(x)$, an analytic function $\frac{1}{1+x^2}$ in a neighborhood of $[0,+\infty)$ but not throughout the complex plane, a C^∞ function e^{-1/x^2} and a nonsmooth function $|x-1|$, respectively (see Fig. 1), where $a_n = \int_0^\infty e^{-x} f(x) L_n(x)dx$ is computed by Gauss–Laguerre quadrature Q_N^{GL} with $N=1500$.

3. Hermite expansions and Gauss–Hermite quadrature

In this section, we restrict our attention to the asymptotics of the coefficients of $f(x)$ expanded in the form of Hermite polynomial series. Assume $f(x)$ is a suitably smooth function in $(-\infty,+\infty)$ of finite regularity and

$$\int_{-\infty}^{+\infty} e^{-x^2} f(x)dx < \infty.$$

Then $f(x)$ can be expanded corresponding to $w(x) = e^{-x^2}$ into

$$f(x) = \sum_{n=0}^{\infty} h_n H_n(x), \tag{3.1}$$

with the expansion coefficient

$$h_n = \frac{1}{\gamma_n} \int_{-\infty}^{+\infty} e^{-x^2} f(x) H_n(x) dx,$$

where $H_n(x)$ is of degree n and

$$\gamma_n = \sqrt{\pi} 2^n n!, \qquad H_n'(x) = 2nH_{n-1}(x), \qquad e^{-x^2} H_n(x) = (-1)^n \frac{d^n e^{-x^2}}{dx^n} \tag{3.2}$$

(see [27, p. 774] and [26, pp. 105–106,110]).

Theorem 3.1. *Suppose $f, f', \ldots, f^{(k-1)}$ are absolutely continuous in $(-\infty, +\infty)$ and satisfies for $j = 0, 1, \ldots, k$ for some $k \geq 1$ that*

$$\lim_{x\to\infty} e^{-x^2/2} f^{(j)}(x) = 0, \qquad U = \sqrt{\int_{-\infty}^{+\infty} e^{-x^2} [f^{(k+1)}(x)]^2 dx} < \infty, \tag{3.3}$$

then for the Hermite expansion it follows that

$$|h_n| \leq \frac{U}{2^{\frac{n+k+1}{2}} \sqrt[4]{\pi} n(n-1)\cdots(n-k)\sqrt{(n-k-1)!}} \tag{3.4a}$$

$$\|f(x) - \mathcal{P}_N^f(x)\|_{L^2_w(-\infty,+\infty)} \leq \frac{U\sqrt{N}}{(k-1)2^{(k+1)/2}\sqrt{(N-1)\cdots(N-k)}}. \tag{3.4b}$$

Proof. Integrating by parts, it establishes by (3.2) and (3.3) and Cramér's inequality [27, p. 787]

$$|e^{-x^2/2} H_n(x)| \leq c_0 2^{n/2} \sqrt{n!}, \quad c_0 \approx 1.086435$$

that

$$\begin{aligned} |h_n| &= \left| \frac{1}{\gamma_n} \int_{-\infty}^{+\infty} f(x) d[e^{-x^2}]^{(n-1)} \right| = \left| \frac{1}{\gamma_n} \int_{-\infty}^{+\infty} f'(x) [e^{-x^2}]^{(n-1)} dx \right| \\ &= \cdots \\ &= \left| \frac{1}{\gamma_n} \int_{-\infty}^{+\infty} f^{(k+1)}(x) [e^{-x^2}]^{(n-k-1)} dx \right| \\ &= \left| \frac{1}{\gamma_n} \int_{-\infty}^{+\infty} f^{(k+1)}(x) e^{-x^2} H_{n-k-1}(x) dx \right| \\ &\leq \frac{U\sqrt{\gamma_{n-k-1}}}{\gamma_n} \quad \text{(Cauchy–Schwarz inequality)} \end{aligned}$$

which yields (3.4a).

Expression (3.4b) directly follows by a similar proof to (2.3b). □

Theorem 3.2 (*Error Bounds for Gauss–Hermite Quadrature*)**.** *Suppose $f(x)$ satisfies (3.3) for some $k \geq 2$, then for each $N \geq k/2 + 1$,*

$$|I[f] - Q_N^{GH}[f]| \leq \frac{1.632\sqrt{\pi(N-1)}U}{(k-1)\sqrt{(2N-3)\cdots(2N-k-2)}}. \tag{3.5}$$

Proof. To easily control the overflow on $H_n(x)$, following [1, p. 506] and [17], we define

$$\overline{H}_n(x) = \frac{1}{\pi^{\frac{1}{4}} 2^{n/2}\sqrt{n!}} H_n(x) := c_n H_n(x), \qquad \widetilde{H}_n(x) = \frac{e^{-x^2/2}}{\pi^{\frac{1}{4}} 2^{n/2}\sqrt{n!}} H_n(x)$$

and consider

$$f(x)=\sum_{n=0}^{\infty}\overline{h}_n\overline{H}_n(x)$$

with the expansion coefficient

$$\overline{h}_n=\frac{1}{c_n\gamma_n}\int_{-\infty}^{+\infty}e^{-x^2}f(x)H_n(x)dx.$$

In the same way as the proof of (3.4a), we have

$$|\overline{h}_n|\le\frac{U\sqrt{\gamma_{n-k-1}}}{c_n\gamma_n}=\frac{U}{2^{\frac{k+1}{2}}\sqrt{n(n-1)\cdots(n-k)}},\tag{3.6}$$

which, together with $I[H_n(x)]=0$ and $Q_N^{GH}[H_{2n-1}(x)]=Q_N^{GH}[\overline{H}_{2n-1}(x)]=0$ for $n\ge 1$, yields

$$|I[f]-Q_N^{GH}[f]|\le\sum_{n=N}^{\infty}\frac{U|Q_N^{GH}[\overline{H}_{2n}(x)]|}{\sqrt{2n(2n-1)\cdots(2n-k)}}.$$

Notice that by Glaser et al. [17] we see that for n even

$$\begin{aligned}|Q_N^{GH}[\overline{H}_n(x)]|&=\sum_{j=1}^{N}\frac{2e^{-x_j^2}\overline{H}_n(x)}{[H_N'(x_j)]^2}=\sum_{j=1}^{N}\frac{2e^{-x_j^2/2}}{[H_N'(x_j)]^2}e^{-x_j^2/2}\overline{H}_n(x_j)\\&\le 0.816Q_N^{GH}[e^{x^2/2}],\end{aligned}$$

where we used $e^{-x^2/2}|\overline{H}_m(x)|=|\widetilde{H}_m(x)|\le 0.816$ for all x [1, p. 506].

Furthermore, noting by (1.2b) that

$$\int_{-\infty}^{+\infty}e^{-x^2}e^{x^2/2}dx=Q_N^{GH}[e^{x^2/2}]+\frac{N!\sqrt{\pi}}{2^N(2N)!}[e^{x^2/2}]^{(2N)}(\xi_0),\quad -\infty<\xi_0<+\infty$$

and observing

$$[e^{x^2/2}]'=xe^{x^2/2},\qquad [e^{x^2/2}]''=(1+x^2)e^{x^2/2},\qquad [e^{x^2/2}]^{(3)}=(3x+x^3)e^{x^2/2},\qquad [e^{x^2/2}]^{(4)}=(3+6x^2+x^4)e^{x^2/2},$$

it is easy to verify by induction that

$$[e^{x^2/2}]^{(2k-1)}=xp_{k-1}(x^2)e^{x^2/2},\qquad [e^{x^2/2}]^{(2k)}=p_k(x^2)e^{x^2/2},$$

where $p_{k-1}(t)$ and $p_k(t)$ are polynomials of degree $k-1$ and k respectively whose coefficients are nonnegative. Thus, $[e^{x^2/2}]^{(2N)}(\xi_0)\ge 0$, $0<Q_N^{GH}[e^{x^2/2}]\le I[e^{x^2}]=\sqrt{2\pi}$ and

$$|Q_N^{GH}[\overline{H}_n(x)]|\le 0.816Q_N^{GH}[e^{x^2/2}]\le 0.816\sqrt{2\pi}.$$

These together yield

$$|I[f]-Q_N^{GH}[f]|\le\sum_{n=N}^{\infty}\frac{0.816\sqrt{2\pi}U}{\sqrt{2n(2n-1)\cdots(2n-k)}}.$$

Then by a similar proof to (2.3b) it directly leads to the desired result. □

Remark 5. For expansion $f(x)=\sum_{n=0}^{\infty}\overline{h}_n\overline{H}_n(x)$, even though $\overline{h}_n$ decays much slower than h_n. However, from the proof of Theorem 3.2, it follows that

$$\left\|f(x)-\sum_{n=0}^{N}\overline{h}_n\overline{H}_n(x)\right\|_{L_w^2(-\infty,+\infty)}\le\frac{U\sqrt{N}}{(k-1)2^{(k+1)/2}\sqrt{(N-1)\cdots(N-k)}},$$

which is the same as (3.4b). Furthermore, from Theorem 3.1, we find that

$$\left\|e^{-x^2/2}f(x)-\sum_{n=1}^{N}\overline{h}_n\widetilde{H}_n(x)\right\|_{\infty}\le 0.816\sum_{n=N+1}^{\infty}|\overline{h}_n|\le\frac{0.816U\sqrt{N}}{(k-1)2^{(k-1)/2}\sqrt{(N-1)\cdots(N-k)}}.\tag{3.7}$$

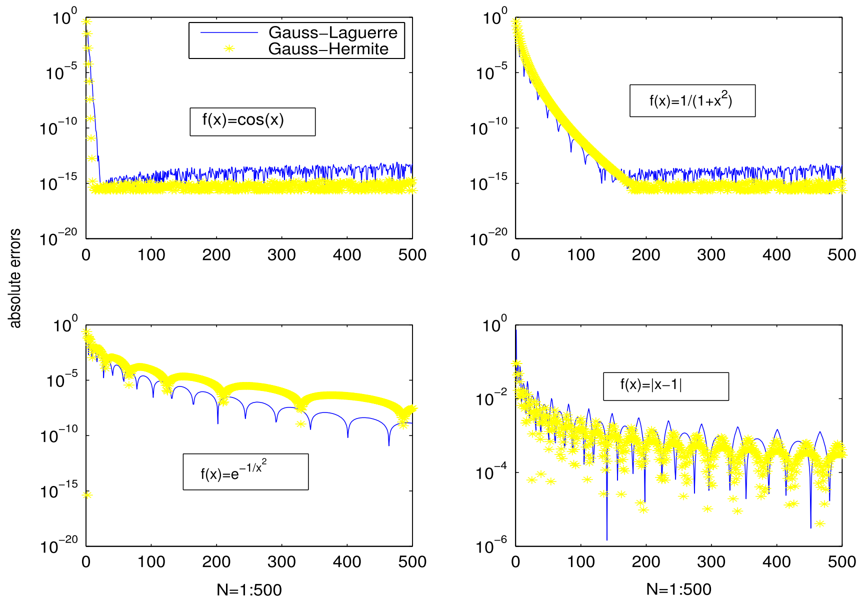

Fig. 2. Absolute errors for Gauss–Laguerre and Gauss–Hermite quadrature: $N = 1 : 500$.

Remark 6. For normalized Hermite functions, Boyd [28] showed that for $f(x) = \sum_{n=0}^{\infty} \widetilde{a}_n \widetilde{H}_n(x)$ with $\widetilde{H}_n(x) = \frac{e^{-x^2/2}}{\pi^{1/4} 2^{n/2} \sqrt{n!}} H_n(x)$

$$\widetilde{a}_n = O(n^{-(k+1)/2})$$

under the condition

$$x^\ell f^{(j)}(x) \quad \text{are bounded and integrable in } (-\infty, +\infty) \text{ for } \ell, j = 0, 1, \ldots, k+1.$$

It is easy to verify that $f(x)$ satisfies (3.3) and then $\widetilde{a}_n = \overline{h}_n$.

Theorem 3.3. *Suppose $f(x) = \sum_{n=0}^{\infty} b_n x^n$ is an entire function and*

$$B_1 = 2e^{-1} \limsup_{n \to \infty} n \sqrt[n]{|b_{2n}|} < 1,$$

then for each $\delta > 0$ with $B_1 + \delta < 1$, there exists $N_0 > 0$ such that for $N > N_0$

$$\left| \int_{-\infty}^{+\infty} e^{-x^2} f(x) dx - Q_N^{GH}[f] \right| \le \frac{(B_1 + \delta)^N}{1 - B_1 - \delta}. \tag{3.8}$$

Proof. From (1.2b), it follows that

$$0 \le Q_N^{GH}[x^{2n}] \le \int_{-\infty}^{+\infty} e^{-x^2} x^{2n} dx = \sqrt{\pi} 2^n n!$$

and then

$$\begin{aligned} \left| \int_{-\infty}^{+\infty} e^{-x^2} f(x) dx - Q_N^{GH}[f] \right| &\le \sum_{n=N}^{\infty} |b_{2n}| \, |I[x^{2n}] - Q_N^{GH}[x^{2n}]| \\ &\le \sum_{n=N}^{\infty} |b_{2n} I[x^{2n}] \\ &= \sum_{n=N}^{\infty} |b_{2n}| \sqrt{\pi} 2^n n! \end{aligned}$$

Applying in the same way to the proof of Theorem 2.3 leads to the desired result. $\square$

Remark 7. Comparing B_1 with B in (1.3), we find that $B_1 \le 2e^{-1} B^2$, which shows that the upper bound in Theorem 3.3 is sharper than that given by Lubinsky [25].

From Theorems 2.2 and 3.2, we see that Gauss–Laguerre quadrature $Q_N^{GL}[f]$ ($\alpha = 0$) and Gauss–Hermite $Q_N^{GH}[f]$ quadrature have nearly the same convergence rates. We illustrate here the convergence rates on Gauss–Laguerre quadrature and Gauss–Hermite quadrature for $f(x)$ being $\cos(x)$, $\frac{1}{1+x^2}$, e^{-1/x^2} and $|x-1|$, respectively (see Fig. 2).

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