J. Math. Anal. Appl. 394 (2012) 391–399

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

General theorem for the existence of iterative roots of homeomorphisms with periodic points

Paweł Solarz

Pedagogical University, Institute of Mathematics, ul. Podchorążych 2, PL-30-084 Kraków, Poland

ARTICLE INFO

Article history:
Received 3 November 2011
Available online 13 April 2012
Submitted by Michael J. Schlosser

Keywords:
Iteration
Iterative root
Periodic point
Rotation number
Orientation-preserving mapping
Unit circle

ABSTRACT

We apply Zdun's factorization theorem (see Zdun (2008) [3]) to give the conditions for the existence and the form of continuous and orientation-preserving iterative roots of homeomorphisms of the circle with a rational rotation number. Our theorem generalizes the previous results given by Jarczyk (2003) in [2], Zdun (2008) in [3] and Solarz (2003, 2009) in [4] and [5].

1. Introduction

In this paper we prove the theorem concerning the existence of continuous and orientation-preserving solutions of the following functional equation

$$G^m(z) = F(z), \quad z \in S^1 := \{z \in \mathbb{C} : |z| = 1\}, \tag{1}$$

where $F : S^1 \to S^1$ is a given orientation-preserving homeomorphism with fixed or periodic points, G^m denotes m-th iterate of G and $m \geq 2$ is an integer. Every solution of (1) is called the *iterative root* od F and m is said to be the *order of iterative root*.

Recall also that $x \in X$ is a *periodic point* of order $n \in \mathbb{N}$, $n > 1$ of a mapping $f : X \to X$ if

$$f^n(x) = x \quad \text{and} \quad f^k(x) \neq x \quad \text{for } k \in \{1, \dots, n-1\}.$$

If $f(x) = x$ then x is said to be a *fixed point of* f. The set of all periodic (fixed) points of f will be denoted by $\operatorname{Per} f$ ($\operatorname{Fix} f$).

The theorem generalizes (see [1]) the results obtained and method used for finding solutions of (1) in some particular cases i.e., $\operatorname{Fix} F = S^1$ (see [2]), $\emptyset \neq \operatorname{Fix} F \neq S^1$ (see [3]), $\operatorname{Per} F = S^1$ (see [4]), $\emptyset \neq \operatorname{Per} F \neq S^1$ (see [5]).

We give the necessary and sufficient conditions under which there exist continuous and orientation-preserving iterative roots of an arbitrary orientation-preserving homeomorphism $F : S^1 \to S^1$ such that $\operatorname{Per} F \cup \operatorname{Fix} F \neq \emptyset$. We also give the description of these roots. It is worth pointing that in contrast to real homeomorphisms (see [6]), there may exist iterative roots for circle mappings with periodic points.

2. Preliminaries

We start with recalling some useful facts and notions. Firstly, set $\mathbb{Z}_n := \{0, 1, \dots, n-1\}$ and $\mathbb{Z}_n^* := \{1, \dots, n-1\}$ for a suitable natural n.

E-mail address: psolarz@up.krakow.pl.

0022-247X/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2012.03.048

Let $u, w \in S^1$ and $u \neq w$, then $u = e^{2\pi i t_1}$ and $w = e^{2\pi i t_2}$ for some $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2 < t_1 + 1$. Put

$$\overrightarrow{(u, w)} := \{e^{2\pi i t},\ t \in (t_1, t_2)\}, \qquad \overrightarrow{[u, w]} := \overrightarrow{(u, w)} \cup \{u, w\}, \qquad \overrightarrow{[u, w)} := \overrightarrow{(u, w)} \cup \{u\}.$$

If $u = w$ we set $\overrightarrow{(u, u)} = S^1 \setminus \{u\}$. We call these sets *arcs*.

For every homeomorphism $F: S^1 \to S^1$ there exists a unique (up to translation by an integer) homeomorphism $f: \mathbb{R} \to \mathbb{R}$, called the *lift* of F, such that

$$F\left(e^{2\pi i x}\right) = e^{2\pi i f(x)}$$

and

$$f(x+1) = f(x) + k$$

for all $x \in \mathbb{R}$, where $k \in \{-1, 1\}$. We call F *orientation-preserving* if f is strictly increasing, which is equivalent to the fact that $k = 1$. Moreover, for every continuous function $G: I \to J$, where $I = \{e^{2\pi i t},\ t \in [a, b]\}$ and $J = \{e^{2\pi i t},\ t \in [c, d]\}$ there exists a unique continuous function $g: [a, b] \to [c, d]$ such that

$$G\left(e^{2\pi i x}\right) = e^{2\pi i g(x)}, \quad x \in [a, b].$$

In this case we also call g the lift of G and we say that G preserves the orientation if g is strictly increasing.

Now assume that F is an orientation-preserving homeomorphism, then the limit

$$\alpha(F) := \lim_{n\to\infty} \frac{f^n(x)}{n} (\mathrm{mod}\ 1), \quad x \in \mathbb{R}$$

always exists and does not depend on the choice of x and f. This number is called the *rotation number* of F (see [7]). It is known that $\alpha(F)$ is a rational number if and only if F has a periodic or fixed point (see for example [7]). If $F: S^1 \to S^1$ is an orientation-preserving homeomorphism such that $\alpha(F) = \frac{q}{n}$, where q, n are positive integers with $0 < q < n$ and $\gcd(q, n) = 1$, then $\mathrm{Per}\, F$ contains only periodic points of order n (see [5,8]). Moreover, there exists a unique number $p \in \mathbb{Z}_n^*$ satisfying $pq = 1 (\mathrm{mod}\ n)$. This number will be called the *characteristic number of* F and denoted $\mathrm{char}\, F := p$ (see [3]). If $\mathrm{Fix}\, F \neq \emptyset$, then $\alpha(F) = 0$ and we define $\mathrm{char}\, F := 1$.

The following result comes from [9] or [5].

Lemma 1. *If $F: S^1 \to S^1$ is an orientation-preserving homeomorphism with periodic points of order n, then for every $z \in \mathrm{Per}\, F$,*

$$\mathrm{Arg} \frac{F^{k\,\mathrm{char}\, F}(z)}{z} < \mathrm{Arg} \frac{F^{(k+1)\mathrm{char}\, F}(z)}{z}, \quad k \in \mathbb{Z}_{n-1}$$

and

$$F[I_k] = I_{(k+q)(\mathrm{mod}\ n)}, \quad k \in \mathbb{Z}_n,$$

where $q = n\alpha(F)$ and

$$I_d = I_d(z) := \overrightarrow{\left[F^{d\,\mathrm{char}\, F}(z), F^{(d+1)\mathrm{char}\, F}(z)\right)}, \quad d \in \mathbb{Z}_n. \tag{2}$$

The starting point for our research was the so called factorization theorem proved by Zdun (see Theorem 5, [3]). In view of it, every orientation-preserving homeomorphism $F: S^1 \to S^1$ possessing periodic points of order n is of the form

$$F(z) = \begin{cases} T^q(F^n(z)), & z \in \overrightarrow{\left[z_0, F^{\mathrm{char}\, F}(z_0)\right)}, \\ T^q(z), & z \in S^1 \setminus \overrightarrow{\left[z_0, F^{\mathrm{char}\, F}(z_0)\right)}, \end{cases} \tag{3}$$

where $z_0 \in \mathrm{Per}\, F$, $q = n\alpha(F)$ and $T: S^1 \to S^1$ is an orientation-preserving homeomorphism such that $\mathrm{Per}\, T = S^1$, $\alpha(T) = \frac{1}{n}$ and $T[I_d] = I_{(d+1)(\mathrm{mod}\ n)}$ for $d \in \mathbb{Z}_n$, where $I_d = I_d(z_0)$ for $d \in Z_n$ are defined by (2). The function $T = T_{z_0}(F)$ is unique up to a periodic point of F and it is called the *Babbage function of* F (see [3]). Let us notice that if $\mathrm{Per}\, F = S^1$, we get $T_{z_0}(F) = F^{\mathrm{char}\, F}$ for every $z_0 \in \mathrm{Per}\, F$, whereas if F is such that $\mathrm{Fix}\, F \neq \emptyset$ we have $\overrightarrow{\left[z_0, F^{\mathrm{char}\, F}(z_0)\right)} = S^1$ and we assume $T_{z_0}(F) = \mathrm{id}_{S^1}$ for every $z_0 \in \mathrm{Fix}\, F$.

In view of (3) and Lemma 1 we have the following property (see [3]).

Theorem 1. *Let $F: S^1 \to S^1$ be an orientation-preserving homeomorphism with periodic points of order n, $z \in \mathrm{Per}\, F$, $T = T_z(F)$ be the Babbage function of F and let $\{I_d\}_{d\in\mathbb{Z}_n}$ be the family defined in (2). Then*

$$F^{k+jn}_{|I_d} = T^{\alpha(F)nk} \circ \begin{cases} T^d \circ (F^n)^{j+1} \circ T^{-d}_{|I_d}, & \text{if } -d\,\mathrm{char}\, F (\mathrm{mod}\ n) \leq k-1, \\ T^d \circ (F^n)^j \circ T^{-d}_{|I_d}, & \text{if } -d\,\mathrm{char}\, F (\mathrm{mod}\ n) > k-1, \end{cases}$$

for $d, k \in \mathbb{Z}_n$ and $j \in \mathbb{N}$.

We end this section by the following remark.

Remark 1. Let $q, n, q', n' \in \mathbb{N}$ be such that $0 \le q < n$, $\gcd(q, n) = 1$ and $0 \le q' < n'$, $\gcd(q', n') = 1$ and let

$$m\frac{q'}{n'} - \frac{q}{n} \in \mathbb{Z} \tag{4}$$

for some integer $m \ge 2$. Then there exists an integer $l \ge 1$ such that $n' = nl$, $l|m$, $q'\frac{m}{l} = q(\mathrm{mod}\ n)$. If moreover $p \in \mathbb{Z}_n^*$ and $p' \in \mathbb{Z}_{n'}^*$ are such that $pq = 1(\mathrm{mod}\ n)$ and $p'q' = 1(\mathrm{mod}\ n')$, then $p' = \frac{m}{l}p(\mathrm{mod}\ n)$.

Proof. Assume that (4) holds for some integer $m \ge 2$. If $q = 0$, then $n = 1$, $\mathbb{Z}_n^* = \emptyset$ and $m\frac{q'}{n'} \in \mathbb{Z}$, thus $n' = l$ and since $\gcd(q', n') = 1$ we have $l|m$. Notice that $q'\frac{m}{n'} = 0(\mathrm{mod}\ 1)$.

Now let $q \neq 0$, then $n > 1$. From (4) we get $q' \neq 0$ and

$$mnq' - n'q = knn' \tag{5}$$

for some integer k. Hence $n(mq' - kn') = n'q$ and since $\gcd(q, n) = 1$ we obtain $n|n'$. Put $l := \frac{n'}{n}$, then (5) yields $mq' = (kn + q)l$ and in consequence $q'\frac{m}{l} = q(\mathrm{mod}\ n)$. Moreover, as $\gcd(q', l) = 1$ we also have $l|m$. Let $p \in \mathbb{Z}_n^*$ and $p' \in \mathbb{Z}_{n'}^*$ be such that $pq = 1(\mathrm{mod}\ n)$ and $p'q' = 1(\mathrm{mod}\ n')$. Clearly $p'q' = 1(\mathrm{mod}\ n)$. The condition $q'\frac{m}{l} = q(\mathrm{mod}\ n)$ now implies $pqq'\frac{m}{l} = qq'p'(\mathrm{mod}\ n)$ which gives $\frac{m}{l}p = p'(\mathrm{mod}\ n)$ as $\gcd(qq', n) = 1$. $\square$

3. Main results

We begin with some properties of homeomorphisms satisfying (1). The following lemma is a consequence of Eq. (1) and Remark 1.

Lemma 2. *Let $F: S^1 \to S^1$ and $G: S^1 \to S^1$ be orientation-preserving homeomorphisms satisfying (1) for some integer $m \ge 2$ and such that $\alpha(F) = \frac{q}{n}$ and $\alpha(G) = \frac{q'}{n'}$ where $0 \le q < n$, $0 \le q' < n'$ and $\gcd(q, n) = \gcd(q', n') = 1$. Then*

(i) *$n' = nl$ and $\frac{m}{l} := m' \in \mathbb{Z}$ for some unique integer $l \ge 1$,*
(ii) *$q'm' = q(\mathrm{mod}\ n)$,*
(iii) *$\mathrm{char}\, G = m'\mathrm{char}\, F(\mathrm{mod}\ n)$,*
(iv) *$(G^{n'})^{m'} = F^n$.*

Proof. Notice that (1) implies $m\alpha(G) = \alpha(F)(\mathrm{mod}\ 1)$, i.e., (4) holds true. From Remark 1 we have (i) and (ii). If $q = 0$, then $n = 1$, $\mathrm{char}\, F = 1$ and (iii) is obvious. If $q \neq 0$, then taking $p := \mathrm{char}\, F$ and $p' := \mathrm{char}\, G$ we get (iii) from Remark 1. Finally, (iv) follows from (1) and (i). $\square$

The following lemma gives the rest of the necessary conditions for (1) to hold.

Lemma 3. *Let $F: S^1 \to S^1$ and $G: S^1 \to S^1$ be orientation-preserving homeomorphisms satisfying (1) for some integer $m \ge 2$ and such that $\alpha(F) = \frac{q}{n}$ and $\alpha(G) = \frac{q'}{n'}$ where $0 \le q < n$, $0 \le q' < n'$ and $\gcd(q, n) = \gcd(q', n') = 1$. Then for every $z \in \mathrm{Per}\, F \cup \mathrm{Fix}\, F$ there exists a partition of $\overrightarrow{[z, F^{\mathrm{char}\, F}(z))}$ onto $l := \frac{n'}{n}$ pairwise disjoint, consecutive closed-open arcs $J_0, J_1, \ldots, J_{l-1}$ such that*

$$F^n[J_k] = J_k, \quad k \in \mathbb{Z}_l$$

and if $l > 1$ there exist orientation-preserving homeomorphisms $V_k: J_k \to J_{k+1}$, $k \in \mathbb{Z}_{l-1}$ satisfying

$$F^n_{|J_{k+1}} \circ V_k = V_k \circ F^n_{|J_k}, \quad k \in \mathbb{Z}_{l-1}.$$

Proof. Fix $z \in \mathrm{Per}\, F \cup \mathrm{Fix}\, F$ (one of these sets must be empty). If $n = n'$ set $J_0 = \overrightarrow{[z, F^{\mathrm{char}\, F}(z))}$, then $F^n[J_0] = J_0$. Let us notice that $\overrightarrow{[z, F^{\mathrm{char}\, F}(z))} = S^1$ if $n = 1$. If $n' > n$, then $l > 1$ and we define

$$J_k = \overrightarrow{[G^{k\mathrm{char}\, G}(z), G^{(k+1)\mathrm{char}\, G}(z))}, \quad k \in \mathbb{Z}_l.$$

This and Lemma 1 imply that $J_0, \ldots, J_{l-1}$ are disjoint and consecutive arcs. Moreover, by Lemma 2(iii) we get $l\mathrm{char}\, G = m\mathrm{char}\, F(\mathrm{mod}\ n')$, hence and by (1) we have

$$\bigcup_{k=0}^{l-1} J_k = \overrightarrow{[z, G^{l\mathrm{char}\, G}(z))} = \overrightarrow{[z, G^{m\mathrm{char}\, F}(z))} = \overrightarrow{[z, F^{\mathrm{char}\, F}(z))}. \tag{6}$$

Since $\operatorname{Per} G = \operatorname{Per} F \cup \operatorname{Fix} F$ we have $z \in \operatorname{Per} G$ and $G^{k\operatorname{char} G}(z) \in \operatorname{Per} G$ for $k \in \mathbb{Z}_{l+1}$. Thus $F^n\left(G^{k\operatorname{char} G}(z)\right) = G^{k\operatorname{char} G}(z)$, $k \in \mathbb{Z}_{l+1}$ and in consequence

$$F^n[J_k] = \overrightarrow{\left[F^n\left(G^{k\operatorname{char} G}(z)\right), F^n\left(G^{(k+1)\operatorname{char} G}(z)\right)\right]} = J_k, \quad k \in \mathbb{Z}_l.$$

Now let $T_z(G)$ be the Babbage homeomorphism of G. Fix $k \in \mathbb{Z}_{l-1}$ and put $V_k := T_z(G)_{|J_k}$. Then by Theorem 1, $V_k : J_k \to J_{k+1}$ satisfies

$$G^{n'}_{|J_{k+1}} \circ V_k = V_k \circ G^{n'}_{|J_k}.$$

Hence

$$\left(G^{n'}\right)^{m'}_{|J_{k+1}} \circ V_k = V_k \circ \left(G^{n'}\right)^{m'}_{|J_k},$$

where $m' = \frac{m}{l}$, and in view of Lemma 2(iv) we have

$$F^n_{|J_{k+1}} \circ V_k = V_k \circ F^n_{|J_k}. \quad \square$$

Before the main theorem let us recall the following result (see [5]).

Lemma 4. *Let $u, w \in S^1$, $u \neq w$ and $I := \overrightarrow{[u, w]}$. For every integer $m \geq 2$ and every orientation-preserving homeomorphism $F : I \to I$ with $\operatorname{Fix} F \neq \emptyset$ there exist infinitely many orientation-preserving homeomorphisms $G : I \to I$ satisfying (1) and such that $\operatorname{Fix} G \neq \emptyset$.*

From Corollary 3 and Lemma 3 in [10] we also have

Lemma 5. *If $F : S^1 \to S^1$ is an orientation-preserving homeomorphism with periodic points of order $n > 1$, $z \in \operatorname{Per} F$, $\{z, F(z), \ldots, F^{n-1}(z)\} = \{z_0, z_1, \ldots, z_{n-1}\}$, where $z_0 = z$ and*

$$\operatorname{Arg}\frac{z_d}{z_0} < \operatorname{Arg}\frac{z_{d+1}}{z_0} < 2\pi, \quad d \in \mathbb{Z}_{n-1}$$

and $F(z_0) = z_q$, then $\alpha(F) = \frac{q}{n}$.

Theorem 2. *Let $m \geq 2$ and $l \geq 1$ be integers and let $F : S^1 \to S^1$ be an orientation-preserving homeomorphism such that $\alpha(F) = \frac{q}{n}$, where $0 \leq q < n$ and $\gcd(q, n) = 1$. F has continuous and orientation-preserving iterative root of order m with periodic points of order ln if and only if the following conditions are fulfilled:*

(i) *$\frac{m}{l} =: m' \in \mathbb{Z}$ and there is $q' \in \mathbb{Z}_{nl}$ such that $\gcd(q', ln) = 1$ and $q'm' = q \pmod n$;*

(ii) *for some $z_0 \in \operatorname{Per} F$ there is a partition of $\overrightarrow{\left[z_0, F^{\operatorname{char} F}(z_0)\right)}$ onto l consecutive disjoint arcs $J_0, \ldots, J_{l-1}$ such that $F^n[J_i] = J_i$, $i \in \mathbb{Z}_l$ and if $l > 1$, then there exist orientation-preserving homeomorphisms $V_i : J_i \to J_{i+1}$, $i \in \mathbb{Z}_{l-1}$ satisfying*

$$F^n_{|J_{i+1}} = V_i \circ F^n_{|J_i} \circ V_i^{-1}, \quad i \in \mathbb{Z}_{l-1}. \tag{7}$$

For any $z_0 \in \operatorname{Per} F \cup \operatorname{Fix} F$, m, l, q', arcs $J_0, \ldots, J_{l-1}$ and homeomorphisms $V_i : J_i \to J_{i+1}$, $i \in \mathbb{Z}_{l-1}$ satisfying (7) the iterative root $G : S^1 \to S^1$ of F is of the form:

$$G(z) := \begin{cases} V^{q'}(G_0(z)), & z \in J_0, \\ V^{q'}(z), & z \in S^1 \setminus J_0, \end{cases} \tag{8}$$

where $G_0 : J_0 \to J_0$ is an orientation-preserving homeomorphism such that $\operatorname{Fix} G_0 \neq \emptyset$ and $G_0^{m'} = F^n_{|J_0}$ and $V = \Psi^{\operatorname{char} F}$ if $l = 1$ or

$$V(z) := \begin{cases} V_i(z), & z \in J_i,\ i \in \mathbb{Z}_{l-1}, \\ V_{l-1}(z) := \Psi^{\operatorname{char} F} \circ V_0^{-1} \circ \cdots \circ V_{l-2}^{-1}(z), & z \in J_{l-1}, \\ \Psi^{d\operatorname{char} F} \circ V_i \circ \Psi^{-d\operatorname{char} F}(z), & z \in F^{d\operatorname{char} F}[J_i],\ i \in \mathbb{Z}_l,\ d \in \mathbb{Z}_n^* \end{cases} \tag{9}$$

if $l > 1$, where $\Psi : S^1 \to S^1$ is given by

$$\Psi(z) := T^q \circ T^d \circ G_j^{\beta_{j,d}} \circ T^{-d}(z), \quad z \in F^{d\operatorname{char} F}[J_j],\ d \in \mathbb{Z}_n,\ j \in \mathbb{Z}_l, \tag{10}$$

where

$$G_j := V_j \circ G_{j-1} \circ V_j^{-1}, \quad j \in \mathbb{Z}_l^*, \tag{11}$$

$T = T_{z_0}(F)$ denotes the Babbage homeomorphism of F and

$$\beta_{i,d} := \begin{cases} m' - \left[\frac{m}{nl}\right] - 1, & \text{if } d = 0,\ i'_i \le m - \left[\frac{m}{nl}\right] nl - 1, \\ m' - \left[\frac{m}{nl}\right], & \text{if } d = 0,\ i'_i > m - \left[\frac{m}{nl}\right] nl - 1, \\ -\left[\frac{m}{nl}\right] - 1, & \text{if } d \in \mathbb{Z}_n^*,\ i'_{i+dl} \le m - \left[\frac{m}{nl}\right] nl - 1, \\ -\left[\frac{m}{nl}\right], & \text{if } d \in \mathbb{Z}_n^*,\ i'_{i+dl} > m - \left[\frac{m}{nl}\right] nl - 1 \end{cases} \tag{12}$$

for $i \in \mathbb{Z}_l$ with $i'_k \in \mathbb{Z}_{nl}$ uniquely determined by $(k + i'_k q')(\text{mod } nl) = 0$ for $k \in \mathbb{Z}_{nl}$.

Moreover, every orientation-preserving iterative root of order m of F with periodic points of order nl (if exists) may be expressed by (8)–(12).

Proof. Clearly, if Eq. (1) holds true then (i) follows from conditions (i) and (ii) of Lemma 2 and (ii) follows from Lemma 3.

Now assume that $m, l \in \mathbb{Z}, m \ge 2, l \ge 1 m' = \frac{m}{l}$ and $F: S^1 \to S^1$ is an orientation-preserving homeomorphism such that $\alpha(F) = \frac{q}{n}$, where $0 \le q < n$ and $\gcd(q, n) = 1$. Let conditions (i) and (ii) of Theorem 1 be fulfilled. We give the proof only for the case $l > 1$, the case $l = 1$ was proved in [5] (let us mention that if $l = 1$ condition (ii) is not the case and (i) is equivalent to the fact that $\gcd(m, n) = 1$). Notice that as $l > 1$ we have $nl > 1$ and therefore $q' > 0$. The proof is divided into four steps.

Firstly, we show that the function Ψ given by (10)–(12) is an orientation-preserving homeomorphism such that $\alpha(\Psi) = \frac{q}{n}$ and $\Psi^n = \mathrm{id}_{S^1}$. To see this set, if $n > 1$,

$$J_{j+dl} := F^{d\,\mathrm{char}\,F}[J_j], \quad d \in \mathbb{Z}_n^*,\ j \in \mathbb{Z}_l. \tag{13}$$

Since F preserves the orientation, J_j for $j \in \mathbb{Z}_{nl}$ are pairwise disjoint and consecutive arcs. Moreover,

$$F^{\mathrm{char}\,F}[J_{j+(n-1)l}] = F^{\mathrm{char}\,F}\left[F^{(n-1)\mathrm{char}\,F}[J_j]\right] = J_j, \quad j \in \mathbb{Z}_l \tag{14}$$

and

$$F^n[J_{j+dl}] = F^n\left[F^{d\,\mathrm{char}\,F}[J_j]\right] = F^{d\,\mathrm{char}\,F}\left[F^n[J_j]\right] = F^{d\,\mathrm{char}\,F}[J_j] = J_{j+dl} \tag{15}$$

for $d \in \mathbb{Z}_n^*$ and $j \in \mathbb{Z}_l$. Let $z_1, z_2, \ldots, z_{nl-1}$ be such that $J_k = [z_k, z_{k+1})$ for $k \in \mathbb{Z}_{nl-1}$. From (15) since F preserves the orientation we have $F^n[z_k] = z_k$ for $k \in \mathbb{Z}_{nl}$, thus z_k for $k \in \mathbb{Z}_{nl}$ are periodic or fixed (if $n = 1$) points of F. Hence and by (13) and (14) we get

$$z_{(k+l)(\text{mod } nl)} = F^{\mathrm{char}\,F}[z_k], \quad k \in \mathbb{Z}_{nl}$$

and as a consequence

$$F[z_k] = F^{q\,\mathrm{char}\,F}[z_k] = z_{(k+ql)(\text{mod } nl)}, \quad k \in \mathbb{Z}_{nl}.$$

This and the fact that F preserves the orientation yield

$$F[J_k] = J_{(k+ql)(\text{mod } nl)}, \quad k \in \mathbb{Z}_{nl}. \tag{16}$$

Now assume that $T = T_{z_0}(F)$, then from (3) one can obtain

$$T^q(z) = \begin{cases} F^{1-n}(z), & z \in \overrightarrow{[z_0, z_l)}, \\ F(z), & z \in S^1 \setminus \overrightarrow{[z_0, z_l)}, \end{cases}$$

hence in view of (16),

$$T^q[J_k] = F[J_k] = J_{(k+ql)(\text{mod } nl)}, \quad k \in \mathbb{Z}_{nl}.$$

This and the equalities $T = T^{q\,\mathrm{char}\,F}$ and $q\,\mathrm{char}\,F = 1(\text{mod } n)$ give

$$T[J_k] = T^{q\,\mathrm{char}\,F}[J_k] = J_{(k+ql\,\mathrm{char}\,F)(\text{mod } nl)} = J_{(k+l)(\text{mod } nl)}, \quad k \in \mathbb{Z}_{nl}. \tag{17}$$

On the other hand, if $G_0: J_0 \to J_0$ is an orientation-preserving homeomorphism such that $\mathrm{Fix}\, G_0 \ne \emptyset$ and $G_0^{m'} = F^n_{|J_0}$, then by (11),

$$G_j[J_j] = J_j, \quad j \in Z_l. \tag{18}$$

Finally, by (10), (17) and (18) we get

$$\begin{aligned}\Psi[J_{j+dl}] &= T^q \circ T^d \circ G_j^{\beta_{j,d}} \circ T^{-d}[J_{j+dl}] \\ &= T^q \circ T^d \circ G_j^{\beta_{j,d}}[J_j] = T^q \circ T^d[J_j] \\ &= J_{(j+(q+d)l)(\mathrm{mod}\ nl)}, \quad j \in \mathbb{Z}_l,\ d \in \mathbb{Z}_n.\end{aligned}$$

Thus

$$\Psi[J_k] = J_{(k+ql)(\mathrm{mod}\ nl)}, \quad k \in \mathbb{Z}_{nl}. \tag{19}$$

Let us notice that as a composition of orientation-preserving homeomorphisms, $\Psi_{|J_k}$ is an orientation-preserving homeomorphism. Hence $\Psi : S^1 \to S^1$ is an orientation-preserving homeomorphism.

To show that $\alpha(\Psi) = \frac{q}{n}$ observe that from (19) and (16) it follows that

$$F^k(z_0) = \Psi^k(z_0), \quad k \in \mathbb{Z},$$

so by the definition of the rotation number $\alpha(\Psi) = \alpha(F) = \frac{q}{n}$ and hence $\operatorname{char}\Psi = \operatorname{char} F$.

Now we are in the position to show that $\Psi^n = \mathrm{id}_{S^1}$. Fix $j \in Z_l$ and $d \in \mathbb{Z}_n$. From (19), (10) and (17), in view of the fact that $T^p = T^{p(\mathrm{mod}\ n)}$ for $p \in \mathbb{N}$, we obtain

$$\begin{aligned}\Psi^n_{|J_{j+dl}} &= (T^q \circ T^{d+(n-1)q} \circ G_j^{\beta_{k,(j+(n-1)q)(\mathrm{mod}\ n)}} \circ T^{-(d+(n-1)q)}) \circ \cdots \circ (T^q \circ T^{d+q} \circ G_j^{\beta_{j,(d+q)(\mathrm{mod}\ n)}} \circ T^{-(d+q)}) \\ &\quad \circ (T^q \circ T^d \circ G_j^{\beta_{j,d}} \circ T^{-d}_{|J_{j+dl}}) \\ &= T^q \circ T^{d+(n-1)q} \circ G_j^{\beta_{j,(d+(n-1)q)(\mathrm{mod}\ n)}+\cdots+\beta_{j,d}} \circ T^{-d}_{|J_{j+dl}}.\end{aligned}$$

Moreover, since $\gcd(q, n) = 1$ we get

$$\{d, (d+q)(\mathrm{mod}\ n), \ldots, (d+(n-1)q)(\mathrm{mod}\ n)\} = \{0, 1, \ldots, n-1\}.$$

We thus have

$$\Psi^n_{|J_{j+dl}} = T^q \circ T^{d+(n-1)q} \circ G^{\beta_{j,n-1}+\cdots+\beta_{j,0}} \circ T^{-d}_{|J_{j+dl}}. \tag{20}$$

We finish the proof of this step by showing $\beta_{j,n-1} + \cdots + \beta_{j,0} = 0$. In order to do this we examine the properties of the mapping $Z_{nl} \ni k \longmapsto i'_k \in Z_{nl}$ defined by $(k + i'_k q')(\mathrm{mod}\ nl) = 0$ for $k \in \mathbb{Z}_{nl}$. As $q' > 0$ there is a unique $p' \in \mathbb{Z}^*_{nl}$ such that $q'p' = 1(\mathrm{mod}\ nl)$. Hence $i'_k = -p'k(\mathrm{mod}\ nl)$ for $k \in \mathbb{Z}_{nl}$. If we had $i'_{k_1} = i'_{k_2}$ for some distinct $k_1, k_2 \in \mathbb{Z}_{nl}$ we would get $k_1 - k_2 = cnl$ for some integer $c \neq 0$, a contradiction. Thus the mapping is an injection, and in consequence a bijection. Moreover,

$$i'_{i+tl} = -p'(i+tl)(\mathrm{mod}\ nl) = -p'i - p'tl(\mathrm{mod}\ nl), \quad t \in \mathbb{Z}_n,\ i \in \mathbb{Z}_l. \tag{21}$$

Now turn to the definition (12) and put $b := \left[\frac{m}{nl}\right]$ and $c := m' - bn$, then $b \geq 0$, $c \in \mathbb{Z}_n$ and $m - \left[\frac{m}{nl}\right] nl = cl$. On the other hand, by (21) we get that $A_i := \{i + tl : t \in \mathbb{Z}_n\}$ is mapped onto $A_{-p'i(\mathrm{mod}\ l)}$ for $i \in \mathbb{Z}_l$. Thus $i'_k \leq cl - 1$ for exactly c elements $k \in A_i$ for $i \in \mathbb{Z}_l$. Consider two cases

(a) $i'_j \leq cl - 1$, then by (12),

$$\beta_{j,n-1} + \cdots + \beta_{j,0} = (n-c)(-b) + (c-1)(-b-1) + m' - b - 1 = 0.$$

(b) $i'_j > cl - 1$, then again by (12),

$$\beta_{j,n-1} + \cdots + \beta_{j,0} = (n-c-1)(-b) + c(-b-1) + m' - b = 0.$$

This finishes the proof that $\Psi^n = \mathrm{id}_{S^1}$.

In the next step we show that $V^l = \Psi^{\operatorname{char} F}$. Immediately from (9) we get that V is an orientation-preserving homeomorphism such that

$$V[J_k] = J_{(k+1)(\mathrm{mod}\ nl)}, \quad k \in \mathbb{Z}_{nl}. \tag{22}$$

Fix $j \in \mathbb{Z}_l$, $d \in Z_n$ and $z \in J_{j+dl}$, then since $\Psi^n = \mathrm{id}_{S^1}$ we get

$$\begin{aligned}V^l(z) &= \Psi^{(d+1)\operatorname{char} F} \circ V_{j-1} \circ \Psi^{-(d+1)\operatorname{char} F} \circ \cdots \circ \Psi^{(d+1)\operatorname{char} F} \circ V_0 \circ \Psi^{-(d+1)\operatorname{char} F} \circ \Psi^{d\operatorname{char} F} \circ V_{l-1} \circ \Psi^{-d\operatorname{char} F} \\ &\quad \circ \Psi^{d\operatorname{char} F} \circ V_{l-2} \circ \Psi^{-d\operatorname{char} F} \circ \cdots \circ \Psi^{d\operatorname{char} F} \circ V_{j+1} \circ \Psi^{-d\operatorname{char} F} \circ \Psi^{d\operatorname{char} F} \circ V_j \circ \Psi^{-d\operatorname{char} F}(z) \\ &= \Psi^{(d+1)\operatorname{char} F} \circ V_{j-1} \circ \cdots \circ V_0 \circ \Psi^{-\operatorname{char} F} \circ V_{l-1} \circ V_{l-2} \circ \cdots \circ V_{j+1} \circ V_j \circ \Psi^{-d\operatorname{char} F}(z) \\ &= \Psi^{(d+1)\operatorname{char} F} \circ V_{j-1} \circ \cdots \circ V_0 \circ \Psi^{-\operatorname{char} F} \circ \Psi^{\operatorname{char} F} \circ V_0^{-1} \circ \cdots \circ V_{l-2}^{-1} \circ V_{l-2} \\ &\quad \circ \cdots \circ V_{j+1} \circ V_j \circ \Psi^{-d\operatorname{char} F}(z) = \Psi^{(d+1)\operatorname{char} F} \circ \Psi^{-d\operatorname{char} F}(z) = \Psi^{\operatorname{char} F}(z).\end{aligned}$$

In the third step we show that the function G given by (8) satisfies (1). Since $V^l = \Psi^{\mathrm{char}\,F}$ and $\Psi^n = \mathrm{id}_{S^1}$ we get $V^{nl} = \Psi^{n\mathrm{char}\,F} = \mathrm{id}_{S^1}$. Moreover, in view of (22) and (8),

$$G(z_k) = z_{(k+q')(\mathrm{mod}\ nl)}, \quad k \in \mathbb{Z}_{nl},$$

where $z_0, z_1, z_2, \ldots, z_{nl-1}$ are such that $J_k = [z_k, z_{k+1})$ for $k \in \mathbb{Z}_{nl-1}$. As $\gcd(q', nl) = 1$ we get that G has periodic points of order nl. Lemma 5 implies now that $\alpha(G) = \frac{q'}{nl}$. Thus $V = T_{z_0}(G)$ is the Babbage homeomorphism of G.

Equations $V^l = \Psi^{\mathrm{char}\,F}$, $\Psi^n = \mathrm{id}_{S^1}$ and equality $q\mathrm{char}\,F = 1(\mathrm{mod}\ n)$ yield also $V^{lq} = \Psi^{q\mathrm{char}\,F} = \Psi$, which together with (10) result in

$$V^{lq}_{|J_{j+dl}} = T^q \circ T^d \circ G_j^{\beta_{j,d}} \circ T^{-d}_{|J_{j+dl}}, \quad d \in \mathbb{Z}_n,\ j \in \mathbb{Z}_l. \tag{23}$$

In virtue of the fact that $T^p = T^{p(\mathrm{mod}\ n)}$ for $p \in \mathbb{Z}$ Eq. (23) is equivalent to

$$V^{lq}_{|J_{(j+pl)(\mathrm{mod}\ nl)}} = T^q \circ T^p \circ G_j^{\beta_{j,p(\mathrm{mod}\ n)}} \circ T^{-p}_{|J_{(j+pl)(\mathrm{mod}\ nl)}}, \quad p \in \mathbb{N},\ j \in \mathbb{Z}_l. \tag{24}$$

Thus

$$\begin{aligned} V^{klq}_{|J_j} &= T^q \circ T^{(k-1)q} \circ G_j^{\beta_{j,q(k-1)(\mathrm{mod}\ n)}} \circ T^{-((k-1)q)} \circ \cdots \circ T^q \circ T^q \circ G_j^{\beta_{j,q(\mathrm{mod}\ n)}} \circ T^{-q} \circ T^q \circ G_j^{\beta_{j,0}} \\ &= T^{kq} \circ G_j^{(\beta_{j,q(k-1)(\mathrm{mod}\ n)} + \beta_{j,q(k-2)(\mathrm{mod}\ n)} + \cdots + \beta_{j,q} + \beta_{j,0})}, \end{aligned}$$

which gives

$$T^{kq}_{|J_j} = V^{klq} \circ G_j^{-(\beta_{j,q(k-1)(\mathrm{mod}\ n)} + \beta_{j,q(k-2)(\mathrm{mod}\ n)} + \cdots + \beta_{j,q} + \beta_{j,0})}, \tag{25}$$

for $k \in \{1, \ldots, n\}$ and $j \in \mathbb{Z}_l$. Now if $n > 1$, fix $d \in \mathbb{Z}_n^*$ and $j \in \mathbb{Z}_l$. Since $\gcd(q, n) = 1$ there is a unique $t \in \mathbb{Z}_n^*$ such that $tq = d(\mathrm{mod}\ n)$. Hence by (24) we have

$$V^{lq}_{|J_{j+dl}} = V^{lq}_{|J_{j+ltq(\mathrm{mod}\ nl)}} = T^q \circ T^{tq} \circ G_j^{\beta_{j,tq(\mathrm{mod}\ n)}} \circ T^{-tq}_{|J_{j+ltq(\mathrm{mod}\ nl)}}.$$

By substituting (25) twice to the above equation we obtain

$$V^{lq}_{|J_{j+dl}} = T^q \circ (V^{tlq} \circ G_j^{-(\beta_{j,q(t-1)(\mathrm{mod}\ n)} + \cdots + \beta_{j,q} + \beta_{j,0})}) \circ G_j^{\beta_{j,tq(\mathrm{mod}\ n)}} \circ (G_j^{(\beta_{j,q(t-1)(\mathrm{mod}\ n)} + \cdots + \beta_{j,q} + \beta_{j,0})} \circ V^{-tlq}_{|J_{j+dl}}).$$

Thus

$$T^q_{|J_{j+dl}} = V^{lq} \circ V^{tlq} \circ G_j^{-\beta_{j,tq(\mathrm{mod}\ n)}} \circ V^{-tlq}_{|J_{j+dl}}$$

and

$$T^q_{|J_{j+dl}} = V^{lq} \circ V^{dl} \circ G_j^{-\beta_{j,d}} \circ V^{-dl}_{|J_{j+dl}}, \tag{26}$$

as $tql = dl(\mathrm{mod}\ nl)$ and $V^{tql} = V^{dl}$. In view of (11) we finally obtain

$$T^q_{|J_{j+dl}} = V^{lq} \circ V^{j+dl} \circ G_0^{-\beta_{j,d}} \circ V^{-(j+dl)}_{|J_{j+dl}}. \tag{27}$$

Now notice that since T is a Babbage homeomorphism of F and $\alpha(F) = \frac{q}{n}$, then by (27) and (3) we get

$$F_{|J_{j+dl}} = V^{lq} \circ \begin{cases} V^j \circ G_0^{-\beta_{j,0}} \circ V^{-j} \circ F^n_{|J_j} & \text{for } j \in \mathbb{Z}_l, \\ V^{j+dl} \circ G_0^{-\beta_{j,d}} \circ V^{-(j+dl)}_{|J_{j+dl}} & \text{for } j \in \mathbb{Z}_l,\ d \in \mathbb{Z}_n^*. \end{cases}$$

This, (7) and the facts that $V_{|J_j} = V_j$, $j \in \mathbb{Z}_l$ and $G_0^{m'} = F^n_{|J_0}$ yield

$$F_{|J_{j+dl}} = V^{lq} \circ \begin{cases} V^j \circ G_0^{-\beta_{j,0}+m'} \circ V^{-j}_{|J_j} & \text{for } j \in \mathbb{Z}_l, \\ V^{j+dl} \circ G_0^{-\beta_{j,d}} \circ V^{-(j+dl)}_{|J_{j+dl}} & \text{for } j \in \mathbb{Z}_l,\ d \in \mathbb{Z}_n^*. \end{cases} \tag{28}$$

We may write $m = i + bnl$, where $b = \left[\frac{m}{nl}\right]$ and $i \in \mathbb{Z}_{nl}$, hence $m'q' = q(\mathrm{mod}\ n)$ and $q'i = q'(i + bnl)(\mathrm{mod}\ nl)$ give $q'i = ql(\mathrm{mod}\ nl)$ and in consequence

$$V^{ql} = V^{q'i}. \tag{29}$$

Furthermore, according to (12),

$$-\beta_{j,0}+m'=\begin{cases} b+1, & i'_j\le i-1,\\ b, & i'_j>i-1\end{cases}$$

for $j\in\mathbb{Z}_l$ and

$$-\beta_{j,d}=\begin{cases} b+1, & i'_{j+dl}\le i-1,\\ b, & i'_{j+dl}>i-1\end{cases}$$

for $j\in\mathbb{Z}_l$ and $d\in\mathbb{Z}_n^*$. Putting these and (29) to (28) we have

$$F_{|J_k}=V^{q'i}\circ\begin{cases} V^k\circ G_0^{b+1}\circ V_{|J_k}^{-k}, & i'_k\le i-1,\\ V^k\circ G_0^{b}\circ V_{|J_k}^{-k}, & i'_k>i-1\end{cases}\tag{30}$$

for every $k\in\mathbb{Z}_{nl}$.

On the other hand, by Theorem 1 we obtain for all $k\in\mathbb{Z}_{nl}$.

$$G_{|J_k}^{i+bnl}=V^{q'i}\circ\begin{cases} V^k\circ G_0^{b+1}\circ V_{|J_k}^{-k}, & \text{if } -kp'(\text{mod } nl)\le i-1,\\ V^k\circ G_0^{b}\circ V_{|J_k}^{-k}, & \text{if } -kp'(\text{mod } nl)>i-1,\end{cases}\tag{31}$$

where $p'=\operatorname{char}G$ is such that $q'p'=1(\text{mod } nl)$, Recall that $i'_k=-p'k(\text{mod } nl)$ for $k\in\mathbb{Z}_{nl}$, thus (30) and (31) result in $G^m=F$ and the proof of the third step is completed.

What is left is to show that every orientation-preserving iterative root of order m of F with periodic points of order nl (if exists) may be expressed by (8)–(12). Suppose that F,G are orientation-preserving homeomorphisms satisfying (1) for some integer $m\ge 2$ and such that $\alpha(F)=\frac{q}{n}$, $\alpha(G)=\frac{q'}{n'}$, where $\gcd(q,n)=\gcd(q',n')=1$. Let moreover $z_0\in\operatorname{Per}F=\operatorname{Per}G$, $V=T_{z_0}(G)$, $T=T_{z_0}(F)$. By (i) of Lemma 2, $n'=nl$ for some integer l. Put

$$J_k=\overrightarrow{\left[G^{k\operatorname{char}G}(z_0),G^{(k+1)\operatorname{char}G}(z_0)\right)},\quad k\in\mathbb{Z}_{nl}^*,$$

then we get (6) with $z=z_0$. Moreover, by Lemma 2(iii) and (1) we obtain

$$\begin{aligned} J_{j+dl}&=\overrightarrow{\left[G^{(j+dl)\operatorname{char}G}(z_0),G^{(j+dl+1)\operatorname{char}G}(z_0)\right)}\\ &=G^{dl\operatorname{char}G}\left[\overrightarrow{\left[G^{j\operatorname{char}G}(z_0),G^{(j+1)\operatorname{char}G}(z_0)\right)}\right]\\ &=G^{dl\operatorname{char}G}\left[J_j\right]=G^{md\operatorname{char}F}\left[J_j\right]=F^{d\operatorname{char}F}[J_j],\quad d\in\mathbb{Z}_n^*,\ j\in\mathbb{Z}_l.\end{aligned}$$

Now let $G_j:=G_{|J_j}^{n'}$ and $V_j:=V_{|J_j}$ for $j\in\mathbb{Z}_l$, then from Lemma 2(iv) we get

$$\left(G_j^{nl}\right)^{m'}=F_{|J_j}^n,\quad j\in\mathbb{Z}_l,\tag{32}$$

where $m'=\frac{m}{l}$. By Theorem 1 we also get that $V_j\colon J_j\to J_{j+1}$ for $j\in\mathbb{Z}_{l-1}$ satisfy

$$G_{|J_{j+1}}^{n'}\circ V_j=V_j\circ G_{|J_j}^{n'},\quad j\in\mathbb{Z}_{l-1},$$

which is equivalent to (11). By Theorem 5 from [3], functions G and F are of the form (8) and (3), respectively. Write $m=i+bnl$, where $b=\left[\frac{m}{nl}\right]$ and $i\in\mathbb{Z}_{nl}$, then Theorem 1 gives (31) with $p'=\operatorname{char}G$. Put $i'_k:=-kp'(\text{mod } nl)$ for $k\in\mathbb{Z}_{nl}$ and let $\beta_{j,d}$ for $j\in\mathbb{Z}_l$ and $d\in\mathbb{Z}_n$ be defined by (12). As (1) is satisfied, (3), (31) and (11) give

$$T^q\circ F_{|J_j}^n=V^{q'i}\circ G_j^{-\beta_{j,0}+m'},\quad j\in\mathbb{Z}_l\tag{33}$$

and

$$T_{|J_{j+dl}}^q=V^{q'i}\circ V^{dl}\circ G_j^{-\beta_{j,d}}\circ V_{|J_{j+dl}}^{-dl},\quad j\in\mathbb{Z}_l,\ d\in\mathbb{Z}_n^*.\tag{34}$$

In view of (32), Eqs. (33) and (34) my be written as

$$T_{|J_{j+dl}}^q=V^{q'i}\circ V^{dl}\circ G_j^{-\beta_{j,d}}\circ V_{|J_{j+dl}}^{-dl},\quad j\in\mathbb{Z}_l,\ d\in\mathbb{Z}_n.\tag{35}$$

From Lemma 2(ii) we get $mq'=ql(\text{mod } nl)$ thus $q'i=ql(\text{mod } nl)$ and we have (29), which together with (35) give (26) for $j\in\mathbb{Z}_l$, $d\in\mathbb{Z}_n$. If $n>1$, for every $d\in\mathbb{Z}_n$ there is a unique $t\in\mathbb{Z}_n$ such that $tq=d(\text{mod } n)$. Hence by (26),

$$T_{|J_{j+tql(\text{mod } nl)}}^q=V^{ql}\circ V^{tql}\circ G_j^{-\beta_{j,tq(\text{mod } n)}}\circ V_{|J_{j+tql(\text{mod } nl)}}^{-tql}\tag{36}$$

for $j \in \mathbb{Z}_l$ and $t \in Z_n^*$. This implies

$$T^{kq}_{|J_j} = V^{ql} \circ V^{(k-1)ql} \circ G_j^{-\beta_{j,(k-1)q(\mathrm{mod}\ n)}} \circ V^{-(k-1)ql} \circ \cdots \circ V^{ql} \circ V^{ql} \circ G_j^{-\beta_{j,q(\mathrm{mod}\ n)}} \circ V^{-ql} \circ V^{ql} \circ G_j^{-\beta_{j,0}}$$
$$= V^{kql} \circ G_j^{-(\beta_{j,(k-1)q(\mathrm{mod}\ n)}+\cdots+\beta_{j,q(\mathrm{mod}\ n)}+\beta_{j,0})}, \quad j \in \mathbb{Z}_l$$

and as a result

$$V^{kql}_{|J_j} = T^{kq} \circ G_j^{\beta_{j,(k-1)q(\mathrm{mod}\ n)}+\cdots+\beta_{j,q(\mathrm{mod}\ n)}+\beta_{j,0}}, \quad j \in \mathbb{Z}_l \tag{37}$$

for $k \in \mathbb{Z}_n$. Putting (37) with $k = t$ into (36) we get

$$T^{q}_{|J_{j+tql(\mathrm{mod}\ nl)}} = V^{ql} \circ T^{tq} \circ G_j^{-\beta_{j,tq(\mathrm{mod}\ n)}} \circ T^{-tq}_{|J_{j+tql(\mathrm{mod}\ nl)}}, \quad j \in \mathbb{Z}_l,\ t \in \mathbb{Z}_n$$

which yields (23), with $tq = d(\mathrm{mod}\ n)$. Let Ψ be defined by (10), then Ψ is an orientation-preserving homeomorphism such that $\Psi^n = \mathrm{id}_{S^1}$, $\alpha(\Psi) = \frac{q}{n}$ and $\mathrm{char}\,\Psi = \mathrm{char}\,F$ (see the first step of this proof). By (10) we obtain $V^{ql} = \Psi$, thus $V^{ql\mathrm{char}\,F} = \Psi^{\mathrm{char}\,F}$ which yields

$$V^l = \Psi^{\mathrm{char}\,F}, \tag{38}$$

as $ql\mathrm{char}\,F = l(\mathrm{mod}\ nl)$ and $V^{nl} = \mathrm{id}_{S^1}$. From (38) we have

$$V \circ \Psi^{\mathrm{char}\,F} = \Psi^{\mathrm{char}\,F} \circ V.$$

This and the fact that $V_j = V_{|J_j}$ for $j \in \mathbb{Z}_l$ give (9). □

References

[1] P. Solarz, Some remarks on the general theorem of the existence of iterative roots of homeomorphisms with a rational rotation number, ESAIM: Proceedings (in press).
[2] W. Jarczyk, Babbage equation on the circle, Publ. Math. Debrecen 63 (3) (2003) 389–400.
[3] M.C. Zdun, On a factorization of homeomorphisms of the circle possessing periodic points, J. Math. Anal. Appl. 342 (2008) 340–348.
[4] P. Solarz, On some iterative roots, Publ. Math. Debrecen 63 (4) (2003) 677–692.
[5] P. Solarz, A note on some iterative roots, Ann. Univ. Paedagog. Crac. Stud. Math. 8 (2009) 57–66.
[6] M. Kuczma, B. Choczewski, R. Ger, Iterative Functional Equations, in: Encyclopaedia Math. Appl., vol. 32, Cambridge Univ. Press, Cambridge, 1990.
[7] I.P. Cornfeld, S.V. Fomin, Y.G. Sinai, Ergodic Theory, in: Grundlehren Math. Wiss., vol. 245, Springer-Verlag, Berlin, Heidelberg, New York, 1982.
[8] W. de Melo, S. van Strien, One-Dimensional Dynamics, in: Ergeb. Math. Grenzgeb. (3), vol. Band 25, Springer-Verlag, Berlin, 1993.
[9] M.C. Zdun, On conjugacy of homeomorphisms of the circle possessing periodic points, J. Math. Anal. Appl. 330 (2007) 51–65.
[10] P. Solarz, On some properties of orientation-preserving surjections on the circle, Math. Slovaca 57 (6) (2007) 547–560.