



# Infinite divisibility of interpolated gamma powers



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## ABSTRACT

This paper is concerned with the distribution properties of the binomial  $aX + bX^\alpha$ , where  $X$  is a gamma random variable. We show in particular that  $aX + bX^\alpha$  is infinitely divisible for all  $\alpha \in [1, 2]$  and  $a, b \in \mathbb{R}_+$ , and that for  $\alpha = 2$  the second order polynomial  $aX + bX^2$  is a generalized gamma convolution whose Thorin density and Wiener–gamma integral representation are computed explicitly. As a byproduct we deduce that fourth order multiple Wiener integrals are in general not infinitely divisible.

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## 1. Introduction

The power  $X_\beta^\alpha$  of order  $\alpha \in (-\infty, -1] \cup [1, \infty)$  of a gamma random variable  $X_\beta$  with shape parameter  $\beta > 0$  is known to be infinitely divisible, and in addition it belongs to the class of generalized gamma convolutions (GGCs), cf. [10], which is made of random variables  $Z$  whose Laplace transform can be expressed as

$$E[e^{-sZ}] = \exp\left(-cs - \int_0^\infty \log\left(1 + \frac{s}{t}\right) \mu(dt)\right), \quad s \geq 0, \quad (1.1)$$

where  $c \geq 0$  and the Thorin measure  $\mu(dt)$  of  $Z$  satisfies

$$\int_{0+}^1 |\log t| \mu(dt) < \infty \quad \text{and} \quad \int_1^\infty \frac{1}{t} \mu(dt) < \infty; \quad (1.2)$$

cf. p. 29 of [1]. We refer the reader to [5] for a complete survey with recent results on generalized gamma convolutions.

In particular, the random variable  $X_\beta^\alpha$  is a GGC for all  $\alpha \geq 1$ , cf. Example 4.3.4 p. 60 of [1], and its Thorin measure  $\mu_{0,\alpha}(dx)$  has total mass

$$\mu_{0,\alpha}([0, \infty)) = \sup \left\{ \nu > 0 : \lim_{x \searrow 0} \frac{f_{X_\beta^\alpha}(x)}{x^{\nu-1}} = 0 \right\} = \frac{\beta}{\alpha}, \quad (1.3)$$

cf. Theorem 4.4 of [1], and cumulative function

$$F_{0,\alpha} : \mathbb{R}_+ \rightarrow [0, \beta/\alpha).$$

In addition,  $X_\beta^\alpha$  is also a GGC with  $\mu_{0,\alpha}([0, \infty)) = \infty$  if  $\alpha < -1$ .

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In this paper we deal with the binomial  $aX_\beta + bX_\beta^\alpha$ ,  $\alpha \geq 0$ , and show in [Proposition 2.1](#) that  $aX_\beta + bX_\beta^\alpha$  is infinitely divisible for all  $\alpha \in [1, 2]$  and  $a, b \in \mathbb{R}_+$ . For  $\alpha = 2$  we consider the problem of addition and interpolation of two *dependent* GGCs by proving that the (non-central) gamma square  $aX_\beta + bX_\beta^2$  is a generalized gamma convolution for all  $\beta > 0$ , whose Thorin density  $\varphi_{a,b}$  is computed explicitly in [Proposition 3.1](#) as

$$\varphi_{a,b}(y) = \frac{\beta(1+ay)e^{(1-ay)^2/(4by)}}{2^\beta b^{1/2} \pi^{3/2} y^{3/2} \left( \Gamma(1/2 + \beta/2) + 2^{2-2\beta} F^2\left(\frac{|1-ay|}{2\sqrt{by}}\right) / \Gamma(1/2 + \beta/2) \right)},$$

$y \in \mathbb{R}_+$ , where  $F$  is the complex error function. In [Proposition 3.4](#) we also compute the Wiener–gamma integral representation

$$aX_\beta + bX_\beta^2 \simeq \int_0^\beta h_{a,b}(s) d\gamma_s,$$

where  $(\gamma_s)_{s \in \mathbb{R}_+}$  is a standard gamma process, and  $h_{a,b}$  is given by [\(3.9\)](#).

The case  $\beta = 1/2$  allows us to consider powers of the square  $Z^2 \simeq X_{1/2}$  of a centered Gaussian random variable  $Z$ . As an application of [Proposition 2.1](#) for  $\beta = 1/2$  we find that  $|Z|^\alpha + aZ^2$  is infinitely divisible for all  $\alpha \in [2, 4]$  and  $a \geq 0$ , and, in particular, that  $Z^4 + aZ^2$  is infinitely divisible for  $a \geq 0$ . In [Section 4](#) we show that  $Z^4 + aZ^2$  is not infinitely divisible for all  $a < 0$ , showing in particular that fourth order multiple stochastic integrals with respect to Brownian motion are not infinitely divisible random variables, although first and second order Brownian stochastic integrals are known to be infinitely divisible.

We proceed as follows. After recalling some basic results on infinite divisibility at the end of this introduction, we consider interpolated gamma powers of the form  $aX_\beta + bX_\beta^\alpha$  in [Section 2](#). The case of second order polynomials of the form  $aX_\beta + bX_\beta^2$  as generalized gamma convolutions is discussed in [Section 3](#) with explicit calculations of Thorin measures and Wiener–gamma representations, based in part on [Theorem 2.3](#) of [\[5\]](#). In [Section 4](#) we consider the infinite divisibility of multiple Wiener integrals. [Section 5](#) contains the complete monotonicity results needed in this paper, and in the [Appendix](#) we extend to the gamma case the results of [\[5\]](#) on the computation of Thorin densities for exponential random variables.

#### Infinite divisibility and complete monotonicity

We close this introduction with a review of the links between infinite divisibility and complete monotonicity. Recall that a nonnegative random variable  $Z \geq 0$  is infinitely divisible if and only if its Laplace transform takes the form

$$E[e^{-sZ}] = \exp\left(-cs - \int_0^\infty (1 - e^{-sx})\nu(dx)\right), \quad s \in \mathbb{R}_+, \quad (1.4)$$

where  $c \geq 0$  and  $\nu(dx)$  is a measure on  $\mathbb{R}_+$  such that

$$\int_0^\infty (1 \wedge x)\nu(dx) < \infty.$$

In this paper we are mainly concerned with the infinite divisibility of lower (or upper) bounded random variables, for which we will use the following criterion; cf. [Theorem XIII.7.1](#) of [\[4\]](#), [Theorem III.4.1](#) of [\[9\]](#), or [Theorem 5.9](#) of [\[8\]](#). Recall that a  $\mathcal{C}^\infty$  function  $f : (0, \infty) \rightarrow \mathbb{R}$  is completely monotone if

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in \mathbb{R}_+,$$

for all integers  $n \geq 0$ .

**Theorem 1.1.** *Let  $Z$  be a nonnegative random variable with Laplace transform*

$$\Psi_Z(s) = E[e^{-sZ}], \quad s \in \mathbb{R}_+.$$

*Then  $Z$  is infinitely divisible if and only if*

$$s \mapsto -\frac{\partial}{\partial s} \log \Psi_Z(s)$$

*is completely monotone on  $\mathbb{R}_+$ .*

**Proof.** By the Bernstein theorem, cf. e.g. [Theorem 3.2](#) of [\[8\]](#), the  $\mathcal{C}^\infty$  function

$$\varphi(s) = -\log \Psi_Z(s), \quad s \in \mathbb{R}_+,$$

has the representation

$$\varphi(s) = cs + \int_0^\infty (1 - e^{-sx})\nu(dx), \quad s \in \mathbb{R}_+, \quad (1.5)$$

where  $c \in \mathbb{R}_+$  and  $\nu(dx)$  is a measure on  $\mathbb{R}_+$  such that  $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$ , if and only if  $\varphi'$  is completely monotone.  $\square$

In addition it follows from (1.1), (1.4), Frullani's identity

$$\log\left(1 + \frac{s}{t}\right) = \int_0^\infty (1 - e^{-sx})e^{-xt} \frac{dx}{x}, \quad s, t \in \mathbb{R}_+,$$

that the Lévy measure  $\nu(dx)$  is linked to the Thorin measure  $\mu(dx)$  by the relation

$$\nu(dx) = \frac{1}{x} \int_0^\infty e^{-xz} \mu(dz) dx, \quad x > 0.$$

## 2. Interpolated gamma powers

Let  $X_\beta$  denote a gamma random variable with density

$$f_{X_\beta}(x) = \frac{1}{\Gamma(\beta)} x^{\beta-1} e^{-x}, \quad x > 0,$$

and shape parameter  $\beta > 0$ . Recall that for any  $\alpha \in \mathbb{R} \setminus \{0\}$  the random variable  $X_\beta^\alpha$  has density

$$f_{X_\beta^\alpha}(x) = \frac{1}{|\alpha| \Gamma(\beta)} x^{-1+\beta/\alpha} e^{-x^{1/\alpha}}, \quad x > 0,$$

which is a Weibull probability density when  $\alpha > 0$  and  $\beta = 1$ . On the other hand,  $X_\beta^\alpha$  is not infinitely divisible for  $\alpha \in (0, 1)$  and it is unknown whether  $X_\beta^\alpha$  is infinitely divisible when  $\alpha \in (-1, 0)$ ; cf. Example 4.3.4 p. 60 and Section III p. 67 of [1].

In this section we prove the following result.

**Proposition 2.1.** *Let  $\beta > 0$  and  $a, b \in \mathbb{R}_+$ . The random variable  $aX_\beta + bX_\beta^\alpha$  is infinitely divisible for all  $\alpha \in [1, 2]$ .*

**Proof.** This result is a consequence of Theorem 1.1, Lemma 2.2, and Corollary 5.2 which states that the function

$$s \mapsto \frac{\eta + a\alpha(1+as)^{\alpha-1}}{\eta s + (1+as)^\alpha} = \frac{\partial}{\partial s} \log(\eta s + (1+as)^\alpha) \quad (2.1)$$

is completely monotone for all  $\alpha \in [0, 2]$  and  $a, \eta \in \mathbb{R}_+$ .  $\square$

Lemma 2.2 has been used in the proof of Proposition 2.1.

**Lemma 2.2.** *Let  $a \in \mathbb{R}_+$  and  $\alpha \geq 1$ . We have*

$$\Psi_{aX_\beta + X_\beta^\alpha}(s) = \exp\left(-\int_0^\infty \log\left((1+as)^\alpha + \frac{s}{t}\right) \mu_{0,\alpha}(dt)\right), \quad s \in \mathbb{R}_+. \quad (2.2)$$

**Proof.** We have

$$\begin{aligned} \Psi_{aX_\beta + X_\beta^\alpha}(s) &= E[e^{-s(X_\beta^\alpha + aX_\beta)}] \\ &= \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-s(x^\alpha + ax)} x^{\beta-1} e^{-x} dx \\ &= \frac{1}{\Gamma(\beta)} \int_0^\infty x^{\beta-1} e^{-sx^\alpha - x(1+as)} dx \\ &= (1+as)^{-\beta} \frac{1}{\Gamma(\beta)} \int_0^\infty x^{\beta-1} e^{-s(1+as)^{-\alpha} x^\alpha - x} dx \\ &= (1+as)^{-\beta} E[e^{-s(1+as)^{-\alpha} X_\beta^\alpha}] \\ &= (1+as)^{-\beta} \Psi_{X_\beta^\alpha}(s(1+as)^{-\alpha}), \quad s \in \mathbb{R}_+. \end{aligned}$$

Hence from (1.1) and (1.3) we get

$$\begin{aligned} \Psi_{aX_\beta + X_\beta^\alpha}(s) &= (1+as)^{-\beta} \Psi_{X_\beta^\alpha}(s(1+as)^{-\alpha}) \\ &= (1+as)^{-\beta} \exp\left(-\int_0^\infty \log\left(1 + \frac{s}{t}(1+as)^{-\alpha}\right) \mu_{0,\alpha}(dt)\right) \\ &= (1+as)^{-\beta} \exp\left(-\int_0^\infty \log\left((1+as)^\alpha + \frac{s}{t}\right) \mu_{0,\alpha}(dt) + \alpha \log(1+as) \mu_{0,\alpha}([0, \infty))\right) \\ &= \exp\left(-\int_0^\infty \log\left((1+as)^\alpha + \frac{s}{t}\right) \mu_{0,\alpha}(dt)\right), \end{aligned}$$

since  $\mu_{0,\alpha}([0, \infty)) = \beta/\alpha$  by (1.3).  $\square$

Note that the transformations of Lemma 2.2 are not applicable to the Lévy–Khintchine formula (1.1); as a consequence the Lévy measure of  $aX_\beta + X_\beta^\alpha$  does not seem to be computable from the Lévy measure of  $X_\beta^\alpha$ .

We close this section with some remarks on the case  $\alpha \notin [1, 2]$ . When  $\alpha \in (0, 1)$  the random variable  $X_\beta^\alpha$  is not infinitely divisible, and from Remark 5.3 we conjecture that  $X_\beta^\alpha + aX_\beta$  is not infinitely divisible for  $\alpha > 2$ .

When  $\alpha \geq 1$  is an integer we may decompose the polynomial  $s \mapsto (1 + as)^\alpha + s/t$  in (2.2),  $t > 0$ , as

$$(1 + as)^\alpha + \frac{s}{t} = \prod_{k=1}^{\alpha} \left( 1 + \frac{s}{g_k^a(t)} \right) \geq 0, \quad (2.3)$$

where  $g_1^a(t), \dots, g_\alpha^a(t)$  are the (complex) roots of  $s \mapsto (1 - as)^\alpha - s/t$ , counted with their multiplicities. Then Frullani's identity yields

$$\log \left( (1 + as)^\alpha + \frac{s}{t} \right) = \sum_{k=1}^{\alpha} \log \left( 1 + \frac{s}{g_k^a(t)} \right) = \int_0^\infty (1 - e^{-sx}) \sum_{k=1}^{\alpha} e^{-xg_k^a(t)} \frac{dx}{x},$$

and the Lévy measure  $\nu(dx)$  of  $X_\beta^\alpha + aX_\beta$ , if it exists, is given from Lemma 2.2 by

$$\nu(dx) = \frac{1}{x} \sum_{k=1}^{\alpha} \int_0^\infty e^{-xg_k^a(t)} \mu_{0,\alpha}(dt) dx,$$

which is real since the roots  $\{g_1^a(t), \dots, g_\alpha^a(t)\}$  are either real or complex conjugate.

In Section 3 we will consider the case  $\alpha = 2$  where both roots are real,  $\sum_{k=1}^2 e^{-xg_k^a(t)}$  is positive, and  $aX_\beta + X_\beta^2$  is infinitely divisible. Numerical computations not presented here have shown that the sum  $\sum_{k=1}^k e^{-xg_k^a(t)}$  is not always positive when  $\alpha \geq 3$ .

### 3. Second order polynomials

In this section we consider the case  $\alpha = 2$  and  $a \in \mathbb{R}_+$ . The probability density function of  $aX_\beta + X_\beta^2$  is given for all  $\beta > 0$  by

$$f_a(x) = \frac{2^{1-\beta}}{\sqrt{a^2 + 4x}} \frac{1}{\Gamma(\beta)} \left( -a + \sqrt{a^2 + 4x} \right)^{\beta-1} e^{-\frac{-a + \sqrt{a^2 + 4x}}{2}}, \quad x > 0,$$

and it is log-convex only for  $\beta \in (0, 1]$ , which shows that  $aX_\beta + X_\beta^2$  is infinitely divisible for  $\beta \in (0, 1]$  and  $a > 0$ , by e.g. Theorem 51.4 of [7].

In case  $\beta = 1$  the density function  $f_a$  is hyperbolically completely monotone (HCM, cf. Section 5.1 of [1]), hence  $aX_1 + X_1^2$  is a GGC by Theorem 5.1.2 p. 71 of [1], or Theorem 5.18 of [9].

That  $f_a$  is HCM follows from the facts that the function  $x \mapsto (a^2 + 4x)^{-1/2}$  is HCM by p. 68 of [1], and the function  $x \mapsto e^{-c\sqrt{x}}$  is HCM and decreasing for  $c > 0$ , cf. Property (iv) p. 68 of [1]. Then, by Property (xi) of [1],  $x \mapsto e^{-c\sqrt{a^2 + 4x}}$  is HCM, and  $f_a$  is HCM since the product of two HCM functions is HCM. Note also that  $x \mapsto e^{-c(\sqrt{a^2 + 4x} - a)}$  is HCM because it is the Laplace transform of a tempered stable distribution with parameter  $1/2$  which is a GGC; cf. Theorem 6.1.1 p. 90 of [1].

In Proposition 3.1 we prove that  $aX_\beta + bX_\beta^2$  is a GGC for all  $\beta > 0$  and  $a, b \geq 0$  by resorting directly to the definition (1.1) of the GGC class.

**Proposition 3.1.** *For all  $a \in \mathbb{R}_+$ ,  $b > 0$  and  $\beta > 0$ , the random variable  $aX_\beta + bX_\beta^2$  is a GGC whose Thorin measure  $\mu_{a,2}(dy)$  has the density*

$$\varphi_{a,b}(y) = \frac{\beta(1 + ay)e^{(1-ay)^2/(4by)}}{2^\beta b^{1/2} \pi^{3/2} y^{3/2} \left( \Gamma(1/2 + \beta/2) + 2^{2-2\beta} F^2 \left( \frac{1-ay}{2\sqrt{by}} \right) / \Gamma(1/2 + \beta/2) \right)}, \quad (3.1)$$

$y > 0$ , where

$$F(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{z^2} dz, \quad x \in \mathbb{R}_+, \quad (3.2)$$

is the complex error function.

**Proof.** We use the expression (3.3) of the Thorin density  $\varphi_{0,1}(y)$  of  $X_\beta^2$  given in Lemma 3.2, and apply Lemma 3.3.  $\square$

Proposition 3.1 follows from Lemmas 3.2 and 3.3. Lemma 3.2 is an application for  $\alpha = 2$  of Proposition A.1 which relies on Theorem 2.3 of [5]; cf. Appendix.

**Lemma 3.2.** The distribution function  $F_{0,2}$  of the Thorin measure  $\mu_{0,2}(dy)$  of  $X_\beta^2$  is given by

$$F_{0,2}(y) = \frac{\beta}{\pi} \arctan \left( \frac{\Gamma(\beta/2 + 1/2)}{2^{1-\beta} F(1/(2\sqrt{y}))} \right), \quad y > 0,$$

for all  $\beta > 0$ . In particular,  $F_{0,2}(y)$  is absolutely continuous and  $\mu_{0,2}(dy)$  admits a density  $\varphi_{0,1}(y)$  with respect to the Lebesgue measure, given by

$$\varphi_{0,1}(y) = \frac{\beta e^{1/(4y)}}{2^\beta \pi^{3/2} y^{3/2} (\Gamma(1/2 + \beta/2) + 2^{2-2\beta} F^2(1/(2\sqrt{y})) / \Gamma(1/2 + \beta/2))}, \quad (3.3)$$

$y > 0$ .

**Lemma 3.3** shows that the Thorin density  $\varphi_{a,b}(x)$  of  $aX_\beta + bX_\beta^2$  can be computed from the Thorin density  $\varphi_{0,1}(x)$  of  $X_\beta^2$ , and relies on **Proposition A.1** in the **Appendix** for the absolute continuity of  $F_{0,2}$  and the existence of  $\varphi_{0,1}$ .

**Lemma 3.3.** For all  $a \in \mathbb{R}_+$ ,  $b > 0$  and  $\beta > 0$ , the random variable  $aX_\beta + bX_\beta^2$  is a GGC whose Thorin density  $\varphi_{a,b}$  satisfies

$$\varphi_{a,b}(x) = b \frac{1+ax}{|1-ax|^3} \varphi_{0,1} \left( \frac{bx}{(1-ax)^2} \right), \quad x > 0,$$

with total mass

$$\mu_{a,2}((0, \infty)) = \beta$$

if  $a > 0$ , while  $\mu_{0,2}((0, \infty)) = \beta/2$ .

**Proof.** Letting

$$g_a^\pm(t) = \frac{1}{2a^2t} + \frac{1}{a} \pm \sqrt{\left( \frac{1}{2a^2t} + \frac{1}{a} \right)^2 - \frac{1}{a^2}} > 0, \quad t > 0, \quad (3.4)$$

with

$$g_a^+ : (0, \infty) \longrightarrow (1/a, \infty) \quad \text{and} \quad g_a^- : (0, \infty) \longrightarrow (0, 1/a), \quad (3.5)$$

$g_a^-(t)g_a^+(t) = 1/a^2$ ,  $t > 0$ , and

$$\lim_{a \rightarrow 0} g_a^-(t) = t, \quad \lim_{a \rightarrow 0} g_a^+(t) = \infty, \quad t \in \mathbb{R}_+.$$

Letting  $\mu_{a,2}^+(dt)$ , resp.  $\mu_{a,2}^-(dt)$  denote the image measures of  $\mu_{0,2}(dt)$  by  $g_a^+$ , resp.  $g_a^-$ , by (2.3) and **Lemma 2.2** we have

$$\begin{aligned} \Psi_{aX_\beta + X_\beta^2}(s) &= E[e^{-s(aX_\beta + X_\beta^2)}] \\ &= \exp \left( - \int_0^\infty \log \left( (1+as)^2 + \frac{s}{t} \right) \mu_{0,2}(dt) \right) \\ &= \exp \left( - \int_0^\infty \log \left( 1 + \left( \frac{1}{t} + 2a \right) s + a^2 s^2 \right) \mu_{0,2}(dt) \right) \\ &= \exp \left( - \int_0^\infty \log \left( \left( 1 + \frac{s}{g_a^+(t)} \right) \left( 1 + \frac{s}{g_a^-(t)} \right) \right) \mu_{0,2}(dt) \right) \\ &= \exp \left( - \int_0^\infty \log \left( 1 + \frac{s}{g_a^-(t)} \right) \mu_{0,2}(dt) - \int_0^\infty \log \left( 1 + \frac{s}{g_a^+(t)} \right) \mu_{0,2}(dt) \right) \\ &= \exp \left( - \int_0^{1/a} \log \left( 1 + \frac{s}{t} \right) \mu_{a,2}^-(dt) - \int_{1/a}^\infty \log \left( 1 + \frac{s}{t} \right) \mu_{a,2}^+(dt) \right), \end{aligned}$$

$s \in \mathbb{R}_+$ , which shows that the Thorin measure  $\mu_{a,2}(dx)$  of  $aX_\beta + X_\beta^2$  satisfies

$$\mu_{a,2}(dx) = \mu_{a,2}^+(dx) + \mu_{a,2}^-(dx).$$

Next, denoting by  $F_{a,2}(x) = \mu_{a,2}([0, x])$  the cumulative distribution function of the Thorin measure  $\mu_{a,2}(dx)$ , we have

$$\begin{aligned}
 F_{a,2}(x) &= \int_0^\infty \mathbf{1}_{[0,x]}(g_a^-(t)) \mu_{0,2}(dt) + \int_0^\infty \mathbf{1}_{[0,x]}(g_a^+(t)) \mu_{0,2}(dt) \\
 &= \mathbf{1}_{\{x \leq 1/a\}} \int_0^\infty \mathbf{1}_{[0,x]}(g_a^-(t)) \mu_{0,2}(dt) + \mathbf{1}_{\{x > 1/a\}} \frac{\beta}{2} + \mathbf{1}_{\{x > 1/a\}} \int_0^\infty \mathbf{1}_{[0,x]}(g_a^+(t)) \mu_{0,2}(dt) \\
 &= \mathbf{1}_{\{x \leq 1/a\}} \int_0^{(g_a^-)^{-1}(x)} \mu_{0,2}(dt) + \mathbf{1}_{\{x > 1/a\}} \left( \frac{\beta}{2} + \int_{(g_a^+)^{-1}(x)}^\infty \mu_{0,2}(dt) \right) \\
 &= \mathbf{1}_{\{x \leq 1/a\}} \int_0^{(g_a^-)^{-1}(x)} \mu_{0,2}(dt) + \mathbf{1}_{\{x > 1/a\}} \left( \beta - \int_0^{(g_a^+)^{-1}(x)} \mu_{0,2}(dt) \right) \\
 &= F_{0,2} \left( \frac{x}{(1-ax)^2} \right) \mathbf{1}_{\{x \leq 1/a\}} + \left( \beta - F_{0,2} \left( \frac{x}{(1-ax)^2} \right) \right) \mathbf{1}_{\{x > 1/a\}}, \tag{3.6}
 \end{aligned}$$

$x \in \mathbb{R}_+$ ,  $a > 0$ . This shows in particular that

$$F_{a,2}(1/a) = F_{0,2}(+\infty) = \mu_{0,2}((0, \infty)) = \beta/2,$$

by (1.3), and

$$F_{a,2}(\infty) = \mu_{a,2}((0, \infty)) = \beta,$$

$a > 0$ .

By Proposition A.1, the function  $F_{0,2}$  is absolutely continuous and by differentiation with respect to  $x$  we obtain

$$\varphi_{a,1}(x) = \frac{1+ax}{|1-ax|^3} \varphi_{0,1} \left( \frac{x}{(1-ax)^2} \right), \quad x > 0,$$

which gives  $\varphi_{a,b}(x)$  in (3.1) by (3.3) and the rescaling relation

$$\varphi_{a,b}(x) = b\varphi_{a/b,1}(bx), \quad x > 0,$$

for  $a \in \mathbb{R}_+$  and  $b > 0$ .

In order to conclude that  $aX_\beta + X_\beta^2$  is a GGC with Thorin measure  $\mu_{a,2}(dt) = \mu_{a,2}^+(dt) + \mu_{a,2}^-(dt)$  it suffices to check that Condition (1.2) holds. We have

$$\begin{aligned}
 \int_{0^+}^{1/a} |\log t| \mu_{a,2}^-(dt) + \int_{1/a}^\infty \frac{1}{t} \mu_{a,2}^+(dt) &= \int_{0^+}^\infty \left| \log \frac{1}{g_a^-(t)} \right| \mu_{0,2}(dt) + \int_{0^+}^\infty \frac{1}{g_a^+(t)} \mu_{0,2}(dt) \\
 &\leq c_a \frac{\beta}{2} + c_a \int_{0^+}^\infty |\log t| \mu_{0,2}(dt) + a\mu_{0,2}((0, \infty)) \\
 &< \infty,
 \end{aligned}$$

by (3.5) for some  $c_a > 0$  since

$$\begin{aligned}
 g_a^-(t) &= \frac{1}{2a^2t} + \frac{1}{a} - \sqrt{\left( \frac{1}{2a^2t} + \frac{1}{a} \right)^2 - \frac{1}{a^2}} \\
 &= \frac{1}{2a^2t} + \frac{1}{a} - \frac{1}{2a^2t} \sqrt{1+4at} \\
 &= \frac{1}{2a^2t} + \frac{1}{a} - \frac{1}{2a^2t} \left( 1 + 2at - \frac{1}{8}4^2a^2t^2 \right) + o(t) \\
 &= t + o(t), \quad t \searrow 0,
 \end{aligned}$$

which implies

$$\begin{aligned}
 \log \frac{1}{g_a^-(t)} &= -\log g_a^-(t) \\
 &= -\log(t + o(t)) \\
 &= -\log t - \log(1 + o(t)/t) \\
 &\leq c + |\log t|, \quad 0 < t < 1,
 \end{aligned}$$

for some  $c > 0$ .  $\square$

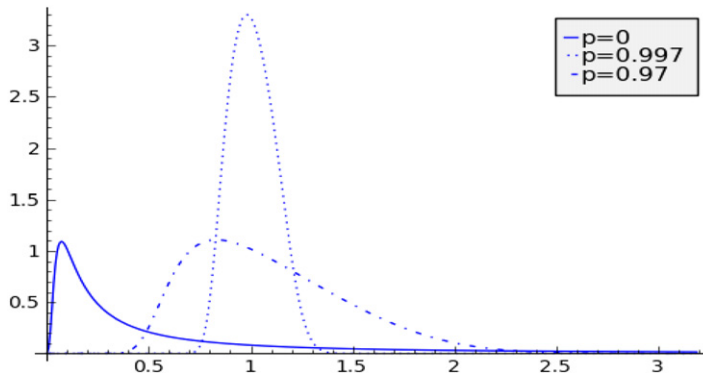


Fig. 1. Graphs of  $x \mapsto \varphi_{p,1-p}(x)$  for  $p = 0$ ,  $p = 0.997$ , and  $p = 0.97$ .

Note that from (3.6) we have

$$F_{a,2}(x/a) = F_{0,2}\left(\frac{x}{a(1-x)^2}\right) \mathbf{1}_{\{x \leq 1\}} + \left(\beta - F_{0,2}\left(\frac{x}{a(1-x)^2}\right)\right) \mathbf{1}_{\{x > 1\}},$$

which converges to

$$\lim_{a \rightarrow \infty} F_{a,2}(x/a) = \beta \mathbf{1}_{[1, \infty)}(x)$$

as  $a$  goes to infinity, and we recover

$$\lim_{b \rightarrow 0} \varphi_{a,b}(x) dx = \beta \delta_{1/a}(dx),$$

which is the Thorin measure of a  $\Gamma(a, \beta)$  random variable.

By (3.1) we also find

$$\varphi_{a,b}(1/a) = \frac{2^{1-\beta} \beta}{\Gamma(1/2 + \beta/2)} \frac{a^{3/2}}{\pi^{3/2} b^{1/2}}, \quad a, b > 0,$$

and when  $\beta = 1$  we have

$$\varphi_{a,b}(x) = \frac{1 + ax}{2\pi^{3/2} b^{1/2} x^{3/2}} \frac{e^{|1-ax|^2/(4bx)}}{1 + F^2(|1-ax|/(2\sqrt{bx}))}, \quad x > 0.$$

Fig. 1 shows a graph of the Thorin density

$$\begin{aligned} \varphi_{p,1-p}(x) &= (1-p) \frac{1+px}{|1-px|^3} \varphi_{0,1}\left(\frac{(1-p)x}{(1-px)^2}\right) \\ &= \frac{1+px}{2\pi^{3/2} (1-p)^{1/2} x^{3/2}} \frac{e^{(1-px)^2/(4(1-p)x)}}{1 + F^2(|1-px|/(2\sqrt{x(1-p)}))}, \quad x > 0, \end{aligned}$$

of  $pX_1 + (1-p)X_1^2$  for  $\beta = 1$  and different values of  $p \in [0, 1]$ , which interpolate between the Thorin measure  $\delta_1(dx)$  of  $X_1$  and the Thorin density  $\varphi_{0,1}(x)$  of  $X_1^2$ .

Here the shape parameter is plotted as a function of the scale parameter. Note that the total mass of the Thorin density  $\varphi_{p,1-p}$  on  $(0, \infty)$  is 1 for  $p > 0$  and  $1/2$  for  $p = 0$ .

**Wiener–gamma representation**

By Proposition 1.1 of [5], the gamma square  $X_\beta^2$  can be written as the Wiener–gamma integral

$$X_\beta^2 \simeq \int_0^{\beta/2} h_{0,1}(s) d\gamma_s = \int_0^\infty \frac{1}{t} d\gamma_{F_{0,2}(t)},$$

with

$$h_{0,1}(s) = \frac{1}{(F_{0,2})^{-1}(s)}, \quad 0 \leq s < \beta/2,$$

where  $(\gamma_s)_{s \in \mathbb{R}_+}$  is a standard gamma process, i.e. the Lévy process whose increments  $\gamma_t - \gamma_s$  are gamma distributed with shape parameter  $t - s$ ,  $0 \leq s \leq t$ , where

$$F_{0,2}(x) = \mu_{0,2}([0, x]), \quad x > 0,$$

is the cumulative distribution function of the Thorin measure  $\mu_{0,2}$  of  $X_\beta^2$ , with total mass  $\beta/2$ . By inverting  $F_{0,2}$  in (A.3), i.e.

$$F_{0,2}(y) = \frac{\beta}{\pi} \arctan \left( \frac{\Gamma(\beta/2 + 1/2)}{2^{1-\beta} F(1/(2\sqrt{y}))} \right),$$

we find

$$h_{0,1}(y) = \frac{1}{(F_{0,2})^{-1}(y)} = 4 \left| F^{-1} \left( 2^{\beta-1} \Gamma(1/2 + \beta/2) \cot(\pi y/\beta) \right) \right|^2, \quad y \in (0, \beta/2), \quad (3.7)$$

where  $F(x)$  is the complex error function (3.2) and  $\cot x = 1/\tan x$ ,  $x \in \mathbb{R} \setminus \{0\}$ .

In particular when  $\beta = 1$  we have

$$h_{0,1}(y) = 4 \left| F^{-1}(\cot(\pi y)) \right|^2, \quad y \in [0, 1/2].$$

The Wiener–gamma integral representation provides a stochastic integral expression of the Lévy process associated to  $aX_\beta + bX_\beta^2$  and indexed by the shape parameter.

**Proposition 3.4.** For all  $a, b \in \mathbb{R}_+$  and  $\beta > 0$  the random variable  $aX_\beta + bX_\beta^2$  admits the representation

$$aX_\beta + bX_\beta^2 \simeq \int_0^\beta h_{a,b}(s) d\gamma_s, \quad (3.8)$$

in law, with

$$h_{a,b}(s) = \begin{cases} a + \frac{b}{2} \left( 1 + \sqrt{1 + \frac{4a}{bh_{0,1}(s)}} \right) h_{0,1}(s), & 0 < s < \beta/2, \\ a + \frac{b}{2} \left( 1 - \sqrt{1 + \frac{4a}{bh_{0,1}(\beta-s)}} \right) h_{0,1}(\beta-s), & \beta/2 < s \leq \beta, \end{cases} \quad (3.9)$$

where  $h_{0,1}(s)$  is given by (3.7).

**Proof.** By (3.6) we have

$$\begin{aligned} F_{a,2}(x) &= F_{0,2} \left( \frac{x}{(1-ax)^2} \right) \mathbf{1}_{\{x \leq 1/a\}} + \left( \beta - F_{0,2} \left( \frac{x}{(1-ax)^2} \right) \right) \mathbf{1}_{\{x > 1/a\}} \\ &= \left( \frac{1}{h_{0,1}} \right)^{-1} \left( \frac{x}{(1-ax)^2} \right) \mathbf{1}_{\{x \leq 1/a\}} + \left( \beta - \left( \frac{1}{h_{0,1}} \right)^{-1} \left( \frac{x}{(1-ax)^2} \right) \right) \mathbf{1}_{\{x > 1/a\}} \\ &= \left( \frac{1}{h_{0,1}} \right)^{-1} ((g_a^-)^{-1}(x)) \mathbf{1}_{\{x \leq 1/a\}} + \left( \beta - \left( \frac{1}{h_{0,1}} \right)^{-1} ((g_a^+)^{-1}(x)) \right) \mathbf{1}_{\{x > 1/a\}}, \end{aligned}$$

$x > 0$ , where

$$\begin{aligned} (g_a^+)^{-1} : (1/a, \infty) &\longrightarrow (0, \infty) \\ x &\longmapsto \frac{x}{(1-ax)^2}, \end{aligned}$$

and

$$\begin{aligned} (g_a^-)^{-1} : (0, 1/a) &\longrightarrow (0, \infty) \\ x &\longmapsto \frac{x}{(1-ax)^2}, \end{aligned}$$

hence the Wiener–gamma representation of  $aX_\beta + X_\beta^2$  is given by

$$\begin{aligned} h_{a,1}(s) &= \frac{1}{(F_{a,2})^{-1}(s)} \\ &= \begin{cases} \frac{1}{g_a^-(1/h_{0,1}(s))}, & 0 < s < F_{a,2}(1/a) = \beta/2, \\ \frac{1}{g_a^+(1/h_{0,1}(\beta-s))}, & \beta/2 = F_{a,2}(1/a) < s \leq \beta, \end{cases} \end{aligned}$$



$$= \begin{cases} a^2 g_a^+ \left( \frac{1}{h_{0,1}(s)} \right), & 0 < s < \beta/2, \\ a^2 g_a^- \left( \frac{1}{h_{0,1}(\beta-s)} \right), & \beta/2 < s \leq \beta, \end{cases} \quad (3.10)$$

$$= \begin{cases} \frac{1}{2} h_{0,1}(s) + a + \sqrt{\frac{h_{0,1}^2(s)}{4} + a h_{0,1}(s)}, & 0 < s < \beta/2, \\ \frac{1}{2} h_{0,1}(\beta-s) + a - \sqrt{\frac{h_{0,1}^2(\beta-s)}{4} + a h_{0,1}(\beta-s)}, & \beta/2 < s \leq \beta. \end{cases}$$

$$= \begin{cases} a + \frac{1}{2} h_{0,1}(s) \left( 1 + \sqrt{1 + \frac{4a}{h_{0,1}(s)}} \right), & 0 < s < \beta/2, \\ a + \frac{1}{2} h_{0,1}(\beta-s) \left( 1 - \sqrt{1 + \frac{4a}{h_{0,1}(\beta-s)}} \right), & \beta/2 < s \leq \beta. \end{cases} \quad (3.11)$$

To conclude the proof we use the rescaling relation

$$h_{a,b}(s) = b h_{a/b,1}(s), \quad s > 0,$$

for  $b > 0$ .  $\square$

We note that  $h_{a,b}(\beta/2) = a$ , and  $h_{a,0}(s) = a$ ,  $s \in (0, \beta]$ . When  $b > 0$  we also have  $h_{0,b}(0) = +\infty$ , and  $h_{a,b}(\beta) = 0$  since

$$\lim_{x \rightarrow \infty} \frac{x}{2} + a - \sqrt{\frac{x^2}{4} + ax} = \lim_{x \rightarrow \infty} \frac{x}{2} \left( 1 + \frac{2a}{x} - \sqrt{1 + \frac{4a}{x}} \right) = 0.$$

Fig. 2 shows a graph of the Wiener-gamma integrand  $x \mapsto h_{p,1-p}(x)$  of  $pX_1 + (1-p)X_1^2$  for  $\beta = 1$  and different values of  $p \in [0, 1]$ . Here the scale parameter is plotted as a function of the shape parameter.

As can be expected, by (3.9) we also have

$$\lim_{b \rightarrow 0} h_{a,b}(y) = h_{a,0}(y) = a, \quad y \in (0, \beta).$$

Finally, denoting by  $(\tilde{\gamma}_s)_{s \in \mathbb{R}_+}$  another standard gamma process independent of  $(\gamma_s)_{s \in \mathbb{R}_+}$ , by (3.10) and (3.11) we have

$$\begin{aligned} aX_\beta + X_\beta^2 &= \int_0^\beta h_{a,1}(s) d\gamma_s \\ &= a^2 \int_0^{\beta/2} g_a^+ \left( \frac{1}{h_{0,1}(s)} \right) d\gamma_s + a^2 \int_{\beta/2}^\beta g_a^- \left( \frac{1}{h_{0,1}(\beta-s)} \right) d\gamma_s \\ &= a^2 \int_0^{\beta/2} g_a^+ \left( \frac{1}{h_{0,1}(s)} \right) d\gamma_s + a^2 \int_0^{\beta/2} g_a^- \left( \frac{1}{h_{0,1}(s)} \right) d\tilde{\gamma}_s \\ &= \frac{1}{2} \int_0^{\beta/2} h_{0,1}(s) d\gamma_s + \int_0^{\beta/2} a d\gamma_s + \int_0^{\beta/2} \sqrt{\frac{h_{0,1}^2(s)}{4} + a h_{0,1}(s)} d\gamma_s \\ &\quad + \frac{1}{2} \int_0^{\beta/2} h_{0,1}(s) d\tilde{\gamma}_s + \int_0^{\beta/2} a d\tilde{\gamma}_s - \int_0^{\beta/2} \sqrt{\frac{h_{0,1}^2(s)}{4} + a h_{0,1}(s)} d\tilde{\gamma}_s \\ &= a(\gamma(\beta/2) + \tilde{\gamma}(\beta/2)) + \int_0^{\beta/2} h_{0,1}(s) d\gamma_s + \int_0^{\beta/2} \frac{h_{0,1}(s)}{2} \left( -1 + \sqrt{1 + a \frac{4}{h_{0,1}(s)}} \right) (d\gamma_s - d\tilde{\gamma}_s) \\ &= a(\gamma(\beta/2) + \tilde{\gamma}(\beta/2)) + \int_0^{\beta/2} h_{0,1}(s) d\gamma_s + 2a \int_0^{\beta/2} \left( 1 + \sqrt{1 + \frac{4a}{h_{0,1}(s)}} \right)^{-1} (d\gamma_s - d\tilde{\gamma}_s), \end{aligned}$$

where we also used (3.8).

This provides a linearization of  $aX_\beta + bX_\beta^2$  in  $\gamma_s$ , into the sum of a  $\Gamma(a, \beta)$  random variable, a squared gamma variable, and a remainder which is an extended GGC in the sense of Chapter 7 of [1], and goes to 0 as  $a$  tends to 0.

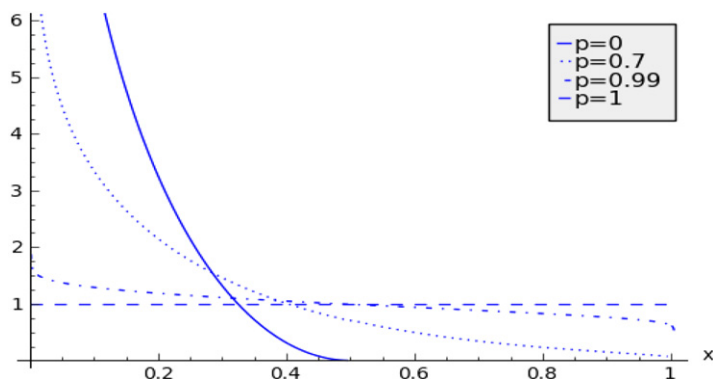


Fig. 2. Graphs of  $x \mapsto h_{p,1-p}(x)$  for  $p = 0$ ,  $p = 0.7$ ,  $p = 0.99$ , and  $p = 1$ .

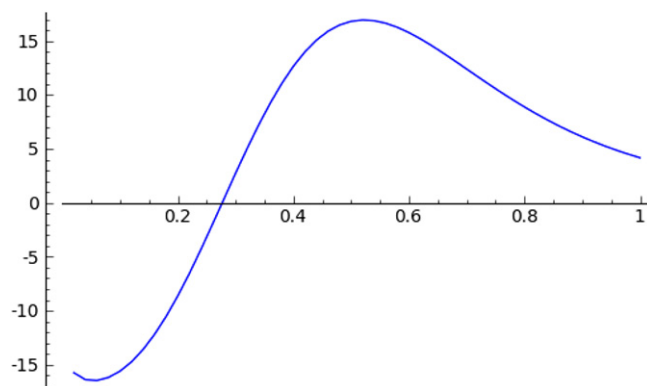


Fig. 3. Graph of  $s \mapsto -\partial^3/\partial s^3 \log \Psi_{-6,0.5}(s)$ .

#### 4. Multiple Wiener integrals

We now consider some examples of non-infinite divisibility of second order polynomials in the case  $a < 0$ , with application to the fourth order multiple Wiener integral of  $f^{\otimes 4}$ , with  $\|f\|_{L^2(\mathbb{R}_+)} = 1$ , which can be written as

$$\begin{aligned} I_4(f^{\otimes 4}) &= 4! \int_0^\infty f(t_4) \int_0^{t_4} f(t_3) \int_0^{t_3} f(t_2) \int_0^{t_2} f(t_1) dB_{t_1} dB_{t_2} dB_{t_3} dB_{t_4} \\ &= H_4(I_1(f)) \\ &= (I_1(f))^4 - 6(I_1(f))^2 + 3, \end{aligned}$$

where  $(B_t)_{t \in \mathbb{R}_+}$  is a standard Brownian motion and  $H_4(x) = x^4 - 6x^2 + 3$  is the Hermite polynomial of degree 4 and  $I_1(f) = \int_0^\infty f(t) dB_t$  is the first order Wiener integral, cf. e.g. Proposition 5.1.3 of [6], and  $X_{1/2} = I_1(f)^2$  is a gamma random variable with shape parameter  $\beta = 1/2$ .

**Proposition 4.1.** *The fourth order multiple Wiener integral  $I_4(f^{\otimes 4})$ , with  $\|f\|_{L^2(\mathbb{R}_+)} = 1$ , is not infinitely divisible.*

**Proof.** It follows from Lemma 4.2 and Fig. 3<sup>1</sup> that the function  $-\frac{\partial \log \Psi_{-6,1/2}}{\partial s}(s)$  is not to be completely monotone since its third derivative does not have a constant sign.

Consequently the fourth order multiple Wiener integral

$$I_4(f^{\otimes 4}) \simeq X_{1/2}^2 - 6X_{1/2} + 3$$

of  $f^{\otimes 4}$ , with  $\|f\|_{L^2(\mathbb{R}_+)} = 1$ , is not infinitely divisible by Theorem 1.1.  $\square$

**Lemma 4.2.** *Letting*

$$\Psi_{a,\beta}(s) := E[e^{-s(X_\beta + a/2)^2}], \quad \text{and} \quad \Gamma_{a,\beta}(s) = \int_0^\infty x^{\beta-1} e^{-s(x^2 + ax) - x} dx,$$

<sup>1</sup> Figs. 3 and 4 have been checked independently using Mathematica and Sage.

$s \in \mathbb{R}_+$ , we have

$$-\frac{\partial}{\partial s} \log \Psi_{a,\beta}(s) = \frac{a^2}{4} + \frac{\int_0^\infty x^\beta (x+a) e^{-s(x^2+ax)-x} dx}{\int_0^\infty x^{\beta-1} e^{-s(x^2+ax)-x} dx} = \frac{a^2}{4} + \frac{\Gamma_{a,\beta+2}(s) + a\Gamma_{a,\beta+1}(s)}{\Gamma_{a,\beta}(s)},$$

$$-\frac{\partial^2}{\partial s^2} \log \Psi_{a,\beta}(s) = -\frac{\Gamma_{a,\beta+4}(s) + 2a\Gamma_{a,\beta+3}(s) + a^2\Gamma_{a,\beta+2}(s)}{\Gamma_{a,\beta}(s)} + \frac{(\Gamma_{a,\beta+2}(s) + a\Gamma_{a,\beta+1}(s))^2}{(\Gamma_{a,\beta}(s))^2},$$

and

$$-\frac{\partial^3}{\partial s^3} \log \Psi_{a,\beta}(s) = \frac{\Gamma_{a,\beta+6}(s) + 3a\Gamma_{a,\beta+5}(s) + 3a^2\Gamma_{a,\beta+4}(s) + a^3\Gamma_{a,\beta+3}(s)}{\Gamma_{a,\beta}(s)}$$

$$- 3 \frac{(\Gamma_{a,\beta+4}(s) + 2a\Gamma_{a,\beta+3}(s) + a^2\Gamma_{a,\beta+2}(s))(\Gamma_{a,\beta+2}(s) + a\Gamma_{a,\beta+1}(s))}{(\Gamma_{a,\beta}(s))^2}$$

$$+ 2 \frac{(\Gamma_{a,\beta+2}(s) + a\Gamma_{a,\beta+1}(s))^3}{(\Gamma_{a,\beta}(s))^3}, \quad s \in \mathbb{R}_+.$$

**Proof.** It suffices to note that we have

$$\Psi_{a,\beta}(s) = e^{-sa^2/4} \frac{\Gamma_{a,\beta}(s)}{\Gamma(\beta)},$$

and the relation

$$-\frac{\partial}{\partial s} \Gamma_{a,\beta}(s) = \Gamma_{a,\beta+2}(s) + a\Gamma_{a,\beta+1}(s), \quad s \in \mathbb{R}_+. \quad \square$$

When  $\beta = 1$  we have

$$\Psi_{a,1}(s) = \sqrt{\frac{\pi}{s}} e^{a/2 + 1/(4s)} \Phi \left( -a\sqrt{\frac{s}{2}} - \frac{1}{\sqrt{2s}} \right),$$

and

$$-\frac{\partial}{\partial s} \log \Psi_{a,1}(s) = \frac{1}{2s} + \frac{1}{4s^2} + \frac{(a - 1/s)e^{-s(\frac{1}{2s} + a/2)^2}}{4\sqrt{\pi s} \Phi \left( -(a + 1/s)\sqrt{s/2} \right)}, \quad s > 0.$$

## 5. Complete monotonicity

In this section we prove the complete monotonicity results used in [Proposition 2.1](#).

**Lemma 5.1.** Let  $a, \eta \in \mathbb{R}_+$ .

(i) Let  $\alpha \in [0, 1]$ . The function

$$s \mapsto \frac{1}{\xi + \eta s + (1 + as)^\alpha}, \quad s \in \mathbb{R}_+,$$

is completely monotone for all  $\xi \in \mathbb{R}_+$ .

(ii) Let  $\alpha \in [1, 2]$ . The function

$$s \mapsto \frac{1 + as}{(1 + as)^\alpha + \eta s}, \quad s \in \mathbb{R}_+,$$

is completely monotone.

**Proof.** If  $\alpha \in [0, 1]$ , consider the nonnegative function

$$h(s) = (1 + as)^\alpha + \eta s + \xi, \quad s \in \mathbb{R}_+,$$

whose derivative

$$h'(s) = \alpha a(1 + as)^{\alpha-1} + \eta$$

is completely monotone on  $\mathbb{R}_+$ . By Criterion XIII.4.2 of [4], the function

$$s \mapsto \frac{1}{h(s)} = \frac{1}{\xi + \eta s + (1 + as)^\alpha}$$

is completely monotone on  $\mathbb{R}_+$ . Next if  $\alpha \in [1, 2]$ , consider the nonnegative function

$$h(s) = (1 + as)^{\alpha-1} + \eta \frac{s}{1 + as},$$

whose derivative

$$h'(s) = (\alpha - 1)a(1 + as)^{\alpha-2} + \frac{\eta}{(1 + as)^2} \geq 0$$

is completely monotone on  $\mathbb{R}_+$ . Again by Criterion XIII.4.2 of [4] the function

$$s \mapsto \frac{1}{h(s)} = \frac{1}{(1 + as)^{\alpha-1} + \eta s/(1 + as)} = \frac{1 + as}{(1 + as)^\alpha + \eta s}$$

is completely monotone on  $\mathbb{R}_+$ .  $\square$

**Remark.** Alternatively we may note that the function

$$s \mapsto (1 + as)^\alpha + \eta s$$

is a complete Bernstein function for all  $\alpha \in [0, 1]$  from Section 15.2.2 of [8]; hence

$$s \mapsto \frac{1}{\xi + \eta s + (1 + as)^\alpha}$$

is a Stieltjes function for all  $\xi \geq 0$  by Theorem 7.5 of [8]; hence it is completely monotone. When  $\alpha \in [1, 2]$ , the function

$$s \mapsto (1 + as)^{\alpha-1} + \eta \frac{s}{1 + as}$$

is also a complete Bernstein function from Section 15.2.4 of [8]; hence

$$s \mapsto \frac{1}{(1 + as)^{\alpha-1} + \eta s(1 + as)^{-1}} = \frac{1 + as}{(1 + as)^\alpha + \eta s}$$

is also a Stieltjes function which is completely monotone.

**Corollary 5.2.** For all  $a, b, c, \eta \in \mathbb{R}_+$  and all  $\alpha \in [0, 2]$ , the function

$$s \mapsto \frac{c + b(1 + as)^{\alpha-1}}{\eta s + (1 + as)^\alpha} \tag{5.1}$$

is completely monotone.

**Proof.** If  $\alpha \in [0, 1]$ , we multiply

$$s \mapsto \frac{1}{\xi + \eta s + (1 + as)^\alpha},$$

of Lemma 5.1-(i) by the completely monotone function

$$s \mapsto a\eta + \alpha(1 + as)^{\alpha-1}.$$

If  $\alpha \in [1, 2]$  we multiply

$$s \mapsto \frac{1 + as}{(1 + as)^\alpha + \eta s}$$

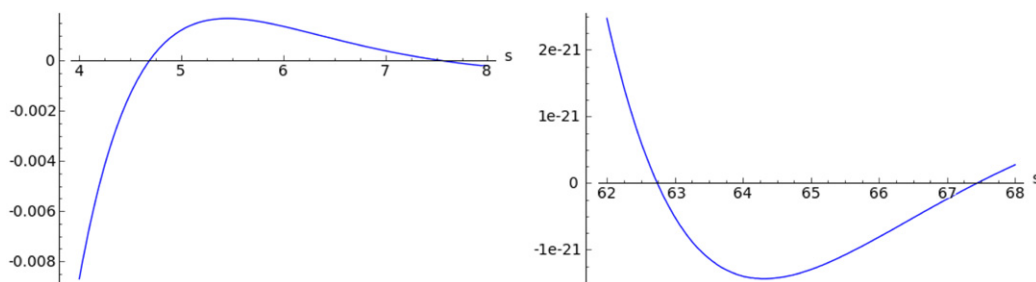
of Lemma 5.1-(ii) by the completely monotone function

$$s \mapsto a\alpha(1 + as)^{\alpha-2},$$

to get that

$$s \mapsto \frac{a\alpha(1 + as)^{\alpha-1}}{(1 + as)^\alpha + \eta s}, \quad s \in \mathbb{R}_+,$$

is completely monotone on  $\mathbb{R}_+$ .



**Fig. 4.** Graphs of  $s \mapsto \partial^2/\partial s^2 \log(100s + (1+s)^4)$  and  $s \mapsto \partial^{17}/\partial s^{17} \log(100s + (1+s)^3)$ .

On the other hand by Lemma 5.1, the function

$$s \mapsto \frac{1+as}{(1+as)^\alpha + \eta s} \times \frac{1}{1+as} = \frac{1}{(1+as)^\alpha + \eta s} \quad s \in \mathbb{R}_+,$$

is also completely monotone on  $\mathbb{R}_+$  for  $\alpha \in [1, 2]$ .  $\square$

**Remark 5.3.** There exist values of  $\alpha > 2$  for which the function (5.1) is not completely monotone.

Remark 5.3 is illustrated for  $a = 1$  and  $\eta = 100$  in Fig. 4, in which differentiation up to the 18th order is required for  $\alpha = 3$ , and the values taken by the derivative are of order  $10^{-20}$ .

## Acknowledgment

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## Appendix

In this appendix we prove Lemma 3.2 by extending to all values of  $\beta > 0$  the results stated for  $\beta = 1$  in Section 2.6a of [5] on the computation of Thorin measures for exponential random variables. The function  $\Lambda_t$ ,  $t > 0$ , is defined by

$$\Lambda_t(y) := 1 - \frac{1}{\pi t} \arctan\left(\frac{\sin(\pi t)}{y + \cos(\pi t)}\right), \quad y > 0.$$

We let  $f_{1/\alpha}$  denote the probability density function of a stable random variable with parameter  $1/\alpha$ .

**Proposition A.1.** Let  $X_\beta$  be a gamma random variable with shape parameter  $\beta > 0$ . Let  $\alpha > 1$ . Then the distribution function  $F_{0,\alpha}$  of the Thorin measure  $\mu_{0,\alpha}$  of  $X_\beta^\alpha$  satisfies

$$F_{0,\alpha}(y) = \frac{\beta}{\alpha} - \frac{\beta}{\alpha} \Lambda_{1/\alpha}\left(\frac{\Gamma(1+\beta/\alpha) \sin(\pi/\alpha)}{\Gamma(1+\beta) \pi y} \int_y^\infty \frac{1}{(x-y)^{1/\alpha}} \frac{f_{1/\alpha}(x)}{f_{1/\alpha}(y)} dx\right), \quad y \in \mathbb{R}_+. \quad (\text{A.1})$$

In particular,  $F_{0,\alpha}(y)$  is absolutely continuous and  $\mu_{0,\alpha}$  admits a density with respect to the Lebesgue measure.

**Proof.** In this proof we use the notation of [5]. The random variable  $X_\beta^\alpha$  is a  $(\beta/\alpha, G)$ -GGC, where  $G$  is a random variable with distribution function  $F_G(x) = \alpha F_{0,\alpha}(x)/\beta$  and such that  $E[\log^+(1/G)] < \infty$ . By Theorem 4.1.1 p. 49 of [1],  $X_\beta^\alpha$  can be written as the gamma mixture

$$X_\beta^\alpha \simeq Y_{\beta/\alpha} D_{1/\alpha}(G),$$

where  $Y_{\beta/\alpha}$  is a gamma random variable with parameter  $\beta/\alpha$ , and  $D_{1/\alpha}(G)$  is a positive independent random variable which is an example of a Dirichlet mean; cf. [3] and Relation (38) in [5].

By the duality Theorem 2.1.3-(ii) in [5], the density

$$f_{X_\beta^\alpha}(x) = \frac{1}{|\alpha| \Gamma(\beta)} x^{\beta/\alpha-1} e^{-x^{1/\alpha}}, \quad x > 0,$$

rewrites as

$$f_{X_\beta^\alpha}(x) = \frac{1}{\Gamma(\beta/\alpha)} x^{\beta/\alpha-1} e^{\beta E[\log(G)]/\alpha} E[e^{-x D_{1/\alpha}(1/G)}], \quad x > 0,$$

which, by identification, yields

$$e^{\beta E[\log(G)]/\alpha} = \frac{\Gamma(\beta/\alpha)}{|\alpha|\Gamma(\beta)} = \frac{\Gamma(1 + \beta/\alpha)}{\Gamma(1 + \beta)}, \quad (\text{A.2})$$

which extends Relation (158) of [5] to  $\beta > 1$ , and

$$E[e^{-x D_{1/\alpha}(1/G)}] = e^{-x^{1/\alpha}}, \quad x \in \mathbb{R}_+,$$

i.e.  $D_{1/\alpha}(1/G) = S_{1/\alpha}$  is a stable random variable with exponent  $1/\alpha$ . For completeness we need to check that the argument of pp. 385–386 of [5], which is stated therein for  $\beta = 1$ , also extends to all  $\beta > 0$ . By Relation (42) in Section 1.4.d-(iii) in [5] we have

$$D_1\left(\frac{1}{GZ_{1/\alpha}}\right) \simeq \beta_{1,1-1/\alpha} D_{1/\alpha}(1/G),$$

where  $Z_{1/\alpha}$  is a Bernoulli random variable with parameter  $1/\alpha$  and  $\beta_{1,1-1/\alpha}$  is a beta random variable, which shows by convolution that the density of  $D_1\left(\frac{1}{GZ_{1/\alpha}}\right)$  is

$$f_{D_1\left(\frac{1}{GZ_{1/\alpha}}\right)}(y) = \frac{\sin(\pi/\alpha)}{\pi y} \int_y^\infty \frac{1}{(x/y - 1)^{1/\alpha}} f_{1/\alpha}(x) dx.$$

Now, by Relation (4.19.2) on p. 112 of [2] we have

$$X_\beta^\alpha \simeq \frac{X_\beta}{S_{1/\alpha}}$$

where  $S_{1/\alpha}$  is a stable random variable with parameter  $1/\alpha$ . Hence, since

$$\frac{X_\beta}{S_{1/\alpha}} \simeq X_\beta^\alpha \simeq Y_{\beta/\alpha} D_{1/\alpha}(G) \simeq Y_\beta \beta_{1,1-1/\alpha} D_{1/\alpha}(G) \simeq Y_\beta D_1\left(\frac{G}{Z_{1/\alpha}}\right),$$

we get

$$D_1\left(\frac{G}{Z_{1/\alpha}}\right) \simeq \frac{1}{S_{1/\alpha}},$$

and by Relation (164) p. 386 of [5] we get

$$f_{D_1\left(\frac{G}{Z_{1/\alpha}}\right)}(x) = \frac{1}{x^2} f_{1/\alpha}(1/x).$$

Next, Theorem 2.3-(4) of [5] shows that the distribution function  $F_G$  of  $G$  satisfies

$$\begin{aligned} 1 - \frac{\alpha}{\beta} F_{0,\alpha}(y) &= F_{1/G}(1/y) \\ &= \Lambda_{1/\alpha} \left( \frac{f_{D_1\left(\frac{1}{GZ_{1/\alpha}}\right)}(y) e^{\beta E[\log(G)]/\alpha}}{y^{1/\alpha-2} f_{D_1\left(\frac{G}{Z_{1/\alpha}}\right)}(1/y)} \right) \\ &= \Lambda_{1/\alpha} \left( \frac{\Gamma(1 + \beta/\alpha)}{\Gamma(1 + \beta)} \frac{\sin(\pi/\alpha)}{\pi y} \int_y^\infty \frac{1}{(x-y)^{1/\alpha}} \frac{f_{1/\alpha}(x)}{f_{1/\alpha}(y)} dx \right), \end{aligned}$$

since  $e^{\beta E[\log G]/\alpha} = \Gamma(1 + \beta/\alpha)/\Gamma(1 + \beta)$  by (A.2).  $\square$

Finally we prove Lemma 3.2 as a corollary of Proposition A.1.

**Proof of Lemma 3.2.** When  $\alpha = 2$  we have

$$\Lambda_{1/2}(y) = 1 - \frac{2}{\pi} \arctan \frac{1}{y} = \frac{2}{\pi} \arctan y, \quad y > 0;$$

hence Relation (A.1) of Proposition A.1 and Legendre's duplication formula (cf. [4, p. 64]):

$$\frac{\Gamma(1 + \beta/2)}{\sqrt{\pi} \Gamma(1 + \beta)} = \frac{1}{2^\beta \Gamma(1/2 + \beta/2)}$$

give

$$\begin{aligned}
 F_{0,2}(y) &= \frac{2\beta}{\alpha\pi} \arctan \left( \frac{2^{-\beta}}{y\sqrt{\pi}\Gamma(1/2 + \beta/2)} \int_y^\infty \frac{1}{(x-y)^{1/2}} \frac{f_{1/2}(x)}{f_{1/2}(y)} dx \right)^{-1} \\
 &= \frac{\beta}{\pi} \arctan \left( \frac{2^{-\beta}\sqrt{y}e^{1/(4y)}}{\sqrt{\pi}\Gamma(1/2 + \beta/2)} \int_y^\infty \frac{1}{\sqrt{x-y}} e^{-1/(4x)} \frac{dx}{x^{3/2}} \right)^{-1} \\
 &= \frac{\beta}{\pi} \arctan \left( \frac{2^{2-\beta}}{\sqrt{\pi}\Gamma(1/2 + \beta/2)} \int_0^{1/(2\sqrt{y})} e^{z^2} dz \right)^{-1} \\
 &= \frac{\beta}{\pi} \arctan \left( \frac{\Gamma(\beta/2 + 1/2)}{2^{1-\beta}F(1/(2\sqrt{y}))} \right), \tag{A.3}
 \end{aligned}$$

where we applied the change of variable  $z^2 = 1/(4y) - 1/(4x)$ ,  $x, y > 0$ .  $\square$

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