



Attractors of asymptotically autonomous quasi-linear parabolic equation with spatially variable exponents



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ARTICLE INFO

Article history:

Received 11 July 2014

Available online 16 January 2015

Submitted by Y. Huang

Keywords:

Non-autonomous quasi-linear

parabolic problems

Variable exponents

Pullback attractors

Asymptotically autonomous systems

ABSTRACT

Some abstract results on the convergence of nonautonomous pullback attractors in asymptotically autonomous problems are established and then applied to quasi-linear parabolic equations with spatially variable exponents in which the parabolic operator is time-dependent. In particular, it is shown that the component subsets of the pullback attractor converge in the Hausdorff semi-distance to the global autonomous attractor.

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1. Introduction

Quasi-linear parabolic equations with variable exponents appear in physical problems like electrorheological fluids [5,10,11], image processing [1,4,7] and porous medium equations [2,12]. A representative example is the system

$$\frac{\partial u}{\partial t}(t) - \operatorname{div}(D(t)|\nabla u(t)|^{p(x)-2}\nabla u(t)) + |u(t)|^{p(x)-2}u(t) = B(u(t)), \quad u(\tau) = \psi_\tau, \quad (1)$$

on a bounded smooth domain Ω in \mathbb{R}^n for some $n \geq 1$ with a homogeneous Neumann boundary condition, where the exponent $p(\cdot) \in C(\bar{\Omega}, \mathbb{R})$ satisfies $p^+ := \max_{x \in \bar{\Omega}} p(x) \geq p^- := \min_{x \in \bar{\Omega}} p(x) > 2$ and the initial condition $u(\tau) \in H := L^2(\Omega)$.

Such problems have attracted considerable attention in the literature in recent years. There is also an established literature on attractors of autonomous and non-autonomous semi-linear parabolic equations,

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e.g., [6,13,14], but there has been very little on non-autonomous pullback attractors for evolution problems involving variable exponents, see [9].

Motivated by problem (1), we study the asymptotic behaviour of an abstract non-autonomous evolution equation in a Hilbert space H of the form

$$\frac{\partial u}{\partial t}(t) + A(t)u(t) = B(u(t)), \quad u(\tau) = \psi_\tau, \quad (2)$$

compared with that of an autonomous evolution equation

$$\frac{\partial v}{\partial t}(t) + A_\infty v(t) = B(v(t)), \quad v(0) = \psi_0, \quad (3)$$

where the operators A , A_∞ and B satisfy the following assumptions.

Assumption A. For each $\tau \in \mathbb{R}$ there exists a function $g_\tau : [0, +\infty) \rightarrow [0, +\infty)$ such that $g_\tau(t) \rightarrow 0$ as $\tau \rightarrow +\infty$ uniformly in t and

$$\langle A(t + \tau)u(t + \tau) - A_\infty v(t), u(t + \tau) - v(t) \rangle \geq -g_\tau(t), \quad \text{for all } t \in \mathbb{R}^+, \tau \in \mathbb{R},$$

for any solution u of (2) and v of (3).

Assumption B. The mapping $B : H \rightarrow H$ is globally Lipschitz, i.e., there exists $L \geq 0$ such that

$$\|B(x_1) - B(x_2)\|_H \leq L\|x_1 - x_2\|_H \quad \text{for all } x_1, x_2 \in H.$$

The autonomous problem (3) is thus the asymptotic autonomous version of the non-autonomous problem (2). It differs from the asymptotic autonomous problems in Section 8.6.2 of [3], where non-autonomous semi-linear problems with the explicit dependence on time occurring only in the external forcing term are investigated. In contrast, in the quasi-linear non-autonomous problems under consideration in this paper the main operator depends explicitly on time.

In particular, we establish the convergence in the Hausdorff semi-distance of the component subsets of the pullback attractor of the non-autonomous problem (2) to the global autonomous attractor of the autonomous problem (3). Definitions and basic results on non-autonomous pullback attractors are given in Section 2. Two general results ensuring the above convergence of attractors are then established in Section 3 for the abstract systems (2) and (3). These are applied to the quasi-linear parabolic system with variable exponents (1) in Section 4 with an additional Assumption D on the diffusion coefficient $D(t)$ that ensures that Assumption A holds.

2. Nonautonomous pullback attractors

Non-autonomous dynamical systems can be formulated abstractly as evolution processes or two-parameter semi-groups. Some definitions and results are recalled here, see [3,8] for more details.

Definition 2.1. An evolution process in a metric space (X, d_X) is a family $\{U(t, \tau) : X \rightarrow X, t \geq \tau \in \mathbb{R}\}$ satisfying:

- i) $U(\tau, \tau) = \mathbf{1}$ (here $\mathbf{1}$ denotes the identity operator);
- ii) $U(t, \tau) = U(t, s)U(s, \tau)$, $\tau \leq s \leq t$.

For greater generality, invariant sets of non-autonomous dynamical systems are defined in terms of families of sets rather than individual sets.

Definition 2.2. Let $\{U(t, \tau) : t \geq \tau \in \mathbb{R}\}$ be an evolution process in a metric space (X, d_X) . Given A and B subsets of X , we say that A pullback attracts B at time t if

$$\lim_{\tau \rightarrow -\infty} \text{dist}_X(U(t, \tau)B, A) = 0,$$

where dist_X denote the Hausdorff semi-distance on X .

Definition 2.3. A family of subsets $\mathfrak{A} = \{A(t) : t \in \mathbb{R}\}$ of (X, d_X) is called a pullback attractor for the evolution process $\{U(t, \tau) : t \geq \tau \in \mathbb{R}\}$ if it is invariant and the component set $A(t)$ is compact and pullback attracts all bounded subsets of X for each $t \in \mathbb{R}$.

The existence of a pullback attractor follows from that of a pullback absorbing family and an appropriate compactness of asymptotic compactness properties of the process. The following result was established in [9] for the quasi-linear parabolic equation with spatially variable exponents (2). It also provides the existence of a global autonomous attractor in the special case that none of the coefficient functions and the external forcing term in (2) depends on time.

Theorem 2.4. *The evolution process associated with problem (2) has a pullback attractor $\mathfrak{A} = \{A(t) : t \in \mathbb{R}\}$ in the Hilbert space $H := L^2(\Omega)$. Moreover, $\overline{\bigcup_{t \in \mathbb{R}} A(t)}$ is a compact subset of H .*

3. Theoretical results

Let $\mathfrak{A} = \{A(t) : t \in \mathbb{R}\}$ a pullback attractor of the non-autonomous process $\{U(t, \tau) : t \geq \tau\}$ on a complete metric state space (X, d_X) and let \mathcal{A}_∞ be the global autonomous attractor of a semigroup $\{T(t) : t \geq 0\}$ on X . We give two abstract results that can be used to establish the convergence in the Hausdorff semi-distance of the component subsets $A(t)$ of the pullback attractor \mathfrak{A} at \mathcal{A}_∞ as $t \rightarrow \infty$.

Theorem 3.1. *Suppose that for each $\epsilon > 0$ there exist $\tau_0 = \tau_0(\epsilon)$ and a bounded set $\mathcal{B}(\tau_0)$ in the complete metric space (X, d_X) such that*

$$\sup_{\psi \in \mathcal{A}(\tau_0)} \text{dist}_X(U(t, \tau_0)\psi, T(t - \tau_0)\psi) < \epsilon \quad \text{for all } t \geq \tau_0, \quad (4)$$

$$\bigcup_{t \geq \tau_0} A(t) \subset \mathcal{B}(\tau_0). \quad (5)$$

Then $\lim_{t \rightarrow +\infty} \text{dist}_X(A(t), \mathcal{A}_\infty) = 0$.

Proof. Let $\epsilon > 0$ be given and let $\tau_1 = \tau_0(\epsilon/3)$. Since the global attractor \mathcal{A}_∞ of the semigroup T attracts bounded sets of X , there exists a positive $t_1 = t_1(\epsilon/3, \mathcal{B}(\tau_1)) > \tau_1$ such that

$$\text{dist}_X(T(t - \tau_1)\mathcal{B}(\tau_1), \mathcal{A}_\infty) < \frac{\epsilon}{3}$$

for $t \geq t_1$.

Then, by (4) with $\epsilon/3$ and τ_1 instead of ϵ and τ_0 ,

$$\begin{aligned} \text{dist}_X(\mathcal{A}(t), \mathcal{A}_\infty) &= \text{dist}_X(U(t, \tau_1)\mathcal{A}(\tau_1), \mathcal{A}_\infty) \\ &= \sup_{\psi \in \mathcal{A}(\tau_1)} \text{dist}_X(U(t, \tau_1)\psi, \mathcal{A}_\infty) \\ &\leq \sup_{\psi \in \mathcal{A}(\tau_1)} \text{dist}_X(U(t, \tau_1)\psi, T(t - \tau_1)\psi) \\ &\quad + \sup_{\psi \in \mathcal{A}(\tau_1)} \text{dist}_X(T(t - \tau_1)\psi, \mathcal{A}_\infty) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon, \end{aligned}$$

for all $t \geq t_1$. \square

The second result is a reformulation of Theorem 3.34 in [8] for skew product flows to non-autonomous processes in the asymptotically autonomous context under consideration.

Theorem 3.2. *Suppose that $\mathcal{C} := \overline{\bigcup_{\tau \in \mathbb{R}} \mathcal{A}(\tau)}$ is a compact subset of X . In addition, suppose that $U(t + \tau, \tau)\psi_\tau \rightarrow T(t)\psi_0$ in X as $\tau \rightarrow +\infty$ uniformly in $t \geq 0$ whenever $\psi_\tau \in \mathcal{A}(\tau)$ and $\psi_\tau \rightarrow \psi_0$ in X as $\tau \rightarrow +\infty$. Then*

$$\lim_{t \rightarrow +\infty} \text{dist}_X(\mathcal{A}(t), \mathcal{A}_\infty) = 0.$$

Proof. Suppose that this is not true. Then there would exist an $\epsilon_0 > 0$ and a real sequence $\{\tau_n\}_{n \in \mathbb{N}}$ with $\tau_n > n$ for $n \in \mathbb{N}$ such that $\text{dist}_X(\mathcal{A}(\tau_n), \mathcal{A}_\infty) \geq 3\epsilon_0$ for all $n \in \mathbb{N}$. Since the sets $\mathcal{A}(\tau_n)$ are compact, there exist $a_n \in \mathcal{A}(\tau_n)$ such that

$$\text{dist}_X(a_n, \mathcal{A}_\infty) = \text{dist}_X(\mathcal{A}(\tau_n), \mathcal{A}_\infty) \geq 3\epsilon_0, \quad (6)$$

for each $n \in \mathbb{N}$. By attraction for the semigroup, $\text{dist}_X(T(\ell)\mathcal{C}, \mathcal{A}_\infty) \leq \epsilon_0$ for $\ell > 0$ large enough. Moreover, by the invariance of the pullback attractor there exist $b_n \in \mathcal{A}(\frac{1}{2}\tau_n) \subset \mathcal{C}$ for $n \in \mathbb{N}$ such that $a_n = U(\tau_n, \frac{1}{2}\tau_n)b_n$ for each $n \in \mathbb{N}$. Since \mathcal{C} is compact, there is a convergent subsequence $b_{n'} \rightarrow b \in \mathcal{C}$. In addition, from the hypotheses, it follows that

$$\text{dist}_X\left(U\left(\tau_{n'}, \frac{1}{2}\tau_{n'}\right)b_{n'}, T\left(\frac{1}{2}\tau_{n'}\right)b\right) < \epsilon_0$$

for n' large enough (since $\frac{1}{2}\tau_{n'} > \frac{1}{2}n'$ is large enough). Hence,

$$\begin{aligned} \text{dist}_X(a_{n'}, \mathcal{A}_\infty) &= \text{dist}_X\left(U\left(\tau_{n'}, \frac{1}{2}\tau_{n'}\right)b_{n'}, \mathcal{A}_\infty\right) \\ &\leq \text{dist}_X\left(U\left(\tau_{n'}, \frac{1}{2}\tau_{n'}\right)b_{n'}, T\left(\frac{1}{2}\tau_{n'}\right)b\right) \\ &\quad + \text{dist}_X\left(T\left(\frac{1}{2}\tau_{n'}\right)b, \mathcal{A}_\infty\right) \leq 2\epsilon_0, \end{aligned}$$

which contradicts (6). \square

The next result is very useful for checking that the hypothesis of asymptotic continuity of the non-autonomous flow in the preceding theorem for problems like (2). For this we suppose that the process

$\{U(t, \tau) : t \geq \tau\}$ generated by problem (2) has a pullback attractor $\mathfrak{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ and that the semigroup $\{T(t) : t \geq 0\}$ generated by problem (3) has a global autonomous attractor \mathcal{A}_∞ in the Hilbert space H .

Lemma 3.3. *Suppose that Assumption A is satisfied. Then $U(t + \tau, \tau)\psi_\tau \rightarrow T(t)\psi_0$ in H as $\tau \rightarrow +\infty$ uniformly in $t \geq 0$ whenever $\psi_\tau \rightarrow \psi_0$ in H as $\tau \rightarrow +\infty$.*

Proof. Subtracting Eq. (3) from Eq. (2) gives

$$\frac{d}{dt}(u(t + \tau) - v(t)) + A(t + \tau)u(t + \tau) - A_\infty v(t) = B(u(t + \tau)) - B(v(t))$$

for a.e. $t \in [\tau, T]$. Multiplying by $u(t + \tau) - v(t)$ and taking the inner product, then using Assumption A, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u(t + \tau) - v(t)\|_H^2 \leq L \|u(t + \tau) - v(t)\|_H^2 + g_\tau(t).$$

Integrating this last inequality from 0 to t , gives

$$\|u(t + \tau) - v(t)\|_H^2 \leq \|\psi_\tau - \psi_0\|_H^2 + 2tg_\tau(t) + 2L \int_0^t \|u(s + \tau) - v(s)\|_H^2 ds.$$

Hence, by the Gronwall inequality, there is a positive constant K such that

$$\|u(t + \tau) - v(t)\|_H^2 \leq (\|\psi_\tau - \psi_0\|_H^2 + g_\tau(t))K.$$

Since $\psi_\tau \rightarrow \psi_0$ in H and $g_\tau(t) \rightarrow 0$ as $\tau \rightarrow +\infty$ uniformly in $t \geq 0$, the result follows. \square

4. Application to equations with spatially variable exponents

The above results are applied here to the quasi-linear parabolic equation with spatially variable exponents (1) in the Hilbert space $H := L^2(\Omega)$.

We assume that B satisfies Assumption B and that the coefficient D satisfies the following assumption:

Assumption D. $D : [\tau, T] \times \Omega \rightarrow \mathbb{R}$ is a function in $L^\infty([\tau, T] \times \Omega)$ such that:

- (D1) There are positive constants, β and M such that $0 < \beta \leq D(t, x) \leq M$ for almost all $(t, x) \in [\tau, T] \times \Omega$.
- (D2) $D(t, x) \geq D(s, x)$ for each $x \in \Omega$ and $t \leq s$ in $[0, T]$.
- (D3) $D(t + \tau, \cdot) \rightarrow D^*(\cdot)$ in $L^\infty(\Omega)$ as $\tau \rightarrow +\infty$ uniformly in $t \geq 0$.

Assumptions (D1)–(D2) imply that the pointwise limit $D^*(x)$ as $t \rightarrow \infty$ exists and satisfies $0 < \beta \leq D^*(x) \leq M$ for almost all $x \in \Omega$. Then the problem (1) with $D^*(x)$ is autonomous and has a global autonomous attractor as a particular case of the results in [9].

In this section the question raised in the final remarks section of [9] will be answered, i.e., it will be shown that the dynamics of the original non-autonomous problem is asymptotically autonomous and its pullback attractor converges upper-semi continuously to the autonomous global attractor \mathcal{A}_∞ of the problem

$$\frac{\partial v}{\partial t}(t) - \operatorname{div}(D^* |\nabla v(t)|^{p(x)-2} \nabla v(t)) + |v(t)|^{p(x)-2} v(t) = B(v(t)), \quad v(0) = \psi_0. \quad (7)$$

In particular, consider the operators

$$\begin{aligned} A(t)u &:= -\operatorname{div}(D(t)|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u, \\ A_\infty v &:= -\operatorname{div}(D^*|\nabla v|^{p(x)-2}\nabla v) + |v|^{p(x)-2}v, \end{aligned}$$

and apply [Theorem 3.2](#) to the quasi-linear parabolic problem with variable exponents [\(1\)](#).

Applying [Theorem 3.1](#) in [\[9\]](#) for the particular case $B(t, u) \equiv B(u)$, there exist positive constants T_1, K_1 such that $\|u(t)\|_H \leq K_1$ for all $t \geq T_1 + \tau$. From the invariance of the pullback attractors $\bigcup_{t \in \mathbb{R}} \mathcal{A}(t) \subset B_0 := B_H(0, K_1)$. Therefore, hypothesis [\(5\)](#) in [Theorem 3.1](#) is satisfied. Applying [Theorem 3.2](#) in [\[9\]](#) for the particular case $B(t, u) \equiv B(u)$ and space $X = W_0^{1,p(x)}(\Omega)$, there exist positive constants T_2, K_2 such that

$$\|u(t)\|_X \leq K_2, \quad \forall t \geq T_2 + \tau. \quad (8)$$

Since, also $\|v(t)\|_X \leq K_2$ for all $t \geq T_2 + \tau$, it follows that

Corollary 4.1. $\overline{\bigcup_{\tau \in \mathbb{R}} \mathcal{A}(\tau)}$ is a compact subset of H .

Hypothesis [\(4\)](#) of [Theorem 3.1](#) seems harder to check directly. Instead, the hypothesis in [Theorem 3.2](#) will be verified, which requires the following result.

Theorem 4.2. If $\{\psi_\tau : \tau \in \mathbb{R}\}$ is a bounded set in X and $\psi_\tau \rightarrow \psi_0$ in H as $\tau \rightarrow +\infty$, then [Assumption A](#) is satisfied.

Proof. If u is the solution of [\(1\)](#) and v is the solution of [\(7\)](#), then

$$\begin{aligned} &\langle A(t+\tau)u(t+\tau) - A_\infty v(t), u(t+\tau) - v(t) \rangle \\ &\geq \int_{\Omega} [D(t+\tau, x) - D^*(x)] [|\nabla v(t, x)|^{p(x)-2} \nabla v(t, x)] [\nabla u(t+\tau, x) - \nabla v(t, x)] dx. \end{aligned}$$

Note that

$$\begin{aligned} &-\int_{\Omega} [D(t+\tau, x) - D^*(x)] [|\nabla v(t, x)|^{p(x)-2} \nabla v(t, x)] [\nabla u(t+\tau, x) - \nabla v(t, x)] dx \\ &\leq \|D(t+\tau, \cdot) - D^*(\cdot)\|_{L^\infty(\Omega)} \int_{\Omega} |||\nabla v(t, x)|^{p(x)-2} \nabla v(t, x)||| |\nabla u(t+\tau, x) - \nabla v(t, x)| dx. \end{aligned}$$

The following assertion will be proved.

Assertion. There exists a positive constant K such that

$$J_\tau(t) := \int_{\Omega} |\nabla v(t, x)|^{p(x)-1} |\nabla u(t+\tau, x)| dx + \int_{\Omega} |\nabla v(t, x)|^{p(x)} dx \leq K.$$

Indeed, by Young's inequality and [Theorem 2.1](#) in [\[9\]](#),

$$J_\tau(t) \leq 2 \max\{\|\nabla v(t)\|_{p(x)}^{p^+}, \|\nabla v(t)\|_{p(x)}^{p^-}\} + \frac{1}{p^-} \max\{\|\nabla u(t+\tau)\|_{p(x)}^{p^+}, \|\nabla u(t+\tau)\|_{p(x)}^{p^-}\}.$$

Then, the fact that $\{\psi_\tau : \tau \in \mathbb{R}\}$ is a bounded set in X and the estimate (8) give $\|\nabla u(t)\|_{p(x)} \leq \|u(t)\|_X \leq K_2$ for all $t \geq 0$. Also, $\|\nabla v(t)\|_{p(x)} \leq \|v(t)\|_X \leq K_2$ for all $t \geq 0$. This proves the assertion.

Thus,

$$\begin{aligned} & - \int_{\Omega} [D(t+\tau, x) - D^*(x)] [|\nabla v(t, x)|^{p(x)-2} \nabla v(t, x)] [\nabla u(t+\tau, x) - \nabla v(t, x)] dx \\ & \leq K \|D(t+\tau, \cdot) - D^*(\cdot)\|_{L^\infty(\Omega)}. \end{aligned}$$

Defining $g_\tau(t) := K \|D(t+\tau, \cdot) - D^*(\cdot)\|_{L^\infty(\Omega)}$, it follows that

$$\langle A(t+\tau)u(t+\tau) - A_\infty v(t), u(t+\tau) - v(t) \rangle \geq -g_\tau(t).$$

Obviously, the function $g_\tau : [0, +\infty) \rightarrow [0, +\infty)$ satisfies $g_\tau(t) \rightarrow 0$ as $\tau \rightarrow +\infty$ uniformly in $t \geq 0$. \square

The next result gives the desired asymptotic upper semi-continuous convergence.

Theorem 4.3. $\lim_{t \rightarrow +\infty} \text{dist}(\mathcal{A}(t), \mathcal{A}_\infty) = 0$.

Proof. Suppose that $\psi_\tau \in \mathcal{A}(\tau)$ and $\psi_\tau \rightarrow \psi_0$ in H . Using the invariance of the pullback attractor and the estimate (8) it follows that $\{\psi_\tau : \tau \in \mathbb{R}\}$ is a bounded set in X . Theorem 4.2 then guarantees that Assumption A is satisfied. Thus, from Lemma 3.3, $U(t+\tau, \tau)\psi_\tau \rightarrow T(t)\psi_0$ in H as $\tau \rightarrow +\infty$, uniformly in $t \geq 0$. Theorem 3.2 then yields $\lim_{t \rightarrow +\infty} \text{dist}(\mathcal{A}(t), \mathcal{A}_\infty) = 0$. \square

Acknowledgments

The first author was partly supported by the Spanish Ministerio de Economía y Competitividad project MTM2011-22411, the Consejería de Innovación, Ciencia y Empresa (Junta de Andalucía) under the Ayuda 2009/FQM314, the Proyecto de Excelencia: P12-FQM-1492. The second author was partly supported by the Brazilian research agencies FAPEMIG grant ETC-00018-13 and Science without Borders – CAPES – PVE – Process 88881.0303888/2013-01.

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