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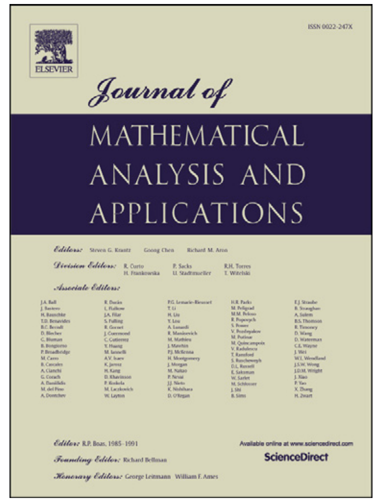

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On ground states for the Kirchhoff-type problem with a general critical nonlinearity☆

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Abstract

In this paper, we consider the following Kirchhoff-type problem

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2)\triangle u = f(u), & \text{in } \mathbb{R}^3, \\ u\in H^1(\mathbb{R}^3), \quad u>0, & \text{in } \mathbb{R}^3, \end{cases}$$

where $a, b > 0$ are constants, and f has a critical growth. The aim of this paper is to study the existence of ground state solutions for Kirchhoff-type equations with a general nonlinearity in the critical growth, without the assumption of the monotonicity of the function $t \to \frac{f(t)}{t^3}$. Moveover, we will show that the mountain pass value gives the least energy level and also obtain a mountain pass solution.

Keywords: Kirchhoff type problem; Critical nonlinearity; Ground state; Variational methods
2000 MSC: 35J60, 35J50, 53C35.

1. Introduction

Consider the following Kirchhoff-type problem:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2)\triangle u = f(u), & \text{in } \mathbb{R}^3, \\ u\in H^1(\mathbb{R}^3), \quad u>0, & \text{in } \mathbb{R}^3, \end{cases} \tag{K}$$

where $a, b > 0$ are constants and $f \in C(\mathbb{R},\mathbb{R})$ satisfies the following assumptions.

(f$_1$) $f \in C(\mathbb{R},\mathbb{R})$ is odd;

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(f$_2$) $\lim\limits_{s\to 0}\frac{f(s)}{s} = -m < 0$;

(f$_3$) $\lim\limits_{|s|\to +\infty}\frac{f(s)}{s^5} = k > 0$;

(f$_4$) There exist $D > 0$ and $q \in (2, 6)$ such that $f(s) + ms \geq ks^5 + Ds^{q-1}$ for all $s > 0$.

For the sake of simplicity, in this paper we always assume that $k = 1$. We recall that $u(x)$ is said to be the ground state (or the least energy solution) of (K) if and only if $I(u) = l$, where $l := \inf\{I(u); u \in H^1(\mathbb{R}^3) \setminus \{0\}$ is a solution of (K)$\}$. Here, $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ is the natural functional corresponding to (K), that is,

$$I(u) = \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2 - \int_{\mathbb{R}^3}F(u), \tag{1.1}$$

where $H^1(\mathbb{R}^3)$ is the usual Sobolev space and $F(u) = \int_0^u f(s)ds$.

Note that Kirchhoff type problem on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ takes the form of

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2)\triangle u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega \end{cases} \tag{1.2}$$

and has been studied by many mathematicians. Indeed, such a class of problems is viewed as being nonlocal because of the presence of the term $(\int_\Omega |\nabla u|^2 dx)\triangle u$, which implies that the equation in (1.2) is no longer a pointwise identity and is very different from classical elliptic equations. That is to say, such a phenomenon provokes some mathematical difficulties, which make the study of such a class of problems particularly interesting. On the other hand, problem (1.2) arises in many mathematical biological contexts. It is pointed out in [1] that the problem (1.2) models several biological systems, where u describes a process which depends on the average of itself (for example, population density). Moreover, problem (1.2) is related to the stationary analogue of the following equation

$$\begin{cases} u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\triangle u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

which is proposed by Kirchhoff in [22] as an existence of the classical D'Alembert's wave equations for free vibration of elastic strings, particularly, taking into account the subsequent change in string length caused by oscillations. We have to point out that Equation (1.3) received much attention only after Lions [25] introduced an abstract framework to this problem. Some interesting results can be found, for example, in [14, 6, 3]. More precisely, D'Ancona and Spagnolo [6] proved the existence of a global classical periodic solution for the degenerate Kirchhoff equation with real analytic data which is an example of a quasilinear hyperbolic Cauchy problem that describes the transverse oscillations of a stretched string. In [3], Arosio and Panizzi studied the Cauchy-Dirichlet type problem related to (1.3) in the Hadamard sense as a special case of an abstract second-order Cauchy problem in a Hilbert space. Recently, the solvability of the Kirchhoff type equation (1.2) has been well studied by many authors. In particular, Ma and Rivera [28] obtained positive solutions of such problems via variational methods. Alves, Corrêa and Ma [1] studied problem (1.2) and obtained positive solutions by using the Mountain Pass Theorem. Using the variant version of the Mountain Pass Theorem, Cheng and Wu [12] also proved two existence results of positive solutions for problem (1.2), when f satisfies some asymptotic behaviors near zero and infinity. In [31], Perera and Zhang obtained nontrivial solutions for (1.2) with the aid of the Yang index and critical groups. In [39] and [29], the authors used minimax methods and invariant sets of descent flow to prove the existence of three solutions (a positive solution, a negative solution and a sign-changing solution) for (1.2). Moreover, He and Zou [17] showed the existence of infinitely many solutions by using the local minimum methods and the Fountain Theorems. We refer to [11, 34, 13, 38, 12] for more existence results of (1.2). We also note that there are several existence results for the following Kirchhoff type problem on $\mathbb{R}^N$:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2)\triangle u+V(x)u=f(u), & \text{in } \mathbb{R}^N,\\ u\in H^1(\mathbb{R}^3), & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $N=1,2$ or 3. In fact, under the condition that $f(u)$ is superlinear at infinity, Wu [36] obtained a sequence of high energy solutions for problem (1.4) via a symmetric Mountain Pass Theorem. In [37], Wu also studied the existence of a sequence of high energy solutions for the above system with the help of some new critical point theorems. When the nonlinear term $f(u)$ is asymptotically linear at infinity, Liu and Guo [26] obtained the existence of at least a positive solution for (1.4). If $V(x)$ is radially symmetric, Nie and Wu [30] studied the existence of infinitely many high energy solutions for problem (1.4) by using Mountain Pass Theorem and symmetric Mountain Pass Theorem. Jin and Wu [21] employed the Fountain Theorem and obtained three existence results of infinitely many radial solutions for problem (1.4) with $V(x)\equiv 1$. In addition, Li et al. [24] made use of the cut-off function technique to investigate the existence of positive solution for the following system without assuming compactness:

$$\begin{cases} (a+\lambda\int_{\mathbb{R}^N}|\nabla u|^2+\lambda b\int_{\mathbb{R}^N}u^2)[-\triangle u+bu]=f(u), & \text{in } \mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N), \quad u>0, & \text{in } \mathbb{R}^N, \end{cases}$$

where $N\geq 3$. Alves and Figueiredo [2] studied the following class of Kirchhoff problem

$$M\left(\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)\right)[-\triangle u+V(x)u]=\lambda f(u)+\gamma u^{\tau} \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $\tau=5$ for $N=3$ and $\tau\in(1,+\infty)$ for $N=1,2$; $\lambda>0$ and $\gamma\in\{0,1\}$. Alves and Figueiredo [2] showed that under certain conditions on functions M, V and f, there exists a constant $\lambda^*>0$ such that (1.5) with $\gamma=1$ has at least a positive solution for every $\lambda > \lambda^*$, and (1.5) with $\gamma = 0$ has a positive solution, for every $\lambda > 0$.

We remark that problem (K) with $a = 1, b = 0$ and $\mathbb{R}^3$ replaced by $\mathbb{R}^N$, reduces to the well-known elliptic equation

$$\begin{cases} -\triangle u = f(u), & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{1.6}$$

For (1.6), there is a large quantity of results on the existence and multiplicity of solutions in the literature. We refer to [7, 8, 20, 4, 5, 41] and the more references therein. In particular, for the subcritical growth case, we recall that their celebrated paper [7], Berestycki and Lions studied the nonlinear elliptic equation (1.6) and obtained the following existence result.

Theorem 1.1. *Suppose $N \geq 3$ and that f satisfies the following conditions:*

(H_1) $f \in C(\mathbb{R}, \mathbb{R})$ *is odd;*

(H_2) $-\infty < \liminf_{s\to 0^+} \frac{f(s)}{s} \leq \limsup_{s\to 0^+} \frac{f(s)}{s} = -m < 0$*;*

(H_3) $-\infty \leq \limsup_{s\to 0^+} \frac{f(s)}{s^l} \leq 0$, *where* $l = \frac{N+2}{N-2}$*;*

(H_4) *there exists* $\zeta > 0$ *such that* $F(\zeta) := \int_0^\zeta f(s)ds > 0$.

Then (1.6) possesses a positive least energy solution u such that

(i) *u is spherically symmetric: $u(x) = u(r)$, where $r = |x|$, and u decreases with respect to r;*

(ii) $u \in C^2(\mathbb{R}^N)$.

(iii) *u together with its derivatives up to order 2 has exponential decay at infinity:*

$$|D^\alpha u(x)| \leq Ce^{-\delta|x|}, \quad x \in \mathbb{R}^N,$$

for some C, $\delta > 0$ and for $|\alpha| \leq 2$.

Under the assumptions that f satisfies (H_1)-(H_4), Berestycki and Lions [8] also investigated the existence of infinitely many bound state solutions of (1.6); Jeanjean and Tanaka [20] showed that the mountain pass value gives the least energy level for problem (1.6); Azzollini, d'Avenia and Pomponio [4] studied a class of Schrödinger-Poisson problems and obtained the existence of at least a radial positive solution; Azzollini [5] extended Theorem 1.1 to problem (K) and obtained the existence of ground state solutions by using minimizing arguments on a suitable natural constraint (the Pohozaev's manifold $\mathcal{P}$, see Section 4).

Zhang and Zou [41] investigated the existence of the least energy solutions of the problem (1.6) under the critical growth assumption on f and obtained the following result.

Theorem 1.2. *Suppose $N = 3$, and that (f_1)-(f_4) hold. If $q \in (4, 6)$, then (1.6) possesses a positive least energy solution $u \in H^1(\mathbb{R}^N)$ satisfying* (i)-(ii) *and* (iii') *u and its first derivatives decay exponentially at infinity:*

$$|D^{\alpha}u(x)| \le Ce^{-\delta|x|}, \quad x \in \mathbb{R}^N$$

for some C, $\delta > 0$ and for $|\alpha| = 0, 1$.

Moreover, motivated by Jeanjean and Tanaka [20], Zhang and Zou [41] also showed the fact that the mountain pass value is equal to the least energy level.

Now we give the fifth assumption on the nonlinear term f:

(f_5) Set $g(s) = f(s) + ms$, then there exists $\gamma > 3$ such that $\frac{1}{\gamma}g(s)s - G(s) \ge 0$ for all $s \in \mathbb{R}$, where $G(s) = \int_0^s g(t)dt$.

Note that, under the assumptions (f_1)-(f_5), Zhang [40] studied a class of Schrödinger-Poisson problems and proved the existence of ground state solutions for $q \in (2, 4]$ with D sufficiently large, or $q \in (4, 6)$.

Now, an interesting question is whether the same existence results occur to the nonlocal problem (K). It is worth noticing that, to our best knowledge, there is no work on the existence of positive solutions for problem (K) where f stands for a general nonlinearity in the critical growth; that is, f satisfies the conditions (f_1)-(f_4). In the present paper, we are interested in studying the existence of positive ground state solutions for this class of problems. In fact, we can not only obtain the existence of positive ground state solutions, but also prove that problem (K) has a mountain pass solution. Moreover, it is easy to see that the mountain pass solution is also a ground state solution, which is very important for us to study singular perturbation problems of Kirchhoff-type equations, i.e., the search of peak solutions. The ground state often serves as the scaled limit profile of solutions near the spike in singular perturbation problem

$$\begin{cases} -(\varepsilon^2 a + \varepsilon b \int\limits_{\mathbb{R}^3} |\nabla u|^2)\triangle u + V(x)u = f(u), & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u > 0, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.7}$$

The peak solutions of (1.7), sometimes called semi-classical states, are families of solutions $u_\varepsilon(x)$ which concentrate and develop a spike shape around certain points in $\mathbb{R}^3$ while vanishing elsewhere as $\varepsilon \to 0$. For this subject we refer, for example, to [18, 35, 27, 15]. By minimizing the energy functional restricted to the Nehari manifold $\mathcal{N}$, the monotonicity assumption $t \to \frac{f(t)}{t^3}$ is often used to ensure that the mountain pass value gives the least energy. See, for example, [18] and [24]. However, in our paper we also get the same result without the monotonicity assumption. In addition, we obtain a positive ground state solution (see Proposition 4.2) by minimizing the energy functional restricted to the Pohozaev's manifold $\mathcal{P}$. Unfortunately, we can not distinguish between the mountain pass solution and the positive ground state solution.

Now, we state our main result.

Theorem 1.3. *Under assumptions* (f_1)-(f_4)*, assume that either $q \in (4, 6)$ or $q \in (2, 4]$ and D is sufficiently large. Then problem* (K) *admits at least a positive ground state solution $u \in H^1(\mathbb{R}^3)$. Moreover, the least energy level can be given by the mountain pass value.*

Remark 1.1. *Note that our result is very important in the study of semi-classical states of Kirchhoff problem due to the facts mentioned above.*

Remark 1.2. In the present paper, we can also obtain conclusions (i), (ii) of Theorem 1.1, and (iii') of Theorem 1.2. Indeed, in Section 2 we shall prove the solution u is spherically symmetric and decreases with respect to r. As for the regularity, set

$$-\triangle u = q(x)u \text{ in } \mathbb{R}^3,$$

where $q(x) = \frac{f(u(x))}{(a+b\int_{\mathbb{R}^3}|\nabla u|^2)u(x)}$. By using similar arguments as that in [41], we have $u \in C^2(\mathbb{R}^3)$ and $u'(r) < 0$ for any $r > 0$. The exponential decay of u at infinity follows from a standard argument of ordinary differential equations. See also [41] for the details of the proof.

Remark 1.3. *Assumption (f_4) plays a significant role in ensuring the existence of ground state solution to problem (K). Without (f_4), the assumptions (f_1)-(f_3) are not enough to guarantee the ground state solution of problem (K). Indeed, we can give a counterexample, i.e., $f(u) = -ms + k|s|^4 s$. It is not difficult to see that f satisfies the assumptions (f_1)-(f_3) except (f_4). But it is easy to verify that problem (K) has not any nontrivial solutions, see [23].*

In order to prove Theorem 1.3, we have to solve four difficulties. Firstly, in our general assumptions, we don't assume the following global Ambrosetti-Rabinowitz growth hypothesis on f:

$$\text{there exists } \mu > 4 \text{ such that } 0 < \mu F(s) \leq f(s)s \text{ for all } s \in \mathbb{R},$$

which makes the proof of the boundedness of Palais-Smale sequences very tough. Thus, we will use an indirect approach developed by Jeanjean [20] to get the boundedness. Secondly, as we deal with the critical problem (K) in $H^1(\mathbb{R}^3)$, the Sobolev embeddings $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is not compact. The functional I does not satisfy $(\mathrm{PS})_c$ condition at every energy level c. To overcome this difficulty, in the spirit of Brezis and Nirenberg's celebrated paper [10], we try to pull the energy level down below some critical level to recover the compactness. Thirdly, it is very difficult to prove the weak sequential continuity of I' by direct calculations since problem (K) is no longer a pointwise identity. Indeed, in general, we do not know whether $\int_{\mathbb{R}^3}|\nabla u_n|^2 \to \int_{\mathbb{R}^3}|\nabla u|^2$ follows from $u_n \hookrightarrow u$ in $H^1(\mathbb{R}^3)$. Fortunately, by citing the functional

$$J(u) = \frac{a + bA^2}{2}\int_{\mathbb{R}^3}|\nabla u|^2 - \int_{\mathbb{R}^3}F(u),$$

where $A^2 = \lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2$, we do easily see that J' is weakly sequentially continuous at the weak limit point u of the $(PS)_c$ sequence $\{u_n\}$ of I, which helps us to find the nontrivial critical point of I by pulling the mountain-pass level c down below some critical energy level c_1^* (see Section 3). Finally, Berestycki and Lions [7] and Zhang and Zou [41] obtained the existence of ground state solutions by studying the minimization problem

$$\text{Minimize}\left\{\int_{\mathbb{R}^3}|\nabla u|^2;\quad \int_{\mathbb{R}^3}F(u)=1\right\}.$$

This method doesn't work in our case due to the nonlocal term $(\int_\Omega|\nabla u|^2dx)\triangle u$. Thus, we have to find another approach in order to obtain the existence of ground state solutions. In fact, our plan is to minimize I restricted to the Pohozaev's manifold $\mathcal{P}$ and then use Lagrange multiplier technique to obtain a ground state solution. However, the functional $J(u)$ is not applicable here, since the minimizing sequence on the manifold $\mathcal{P}$ takes the place of $(PS)_c$ sequence. Fortunately, we can prove the minimum point of I can be attained on the manifold $\mathcal{P}$ with the help of a Pohozaev-type identity of problem (K). Moreover, motivated by Jeanjean and Tanaka' arguments in [20], we can show that the mountain pass solution is also actually a ground state solution.

Throughout this paper, $C>0$ denotes a universal positive constant. The remainder of this paper is organized as follows. We will give some notations and preliminaries in Section 2. The existence of the mountain pass solution are presented in Section 3, and Section 4 is devoted to the existence of positive ground state solutions. Finally, in Section 5, we prove Theorem 1.3.

2. Notations and preliminaries

Hereafter, let us fix some notations. For every $\rho>0$ and every $z\in\mathbb{R}^3$, $B_\rho(z)$ denotes the ball of radius ρ centered at z and $|B_\rho(z)|$ denotes its Lebesgue measure. For any $1\le s\le+\infty$, we denote by $\|\cdot\|_s$ the usual norm of the Lebesgue space $L^s(\mathbb{R}^3)$. Let $D^{1,2}(\mathbb{R}^3):=\{u\in L^6(\mathbb{R}^3):|\nabla u|\in L^2(\mathbb{R}^3)\}$ be the Sobolev space equipped with the norm $\|u\|^2_{D^{1,2}}:=\int_{\mathbb{R}^3}|\nabla u|^2$. Recall that

$$S=\inf_{u\in D^{1,2}(\mathbb{R}^3)\setminus\{0\}}\frac{\int_{\mathbb{R}^3}|\nabla u|^2}{(\int_{\mathbb{R}^3}u^6dx)^{1/3}},$$

where S is the best constant in the Sobolev inclusion. For fixed $a,m>0$, we introduce an equivalent norm on $H^1(\mathbb{R}^3)$, that is, the norm of $u\in H^1(\mathbb{R}^3)$ is defined as

$$\|u\|:=\left(\int_{\mathbb{R}^3}(a|\nabla u|^2+mu^2)\right)^{\frac{1}{2}},$$

which is induced by the associated inner product on $H^1(\mathbb{R}^3)$. Set $g(s)=f(s)+ms$, so functional I is reduced as

$$I(u)=\frac{1}{2}\|u\|^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2-\int_{\mathbb{R}^3}G(u),\tag{2.1}$$

where $u \in H^1(\mathbb{R}^3)$ and $G(s) = \int_0^s g(t)dt$. The conditions (f_1)-(f_3) imply that the functional $I : H^1(\mathbb{R}^3) \mapsto \mathbb{R}$ is of class C^1. It can be proved that u is a solution of (K) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of the functional I. Let $H_r^1(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3)\colon u \text{ is radial}\}$. In this paper, we will look for critical points of I on $H_r^1(\mathbb{R}^3)$, which is a natural constraint.

In the following, we give the abstract critical point theorem developed by Jeanjean [19].

Theorem 2.1. ([19]) *Let $(E, \|\cdot\|)$ be a real Banach space with its dual space E^{-1} and $J \in \mathbb{R}^+$ an interval. Consider the family of C^1 functionals on E:*

$$I_\lambda = A(u) - \lambda B(u),\ \forall \lambda \in J,$$

with B nonnegative and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$, satisfying $I_\lambda(0) = 0$. For any $\lambda \in J$ we set

$$\Gamma_\lambda = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0, \gamma(1) < 0\}.$$

If for every $\lambda \in J$ the set Γ_λ is nonempty and

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} I(\gamma(s)) > 0, \tag{2.2}$$

then for almost every $\lambda \in J$ there is a bounded Palais-Smale sequence $\{u_n\}$, i.e, $\{u_n\}$ is bounded and satisfies that $I_\lambda(u_n) \to c_\lambda$ and $I'_\lambda(u_n) \to 0$ in E^{-1}.

In our case, $E = H_r^1(\mathbb{R}^3)$,

$$A(u) := \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2,$$
$$B(u) := \int_{\mathbb{R}^3} G(u),$$

and the associated perturbed functional we study is

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \lambda \int_{\mathbb{R}^3} G(u) \tag{2.3}$$

for $u \in E$ and $\lambda \in J := [\frac{1}{2}, 1]$. It is clear that this functional is of C^1-functional defined on the whole space E and for every $u, v \in E$,

$$I'_\lambda(u)v = a\int_{\mathbb{R}^3} \nabla u \nabla v + m\int_{\mathbb{R}^3} uv + b\int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla u \nabla v - \lambda \int_{\mathbb{R}^3} g(u)v. \tag{2.4}$$

Note that $B(u) \geq 0, \forall u \in E$ and $A(u) \to +\infty$ as $\|u\| \to \infty$ which is suitable for some conditions of Theorem 2.1.

We will make use of the following Pohozaev type identity, whose proof is standard and can be found in [7, 5].

Lemma 2.1. *Let u be a critical point of I_λ in E for $\lambda \in J$, then*

$$\frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + \frac{3m}{2}\int_{\mathbb{R}^3} u^2 - 3\lambda\int_{\mathbb{R}^3} G(u) = 0.$$

3. A Mountain Pass solution

Lemma 3.1. *Suppose that* (f_1)-(f_4) *hold, then*
(i) $\Gamma_\lambda \neq \emptyset$ *for every* $\lambda \in J$*;*
(ii) *there exists a constant* $\eta > 0$ *such that* $c_\lambda \geq \eta > 0$ *for every* $\lambda \in J$*.*

Proof (i) For every $\lambda \in J$, using (f_4) and (2.3), we have

$$I_\lambda(u) \leq \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \frac{1}{12}\int_{\mathbb{R}^3} u^6 - \frac{D}{2q}\int_{\mathbb{R}^3} |u|^q.$$

Set $w \in E \setminus \{0\}$ such that $w \not\equiv 0$. Then $\lim_{t\to+\infty} I_\lambda(tw) = -\infty$. Thus there exists $t_0 > 0$ such that $I_\lambda(t_0 w) < 0$ for every $\lambda \in J$. Define $\gamma_1 : [0,1] \to E$ as

$$\gamma_1(t) = tt_0 w, \quad 0 \leq t \leq 1.$$

It is easy to see that γ_1 is a continuous path from 0 to $t_0 w$. Furthermore, we have $I_\lambda(\gamma_1(1)) < 0$ and $I_\lambda(\gamma_1(0)) = 0$ for every $\lambda \in J$.

(ii) The conditions (f_1)-(f_3) imply that, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|g(u)| \leq \epsilon|u| + C_\epsilon|u|^5. \tag{3.1}$$

In view of (2.3) and (3.1), there holds

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\epsilon}{2}\int_{\mathbb{R}^3} u^2 - \frac{C_\epsilon}{6}\int_{\mathbb{R}^3} u^6 \\ &\geq \frac{m-\epsilon}{2m}\|u\|^2 - CC_\epsilon\|u\|^6. \end{aligned}$$

So, by fixing $\epsilon \in (0,m)$ and letting $\|u\| = \rho > 0$ small enough, it is easy to see that there is $\eta > 0$ such that $I_\lambda(u) \geq \eta$ for every $\lambda \in J$. Now fix $\lambda \in J$ and $\gamma \in \Gamma_\lambda$. Since $\gamma(0) = 0$ and $I_\lambda(\gamma(1)) < 0$, certainly $\|\gamma(1)\| > \rho$. By continuity, we conclude that there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Therefore, for every $\lambda \in J$,

$$c_\lambda \geq \inf_{\gamma\in\Gamma_\lambda} I_\lambda(\gamma(t_\gamma)) \geq \eta > 0.$$

This completes the proof of this lemma. □

It follows from Lemma 3.1 that the conclusions of Theorem 2.1 hold. In the following, we will give an upper bounded estimate on c_λ.

Lemma 3.2. *Assume* (f_1) -(f_4) *hold and if* $q \in (4,6)$ *or* $q \in (2,4]$ *and* D *is sufficiently large, then* $c_\lambda < c_\lambda^* := \frac{ab}{4\lambda}S^3 + \frac{[b^2S^4+4\lambda aS]^{\frac{3}{2}}}{24\lambda^2} + \frac{b^3S^6}{24\lambda^2}$*.*

Proof For $\epsilon, r > 0$, take $U_\epsilon(x) = \frac{\phi(x)\epsilon^{1/4}}{(\epsilon+|x|^2)^{1/2}}$, where $\phi \in C_0^\infty(B_{2r}(0))$ satisfies $0 \leq \phi(x) \leq 1$ and $\phi(x) \equiv 1$ on $B_r(0)$. Recall that S is attained by the functions $\frac{\epsilon^{1/4}}{(\epsilon+|x|^2)^{1/2}}$. Direct calculation shows that

$$\int_{\mathbb{R}^3} |\nabla U_\epsilon|^2 dx = K_1 + O(\epsilon^{\frac{1}{2}}), \quad \int_{\mathbb{R}^3} |U_\epsilon|^6 dx = K_2 + O(\epsilon^{\frac{3}{2}}) \tag{3.2}$$

and

$$\int_{\mathbb{R}^3} |U_\epsilon|^t dx = \begin{cases} O(\epsilon^{\frac{t}{4}}), & t \in [2,3); \\ O(\epsilon^{\frac{3}{4}}|ln\epsilon|), & t = 3; \\ O(\epsilon^{\frac{6-t}{4}}), & t \in (3,6), \end{cases} \tag{3.3}$$

where K_1, K_2 are positive constants. Moreover, $S = \frac{K_1}{K_2^{1/3}}$. Using (3.2) and (3.3) we have

$$\frac{\int_{\mathbb{R}^3} |\nabla U_\epsilon|^2 dx}{(\int_{\mathbb{R}^3} |U_\epsilon|^6 dx)^{1/3}} = S + O(\epsilon^{\frac{1}{2}}). \tag{3.4}$$

In view of the definition of c_λ, we have $c_\lambda \leq \max_{t\geq 0} I(tU_\epsilon)$. Define function $y(t) := \frac{t^2}{2}\|U_\epsilon\|^2 + \frac{bt^4}{4}(\int_{\mathbb{R}^3} |\nabla U_\epsilon|^2)^2 - \frac{\lambda t^6}{6}\int_{\mathbb{R}^3} |U_\epsilon|^6$. It is clear that $y(t)$ attains its maximum at

$$t_0 = \left(\frac{b(\int_{\mathbb{R}^3} |\nabla U_\epsilon|^2)^2 + \sqrt{b^2(\int_{\mathbb{R}^3} |\nabla U_\epsilon|^2)^4 + 4\lambda \|U_\epsilon\|^2 \int_{\mathbb{R}^3} |U_\epsilon|^6}}{2\lambda \int_{\mathbb{R}^3} |U_\epsilon|^6} \right)^{1/2}$$

and

$$y(t_0) = \frac{b\|U_\epsilon\|^2(\int_{\mathbb{R}^3} |\nabla U_\epsilon|^2)^2}{4\lambda \int_{\mathbb{R}^3} |U_\epsilon|^6} + \frac{[b^2(\int_{\mathbb{R}^3} |\nabla U_\epsilon|^2)^4 + 4\lambda\|U_\epsilon\|^2 \int_{\mathbb{R}^3} |U_\epsilon|^6]^{\frac{3}{2}}}{24(\lambda \int_{\mathbb{R}^3} |U_\epsilon|^6)^2} + \frac{b^3(\int_{\mathbb{R}^3} |\nabla U_\epsilon|^2)^6}{24(\lambda \int_{\mathbb{R}^3} |U_\epsilon|^6)^2}.$$

Observe that there exists $t' \in (0,1)$ such that for $\epsilon < 1$, we have

$$\begin{aligned} \max_{t'\geq t\geq 0} I_\lambda(tU_\epsilon(x)) &\leq \max_{t'\geq t\geq 0} \left(\frac{t^2}{2}\|U_\epsilon\|^2 + \frac{bt^4}{4}(\int_{\mathbb{R}^3} |\nabla U_\epsilon|^2)^2 \right) \\ &\leq \max_{t'\geq t\geq 0} \left(\frac{t^2}{2}\|U_\epsilon\|^2 + Ct^4\|U_\epsilon\|^4 \right) < c_\lambda^*. \end{aligned} \tag{3.5}$$

By (f_4) and (2.3), one has

$$\begin{aligned} I_\lambda(tU_\epsilon(x)) &\leq y(t) - \frac{\lambda D}{q} t^q \int_{\mathbb{R}^3} |U_\epsilon|^q \\ &\leq y(t) - CDt^q \int_{\mathbb{R}^3} |U_\epsilon|^q. \end{aligned} \tag{3.6}$$

Now we claim that there exists $\epsilon_0 \in (0,1)$ such that $\lim_{t\to+\infty} I_\lambda(tU_\epsilon(x)) < 0$ uniformly in $\epsilon \in (0,\epsilon_0)$. Indeed, It follows from Lemma 3.1 and (3.6) that $\lim_{t\to+\infty} I_\lambda(tU_\epsilon(x)) = -\infty$ and $I_\lambda(tU_\epsilon(x)) > 0$ as t is close to 0. Define

$$e(t) := \frac{t^2}{2}\|U_\epsilon\|^2 + \frac{bt^4}{4}(\int_{\mathbb{R}^3} |\nabla U_\epsilon|^2)^2 - \frac{\lambda t^6}{6}\int_{\mathbb{R}^3} |U_\epsilon|^6 - \frac{\lambda D}{q} t^q \int_{\mathbb{R}^3} |U_\epsilon|^q,$$

then there exists $t_\epsilon > 0$ such that $e(t_\epsilon) = 0$ and $e(t) < 0$ for $t > t_\epsilon$. From

$$e(t_\epsilon) = t_\epsilon^2 \left(\frac{1}{2}\|U_\epsilon\|^2 + \frac{bt_\epsilon^2}{4}(\int_{\mathbb{R}^3} |\nabla U_\epsilon|^2)^2 - \frac{\lambda t_\epsilon^4}{6}\int_{\mathbb{R}^3} |U_\epsilon|^6 - \frac{\lambda D}{q} t^{q-2} \int_{\mathbb{R}^3} |U_\epsilon|^q \right) = 0,$$

we have

$$\frac{1}{2}\|U_\epsilon\|^2+\frac{bt_\epsilon^2}{4}(\int_{\mathbb{R}^3}|\nabla U_\epsilon|^2)^2=\frac{\lambda t_\epsilon^4}{6}\int_{\mathbb{R}^3}|U_\epsilon|^6+\frac{\lambda D}{q}t^{q-2}\int_{\mathbb{R}^3}|U_\epsilon|^q$$
$$\geq\frac{\lambda t_\epsilon^4}{6}\int_{\mathbb{R}^3}|U_\epsilon|^6.$$

Furthermore, using (3.2), (3.3) one has

$$\begin{aligned}\frac{\lambda t_\epsilon^4}{6}&\leq\frac{1}{2}\frac{\|U_\epsilon\|^2}{\int_{\mathbb{R}^3}|U_\epsilon|^6}+\frac{bt_\epsilon^2}{4}\frac{(\int_{\mathbb{R}^3}|\nabla U_\epsilon|^2)^2}{\int_{\mathbb{R}^3}|U_\epsilon|^6}\\&\leq\frac{1}{2}\frac{aK_1+O(\epsilon^{\frac{1}{2}})}{K_2+O(\epsilon^{\frac{3}{2}})}+\frac{bt_\epsilon^2}{4}\frac{K_1^2+O(\epsilon^{\frac{3}{2}})}{K_2+O(\epsilon^{\frac{3}{2}})}+O(\epsilon^{\frac{1}{2}})\\&\leq\frac{aK_1+|O(\epsilon_0^{\frac{1}{2}})|}{K_2}+\frac{bt_\epsilon^2}{2}\frac{K_1^2+|O(\epsilon_0^{\frac{3}{2}})|}{K_2}+|O(\epsilon_0^{\frac{1}{2}})|\end{aligned}\tag{3.7}$$

for $\epsilon\in(0,\epsilon_0)$, where $\epsilon_0>0$ is small enough. (3.7) implies that t_ϵ is bounded from above by some $t^*>0$ uniformly for $\epsilon\in(0,\epsilon_0)$, where t^* is independent of ϵ. Combining the above fact and (3.6), it is easy to see that there exists $\epsilon_0\in(0,1)$ such that $\lim\limits_{t\to+\infty}I_\lambda(tU_\epsilon(x))<0$ uniformly in $\epsilon\in(0,\epsilon_0)$. Thus there exists $t''>t^*$ such that for $\epsilon\in(0,\epsilon_0)$,

$$\max_{t\geq t''}I_\lambda(tU_\epsilon(x))<c_\lambda^*.\tag{3.8}$$

It follows from (3.3), (3.4) and (3.6) that

$$\begin{aligned}&\max_{t''\geq t\geq t'}I_\lambda(tU_\epsilon(x))\\&\leq y(t_0)-CD\int_{\mathbb{R}^3}|U_\epsilon|^q\\&=\frac{ab}{4\lambda}S^3+\frac{(b^2S^4+4\lambda aS)^{\frac{3}{2}}}{24\lambda^2}+\frac{b^3S^6}{24\lambda^2}+O(\epsilon^{\frac{1}{2}})-CD\int_{\mathbb{R}^3}|U_\epsilon|^q.\end{aligned}\tag{3.9}$$

For $q\in(2,4]$ and D sufficiently large, $\epsilon\in(0,\epsilon_0)$ fixed, we derive from (3.9) that

$$\max_{t''\geq t\geq t'}I(t^2U_\epsilon(tx))<\frac{ab}{4\lambda}S^3+\frac{(b^2S^4+4\lambda aS)^{\frac{3}{2}}}{24\lambda^2}+\frac{b^3S^6}{24\lambda^2}.\tag{3.10}$$

For $q\in(4,6)$, observe that $\frac{6-q}{4}<\frac{1}{2}$, then it follows from (3.3) and (3.9) that, there exists $\epsilon_1\in(0,\epsilon_0)$ small enough such that for $\epsilon\in(0,\epsilon_1)$,

$$\max_{t''\geq t\geq t'}I(t^2U_\epsilon(tx))<\frac{ab}{4\lambda}S^3+\frac{(b^2S^4+4\lambda aS)^{\frac{3}{2}}}{24\lambda^2}+\frac{b^3S^6}{24\lambda^2}.\tag{3.11}$$

Combining (3.5), (3.8) and (3.10), (3.11), we deduce that $c_\lambda<c_\lambda^*$.

We cite a variant of Strauss' compactness result [33] which plays a crucial role in our arguments:

Theorem 3.1. ([33]) *Let P and $Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying*

$$\lim_{s\to\infty} \frac{P(s)}{Q(s)} = 0.$$

Let $\{v_n\}_n$, v and w be measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$, with z bounded, such that

$$\sup_n \int_{\mathbb{R}^N} |Q(v_n)w| < +\infty$$

and

$$P(v_n(x)) \to v(x) \ a.e. \text{ in } \mathbb{R}^N, \quad \text{as} \quad n \to +\infty.$$

Then for any bounded Borel set B one has $\|(P(v_n) - v)w\|_{L^1(B)} \to 0$. Furthermore, if

$$\lim_{s\to 0} \frac{P(s)}{Q(s)} = 0$$

and

$$\lim_{|x|\to+\infty} \sup_n |v_n(x)| = 0,$$

then $\|(P(v_n) - v)w\|_{L^1(\mathbb{R}^N)} \to 0$ as $n \to +\infty$.

Lemma 3.3. *For $t, s > 0$ and $\lambda \in [\frac{1}{2}, 1]$, the following system*

$$\begin{cases} \Phi(t,s) = t - aS\lambda^{\frac{-1}{3}}(t+s)^{\frac{1}{3}} = 0, \\ \Psi(t,s) = s - bS^2\lambda^{\frac{-2}{3}}(t+s)^{\frac{2}{3}} = 0 \end{cases}$$

has a unique solution (t_0, s_0). Moreover, if $\Phi(t,s) \geq 0$ and $\Psi(t,s) \geq 0$, then $t \geq t_0, s \geq s_0$.

Proof The proof is similar to that of Lemma 3.6 in [37], we now provide it for the completeness. If $\Phi(t_0, s_0) = \Psi(t_0, s_0) = 0$, then $t_0 + s_0 = \frac{\lambda t_0^3}{a^3 S^3}$. Hence, it follows from the second equality of system that

$$\left(\frac{\lambda t_0^3 - a^3 S^3 t_0}{a^3 S^3}\right)^3 = b^3\lambda^{-2}S^6\left(\frac{\lambda t_0^3}{a^3 S^3}\right)^2,$$

which implies that

$$t_0 = \frac{abS^3 + a\sqrt{b^2S^6 + 4a\lambda S^3}}{2\lambda}$$

and

$$s_0 = \frac{b^2S^3\sqrt{b^2S^6 + 4a\lambda S^3}}{2\lambda^2} + \frac{b^3S^6}{2\lambda^2} + \frac{abS^3}{\lambda}.$$

If $\Phi(t,s) \geq 0$ and $\Psi(t,s) \geq 0$, then $t+s \geq aS\lambda^{\frac{-1}{3}}(t+s)^{\frac{1}{3}} + bS^2\lambda^{\frac{-2}{3}}(t+s)^{\frac{2}{3}}$. Let $K(l) := l - aS\lambda^{\frac{-1}{3}}l^{\frac{1}{3}} - bS^2\lambda^{\frac{-2}{3}}l^{\frac{2}{3}}$, $l > 0$. Then, $K(l)$ has a unique zero point

$l_0 > 0$ and $K(l) \geq 0$ and hence $l \geq l_0$. Namely, $t + s \geq t_0 + s_0$. Suppose that $t < t_0$, then

$$\Phi(t,s) = t - aS\lambda^{\frac{-1}{3}}(t+s)^{\frac{1}{3}} < t_0 - aS\lambda^{\frac{-1}{3}}(t_0+s_0)^{\frac{1}{3}} = 0,$$

which contradicts with $\Phi(t,s) \geq 0$, so $t \geq t_0$. Similarly, $s \geq s_0$. □

Lemma 3.4. *Under the conditions of Theorem 1.3, for almost every $\lambda \in J$, there exists $u \in E$, $u \neq 0$, such that $I'_\lambda(u) = 0$ and $I_\lambda(u) = c_\lambda$.*

Proof By Lemma 3.1 and Theorem 2.1, for almost every $\lambda \in J$, there exists a bounded sequence $\{u_n\} \subset E$ such that

$$I_\lambda(u_n) \to c_\lambda; \quad I'_\lambda(u_n) \to 0 \text{ in } E^{-1}, \tag{3.12}$$

where E^{-1} is the dual space of E. Up to a subsequence, we can suppose that there exist $u \in E$ and $A \in \mathbb{R}$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } E,\\ u_n &\to u \quad \text{in } L^p(\mathbb{R}^3),\ 2<p<6,\\ u_n &\to u \quad \text{a.e. in } \mathbb{R}^3 \end{aligned} \tag{3.13}$$

and

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 \to A^2. \tag{3.14}$$

Set $h(t) = g(t) - t^5$. If we apply Theorem 3.1 to $P(t) = h(t)$, $Q(t) = |t|^5$, $\{v_n\}_n = \{u_n\}_n$, $v = h(u)$, and $w \in C_0^\infty(\mathbb{R}^3)$, then it follows from Sobolev inequality, (3.13) and (f_1)-(f_4) that

$$\int_{\mathbb{R}^3} h(u_n)w \to \int_{\mathbb{R}^3} h(u)w. \tag{3.15}$$

As a consequence, from (3.12)-(3.15), we deduce that

$$\int_{\mathbb{R}^3}(a\nabla u\nabla w + muw) + bA^2\int_{\mathbb{R}^3}\nabla u\nabla w - \lambda\int_{\mathbb{R}^3}(h(u)+u^5)w = 0, \quad \forall w \in C_0^\infty(\mathbb{R}^3),$$

i.e., $J'_\lambda(u) = 0$, where $u \in E$ and

$$J_\lambda(u) := \frac{1}{2}\|u\|^2 + \frac{bA^2}{2}\int_{\mathbb{R}^3}|\nabla u|^2 - \lambda\int_{\mathbb{R}^3} H(u) - \frac{\lambda}{6}\int_{\mathbb{R}^3} u^6,$$

and

$$H(u) = \int_0^u h(t)dt.$$

Note that J'_λ is weakly sequence continuous in E. From Lemma 2.1, we can easily conclude that the following Pohozaev type identity

$$P_\lambda(u) := \frac{3m}{2}\int_{\mathbb{R}^3} u^2 + \frac{a+bA^2}{2}\int_{\mathbb{R}^3}|\nabla u|^2 - 3\lambda\int_{\mathbb{R}^3} H(u) - \frac{\lambda}{2}\int_{\mathbb{R}^3} u^6 = 0$$

holds. It follows from (3.12) and (3.14) that $\{u_n\}$ is a bounded $(\mathrm{PS})_{\mathrm{c}_\lambda+\frac{\mathrm{bA}^4}{4}}$ sequence of J_λ and

$$J_\lambda(u) - \frac{1}{3}P_\lambda(u) = \frac{a+bA^2}{3}\int_{\mathbb{R}^3}|\nabla u|^2 \geq 0. \tag{3.16}$$

Set $v_n = u_n - u$, then we have $v_n \rightharpoonup 0$ in E. By the well-known Brezis-Lieb lemma [9], one has

$$\begin{aligned}
&\|v_n\|_2^2 = \|u_n\|_2^2 - \|u\|_2^2 + o(1),\\
&A^2 + o(1) = \|\nabla u_n\|_2^2 = \|\nabla v_n\|_2^2 + \|\nabla u\|_2^2 + o(1),\\
&\|v_n\|_6^6 = \|u_n\|_6^6 - \|u\|_6^6 + o(1).
\end{aligned} \tag{3.17}$$

where $o(1) \to 0$ as $n \to \infty$. If we apply Theorem 3.1 to $P(t) = h(t)t$, $Q(t) = t^2 + t^6$, $\{v_n\}_n = \{u_n\}_n$, $v = h(u)u$, and $w = 1$, then it follows from (3.13) and (f_1)-(f_4) that

$$\int_{\mathbb{R}^3} h(u_n)u_n \to \int_{\mathbb{R}^3} h(u)u. \tag{3.18}$$

Similarly, we also have

$$\int_{\mathbb{R}^3} H(u_n) \to \int_{\mathbb{R}^3} H(u). \tag{3.19}$$

Then, by (3.17), (3.18) and $J'_\lambda(u) = 0$, we have

$$\begin{aligned}
o(1) &= \langle J'_\lambda(u_n), u_n\rangle\\
&= \langle J'_\lambda(u), u\rangle + (a+bA^2)\int_{\mathbb{R}^3}|\nabla v_n|^2 + m\int_{\mathbb{R}^3}|v_n|^2 - \lambda\int_{\mathbb{R}^3}|v_n|^6 + o(1)\\
&= \|v_n\|^2 + b(\int_{\mathbb{R}^3}|\nabla v_n|^2)^2 + b\int_{\mathbb{R}^3}|\nabla v_n|^2\int_{\mathbb{R}^3}|\nabla u|^2 - \lambda\int_{\mathbb{R}^3}|v_n|^6 + o(1),
\end{aligned}$$

which yields

$$\|v_n\|^2 + b(\int_{\mathbb{R}^3}|\nabla v_n|^2)^2 + b\int_{\mathbb{R}^3}|\nabla v_n|^2\int_{\mathbb{R}^3}|\nabla u|^2 - \lambda\int_{\mathbb{R}^3}|v_n|^6 = o(1). \tag{3.20}$$

Up to a subsequence, we assume that there exists $l_i \geq 0$ $(i = 1,2,3)$ such that

$$\|v_n\|^2 \to l_1, \quad b(\int_{\mathbb{R}^3}|\nabla v_n|^2)^2 + b\int_{\mathbb{R}^3}|\nabla v_n|^2\int_{\mathbb{R}^3}|\nabla u|^2 \to l_2, \quad \lambda\int_{\mathbb{R}^3}|v_n|^6 \to l_3,$$

then $l_1 + l_2 = l_3$. If $l_1 > 0$, then $l_2, l_3 > 0$. In view of (3.13), (3.19), (3.20) and the definition of J_λ, we conclude that

$$
\begin{aligned}
J_\lambda(u_n) =& J_\lambda(u) + \frac{a+bA^2}{2}\int_{\mathbb{R}^3}|\nabla v_n|^2 + \frac{m}{2}\int_{\mathbb{R}^3}|v_n|^2 - \frac{\lambda}{6}\int_{\mathbb{R}^3}|v_n|^6 + o(1)\\
=& J_\lambda(u) + \frac{1}{2}\|v_n\|^2 + \frac{b}{4}[(\int_{\mathbb{R}^3}|\nabla v_n|^2)^2 + \int_{\mathbb{R}^3}|\nabla v_n|^2\int_{\mathbb{R}^3}|\nabla u|^2]\\
&-\frac{\lambda}{6}\int_{\mathbb{R}^3}|v_n|^6 + \frac{bA^2}{4}\int_{\mathbb{R}^3}|\nabla v_n|^2 + o(1)\\
=& J_\lambda(u) + \frac{1}{3}\|v_n\|^2 + \frac{b}{12}[(\int_{\mathbb{R}^3}|\nabla v_n|^2)^2 + \int_{\mathbb{R}^3}|\nabla v_n|^2\int_{\mathbb{R}^3}|\nabla u|^2]\\
&+\frac{bA^2}{4}\int_{\mathbb{R}^3}|\nabla v_n|^2 + o(1).
\end{aligned}
$$

Letting $n\to\infty$, we have

$$
c_\lambda + \frac{bA^4}{4} \geq J_\lambda(u) + \frac{1}{3}l_1 + \frac{1}{12}l_2 + \frac{bA^4}{4}\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla v_n|^2. \tag{3.21}
$$

Note that, by Sobolev imbedding inequality we have

$$
\int_{\mathbb{R}^3}|\nabla v_n|^2 \geq S\left(\int_{\mathbb{R}^3}|v_n|^6\right)^{\frac{1}{3}} \quad \text{and} \quad b\left(\int_{\mathbb{R}^3}|\nabla v_n|^2\right)^2 \geq bS^2\left(\int_{\mathbb{R}^3}|v_n|^6\right)^{\frac{2}{3}}.
$$

Then

$$
l_1 \geq aS\lambda^{\frac{-1}{3}}(l_1+l_2)^{\frac{1}{3}} \quad \text{and} \quad l_2 \geq bS^2\lambda^{\frac{-2}{3}}(l_1+l_2)^{\frac{2}{3}}.
$$

By Lemma 3.3, we have

$$
\begin{aligned}
\frac{1}{3}l_1 + \frac{1}{12}l_2 \geq& \frac{1}{3}\frac{abS^3 + a\sqrt{b^2S^6+4a\lambda S^3}}{2\lambda} + \frac{1}{12}(\frac{b^2S^3\sqrt{b^2S^6+4a\lambda S^3} + b^3S^6 + 2\lambda abS^3}{2\lambda^2})\\
=& \frac{ab}{4\lambda}S^3 + \frac{[b^2S^4+4\lambda aS]^{\frac{3}{2}}}{24\lambda^2} + \frac{b^3S^6}{24\lambda^2} = c_\lambda^*.
\end{aligned}
$$

Hence, it follows from (3.16) and (3.21) that

$$
\begin{aligned}
c_\lambda + \frac{bA^4}{4} &\geq J_\lambda(u) + \frac{1}{3}l_1 + \frac{1}{12}l_2 + \frac{bA^4}{4}\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla v_n|^2\\
&\geq c_\lambda^* + \frac{bA^2}{4}\lim_{n\to\infty}\left(\int_{\mathbb{R}^3}|\nabla v_n|^2 + \int_{\mathbb{R}^3}|\nabla u|^2\right)\\
&= c_\lambda^* + \frac{bA^4}{4},
\end{aligned} \tag{3.22}
$$

that is, $c_\lambda \geq c_\lambda^*$ which contradicts with Lemma 3.2. Therefore, $l_1 = 0$, i.e. $\|v_n\|^2 = o(1)$, hence $u_n \to u$ in E. □

Proposition 3.1. *Assume* (f$_1$)- (f$_4$) *hold and if* $q\in(4,6)$ *or* $q\in(2,4]$ *and* D *is sufficiently large, then* I *has a critical point* $u\in E$ *with* $I(u)=c_1$.

Proof In view of Lemma 3.4, in order to show that I has a nontrivial critical point, it suffices to construct a bounded $(\mathrm{PS})_{c_1}$ sequence for I, where c_1 is the mountain pass value of $I_1 = I$. Lemma 3.4 implies that there exists $u_\lambda \in E, u_\lambda \neq 0$, such that

$$I'_\lambda(u_\lambda) = 0, \quad I_\lambda(u_\lambda) = c_\lambda \quad \text{for } a.e.\ \lambda \in J.$$

Choosing $\lambda_n \to 1$, we have a sequence of $\{u_{\lambda_n}\}$, denoted by $\{u_n\}$, which are the critical points of I_{λ_n}. Next, we show that $\{u_n\}$ is bounded. Indeed, by Lemma 2.1 and $I_{\lambda_n}(u_n) = c_{\lambda_n}$, we have

$$\begin{cases} \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 + \frac{3m}{2}\int_{\mathbb{R}^3} u_n^2 - 3\lambda_n \int_{\mathbb{R}^3} G(u_n) = 0, \\ \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 + \frac{m}{2}\int_{\mathbb{R}^3} u_n^2 - \lambda_n \int_{\mathbb{R}^3} G(u_n) = c_{\lambda_n} \leq c_{\frac{1}{2}}. \end{cases}$$

From these relations, we have

$$a\int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 \leq 3c_{\frac{1}{2}}. \tag{3.23}$$

Moreover, by Lemma 2.1 and (3.1), for any $\epsilon \in (0, m)$, there exists $C_\epsilon > 0$ such that

$$\begin{aligned} \frac{3m}{2}\int_{\mathbb{R}^3} u_n^2 &\leq 3\int_{\mathbb{R}^3} G(u_n) \\ &\leq \frac{3\epsilon}{2}\int_{\mathbb{R}^3} u_n^2 + \frac{C_\epsilon}{6}\int_{\mathbb{R}^3} u_n^6 \\ &\leq \frac{3\epsilon}{2}\int_{\mathbb{R}^3} u_n^2 + S^{-3}\frac{C_\epsilon}{6}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^3. \end{aligned} \tag{3.24}$$

Combining (3.23) and (3.24), we can deduce that $\{u_n\}$ is a bounded sequence in E. Therefore, by using the fact that the map $\lambda :\to c_\lambda$ is left-continuous (see [19]), we have

$$\lim_{n\to\infty} I(u_n) = \lim_{n\to\infty}\left(I_{\lambda_n}(u_n) + (\lambda_n - 1)\int_{\mathbb{R}^3} G(u_n)\right) = \lim_{n\to\infty} c_{\lambda_n} = c_1 < c_1^*$$

and

$$\lim_{n\to\infty} I'(u_n)w = \lim_{n\to\infty}\left(I'_{\lambda_n}(u_n)w + (\lambda_n - 1)\int_{\mathbb{R}^3} g(u_n)w\right) = 0$$

for all $w \in C_0^\infty(\mathbb{R}^3)$. Then it is easy to see that $I'(u_n) \to 0$ in E^{-1} and Lemma 3.4 yields that there exists $u_0 \in E, u_0 \neq 0$, being a critical point of I.

Let us note that all the calculations above can be repeated word by word, replacing I with the functional

$$I^+(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \int_{\mathbb{R}^3} H(u) - \int_{\mathbb{R}^3} (u^+)^6,$$

where $u^+ = \max\{u, 0\}$ is the positive part of u. Therefore, there exists nonzero function u_0 can solve the equation

$$-(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx)\triangle u + mu = h(u) + (u^+)^5 \tag{3.25}$$

In (3.25), using $u_0^- = \max\{-u_0, 0\}$ as a text function and integrating by parts, we obtain

$$0 = \int_{\mathbb{R}^3} (a|\nabla u_0^-|^2 + m|u_0^-|^2)dx + b\left(\int_{\mathbb{R}^3} |\nabla u_0|^2 dx\right)\int_{\mathbb{R}^3} |\nabla u_0^-|^2 dx - \int_{\mathbb{R}^3} h(u_0)u_0^- dx. \tag{3.26}$$

It follows from (f_1) and (f_4) that h is an odd function and $h(t) > 0$ for $t > 0$. So from (3.26) we have

$$0 = \int_{\mathbb{R}^3} (a|\nabla u_0^-|^2 + m|u_0^-|^2)dx + b\left(\int_{\mathbb{R}^3} |\nabla u_0|^2 dx\right)\int_{\mathbb{R}^3} |\nabla u_0^-|^2 dx.$$

Thus $u_0^- = 0$, and $u_0 \geq 0$ is a solution of problem (K). From Harnack's inequality (see[16]), we can infer that $u_0 > 0$ for all $x \in \mathbb{R}^3$. Therefore, u_0 is a positive solution of (SK). The proof is complete.

□

4. A ground state solution

Based on the previous sections, in this section we will find a ground state solution. As mentioned before, we plan to consider energy functional I on the Pohozaev manifold. By Lemma 2.1, we know that if u is a solution of problem (K), then u satisfies the following Pohozaev type identity

$$J^*(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - 3\int_{\mathbb{R}^3} F(u) = 0.$$

We introduce the Pohozaev Manifold $\mathcal{P}$:

$$\mathcal{P} := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J^*(u) = 0\}.$$

By Proposition 3.1, we know that $\mathcal{P}$ is nonempty.

Proposition 4.2. *Assume that* (f_1) -(f_4) *hold, and if* $q \in (4, 6)$ *or* $q \in (2, 4]$ *and* D *is sufficiently large, then there exists a minimizer* $u \in \mathcal{P}$ *of* $I|_{\mathcal{P}}$ *(*I *is defined on the manifold* $\mathcal{P}$*). Moreover,* u *is positive and* $I'(u) = 0$ *in* $H^1(\mathbb{R}^3)$.

Proof Similarly to [32], the proof will be developed in several steps.

Step 1. There exists a positive constant C such that $\|u\| \geq C$ for all $u \in \mathcal{P}$. For every $u \in \mathcal{P}$ and $0 < \epsilon < 1$, by (3.1), there exists $C_\epsilon > 0$ such that

$$\begin{aligned} 0 = J^*(u) &= \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + \frac{3m}{2}\int_{\mathbb{R}^2} u^2 - 3\int_{\mathbb{R}^3} G(u) \\ &\geq \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3m}{2}(1-\epsilon)\int_{\mathbb{R}^3} u^2 - \frac{1}{2}C_\epsilon \int_{\mathbb{R}^3} u^6 \qquad (4.1) \\ &\geq \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 - C_\epsilon C(\int_{\mathbb{R}^3} |\nabla u|^2)^3. \end{aligned}$$

We recall that $u \neq 0$ whenever $u \in \mathcal{P}$ and (4.1) implies

$$\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \geq \frac{a}{2CC_\epsilon} > 0. \qquad (4.2)$$

Hence every limit point of a sequence in the Pohozaev manifold is different from zero.

Step 2. We claim that I is bounded from below on $\mathcal{P}$. For every $u \in \mathcal{P}$, the following formula holds:

$$I(u) = I(u) - \frac{1}{3}J^*(u) = \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{12}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2. \qquad (4.3)$$

It follows from that (4.3) and (4.2) that

$$I(u) = \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{12}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 > \frac{ab}{24C_\epsilon C} > 0. \qquad (4.4)$$

for all $u \in \mathcal{P}$. Therefore, I is bounded from below on $\mathcal{P}$.

Define

$$l = \inf\{I(u), u \in \mathcal{P}\}$$

and $l > 0$. Let a sequence $\{u_n\} \subset \mathcal{P}$ be such that $I(u_n) \to l$. Let u_n^* denote the Schwarz spherical rearrangement of $|u_n|$ (the definition and some properties of the Schwarz symmetrization are recalled in [7]). One has $\int_{\mathbb{R}^3} F(u_n^*) = \int_{\mathbb{R}^3} F(u_n)$, and $\int_{\mathbb{R}^3} |\nabla u_n^*|^2 \leq \int_{\mathbb{R}^3} |\nabla u_n|^2$. This means that

$$\begin{aligned} &\frac{a}{2}\int_{\mathbb{R}^3} |\nabla u_n^*|^2 + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla u_n^*|^2\right)^2 - 3\int_{\mathbb{R}^3} F(u_n^*) \\ &\leq \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 - 3\int_{\mathbb{R}^3} F(u_n) \\ &= 0. \end{aligned}$$

Hence, for each $n \in \mathbb{N}$, we can find a positive constant $0 < \omega_n \leq 1$ such that

$w_n(x) := u_n^*(\frac{x}{\omega_n})$ solves the following equation

$$\begin{aligned}
&\frac{a}{2}\int_{\mathbb{R}^3}|\nabla w_n|^2+\frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla w_n|^2\right)^2-3\int_{\mathbb{R}^3}F(w_n)\\
&=\frac{a\omega_n}{2}\int_{\mathbb{R}^3}|\nabla u_n^*|^2+\frac{b\omega_n^2}{2}\left(\int_{\mathbb{R}^3}|\nabla u_n^*|^2\right)^2-3\omega_n^3\int_{\mathbb{R}^3}F(u_n^*)\\
&=0.
\end{aligned}$$

Obviously, $\{w_n\}\subset\mathcal{P}$. From $\int_{\mathbb{R}^3}F(u_n^*)=\int_{\mathbb{R}^3}F(u_n)$, it is easy to conclude that

$$\begin{aligned}
I(w_n)&=\frac{a\omega_n}{2}\int_{\mathbb{R}^3}|\nabla u_n^*|^2+\frac{b\omega_n^2}{4}\left(\int_{\mathbb{R}^3}|\nabla u_n^*|^2\right)^2-\omega_n^3\int_{\mathbb{R}^3}F(u_n^*)\\
&\le\frac{a\omega_n}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2+\frac{b\omega_n^2}{4}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2-\omega_n^3\int_{\mathbb{R}^3}F(u_n)\\
&\le\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2-\int_{\mathbb{R}^3}F(u_n)=I(u_n).
\end{aligned}$$

Combining the above inequality and $I(u_n)\to l$, we have $I(w_n)\to l$. Replacing $\{w_n\}$ by $\{u_n\}$, we will assume henceforth that, for all $n\in\mathbb{N}$, u_n is nonnegative and spherically symmetric.

Similarly to the proof of boundedness in Proposition 3.1. we can prove that $\{u_n\}$ is bounded. Assume, passing to a subsequence, that $u_n\rightharpoonup u$ in $\mathcal{P}$.

Step 3. We claim that $u\in\mathcal{P}$ and $u_n\to u$ strongly in $H^1(\mathbb{R}^3)$. Thus, it is clear that $I|_{\mathcal{P}}$ attains its minimum at u. Actually, in view of Proposition 3.1, we know that problem (K) has a mountain pass solution u^* satisfying $0<I(u)=c_1<c_1^*$, where $c_1^*=\frac{ab}{4}S^3+\frac{[b^2S^4+4aS]^{\frac{3}{2}}}{24}+\frac{b^3S^6}{24}$. Hence, it is easy to see that $l<c_1^*$. By (3.19), the lower semi-continuity of the $\mathcal{D}^{1,2}(\mathbb{R}^3)$-norm and the Fatou lemma, we have

$$\begin{aligned}
&\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2+\frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2+\frac{3m}{2}\int_{\mathbb{R}^3}u^2-3\int_{\mathbb{R}^3}H(u)\\
&\le\liminf_{n\to+\infty}\left(\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2+\frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2+\frac{3m}{2}\int_{\mathbb{R}^3}u_n^2-3\int_{\mathbb{R}^3}H(u_n)\right)\\
&=\frac{1}{2}\liminf_{n\to+\infty}\int_{\mathbb{R}^3}u_n^6.
\end{aligned}$$

Therefore, we have

$$J^*(u)\le\frac{1}{2}\liminf_{n\to+\infty}\int_{\mathbb{R}^3}(u_n^6-u^6). \tag{4.5}$$

If $J^*(u)<0$, then, from **Step 2** we know that there exists $0<\theta<1$ such that $\bar{u}=u(\frac{\cdot}{\theta})\in\mathcal{P}$. Using the lower semi-continuity of the $\mathcal{D}^{1,2}(\mathbb{R}^3)$-norm and

(4.3), we infer that

$$\begin{aligned}I(\bar u) = I(\bar u) - \frac{1}{3}J^*(\bar u) &= \frac{a}{3}\int_{\mathbb{R}^3}|\nabla \bar u|^2 + \frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla \bar u|^2\right)^2\\ &= \frac{a\theta}{3}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{b\theta^2}{12}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2\\ &< \liminf_{n\to+\infty}\left(\frac{a}{3}\int_{\mathbb{R}^3}|\nabla u_n|^2 + \frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2\right) = \lim_{n\to+\infty} I(u_n) = l,\end{aligned}$$

which contradicts with $I(\bar u) \geq l$. Therefore, from (4.5), we infer that $J^*(u) \in [0, \frac{1}{2}\liminf\limits_{n\to+\infty}\int_{\mathbb{R}^3}(u_n^6 - u^6)]$. Set $v_n := u_n - u$, then $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. It follows from (3.17) and (3.19) that

$$\begin{aligned}0 &\geq J^*(u_n) - J^*(u)\\ &= \frac{a}{2}\int_{\mathbb{R}^3}|\nabla v_n|^2 + \frac{3m}{2}\int_{\mathbb{R}^3} v_n^2\\ &+ \frac{b}{2}\left(\left(\int_{\mathbb{R}^3}|\nabla v_n|^2\right)^2 + 2\int_{\mathbb{R}^3}|\nabla v_n|^2\int_{\mathbb{R}^3}|\nabla u|^2\right) - \frac{1}{2}\int_{\mathbb{R}^3} v_n^6 + o(1).\end{aligned}$$

Hence,

$$a\int_{\mathbb{R}^3}|\nabla v_n|^2 + b\left(\int_{\mathbb{R}^3}|\nabla v_n|^2\right)^2 + 2b\int_{\mathbb{R}^3}|\nabla v_n|^2\int_{\mathbb{R}^3}|\nabla u|^2 - \int_{\mathbb{R}^3} v_n^6 \leq o(1). \quad (4.6)$$

Up to a subsequence, we may assume that there exists $l_i \geq 0 (i = 1, 2, 3)$ such that

$$a\int_{\mathbb{R}^3}|\nabla v_n|^2 \to l_1, \quad b(\int_{\mathbb{R}^3}|\nabla v_n|^2)^2 + 2b\int_{\mathbb{R}^3}|\nabla v_n|^2\int_{\mathbb{R}^3}|\nabla u|^2 \to l_2, \quad \int_{\mathbb{R}^3}|v_n|^6 \to l_3,$$

then $l_1 + l_2 \leq l_3$. If $l_1 > 0$, then $l_2, l_3 > 0$. By (4.6) and the definition of I, we conclude that

$$\begin{aligned}I(u_n) =& I(u_n) - \frac{1}{3}J^*(u_n) = \frac{a}{3}\int_{\mathbb{R}^3}|\nabla u_n|^2 + \frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2\\ =& \frac{a}{3}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2\\ &+ \frac{b}{12}\left(\left(\int_{\mathbb{R}^3}|\nabla v_n|^2\right)^2 + 2\int_{\mathbb{R}^3}|\nabla v_n|^2\int_{\mathbb{R}^3}|\nabla u|^2\right) + \frac{a}{3}\int_{\mathbb{R}^3}|\nabla v_n|^2 + o(1).\end{aligned}$$

Letting $n \to \infty$, we have that

$$c_1^* > l = \frac{a}{3}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2 + \frac{1}{3}l_1 + \frac{1}{12}l_2.$$

Similarly to the proof of (3.22), we see that

$$c_1^* > l = \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{12}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + c_1^*.$$

Obviously, it is impossible. Therefore, $l_1 = 0$, i.e. $\|v_n\| \to 0$, hence $u_n \to u$ in $H^1(\mathbb{R}^3)$ and $I(u) = l, u \in \mathcal{P}$.

Step 4. We claim that $J'(u) \neq 0$, where $u \in \mathcal{P}$ is the minimum point of $I|_{\mathcal{P}}$ found above. Again reasoning by contradiction, suppose that $J'(u) = 0$. Define

$$A = a\int_{\mathbb{R}^3} |\nabla u|^2, \quad B = b(\int_{\mathbb{R}^3} |\nabla u|^2)^2, \quad Z = \int_{\mathbb{R}^3} F(u). \tag{4.7}$$

In a weak sense, the equation $J'(u) = 0$ can be written as

$$-(a + 2b\int_{\mathbb{R}^3} |\nabla u|^2)\triangle u - 3f(u) = 0. \tag{4.8}$$

Then, the following equalities hold:

$$\begin{cases} \frac{1}{2}A + \frac{1}{2}B - 3Z = 0, \\ \frac{1}{2}A + B - 9Z = 0. \end{cases} \tag{4.9}$$

The first equation comes from the fact that $J(u) = 0$. The second one is the Pohozaev equality (see [7]) applied to (4.8). From (4.9), we get $A + \frac{B}{2} = 0$. It is not possible. Therefore $J'(u) \neq 0$.

Step 5. We show that $I'(u) = 0$ in $H^1(\mathbb{R}^3)$. Thanks to the Lagrange multiplier rule, there exists $\mu \in \mathbb{R}$ so that $I'(u) = \mu J'(u)$. We claim that $\mu = 0$. As above, the equation $I'(u) = \mu J'(u)$ can be written, in a weak sense, as

$$-(a + b\int_{\mathbb{R}^3} |\nabla u|^2)\triangle u - f(u) = \mu[-(a + 2b\int_{\mathbb{R}^3} |\nabla u|^2)\triangle u - 3f(u)].$$

Then u solves the equation

$$-[(\mu - 1)a + (2\mu - 1)b\int_{\mathbb{R}^3} |\nabla u|^2)]\triangle u - (3\mu - 1)f(u) = 0. \tag{4.10}$$

Recall the definitions of A, B, Z; arguing as above, we have

$$\begin{cases} \frac{1}{2}A + \frac{1}{2}B - 3Z = 0, \\ \frac{\mu-1}{2}A + \frac{2\mu-1}{2}B - 3(3\mu - 1)Z = 0. \end{cases} \tag{4.11}$$

The first equation holds since $J(u) = 0$. The second one is the Pohozaev equality (see [7]) applied to (4.10). It follows from (4.9) that $\mu(A + \frac{B}{2}) = 0$. Obviously, $\mu = 0$. Therefore, $I'(u) = 0$ in $H^1(\mathbb{R}^3)$. The proof is complete. $\square$

5. Proof of Theorem 1.3

Let $u_0 \in H^1(\mathbb{R}^3)$ be the mountain pass solution of problem (K) satisfying $I(u_0) = c_1$, where c_1 has been defined in Theorem 2.1, then $u_1 \in \mathcal{P}$ and $c_1 \geq l$, where l is defined in Proposition 4.2. Let $u \in H^1(\mathbb{R}^3)$ be a ground state solution of problem (K) found in Proposition 4.2, then $I(u) = l$. It follows from Pohozaev type identity of I that $\lim_{t\to+\infty} I(u(\frac{\cdot}{t})) = -\infty$. So there is $t^* > 1$ such that $I(u(\frac{\cdot}{t^*})) < 0$. In the following, we define a continuous curve $\gamma \in \Gamma$: $\gamma : [0,1] \to E$ as

$$\gamma(t) = u(\frac{\cdot}{tt^*}), \quad 0 \leq t \leq 1.$$

So following the definition of c_1, we have $c_1 \leq I(u) = l$. Now it is easy to see that $l = c_1$. Furthermore, $u_0 \in H^1(\mathbb{R}^3)$ is also a ground state solution of problem (K). The proof is complete. □

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