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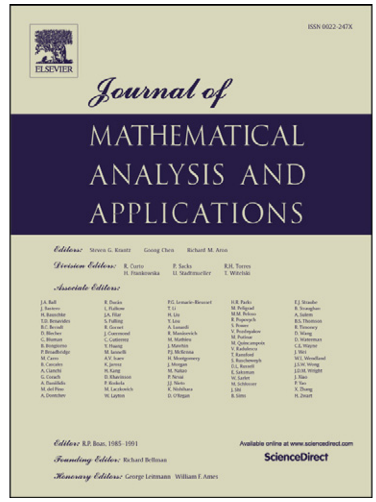

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Existence and Decay of Global Smooth Solutions to the Coupled Chemotaxis-Fluid Model*

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Abstract. In this paper, we consider a coupled chemotaxis-fluid model in $\mathbb{R}^3$, which describes the so-called "chemotaxis Boycott effect" arising from the interplay of chemotaxis and diffusion of nutrients or signaling chemicals in bacterial suspensions. It is shown that the Cauchy problem has a unique global-in-time solution $(n, c, u)(x, t)$ on $\mathbb{R}^3 \times (0, \infty)$, provided the invariant initial norm $\|(u_0, \nabla c_0)\|_{L^3} + \|n_0\|_{L^{3/2}}$ is suitably small, or the diffusion coefficients of cells, substrate and fluid (i.e. λ, ν, μ) are large enough. We also show that the invariant norm $\|(u, \nabla c)(t)\|_{L^3}^3 + \|n(t)\|_{L^{3/2}}^{3/2}$ is monotone decreasing in t for all $t \geq 0$.

Keywords. chemotaxis; Navier-Stokes; global existence; monotonicity

AMS Subject Classifications (2000). 35Q30, 35Q35, 76D03, 92C17

1 Introduction

Chemotaxis is a biological process in which cells (e.g., bacteria) move towards a chemically more favorable environment. For example, bacteria often swim towards higher concentration of oxygen to survive. A typical model describing chemotaxis is Keller-Segel equations which were derived by Keller and Segel in [10] and have been one of the best-known models in mathematical biology. In nature, cells often live in a viscous fluid, and thus, the biology of chemotaxis is intimately related to the surrounding physics. In other words, the cells and chemical substrates are always transported with the fluid, and meanwhile, the motion of the fluid is also affected by gravitational forcing generated by aggregation of cells. The motion of the fluid is usually determined by the well-known incompressible Navier-Stokes/Stokes equations. Recently, the following coupled Keller-Segel-Navier-Stokes (KSNS) equations were proposed in [22] to model the interaction of swimming bacteria, oxygen and viscous incompressible fluids on $\Omega \times \mathbb{R}^+$:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \lambda \Delta n - \nabla \cdot (\chi(c) n \nabla c), \\ \partial_t c + u \cdot \nabla c = \nu \Delta c - \kappa(c) n, \\ \partial_t u + u \cdot \nabla u + \nabla P = \mu \Delta u - n \nabla \phi, \\ \nabla \cdot u = 0 \end{cases} \tag{1.1}$$

with $t > 0$ and $x \in \Omega \subset \mathbb{R}^3$. Here, the unknown funcitons $n = n(t, x) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$, $c = c(t, x) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$, $u = u(t, x) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^3$, and $P = P(t, x) : \mathbb{R}^+ \times \Omega \to \mathbb{R}$

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are the cell density, the substrate (for example, oxygen) concentration, the velocity field and the pressure of the fluid, respectively. The positive constants λ, ν and μ are the corresponding diffusion coefficients of the cells, substrate and fluid. The function $\chi(c)$ denotes the chemotactic sensitivity and $\kappa(c)$ is the consumption rate of the substrate by the cells. The function $\phi = \phi(x)$ is a given potential function produced by different physical mechanism, e.g., the gravitational, centrifugal, electrically magnetic force, and so on.

It is clear from (1.1) that the coupling of chemotaxis and fluid is realized through both the transport of cells and chemical substrates $u\cdot\nabla n, u\cdot\nabla c$, and the external force $-n\nabla\phi$ exerted on the fluid by cells. In particular, if the chemotaxis effects are ignored, then it becomes the famous Navier-Stokes (NS) system governing the motion of an incompressible viscous fluid, which is one of the most important equations in fluid mechanics and has been extensively studied by many people (see, e.g., [12, 21, 13]). On the other hand, when the hydrodynamic effect is dropped, system (1.1) reduces to the classical Keller-Segel (KS) chemotaxis model (cf. [10, 11]). A huge number of studies in the literature have been devoted to KS model by postulating different biologically relevant chemotactic sensitivity function $\chi(c)$ and consumption rate function $\kappa(c)$, see the surveys [7, 8, 6].

For the coupled chemotaxis-fluid model (1.1) subject to large smooth data, the local existence of regular solutions in two and three spatial dimensions was established in [2, 15]. Under certain structural conditions on the consumption rate function $\kappa(\cdot)$ and the chemotactic sensitivity function $\chi(\cdot)$, the global existence of weak (resp. classical) solutions of (1.1) with large initial data was proved in dimensions three (resp. two) (see [4, 14, 2]). However, similar to that for the incompressible Navier-Stokes equations, the problem of regularity and uniqueness of weak solutions of three-dimensional equations is full of mathematical challenge and remains unknown. Here, we refer to [2, 3] for various blowup criteria of local classical solutions of (1.1). Recently, Duan, Lorz and Markowich [4] proved the global-in-time existence of smooth solutions of (1.1) when the initial data are close to the constant equibrium states in H^3. Chae, Kang and Lee [3] showed the global existence of classical solutions of the Keller-Segel-Stokes (KSS) equations (i.e., the convection term $u\cdot\nabla u$ in $(1.1)_3$ is dropped in the case when the flow is slow), if either $\|c_0\|_{L^\infty}$ or $\|n_0\|_{L^{3/2}}$ is sufficiently small. A simplified model with $\kappa(c) = \chi(c) = 0$ was studied in [5] and the large-time behavior of large amplitude classical solutions was announced.

Let $\Omega = \mathbb{R}^3$. The main purpose of this paper is to study the global existence of classical solutions to the Cauchy problem of KSNS model (1.1) with the following initial data

$$(u, n, c)(0, x) = (u_0, n_0, c_0)(x), \quad x \in \mathbb{R}^3. \tag{1.2}$$

To state our main result precisely, we define

$$\mathcal{E}(t) \triangleq \int_{\mathbb{R}^3} \left(|u(t)|^3 + |\nabla c(t)|^3 + |n(t)|^{3/2}\right) dx, \tag{1.3}$$

where (u, P, c, n) is a solution of (1.1), (1.2). Indeed, let

$$\begin{aligned} u_R(t,x) \triangleq Ru(R^2t, Rx), \quad n_R(t,x) \triangleq R^2 n(R^2t, Rx), \quad c_R(t,x) \triangleq c(R^2t, Rx),\\ P_R(t,x) \triangleq R^2 P(R^2t, Rx), \quad \text{and} \quad \phi_R(t,x) \triangleq \phi(R^2t, Rx). \end{aligned} \tag{1.4}$$

Then it is easily checked that (1.1) and (1.3) are scaling invariant under (1.4). So, similar to the classical result of Navier-Stokes equations due to Kato [9], it is natural to seek global classical solutions of (1.1), (1.2) with small initial energy $\mathcal{E}_0 \triangleq \mathcal{E}(0)$.

Our main result in this paper is the following theorem.

Theorem 1.1 *(i) Assume that* $\phi \in H^2(\mathbb{R}^3), \chi(\cdot) \in C^2([0,\infty))$ *and*

$$\kappa(\cdot) \in C^1([0,\infty)) \quad with \quad \kappa(0)=0, \quad \kappa'(c) \geq 0 \quad for \quad c \geq 0. \tag{1.5}$$

Then for given initial data $(n_0, c_0, u_0) \in H^2 \times H^2 \times H^2$ *satisfying* $n_0, c_0 \geq 0$*, there exists a unique strong solution* $(n,c,u)(x,t)$ *of (1.1), (1.2) on* $\mathbb{R}^3 \times (0,\infty)$ *such that* $\mathcal{E}(t)$ *is monotone decreasing in* t *(i.e.,* $\mathcal{E}'(t) \leq 0$*),*

$$(n,c,u) \in L^\infty(0,\infty; H^2(\mathbb{R}^3)), \quad (\nabla n, \nabla c, \nabla u) \in L^2(0,\infty; H^2(\mathbb{R}^3)), \tag{1.6}$$

and there exists a positive constant K *independent of* t *such that for any* $t \geq 0$*,*

$$\begin{cases} \|(\nabla c, \nabla u)(t)\|_{L^2}^2 \leq K(1+t)^{-1}, \\ \|\nabla^l n(t)\|_{L^2}^2 \leq K(1+t)^{-(l+\alpha)}, \\ \|(\Delta c, \Delta u)(t)\|_{L^2}^2 \leq K(1+t)^{-(3+2\alpha)/2} \end{cases} \tag{1.7}$$

with $l = 0,1,2$ *and* $\alpha \in [0,1/2)$*, provided*

$$\mathcal{E}_0 \triangleq \mathcal{E}(0) = \int \left(n_0^{3/2} + |\nabla c_0|^3 + |u_0|^3 \right) dx \leq \min\left\{ \frac{\Lambda_0}{2}, \tilde{\Lambda}_0 \right\}, \tag{1.8}$$

where

$$\Lambda_0 \triangleq \min\Bigg\{ \left(\frac{\lambda}{4C_1\left(\lambda^{-1}M^2 + \nu^{-1}M^2 + \mu^{-1}\|\nabla\phi\|_{L^3}^2\right)} \right)^{3/2}, \left(\frac{\lambda}{4C_2\|\nabla\phi\|_{L^3}} \right)^3, \\ \left(\frac{\nu}{2C_3\nu^{-1}} \right)^{3/2}, \left(\frac{\mu}{4C_4\|\nabla\phi\|_{L^3}} \right)^3, \left(\frac{\mu}{4C_5\mu^{-1}} \right)^{3/2} \Bigg\}, \tag{1.9}$$

and

$$\tilde{\Lambda}_0 \triangleq \min\left\{ \left(\frac{\lambda}{2C_6 M} \right)^3, \left(\frac{\mu}{2C_7} \right)^3, \left(\frac{\nu}{2C_8} \right)^3 \right\}, \tag{1.10}$$

where M *is a positive constant depending only on* $\|c_0\|_{L^\infty}$ *(such that* $\chi(c), \kappa(c) \leq M$ *for all* $0 \leq c \leq \|c_0\|_{L^\infty}$*), and* C_i $(i = 1, \ldots, 8)$ *are absolutely positive constants depending only on various Sobolev's constants.*

(ii) Assume further that $\chi(\cdot) \in C^k([0,\infty))$*,* $\kappa(\cdot) \in C^{k-1}([0,\infty))$ *and* $\phi \in H^k(\mathbb{R}^3)$ *with* $3 \leq k \in \mathbb{Z}^+$*. Then for given initial data* $(n_0, c_0, u_0) \in H^k \times H^k \times H^k$ *with* $3 \leq k \in \mathbb{Z}^+$*, there exists a unique global smooth solutions of (1.1), (1.2) satisfying (1.6), (1.7) and*

$$(n,c,u) \in L^\infty(0,\infty; H^k(\mathbb{R}^3)), \quad (\nabla n, \nabla c, \nabla u) \in L^2(0,\infty; H^k(\mathbb{R}^3)), \tag{1.11}$$

provided (1.8), together (1.9) and (1.10), holds.

Remark 1.1 It is easily seen from (1.8)–(1.10) that the given initial data can be arbitrarily large provided λ, ν and μ are large enough. Although the initial norm $\mathcal{E}_0$ is suitably small, yet the oscillations of the solutions can be arbitrary large.

Remark 1.2 There is no any smallness condition on the potential function $\phi(x)$, nor structural conditions on the chemotactic sensitivity and consumption rate functions $\chi(c), \kappa(c)$.

Remark 1.3 The decay of $\|n\|_{L^2}$ is mainly due to the boundedness of $\|n_0\|_{L^{3/2}}$.

Remark 1.4 Since the proof of Theorem 1.1 only relies on energy estimates, we expect that the method herein can be extended to deal with the case of quasilinear degenerate diffusion Δn^m as the one considered in [24, 25]. However, this is left for further investigation.

Theorem 1.1 will be proved by combining the local existence theorem (cf. [2, Theorem 1]) with the global a priori estimates established in sections 2 and 3.

2 A priori estimates

This section is devoted to the global a priori estimates of the solutions to the problem (1.1), (1.2). To do this, we assume that the conditions of (i) Theorem 1.1 hold and $(u, n, c)(t, x)$ is a smooth solution of (1.1), (1.2) defined on $\mathbb{R}^3 \times [0, T)$.

First of all, it is easy to get that

$$n(t,x) \geq 0, \quad 0 \leq c(t,x) \leq \|c_0\|_{L^\infty} \quad \text{for all} \quad t \geq 0,\ x \in \mathbb{R}^3, \tag{2.1}$$

due to $(1.1)_1$, $(1.1)_2$ and the maximal principle. So,

$$0 \leq \chi(c), \kappa(c) \leq M \quad \text{and} \quad 0 \leq \chi'(c), \kappa'(c) \leq \tilde{M}, \tag{2.2}$$

where $M, \tilde{M}$ are positive constants depending only on $\|c_0\|_{L^\infty}$.

The proof of Theorem 1.1 is based on the following key a priori estimates which imply that the quantity $\mathcal{E}(t)$ is monotone decreasing in t, provided $\mathcal{E}_0$ is suitably small.

Lemma 2.1 *Let (n, c, u) be a smooth solution of (1.1), (1.2) on $\mathbb{R}^3 \times [0, T)$. Then $\mathcal{E}(t)$ is decreasing in t on $[0, T)$, and*

$$\mathcal{E}(t) + \frac{1}{2}\int_0^t \int \Big(\lambda n^{-1/2}|\nabla n|^2 + \nu|\nabla c||\nabla^2 c|^2 + \mu|u||\nabla u|^2\Big)\, dxds \leq \mathcal{E}_0, \tag{2.3}$$

provided

$$\mathcal{E}_0 \triangleq \int \Big(n_0^{3/2} + |\nabla c_0|^3 + |u_0|^3\Big)\, dx \leq \frac{\Lambda_0}{2}, \tag{2.4}$$

where Λ_0 is a positive constant defined as (1.9).

Proof. Multiplying $(1.1)_1$, $\nabla(1.1)_2$ and $(1.1)_3$ by $n^{1/2}$, $|\nabla c|\nabla c$ and $|u|u$ respectively, integrating by parts, and adding them together, by $(1.1)_4$, (2.2) we deduce that

$$\begin{aligned}
&\frac{d}{dt}\int \Big(n^{3/2} + |\nabla c|^3 + |u|^3\Big)\, dx + \int \Big(\lambda n^{-1/2}|\nabla n|^2 + \nu|\nabla c||\nabla^2 c|^2 + \mu|u||\nabla u|^2\Big)\, dx \\
&\leq C\int \Big(Mn^{1/2}|\nabla c||\nabla n| + Mn|\nabla c||\nabla^2 c| + |u||\nabla c|^2|\nabla^2 c| + n|\nabla\phi||u|^2 + |P||u||\nabla u|\Big)\, dx \\
&\leq \frac{1}{2}\int \Big(\lambda n^{-1/2}|\nabla n|^2 + \nu|\nabla c||\nabla^2 c|^2 + \mu|u||\nabla u|^2\Big)\, dx + C\lambda^{-1}M^2\int n^{3/2}|\nabla c|^2 dx \\
&\quad + C\nu^{-1}\int \big(M^2 n^2|\nabla c| + |u|^2|\nabla c|^3\big)\, dx + C\int n|\nabla\phi||u|^2 dx + C\mu^{-1}\int |P|^2|u| dx \\
&\triangleq \frac{1}{2}\int \Big(\lambda n^{-1/2}|\nabla n|^2 + \nu|\nabla c||\nabla^2 c|^2 + \mu|u||\nabla u|^2\Big)\, dx + \sum_{i=1}^4 I_i,
\end{aligned} \tag{2.5}$$

where $C > 0$ is an absolute constant independent of λ, ν and μ.

We are now in a position of estimating the terms on the right-hand side of (2.5). First, by the Hölder and Sobolev inequalities we have

$$\begin{aligned} I_1 &\le C\lambda^{-1}M^2\|n\|_{L^{9/2}}^{3/2}\|\nabla c\|_{L^3}^2 = C\lambda^{-1}M^2\|n^{3/4}\|_{L^6}^2\|\nabla c\|_{L^3}^2 \\ &\le C\lambda^{-1}M^2\|\nabla c\|_{L^3}^2\int n^{-1/2}|\nabla n|^2dx. \end{aligned} \tag{2.6}$$

Similarly, we have

$$\begin{aligned} I_2 &\le C\nu^{-1}M^2\|n\|_{L^{3/2}}^{1/2}\|n\|_{L^{9/2}}^{3/2}\|\nabla c\|_{L^3} + C\nu^{-1}\|u\|_{L^3}^2\|\nabla c\|_{L^9}^3 \\ &\le C\nu^{-1}M^2\|\nabla c\|_{L^3}\|n\|_{L^{3/2}}^{1/2}\int n^{-1/2}|\nabla n|^2dx + C\nu^{-1}\|u\|_{L^3}^2\int|\nabla c||\nabla^2 c|^2dx \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} I_3 &\le C\|\nabla\phi\|_{L^3}\|n\|_{L^{3/2}}^{1/4}\|n\|_{L^{9/2}}^{3/4}\|u\|_{L^3}^{1/2}\|u\|_{L^9}^{3/2} \\ &\le C\|\nabla\phi\|_{L^3}\|n\|_{L^{3/2}}^{1/2}\int n^{-1/2}|\nabla n|^2dx + C\|\nabla\phi\|_{L^3}\|u\|_{L^3}\int|u||\nabla u|^2dx. \end{aligned} \tag{2.8}$$

Finally, operating $\nabla\cdot$ to both sides of $(1.1)_3$ and using $(1.1)_4$, we find

$$-\Delta P = \nabla\cdot\nabla\cdot(u\otimes u) + \nabla\cdot(n\nabla\phi). \tag{2.9}$$

So, in view of the Calderón-Zygmund theorem (cf. [18]) and Hardy-Littlewood-Sobolev inequality (see [19, Theorem 1 on Page 119]), we infer from (2.9) that

$$\begin{aligned} \|P\|_{L^3} &\le C\|u\|_{L^6}^2 + C\|n\nabla\phi\|_{L^{3/2}} \le C\|u\|_{L^3}^{1/2}\|u\|_{L^9}^{3/2} + C\|\nabla\phi\|_{L^3}\|n\|_{L^{3/2}}^{1/4}\|n\|_{L^{9/2}}^{3/4} \\ &\le C\|u\|_{L^3}^{1/2}\left(\int|u||\nabla u|^2dx\right)^{1/2} + C\|\nabla\phi\|_{L^3}\|n\|_{L^{3/2}}^{1/4}\left(\int n^{-1/2}|\nabla n|^2dx\right)^{1/2}, \end{aligned}$$

and consequently,

$$\begin{aligned} I_4 &\le C\mu^{-1}\|u\|_{L^3}\|P\|_{L^3}^2 \\ &\le C\mu^{-1}\|u\|_{L^3}^2\int|u||\nabla u|^2dx + C\mu^{-1}\|\nabla\phi\|_{L^3}^2\|u\|_{L^3}\|n\|_{L^{3/2}}^{1/2}\int n^{-1/2}|\nabla n|^2dx. \end{aligned} \tag{2.10}$$

Thus, substituting (2.6)–(2.8) and (2.10) into (2.5), we arrive at

$$\begin{aligned} &\frac{d\mathcal{E}(t)}{dt} + [\lambda-\alpha(t)]\int n^{-1/2}|\nabla n|^2dx \\ &\quad + [\nu-\beta(t)]\int|\nabla c||\nabla^2 c|^2dx + [\mu-\gamma(t)]\int|u||\nabla u|^2dx \le 0, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} \alpha(t) &\triangleq C\lambda^{-1}M^2\|\nabla c(t)\|_{L^3}^2 + C\nu^{-1}M^2\|\nabla c(t)\|_{L^3}\|n(t)\|_{L^{3/2}}^{1/2} \\ &\quad + C\|\nabla\phi\|_{L^3}\|n(t)\|_{L^{3/2}}^{1/2} + C\mu^{-1}\|\nabla\phi\|_{L^3}^2\|u(t)\|_{L^3}\|n(t)\|_{L^{3/2}}^{1/2}, \end{aligned}$$

$$\beta(t) \triangleq C\nu^{-1}\|u(t)\|_{L^3}^2, \quad \text{and} \quad \gamma(t) \triangleq C\|\nabla\phi\|_{L^3}\|u(t)\|_{L^3} + C\mu^{-1}\|u(t)\|_{L^3}^2$$

with $C > 0$ an absolute constant depending only on various Sobolev's constants. Recalling the definition of $\mathcal{E}(t)$, one easily gets that

$$\alpha(t) \le C_1 \left(\lambda^{-1}M^2 + \nu^{-1}M^2 + \mu^{-1}\|\nabla\phi\|_{L^3}^2\right)\mathcal{E}(t)^{2/3} + C_2\|\nabla\phi\|_{L^3}\mathcal{E}(t)^{1/3},$$

$$\beta(t) \le C_3\nu^{-1}\mathcal{E}(t)^{2/3}, \quad \text{and} \quad \gamma(t) \le C_4\|\nabla\phi\|_{L^3}\mathcal{E}(t)^{1/3} + C_5\mu^{-1}\mathcal{E}(t)^{2/3}.$$

Let $\Lambda_0 > 0$ be the positive constant defined in (1.9). To be continued, we set

$$T_{\max} \triangleq \sup\left\{t \in [0,T) \mid \mathcal{E}(s) \le \Lambda_0,\ 0 \le s \le t\right\}.$$

Due to (2.4), $\mathcal{E}(0) = \mathcal{E}_0 \le \Lambda_0/2 < \Lambda_0$. Thus, by continuity arguments we have $T_{\max} > 0$ and

$$\mathcal{E}(t) \le \Lambda_0 \quad \text{for all} \quad 0 \le t \le T_{\max}, \quad \text{and} \quad \mathcal{E}(T_{\max}) = \Lambda_0. \tag{2.12}$$

Suppose $T_{\max} < T$. Then it follows from (2.11) that for all $0 \le t \le T_{\max}$,

$$\frac{d\mathcal{E}(t)}{dt} \le \frac{d\mathcal{E}(t)}{dt} + \int \left(\frac{\lambda}{2}n^{-1/2}|\nabla n|^2 + \frac{\nu}{2}|\nabla c||\nabla^2 c|^2 + \frac{\mu}{2}|u||\nabla u|^2\right)dx \le 0, \tag{2.13}$$

which shows that $\mathcal{E}(t)$ is decreasing on $[0, T_{\max}]$. Consequently,

$$\mathcal{E}(t) \le \mathcal{E}_0 \le \frac{\Lambda_0}{2} < \Lambda_0.$$

This contradicts (2.12). Therefore, $T_{\max} = T$ and $\mathcal{E}(t)$ is decreasing on $[0,T)$. By (2.13), we obtain the desired estimate (2.3) of Lemma 2.1. □

Based on Lemma 2.1, we have the following uniform estimates for (n,c,u).

Lemma 2.2 *Let (n,c,u) be a smooth solution of (1.1), (1.2) on $\mathbb{R}^3 \times [0,T)$. Then,*

$$\begin{aligned}
&\sup_{0\le t<T}\left(\|n\|_{L^2}^2 + \|c\|_{H^1}^2 + \|u\|_{H^1}^2\right) + \int_0^T \left(\lambda\|\nabla n\|_{L^2}^2 + \nu\|\nabla c\|_{H^1}^2 + \mu\|\nabla u\|_{H^1}^2\right)dt \\
&\quad \le C\left(\|n_0\|_{L^2}^2 + \|c_0\|_{L^2}^2 + \|u_0\|_{H^1}^2 + \|\nabla c_0\|_{L^2}^2 \exp\{C\tilde{M}^2\mathcal{E}_0\} + \mathcal{E}_0\|\nabla\phi\|_{H^1}^2\right),
\end{aligned} \tag{2.14}$$

provided

$$\mathcal{E}_0 \le \Lambda \triangleq \min\left\{\frac{\Lambda_0}{2}, \left(\frac{\lambda}{2C_6 M}\right)^3, \left(\frac{\mu}{2C_7}\right)^3, \left(\frac{\nu}{2C_8}\right)^3\right\}, \tag{2.15}$$

where C and C_i $(i = 6,7,8)$ are absolutely positive constants.

Proof. First, multiplying $(1.1)_1$, $(1.1)_2$ and $(1.1)_3$ by n, c, u in L^2 respectively, after integrating by parts we deduce from $(1.1)_4$ and (2.1)–(2.3) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|n\|_{L^2}^2 + \|c\|_{L^2}^2 + \|u\|_{L^2}^2\right) + \left(\lambda\|\nabla n\|_{L^2}^2 + \nu\|\nabla c\|_{L^2}^2 + \mu\|\nabla u\|_{L^2}^2\right) \\
&\quad \le CM\|n\|_{L^6}\|\nabla c\|_{L^3}\|\nabla n\|_{L^2} + C\|n\|_{L^{3/2}}^{1/4}\|n\|_{L^{9/2}}^{3/4}\|\nabla\phi\|_{L^2}\|u\|_{L^6} \\
&\quad \le CM\|\nabla c\|_{L^3}\|\nabla n\|_{L^2}^2 + C\|n\|_{L^{3/2}}^{1/2}\|\nabla u\|_{L^2}^2 + C\|\nabla\phi\|_{L^2}^2\|n\|_{L^{9/2}}^{3/2} \\
&\quad \le C_6 M\mathcal{E}_0^{1/3}\|\nabla n\|_{L^2}^2 + C_7\mathcal{E}_0^{1/3}\|\nabla u\|_{L^2}^2 + C\|\nabla\phi\|_{L^2}^2\|n^{-1/4}\nabla n\|_{L^2}^2,
\end{aligned}$$

so that, it follows from (2.3) that

$$\sup_{0\le t<T}\left(\|n\|_{L^2}^2+\|c\|_{L^2}^2+\|u\|_{L^2}^2\right)+\int_0^T\left(\lambda\|\nabla n\|_{L^2}^2+\nu\|\nabla c\|_{L^2}^2+\mu\|\nabla u\|_{L^2}^2\right)dt \\ \le \left(\|n_0\|_{L^2}^2+\|c_0\|_{L^2}^2+\|u_0\|_{L^2}^2\right)+C\mathcal{E}_0\|\nabla\phi\|_{L^2}^2, \tag{2.16}$$

provided $\mathcal{E}_0$ satisfies

$$\mathcal{E}_0\le\Lambda_1\triangleq\min\left\{\frac{\Lambda_0}{2},\left(\frac{\lambda}{2C_6M}\right)^3,\left(\frac{\mu}{2C_7}\right)^3\right\}.$$

Due to $\kappa(0)=0$, it holds that $\kappa(c)\le\tilde{M}c$ with $\tilde{M}$ being the one in (2.2). So, multiplying $(1.1)_2$ by Δc in L^2 and integrating by parts, we obtain

$$\begin{aligned}&\frac12\frac{d}{dt}\|\nabla c\|_{L^2}^2+\nu\|\Delta c\|_{L^2}^2\\&\le C\int\left(|u||\nabla c||\Delta c|+\kappa(c)n|\Delta c|\right)dx\\&\le C\|u\|_{L^3}\|\nabla c\|_{L^6}\|\Delta c\|_{L^2}+C\tilde{M}\|c\|_{L^6}\|n\|_{L^{3/2}}^{1/4}\|n^{-1/4}\nabla n\|_{L^2}\|\Delta c\|_{L^2}\\&\le C\left(\|u\|_{L^3}+\|n\|_{L^{3/2}}^{1/2}\right)\|\nabla^2c\|_{L^2}^2+C\tilde{M}^2\|\nabla c\|_{L^2}^2\|n^{-1/4}\nabla n\|_{L^2}^2\\&\le C_8\mathcal{E}_0^{1/3}\|\nabla^2c\|_{L^2}^2+C\tilde{M}^2\|\nabla c\|_{L^2}^2\|n^{-1/4}\nabla n\|_{L^2}^2,\end{aligned}\tag{2.17}$$

so that, by Lemma 2.1 and Gronwall's inequality we know that

$$\sup_{0\le t<T}\|\nabla c\|_{L^2}^2+\int_0^T\nu\|\Delta c\|_{L^2}^2dt\le C\|\nabla c_0\|_{L^2}^2\exp\{C\tilde{M}^2\mathcal{E}_0\},\tag{2.18}$$

provided

$$\mathcal{E}_0\le\Lambda_2\triangleq\min\left\{\Lambda_1,\left(\frac{\nu}{2C_8}\right)^3\right\}.$$

In a similar manner, we also have

$$\begin{aligned}&\frac12\frac{d}{dt}\|\nabla u\|_{L^2}^2+\mu\|\Delta u\|_{L^2}^2\\&\le C\|u\|_{L^3}\|\nabla u\|_{L^6}\|\Delta u\|_{L^2}+C\|n\|_{L^{3/2}}^{1/4}\|n\|_{L^{9/2}}^{3/4}\|\nabla\phi\|_{L^6}\|\Delta u\|_{L^2}\\&\le C_9\mathcal{E}_0^{1/3}\|\nabla^2u\|_{L^2}^2+C\|\nabla\phi\|_{H^1}^2\|n^{-1/4}\nabla n\|_{L^2}^2,\end{aligned}$$

and consequently,

$$\sup_{0\le t<T}\|\nabla u\|_{L^2}^2+\int_0^T\mu\|\Delta u\|_{L^2}^2dt\le\|\nabla u_0\|_{L^2}^2+C\mathcal{E}_0\|\nabla\phi\|_{H^1}^2,\tag{2.19}$$

provided

$$\mathcal{E}_0\le\Lambda\triangleq\min\left\{\Lambda_2,\left(\frac{\mu}{2C_9}\right)^3\right\}.$$

Therefore, combining (2.16), (2.18) and (2.19), we obtain Lemma 2.2. □

3 Proof of Theorem 1.1

The section is devote to the proof of Theorem 1.1. To do this, we still need some global higher-order estimates which are necessary for the existence of strong/smooth solutions. So, from now on we assume that $\mathcal{E}_0 \leq \Lambda$ with $\Lambda > 0$ being the same one as in (2.15), so that, the global estimates established in section 2 hold. Throughout this section, for simplicity we use the same letter K to denote the positive constant, which may depend on $\lambda, \nu, \mu, \phi, \chi(\cdot), \kappa(\cdot)$ and the norms of (n_0, c_0, u_0), but is independent of T.

We start with the following estimates.

Lemma 3.1 *Assume that the conditions of (i) Theorem 1.1 hold. Let (n, c, u) be a smooth solution of (1.1), (1.2) on $\mathbb{R}^3 \times [0, T)$. Then,*

$$\sup_{0\leq t<T} \|(n,c,u)(t)\|_{H^2}^2 + \int_0^T \|(\nabla n, \nabla c, \nabla u)(t)\|_{H^2}^2 \leq K. \tag{3.1}$$

Proof. First, operating Δ to both sides of $(1.1)_2$, multiplying it by Δc in L^2, and integrating by parts, by (2.1)–(2.3) we find

$$\begin{aligned}
&\frac{d}{dt}\|\Delta c\|_{L^2}^2 + \|\nabla\Delta c\|_{L^2}^2 \\
&\leq K\int \left(|u|^2|\nabla^2 c|^2 + |\nabla u|^2|\nabla c|^2 + \kappa'(c)^2|n|^2|\nabla c|^2 + \kappa(c)^2|\nabla n|^2\right) dx \\
&\leq K\left(\|\nabla u\|_{L^2}^2\|\nabla^2 c\|_{L^2}\|\nabla\Delta c\|_{L^2} + \|n\|_{L^{3/2}}^{1/2}\|n\|_{L^{9/2}}^{3/2}\|\nabla c\|_{L^6}^2 + \|\nabla n\|_{L^2}^2\right) \\
&\leq \frac{1}{2}\|\nabla\Delta c\|_{L^2}^2 + K\|\nabla^2 c\|_{L^2}^2\left(\|\nabla u\|_{L^2}^4 + \|n^{-1/4}\nabla n\|_{L^2}^2\right) + K\|\nabla n\|_{L^2}^2.
\end{aligned} \tag{3.2}$$

Thanks to (2.3) and (2.14), one has

$$\int_0^T \left(\|\nabla u\|_{L^2}^4 + \|n^{-1/4}\nabla n\|_{L^2}^2 + \|\nabla n\|_{L^2}^2\right) dt \leq K + K\sup_{0\leq t<T}\|\nabla u\|_{L^2}^2 \leq K,$$

so that, it follows from (3.2) and Gronwall's inequality that

$$\sup_{0\leq t<T}\|\Delta c\|_{L^2}^2 + \int_0^T \|\nabla\Delta c\|_{L^2}^2 \leq K. \tag{3.3}$$

Next, multiplying $(1.1)_1$ by Δn in L^2 and integrating by parts give

$$\begin{aligned}
&\frac{d}{dt}\|\nabla n\|_{L^2}^2 + \|\Delta n\|_{L^2}^2 \\
&\leq K\int \left(|u|^2|\nabla n|^2 + |\nabla n|^2|\nabla c|^2 + |n|^2|\nabla^2 c|^2 + |n|^2|\nabla c|^4\right) dx \\
&\leq K\left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2 + \|\nabla c\|_{L^3}^2\|\nabla^2 c\|_{L^2}^2\right)\|\nabla n\|_{L^2}\|\nabla^2 n\|_{L^2} \\
&\leq \frac{1}{4}\|\nabla^2 n\|_{L^2}^2 + K\|\nabla n\|_{L^2}^2\left(\|\nabla u\|_{L^2}^4 + \|\nabla^2 c\|_{L^2}^4\right),
\end{aligned} \tag{3.4}$$

where we have used (2.1)–(2.3) and the following Sobolev's inequalities:

$$\|n\|_{L^\infty}^2 \leq K\|\nabla n\|_{L^2}\|\nabla^2 n\|_{L^2}, \quad \|\nabla c\|_{L^4}^2 \leq K\|\nabla c\|_{L^3}\|\nabla^2 c\|_{L^2}.$$

In view of (2.14) and (3.3), we have

$$\begin{aligned}&\int_0^T \left(\|\nabla u\|_{L^2}^4 + \|\nabla^2 c\|_{L^2}^4\right) dt\\&\le \sup_{0\le t<T} \left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2\right) \int_0^T \left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2\right) dt\\&\le K,\end{aligned}$$

and hence, we deduce from (3.4) that

$$\sup_{0\le t<T} \|\nabla n\|_{L^2}^2 + \int_0^T \|\Delta n\|_{L^2}^2 dt \le K. \tag{3.5}$$

Analogously to the derivation of (3.2), we have

$$\begin{aligned}&\frac{d}{dt}\|\Delta n\|_{L^2}^2 + \|\nabla \Delta n\|_{L^2}^2\\&\le K \int \left(|u|^2|\nabla^2 n|^2 + |\nabla u|^2|\nabla n|^2 + |\nabla c|^2|\nabla^2 n|^2 + |\nabla n|^2|\nabla^2 c|^2\right) dx\\&\quad +K \int \left(|\nabla n|^2|\nabla c|^4 + n^2|\nabla \Delta c|^2 + n^2|\nabla^2 c|^2|\nabla c|^2 + n^2|\nabla c|^6\right) dx\\&\le K \left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2 + \|\nabla c\|_{L^3}^2\|\nabla^2 c\|_{L^2}^2\right) \|\nabla^2 n\|_{L^2}\|\nabla \Delta n\|_{L^2}\\&\quad +K \left(\|\nabla \Delta c\|_{L^2}^2 + \|\nabla c\|_{L^3}^2\|\nabla \Delta c\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^6\right) \|\nabla n\|_{L^2}\|\nabla^2 n\|_{L^2}.\end{aligned} \tag{3.6}$$

By (2.3), (2.14), (3.3), (3.5) and the Cauchy-Schwarz inequality, we deduce from (3.6) that

$$\begin{aligned}\frac{d}{dt}\|\Delta n\|_{L^2}^2 + \|\nabla \Delta n\|_{L^2}^2 \le{}& K\|\nabla^2 n\|_{L^2}^2 \left(\|\nabla u\|_{L^2}^4 + \|\nabla^2 c\|_{L^2}^4 + \|\nabla \Delta c\|_{L^2}^2\right)\\&+K \left(\|\nabla \Delta c\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2\right),\end{aligned}$$

where we have used the facts that $\|\nabla n\|_{L^2}$ and $\|\nabla^2 c\|_{L^2}$ are uniformly bounded in time, due to (3.3) and (3.5). This, together with (2.14), (3.3) and Gronwall's inequality, yields

$$\sup_{0\le t<T} \|\Delta n\|_{L^2}^2 + \int_0^T \|\nabla \Delta n\|_{L^2}^2 dt \le K. \tag{3.7}$$

Noting that $n\nabla\phi = \nabla(n\phi) - \phi\nabla n$, we infer from $(1.1)_3$ that

$$\begin{aligned}&\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \|\nabla \Delta u\|_{L^2}^2\\&\le K \int \left(|u|^2|\nabla^2 u|^2 + |\nabla u|^4 + |\nabla^2 n|^2|\phi|^2 + |\nabla n|^2|\nabla \phi|^2\right) dx\\&\le K \left(\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}\|\nabla \Delta u\|_{L^2} + \|\nabla^2 n\|_{L^2}\|\nabla \Delta n\|_{L^2}\|\nabla \phi\|_{L^2}^2\right)\\&\le \frac{1}{2}\|\nabla \Delta u\|_{L^2}^2 + K\|\nabla^2 u\|_{L^2}^2\|\nabla u\|_{L^2}^4 + K\|\nabla^2 n\|_{L^2}\|\nabla \Delta n\|_{L^2},\end{aligned}$$

so that, by (2.14), (3.5) and (3.7) we see that

$$\sup_{0\le t<T} \|\Delta u\|_{L^2}^2 + \int_0^T \|\nabla \Delta u\|_{L^2}^2 dt \le K. \tag{3.8}$$

Now, (3.1) readily follows from (3.3), (3.5), (3.7), (3.8) and (2.14). □

To proceed, we need the following Moser-type calculus inequalities.

Lemma 3.2 *([16]) (Moser-Type Calculus Inequalities)*
(i) For $f, g \in H^s \cap L^\infty$ and $|\alpha| \le s$,

$$\|D^\alpha(fg)\|_{L^2} \le C(s)\left(\|f\|_{L^\infty}\|D^s g\|_{L^2} + \|g\|_{L^\infty}\|D^s f\|_{L^2}\right).$$

(ii) For $f \in H^s, Df \in L^\infty, g \in H^{s-1} \cap L^\infty$ and $|\alpha| \le s$,

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^2} \le C(s)\left(\|Df\|_{L^\infty}\|D^{s-1}g\|_{L^2} + \|g\|_{L^\infty}\|D^s f\|_{L^2}\right).$$

(iii) Assume that $h(v)$ with $v \in W$ is a smooth function on W, and that $u(x)$ is a continuous function satisfying $u(x) \in W_1$, $\bar{W}_1 \subset\subset W$ and $u \in L^\infty \cap H^s$. Then for $s \ge 1$,

$$\|D^s h(u)\|_{L^2} \le C(s)\|h'(u)\|_{C^{s-1}(\bar{W}_1)}\|u\|_{L^\infty}^{s-1}\|D^s u\|_{L^2}.$$

Based on Lemmas 3.1, 3.2 and induction arguments, we have

Lemma 3.3 *Assume that the conditions of (ii) Theorem 1.1 hold. Then,*

$$\sup_{0\le t<T}\|(n,c,u)(t)\|_{H^k}^2 + \int_0^T \|(\nabla n, \nabla c, \nabla u)\|_{H^k}^2 dt \le K. \tag{3.9}$$

Proof. The proof of (3.9) will be done by induction arguments. First, it follows from Lemmas 2.2 and 3.1 that (3.9) holds for $k = 2$. Now, assume that (3.9) holds for $k = m - 1$ with $m \ge 3$,

$$\sup_{0\le t<T}\|(n,c,u)(t)\|_{H^{m-1}}^2 + \int_0^T \|(\nabla n, \nabla c, \nabla u)\|_{H^{m-1}}^2 dt \le K. \tag{3.10}$$

Next, we shall show that (3.9) holds for $k = m$ by using the induction assumption (3.10). To do so, operating D^m with $m \ge 3$ to both sides of $(1.1)_1$, multiplying it by $D^m n$ in L^2, and integrating by parts, by (3.10) and Lemma 3.2 we obtain

$$\begin{aligned}
&\frac{d}{dt}\|D^m n\|_{L^2}^2 + \|\nabla D^m n\|_{L^2}^2 \\
&\quad \le K\left(\|u\|_{L^\infty}^2\|\nabla n\|_{H^{m-1}}^2 + \|\nabla n\|_{L^\infty}^2\|u\|_{H^{m-1}}^2\right) \\
&\qquad + K\left(\|n\|_{L^\infty}^2\|\nabla c\|_{H^m}^2 + \|\nabla c\|_{L^\infty}^2\|n\|_{H^m}^2\right) \\
&\quad \le K_1\left(\|\nabla n\|_{H^{m-1}}^2 + \|\nabla c\|_{H^m}^2 + \|\nabla c\|_{H^{m-1}}^2\|n\|_{H^m}^2\right),
\end{aligned} \tag{3.11}$$

where we have used $\|(n,c,u)\|_{L^\infty} \le K$ and $\|\nabla n\|_{L^\infty} \le K\|\nabla n\|_{H^{m-1}}$, due to Lemma 3.1, Sobolev embedding inequality and the fact that $m \ge 3$.

Similarly, it follows from $(1.1)_2$ that

$$\begin{aligned}
\frac{d}{dt}\|D^m c\|_{L^2}^2 + \|\nabla D^m c\|_{L^2}^2 &\le K\left(\|u\|_{L^\infty}^2\|\nabla c\|_{H^{m-1}}^2 + \|\nabla c\|_{L^\infty}^2\|u\|_{H^{m-1}}^2\right) \\
&\quad + K\left(\|n\|_{L^\infty}^2\|c\|_{H^{m-1}}^2 + \|c\|_{L^\infty}^2\|n\|_{H^{m-1}}^2\right) \\
&\le K_2\left(\|\nabla n\|_{H^{m-1}}^2 + \|\nabla c\|_{H^{m-1}}^2\right),
\end{aligned} \tag{3.12}$$

and using the fact that $n\nabla\phi = \nabla(n\phi) - \phi\nabla n$, one gets from $(1.1)_3$ that

$$\begin{aligned}
\frac{d}{dt}\|D^m u\|_{L^2}^2 + \|\nabla D^m u\|_{L^2}^2 &\le K\left(\|u\|_{L^\infty}^2\|\nabla u\|_{H^{m-1}}^2 + \|\nabla u\|_{L^\infty}^2\|u\|_{H^{m-1}}^2\right) \\
&\quad + K\left(\|\phi\|_{L^\infty}^2\|\nabla n\|_{H^{m-1}}^2 + \|\nabla n\|_{L^\infty}^2\|\phi\|_{H^{m-1}}^2\right)
\end{aligned}$$

$$\leq K_3 \left(\|\nabla n\|_{H^{m-1}}^2 + \|\nabla u\|_{H^{m-1}}^2 \right), \tag{3.13}$$

where we have also used $\|(n,c)\|_{L^\infty} \leq K\|(\nabla n, \nabla c)\|_{H^{m-1}}$ for any $m \geq 2$.

Thus, adding (3.11), $2K_1\times$(3.12) and (3.13) together, by (3.10) and Gronwall's inequality we arrive at

$$\sup_{0\leq t<T} \|(D^m n, D^m c, D^m u)(t)\|_{L^2}^2 + \int_0^T \|(\nabla D^m n, \nabla D^m c, \nabla D^m u)\|_{L^2}^2 dt \leq K,$$

which, together with (3.10), leads to (3.9). □

Proof of Theorem 1.1. Combining the local existence result [2, Theorem 1] and the global a priori estimates established in Lemmas 2.1, 2.2, 3.1 and 3.3, by continuity arguments we obtain the global existence result of (1.1), (1.2) stated in (i), (ii) of Theorem 1.1, provided $\mathcal{E}_0 \leq \Lambda$. The uniqueness of smooth solutions can be proved in a standard manner as the one in [2].

The decay estimates in (1.7) are immediate results of the following proposition.

Proposition 3.1 *Assume that $\chi(\cdot), \kappa(\cdot), \phi$ and the initial data (n_0, c_0, u_0) satisfy the conditions stated in (i) of Theorem 1.1. Let (u, n, c) be the unique strong solution of (1.1), (1.2) on $\mathbb{R}^3 \times (0,\infty)$. Then, there exists a positive constant K, independent of t, such that*

$$\begin{cases} \sup\limits_{t\geq 0} \left((1+t)\|(\nabla c, \nabla u)\|_{L^2}^2\right) + \displaystyle\int_0^\infty (1+t)\|(\Delta c, \Delta u)\|_{L^2}^2 dt \leq K, \\ \sup\limits_{t\geq 0} \left((1+t)^{l+\alpha}\|\nabla^l n\|_{L^2}^2\right) + \displaystyle\int_0^\infty (1+t)^{l+\alpha}\|\nabla^{l+1} n\|_{L^2}^2 dt \leq K, \\ \sup\limits_{t\geq 0} \left((1+t)^{(3+2\alpha)/2}\|(\Delta c, \Delta u)\|_{L^2}^2\right) + \displaystyle\int_0^\infty (1+t)^{(3+2\alpha)/2}\|(\nabla\Delta c, \nabla\Delta u)\|_{L^2}^2 dt \leq K, \end{cases} \tag{3.14}$$

where $l \in \{0,1,2\}$ and $\alpha \in [0,1/2)$.

Proof. First, since $\|n^{-1/4}\nabla n\|_{L^2}^2 \in L^1(0,\infty)$ due to (2.3), we easily deduce after multiplying (2.17) by $(1+t)$ and integrating it over $(0,\infty)$ that

$$\sup_{t\geq 0} \left((1+t)\|\nabla c\|_{L^2}^2\right) + \int_0^\infty (1+t)\|\Delta c\|_{L^2}^2 dt \leq K. \tag{3.15}$$

As for the decay of $\|n\|_{L^2}$, we multiply $(1.1)_1$ by n in L^2 and integrate by parts to get

$$\frac{d}{dt}\|n\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 \leq K\|n\|_{L^\infty}^2 \|\nabla c\|_{L^2}^2 \leq K\|\nabla n\|_{L^2}\|\nabla^2 n\|_{L^2}\|\nabla c\|_{L^2}^2,$$

which, multiplied by $(1+t)^\alpha$ with $\alpha \in [0,1/2)$ and integrated over $(0,\infty)$, yields

$$\begin{aligned} &\sup_{t\geq 0}\left((1+t)^\alpha\|n\|_{L^2}^2\right) + \int_0^\infty (1+t)^\alpha \|\nabla n\|_{L^2}^2 dt \\ &\leq K + K\int_0^\infty (1+t)^{\alpha-1}\|n\|_{L^2}^2 dt \\ &\quad + K\sup_{t\geq 0}\left((1+t)\|\nabla c\|_{L^2}^2\right)\left(\int_0^\infty \|\nabla n\|_{L^2}^2 dt\right)^{1/2}\left(\int_0^\infty \|\nabla^2 n\|_{L^2}^2 dt\right)^{1/2} \\ &\leq K + K\int_0^\infty (1+t)^{\alpha-1}\|n\|_{L^{3/2}}^{4/3}\|\nabla n\|_{L^2}^{2/3} dt \end{aligned}$$

$$\leq K + K\left(\int_0^\infty (1+t)^\alpha \|\nabla n\|_{L^2}^2 dt\right)^{1/3}\left(\int_0^\infty (1+t)^{\alpha-3/2} dt\right)^{2/3}$$
$$\leq K + K\left(\int_0^\infty (1+t)^\alpha \|\nabla n\|_{L^2}^2 dt\right)^{1/3},$$

which, combined with Young's inequality, gives

$$\sup_{t\geq 0}\left((1+t)^\alpha \|n\|_{L^2}^2\right) + \int_0^\infty (1+t)^\alpha \|\nabla n\|_{L^2}^2 dt \leq K. \tag{3.16}$$

Here, we have also used (3.1) and (3.15). Moreover, with the help of (3.1) and (3.16), it is easily deduced from (3.4) that

$$\sup_{t\geq 0}\left((1+t)^{1+\alpha} \|\nabla n\|_{L^2}^2\right) + \int_0^\infty (1+t)^{1+\alpha} \|\Delta n\|_{L^2}^2 dt \leq K. \tag{3.17}$$

Noting that $\kappa(c) \leq \tilde{M}c$ due to $\kappa(\cdot) \in C^1([0,\infty))$, by (2.2) we deduce in a manner similar to the one used in (3.2) that

$$\begin{aligned}
&\frac{d}{dt}\|\Delta c\|_{L^2}^2 + \|\nabla \Delta c\|_{L^2}^2 \\
&\quad \leq K\|\nabla^2 c\|_{L^2}^2\left(\|\nabla u\|_{L^2}^4 + \|n^{-1/4}\nabla n\|_{L^2}^2\right) + K\|c\|_{L^\infty}^2\|\nabla n\|_{L^2}^2 \\
&\quad \leq K\|\nabla^2 c\|_{L^2}^2\left(\|\nabla u\|_{L^2}^4 + \|n^{-1/4}\nabla n\|_{L^2}^2 + \|\nabla c\|_{L^2}^2\right) + K\|\nabla n\|_{L^2}^4,
\end{aligned}$$

so that, using (2.3), (3.1), (3.15) and (3.17), we find

$$\begin{aligned}
&\sup_{t\geq 0}\left((1+t)\|\Delta c\|_{L^2}^2\right) + \int_0^\infty (1+t)\|\nabla \Delta c\|_{L^2}^2 dt \\
&\quad \leq K + K\int_0^\infty \|\Delta c\|_{L^2}^2 dt + K\sup_{t\geq 0}\left((1+t)\|\nabla n\|_{L^2}^2\right)\int_0^\infty \|\nabla n\|_{L^2}^2 dt \\
&\quad \leq K.
\end{aligned} \tag{3.18}$$

By virtue of (2.3), (3.1) and the Cauchy-Schwarz inequality, we deduce from (3.6) that

$$\begin{aligned}
&\frac{d}{dt}\|\Delta n\|_{L^2}^2 + \|\nabla \Delta n\|_{L^2}^2 \\
&\quad \leq K\|\nabla^2 n\|_{L^2}^2\left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2\right) + K\left(\|\nabla \Delta c\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2\right)\|\nabla n\|_{L^2}\|\nabla^2 n\|_{L^2} \\
&\quad \leq K\|\nabla^2 n\|_{L^2}^2\left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2 + \|\nabla \Delta c\|_{L^2}^2\right) + K\|\nabla n\|_{L^2}^2\left(\|\nabla \Delta c\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2\right),
\end{aligned}$$

which, combined with (3.1) and (3.15)–(3.18), yields

$$\begin{aligned}
&\sup_{t\geq 0}\left((1+t)^{2+\alpha}\|\Delta n\|_{L^2}^2\right) + \int_0^\infty (1+t)^{2+\alpha}\|\nabla \Delta n\|_{L^2}^2 dt \\
&\quad \leq K + K\int_0^\infty (1+t)^{1+\alpha}\|\Delta n\|_{L^2}^2 dt \\
&\qquad + K\int_0^\infty (1+t)^{2+\alpha}\|\nabla n\|_{L^2}^2\left(\|\nabla \Delta c\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2\right) dt
\end{aligned}$$

$$\leq K + K\sup_{t\geq 0}\left((1+t)^{1+\alpha}\|\nabla n\|_{L^2}^2\right)\int_0^\infty (1+t)\left(\|\nabla\Delta c\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2\right)dt$$
$$\leq K. \tag{3.19}$$

By using (3.17) and (3.19), we can improve the decay rate of $\|\Delta c\|_{L^2}$. Indeed, similarly to the derivation of (3.2), one has (keeping in mind that $\kappa(c) \leq \tilde{M}c$)

$$\begin{aligned}&\frac{d}{dt}\|\Delta c\|_{L^2}^2 + \|\nabla\Delta c\|_{L^2}^2\\ &\quad\leq K\|\nabla^2 c\|_{L^2}^2\left(\|\nabla u\|_{L^2}^4 + \|n^{-1/4}\nabla n\|_{L^2}^2\right) + K\|\nabla c\|_{L^2}^2\|\nabla n\|_{L^3}^2\\ &\quad\leq K\|\nabla^2 c\|_{L^2}^2\left(\|\nabla u\|_{L^2}^4 + \|n^{-1/4}\nabla n\|_{L^2}^2\right) + K\|\nabla c\|_{L^2}^2\|\nabla n\|_{L^2}\|\nabla^2 n\|_{L^2},\end{aligned}$$

and thus, it follows from (2.3), (3.1), (3.15), (3.17) and (3.19) that

$$\begin{aligned}&\sup_{t\geq 0}\left((1+t)^{(3+2\alpha)/2}\|\Delta c\|_{L^2}^2\right) + \int_0^\infty (1+t)^{(3+2\alpha)/2}\|\nabla\Delta c\|_{L^2}^2 dt\\ &\quad\leq K + K\sup_{t\geq 0}\left((1+t)^{(1+\alpha)/2}\|\nabla n\|_{L^2}(1+t)^{(2+\alpha)/2}\|\nabla^2 n\|_{L^2}\right)\int_0^\infty \|\nabla c\|_{L^2}^2 dt\\ &\quad\leq K.\end{aligned} \tag{3.20}$$

Finally, we derive the decay estimates for u. First, noting that $n\nabla\phi = \nabla(n\phi) - \phi\nabla n$, we deduce in a manner similar to the one used for (2.19) that

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \leq K\|\nabla n\|_{L^\infty}^2\|\phi\|_{L^2}^2 \leq K\|\nabla^2 n\|_{L^2}\|\nabla\Delta n\|_{L^2}$$

from which and (3.1), (3.17), (3.19), it is easily seen that

$$\sup_{t\geq 0}\left((1+t)\|\nabla u\|_{L^2}^2\right) + \int_0^\infty (1+t)\|\Delta u\|_{L^2}^2 dt \leq K. \tag{3.21}$$

Similarly to the derivation of (3.8) , we have from (3.1), (3.17), (3.19) and (3.21) that

$$\begin{aligned}&\sup_{t\geq 0}\left((1+t)^{(3+2\alpha)/2}\|\Delta u\|_{L^2}^2\right) + \int_0^\infty (1+t)^{(3+2\alpha)/2}\|\nabla\Delta u\|_{L^2}^2 dt\\ &\quad\leq K + K\int_0^\infty (1+t)^{(1+2\alpha)/2}\|\Delta u\|_{L^2}^2 dt + K\int_0^\infty (1+t)^{(3+2\alpha)/2}\|\nabla^2 n\|_{L^2}\|\nabla\Delta n\|_{L^2} dt\\ &\quad\leq K + K\left(\int_0^\infty (1+t)^{1+\alpha}\|\nabla^2 n\|_{L^2}^2 dt\right)^{1/2}\left(\int_0^\infty (1+t)^{2+\alpha}\|\nabla\Delta n\|_{L^2}^2 dt\right)^{1/2}\\ &\quad\leq K.\end{aligned} \tag{3.22}$$

Collecting (3.15)–(3.17) and (3.19)–(3.22) together finishes the proof of (3.14). The proof of Theorem 1.1 is therefore complete. □

Remark. The derivations of the decay estimates for the second-order derivatives can be made rigorously by mollifying the initial data and passing to the limits. Moreover, by using the similar method, one can obtain the decay estimates of higher-order derivatives of the solutions, which will be a bit more complicated.

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