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The theorem of Halmos and Savage under finite additivity

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ABSTRACT

We prove an extension to the finitely additive setting of the theorem of Halmos and Savage. From this we deduce extensions of classical results of Drewnowski and of Yan as well as a new characterization of weak compactness in the space of finitely additive set functions.

1. Introduction and preliminaries

In a paper that soon became a classic in statistics [10], Halmos and Savage illustrated the powerful implications of the Radon Nikodym theorem for the theory of sufficient statistics. One of their results, Lemma 7, deals with dominated sets of probability measures and states that each such set admits an equivalent, countable subset. This lemma rapidly obtained its own popularity, proving to be very useful in a variety of different contexts, such as the proof of Yan Theorem, another classical result in probability and in mathematical finance.

In their proof, Halmos and Savage exploit extensively countable additivity and the fact that the underlying family is a σ-algebra. Both properties are essential as they allow, loosely speaking, for the possibility of taking limits. For this reason their method of proof cannot be adapted to the case in which probability is defined on an *algebra* and is just *finitely* additive, a situation of interest for the subjective theory of probability originating from the seminal work of de Finetti [6] and, more generally, for decision theory in which countable additivity is more an exception than a rule. Finite additivity is also unavoidable in many classical problems in which it is needed to take extensions of the given set function.

In this short note we extend the original result of Halmos and Savage to the case of finitely additive measures. The proof is, somehow surprisingly, straightforward and does not make use but of classical

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decomposition results of set functions, ultimately due to Bochner and Phillips. Before exploring its corollaries we obtain in Theorems 2 and 3 necessary and sufficient conditions for a family of bounded additive set functions to be dominated in both the finitely and the countably additive framework. Despite its simplicity, Halmos and Savage theorem has interesting implications in identifying special countable families of sets. Thus, in Theorem 4 we prove that, even when countable additivity fails on a given algebra of sets, it necessarily holds on some appropriate sub algebra. A related result is the extension of a classical finding of Drewnowski – that each disjoint sequence of sets may be refined to a subsequence on which countable additivity obtains – to the case in which the given family of sets is just an algebra. We also obtain a new characterization of weak compactness in the space of additive set functions. Eventually, Halmos–Savage theorem delivers, as a corollary, an extension of the theorem of Yan [12] to the case of finite additivity.

In the following, Ω will be a fixed, nonempty set and $\mathscr{A}$ an algebra of subsets of Ω. Also given is a positive, additive, bounded set function: in symbols $\lambda \in ba(\mathscr{A})_+$. The symbol $ca(\mathscr{A})$ describes bounded, countably additive set functions on $\mathscr{A}$. A set $\mathscr{M} \subset ba(\mathscr{A})$ is said to be dominated by λ if $\mu \ll \lambda$ for every $\mu \in \mathscr{M}$ (in symbols $\mathscr{M} \ll \lambda$), i.e. if

$$\lim_{|\lambda|(A)\to 0} |\mu|(A) = 0 \qquad \mu \in \mathscr{M} \tag{1}$$

Let us recall that, under finite additivity, the property that μ vanishes on λ null sets is necessary but not sufficient to conclude that $\mu \ll \lambda$, see [2, Chapter 6].

For the theory of finitely additive measures and integrals we mainly borrow notation, definitions and terminology from Dunford and Schwartz [9], although we prefer the symbol $|\mu|$ to denote the total variation measure generated by μ and we write μ_f (or μ_H when $f = \mathbf{1}_H$ and $H \in \mathscr{A}$) to denote that element of $ba(\mathscr{A})$ defined implicitly by letting

$$\mu_f(A) = \int \mathbf{1}_A f d\mu \qquad A \in \mathscr{A} \tag{2}$$

whenever $f \in L^1(\mu)$. We often write $\mu(f)$ rather than $\int f d\mu$.

$\mathscr{S}(\mathscr{A})$ designates the family of $\mathscr{A}$-simple functions. A real valued function h on Ω is said to be λ measurable if and only if there is a sequence $\langle h_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathscr{A})$ that λ-converges to h i.e.

$$\lim_n \lambda^*(|h_n - h| > \eta) = 0 \qquad \eta > 0 \quad \text{where} \quad \lambda^*(B) = \inf_{\{A\in\mathscr{A}:B\subset A\}} \lambda(A) \qquad B \subset \Omega \tag{3}$$

If $\mu, \nu \in ba(\mathscr{A})$ are such that for each $\eta > 0$ there exists $A \in \mathscr{A}$ such that $|\mu|(A) + |\nu|(A^c) < \eta$ then we write $\mu \perp \nu$. The orthogonal complement of $\mathscr{M} \subset ba(A)$ is defined correspondingly as

$$\mathscr{M}^\perp = \{\nu \in ba(\mathscr{A}) : \nu \perp \mu \text{ for every } \mu \in \mathscr{M}\}$$

and is known to be a normal sublattice of $ba(\mathscr{A})$, see e.g. [2, Definition 1.5.6 and Theorem 1.5.8].

2. A decomposition

We associate with $\mathscr{M} \subset ba(\mathscr{A})$ the collections

$$\mathbf{A}(\mathscr{M}) = \left\{ \sum_n \alpha_n \frac{|\mu_n|}{1 \vee \|\mu_n\|} : \ \mu_n \in \mathscr{M}, \ \alpha_n \geq 0 \text{ for } n = 1, 2, \dots \text{ and } \sum_n \alpha_n = 1 \right\} \tag{4}$$

$$\mathbf{L}(\mathscr{M}) = \{\nu \in ba(\mathscr{A}) : \nu \ll m \text{ for some } m \in \mathbf{A}(\mathscr{M})\} \tag{5}$$

To obtain a simple generalization of Lebesgue decomposition, we start remarking that $\mathbf{L}(\mathscr{M})$ is a normal sublattice of $ba(\mathscr{A})$ and so, by Riesz decomposition Theorem [2, 1.5.10], $ba(\mathscr{A}) = \mathbf{L}(\mathscr{M}) + \mathbf{L}(\mathscr{M})^{\perp}$. To see this, take an increasing net $\langle \nu_\alpha \rangle_{\alpha \in \mathfrak{A}}$ in $\mathbf{L}(\mathscr{M})$ with $\nu = \lim_\alpha \nu_\alpha \in ba(\mathscr{A})$, extract a sequence $\langle \nu_{\alpha_n} \rangle_{n \in \mathbb{N}}$ such that $\|\nu - \nu_{\alpha_n}\| = (\nu - \nu_{\alpha_n})(\Omega) < 2^{-n-1}$, choose $m_n \in \mathbf{A}(\mathscr{M})$ such that $m_n \gg \nu_{\alpha_n}$ and define $m = \sum_n 2^{-n} m_n \in \mathbf{A}(\mathscr{M})$. Since $m \gg m_n \gg \nu_{\alpha_n}$ for each $n \in \mathbb{N}$, there is $\delta_n > 0$ such that $m(A) < \delta_n$ implies $|\nu_{\alpha_n}|(A) < 2^{-n-1}$ and, therefore, $|\nu|(A) \le |\nu_{\alpha_n}|(A) + 2^{-n-1} \le 2^{-n}$. This proves that if $\{\nu_\alpha : \alpha \in \mathfrak{A}\}$ is a nonempty family in $\mathbf{L}(\mathscr{M})$ and if $\bigvee_{\alpha \in \mathfrak{A}} \nu_\alpha$ exists in $ba(\mathscr{A})$, then necessarily $\bigvee_{\alpha \in \mathfrak{A}} \nu_\alpha \in \mathbf{L}(\mathscr{M})$. Moreover, $|\nu_1| \le |\nu|$ and $\nu \in \mathbf{L}(\mathscr{M})$ imply $\nu_1 \in \mathbf{L}(\mathscr{M})$. Noting that $\mathbf{L}(\mathscr{M})^{\perp} = \mathbf{A}(\mathscr{M})^{\perp}$ we obtain the following:

Lemma 1. *For each $\lambda \in ba(\mathscr{A})$ and $\mathscr{M} \subset ba(\mathscr{A})$ there is a unique way of writing*

$$\lambda = \lambda^c_{\mathscr{M}} + \lambda^{\perp}_{\mathscr{M}} \tag{6}$$

with $\lambda^c_{\mathscr{M}} \in \mathbf{L}(\mathscr{M})$ and $\lambda^{\perp}_{\mathscr{M}} \perp \mathbf{A}(\mathscr{M})$. If λ is positive or countably additive then so are $\lambda^{\perp}_{\mathscr{M}}$ and $\lambda^c_{\mathscr{M}}$.

3. The Halmos–Savage theorem and its implications

The original result of Halmos and Savage follows quickly from Lemma 1 which helps circumventing the lack of countable additivity and its implications.

Theorem 1 *(Halmos and Savage). $\mathscr{M} \subset ba(\mathscr{A})$ is dominated if and only if $\mathscr{M} \ll m$ for some $m \in \mathbf{A}(\mathscr{M})$.*

Proof. λ dominates $\mathscr{M}$ if and only if $\lambda^c_{\mathscr{M}}$ does. In fact, choose $\mu \in \mathscr{M}$ and $\varepsilon > 0$ and let δ be such that $\lambda(A) < \delta$ implies $|\mu|(A) < \varepsilon$. Pick $B \in \mathscr{A}$ such that $|\mu|(B^c) + \lambda^{\perp}_{\mathscr{M}}(B) < (\delta/2) \wedge \varepsilon$. Then $\lambda^c_{\mathscr{M}}(A) < \delta/2$ implies $\lambda(A \cap B) < \delta$ and thus $|\mu|(A) \le |\mu|(A \cap B) + \varepsilon \le 2\varepsilon$. $\square$

To rephrase the above theorem in the language of Halmos and Savage, observe that if $\mathscr{M}_0 = \{\mu_1, \mu_2, \ldots\}$ is the subfamily of $\mathscr{M}$ generating $m = \sum_n 2^{-n} |\mu_n| / (1 \vee \|\mu_n\|)$ and $\langle A_k \rangle_{k \in \mathbb{N}}$ is a sequence in $\mathscr{A}$, then $\lim_k |\mu_n|(A_k) = 0$ for $n = 1, 2, \ldots$ if and only if $\lim_k |\mu|(A_k) = 0$ for all $\mu \in \mathscr{M}$ and $\mathscr{M}_0$ may then be said to be *equivalent* to $\mathscr{M}$. Thus, $\mathscr{M}$ is dominated if and only if it admits an equivalent, countable subset. In the original work of Halmos and Savage, the role of the set $\mathscr{M}$ is played by the family of probability measures relatively to which a given statistic may or may not be sufficient.

Dominated subsets of $ba(\mathscr{A})$ are easily constructed starting from some $\lambda \in ba(\mathscr{A})$ and a given subset $\mathcal{K} \subset L^1(\lambda)$, by letting $\mathscr{M} = \{\lambda_f : f \in \mathcal{K}\}$. In the following section 4, e.g., $\mathcal{K}$ will be a family of indicator functions. In case λ is countably additive this is in fact the only possible case, by the Radon Nikodym theorem. This classical conclusion goes together with the characterization of absolute continuity in terms of null sets. Under finite additivity, however, neither of these properties hold.

Example 1. Let $\mathscr{A}$ be the algebra generated by the finite subsets of $\mathbb{N}$ and define $\lambda \in ba(\mathbb{N})$ by letting $\lambda(A) = 1 - \lambda(A^c) = 0$ when $A \in \mathscr{A}$ is finite or empty. Given that $\mathscr{A}$ consists of finite sets or of their complements, λ is well defined. Denote by $\mathscr{M}$ the collection of all $\mu \in ba(\mathscr{A})_+$ vanishing on the finite subsets of $\mathbb{N}$. It is then obvious that for any $A \in \mathscr{A}$ and $1 > \varepsilon > 0$, $\lambda(A) < \varepsilon$ implies $\lambda(A) = 0$ and thus that A is a finite set so that $\mu(A) = 0$ for all $\mu \in \mathscr{M}$. In other words $\mathscr{M} \ll \lambda$. Define $\nu \in ba(\mathscr{A})$ implicitly by letting $\nu(A) = \sum_{n \in A} 2^{-n}$ for all $A \in \mathscr{A}$. Then, $\nu(A) = 0$ implies $A = \varnothing$ and thus $\mu(A) = 0$. However this is not enough to conclude that $\nu \gg \mathscr{M}$ since $\nu(\{n \in \mathbb{N} : n > N\}) = 2^{-N}$ while $\mu(\{n \in \mathbb{N} : n > N\}) = \|\mu\|$.

Example 2. Let Ω be a separable metric space, $\mathscr{A}$ its Baire σ-algebra and $\Omega_0 = \{\omega_1, \omega_2, \dots\}$ a dense, countable subset. Define

$$\lambda(A) = \sum_{\omega_n \in A} 2^{-n} \qquad A \in \mathscr{A}$$

It is easily seen that λ is countably additive and that $\lambda(A) > 0$ whenever A is open (as then $\Omega_0 \cap A \neq \varnothing$). With each decreasing sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ of open sets one may associate

$$\mu(A) = \operatorname*{LIM}_n \lambda(A \cap A_n)/\lambda(A_n) \qquad A \in \mathscr{A}$$

(with LIM denoting a Banach limit) and let $\mathscr{M}$ be the corresponding collection. Then clearly $\lambda(A) = 0$ implies $\mu(A) = 0$ for each $\mu \in \mathscr{M}$, while μ is either countably additive (and thus $\mu \ll \lambda$), if $\lim_n \lambda(A_n) > 0$, or purely finitely additive (and thus $\mu \perp \lambda$), if $\lim_n \lambda(A_n) = 0$, as $\mu(A_n) = 1$ for all $n \in \mathbb{N}$.

Common to the preceding examples and to the original applications imagined by Halmos and Savage (see also [1]), is the fact that the dominating λ is *given*. A natural question is then how to characterize dominated sets with no previous knowledge of the dominating element. An easy example in which the answer is well known is when $\mathscr{M}$ is relatively weakly compact. In the next two results we provide a separate answer for the cases $ba(\mathscr{A})$ and $ca(\mathscr{A})$, respectively.

Theorem 2. *For $\mathscr{M} \subset ba(\mathscr{A})$ to be dominated it is necessary and sufficient that every subset of*

$$\mathscr{N} = \{\nu \in ba(\mathscr{A})_+ : 0 < \nu \le |\mu| \text{ for some } \mu \in \mathscr{M}\} \tag{7}$$

which consists of pairwise orthogonal elements is at most countable.

Proof. (Necessity). If $\mathscr{M}$ is dominated then so is $\mathscr{N}$. By Theorem 1, if $\mathscr{N}_0 \subset \mathscr{N}$ there exist $\nu_1, \nu_2, \ldots \in \mathscr{N}_0$ such that $\nu \ll \sum_n 2^{-n}\nu_n/(1 \vee \|\nu_n\|) = \nu_0$ for every $\nu \in \mathscr{N}_0$. However, if the elements of $\mathscr{N}_0$ are pairwise disjoint then any $\nu \in \mathscr{N}_0$ other than $\nu_1, \nu_2, \ldots$ is necessarily orthogonal to ν_0 and thus null, contradicting the inclusion $\mathscr{N}_0 \subset \mathscr{N}$. We conclude that $\mathscr{N}_0 = \{\nu_n : n \in \mathbb{N}\}$.

(Sufficiency). Let $\mathfrak{M} = \{(\mathscr{M}_\alpha, \mathscr{M}'_\alpha) : \alpha \in \mathfrak{A}\}$ be the family of all pairs $(\mathscr{M}_\alpha, \mathscr{M}'_\alpha)$ of countable subsets of $\mathscr{M}$ with $\mathscr{M}_\alpha \subset \mathscr{M}'_\alpha$ and with $\sup_{\mu\in\mathscr{M}'_\alpha} \|\mu^\perp_{\mathscr{M}_\alpha}\| > 0$. We can assume that the family $\mathfrak{M}$ is non-empty, since otherwise $\mathscr{M}$ is dominated by any of its elements. Define the binary relationship $\prec$ on $\mathfrak{M}$ and $<$ on $\mathfrak{A}$ by writing $\alpha < \beta$ and $(\mathscr{M}_\alpha, \mathscr{M}'_\alpha) \prec (\mathscr{M}_\beta, \mathscr{M}'_\beta)$ whenever $\mathscr{M}'_\alpha \subset \mathscr{M}_\beta$. One easily sees that $\prec$ is transitive and antisymmetric. By Hausdorff principle we obtain a maximal linearly ordered subfamily $\mathfrak{M}_0$. Define,

$$\mathfrak{A}_0 = \{\alpha \in \mathfrak{A} : (\mathscr{M}_\alpha, \mathscr{M}'_\alpha) \in \mathfrak{M}_0\} \quad \text{and} \quad \mathscr{N}_0 = \{\nu_\alpha : \alpha \in \mathfrak{A}_0\} \tag{8}$$

where $\nu_\alpha = \left|(\mu_\alpha)^\perp_{\mathscr{M}_\alpha}\right|$ for some $\mu_\alpha \in \mathscr{M}'_\alpha$ such that $\left\|(\mu_\alpha)^\perp_{\mathscr{M}_\alpha}\right\| > 0$. In passing, we observe for the sake of future reference that, independently of the current assumptions, the collection $\{\mu_{\alpha'} : \alpha' \in \mathfrak{A}_0,\ \alpha' < \alpha\}$ so obtained, with $\alpha \in \mathfrak{A}_0$, consists of distinct elements and is contained in the countable set $\mathscr{M}'_\alpha$ so that each $\alpha \in \mathfrak{A}_0$ admits at most countably many predecessors relatively to the order $<$. More importantly, since $\mathscr{N}_0 \subset \mathscr{N}$ and its elements are pairwise orthogonal, by assumption $\mathfrak{A}_0$ must be at most countable, so $\mathscr{M}_\infty = \bigcup_{\alpha\in\mathfrak{A}_0} \mathscr{M}_\alpha$ is a countable subset of $\mathscr{M}$. Assume the existence of $\mu_\infty \in \mathscr{M}$ such that $\left\|(\mu_\infty)^\perp_{\mathscr{M}_\infty}\right\| > 0$. Then, given that $\mathscr{M}_\infty$ is countable and letting $\mathscr{M}'_\infty = \mathscr{M}_\infty \cup \{\mu_\infty\}$, the pair $(\mathscr{M}_\infty, \mathscr{M}'_\infty)$ belongs to $\mathfrak{M}$ and the collection $\mathfrak{M}_0 \cup \{(\mathscr{M}_\infty, \mathscr{M}'_\infty)\}$ is linearly ordered and contains $\mathfrak{M}_0$ properly, a contradiction. One concludes from Lemma 1 that $\mathscr{M} \subset \mathbf{L}(\mathscr{M}_\infty)$ so that, $\mathscr{M}_\infty$ being dominated because countable, $\mathscr{M}$ is a dominated set. $\square$

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This characterization is fairly intuitive and becomes even more so if formulated in the language of decision theory where each $\mu \in \mathscr{M}$ may be interpreted as a subjective belief. The statement suggests then, loosely speaking, that the individual beliefs of a group of agents admit a synthesis if and only if the extent of radical disagreement within the group is limited.

A version of the preceding claim can be formulated for the countably additive case.[2]

Theorem 3. *Let $\mathscr{A}$ be a σ-algebra. Then for $\mathscr{M} \subset ca(\mathscr{A})$ to be dominated it is necessary and sufficient that every collection $\mathscr{H} \subset \mathscr{A}$ of pairwise disjoint sets with $\sup_{\mu\in\mathscr{M}} |\mu|(H) > 0$ for each $H \in \mathscr{H}$ is at most countable.*

Proof. Necessity is obvious. The proof of sufficiency follows from that given for Theorem 2 in which the domination property was shown to be a consequence of the fact that the special, given family $\mathscr{N}_0 = \{\nu_\alpha : \alpha \in \mathfrak{A}_0\} \subset \mathscr{N}$ of pairwise orthogonal elements in (8), is countable. This conclusion will be reached by first showing that, under the present assumption, to any family $\mathscr{N}_1 \subset \mathscr{N}$ of pairwise orthogonal elements one can associate a family $\{A_\nu : \nu \in \mathscr{N}_1\} \subset \mathscr{H}$ with the property that $\nu(A_\nu^c) = \nu'(A_\nu) = 0$ whenever $\nu, \nu' \in \mathscr{N}_1$ are different.

In fact, if $\mathscr{N}_1$ is as above, $\nu \in \mathscr{N}_1$ and $\nu' \in \mathscr{N}_1 \backslash \{\nu\}$ let $B_{\nu'} \in \mathscr{A}$ be such that $\nu(B_{\nu'}^c) = \nu'(B_{\nu'}) = 0$. If a is a countable subset of $\mathscr{N}_1$ not including ν (and $\mathbf{A}$ the corresponding collection) write

$$B_a = \bigcap_{\nu'\in a} B_{\nu'} \in \mathscr{A} \tag{9}$$

with the convention $B_\varnothing = \Omega$. The net $\langle B_a \rangle_{a\in\mathbf{A}}$ is clearly decreasing if $\mathbf{A}$ is directed by inclusion. Consider the collection

$$\mathfrak{X} = \Big\{(a, a') \in \mathbf{A} \times \mathbf{A} : a \subset a' \text{ and } \sup_{\mu\in\mathscr{M}} |\mu|(B_a \backslash B_{a'}) > 0\Big\}$$

partially ordered by writing $(a, a') \prec (b, b')$ whenever $a' \subset b$. We can assume that $\mathfrak{X}$ is nonempty since otherwise for given $a_0 \in \mathbf{A}$ and any $\nu' \in \mathscr{N}_1 \backslash \{\nu\}$ we would have $\nu'(B_{a_0}) = 0$ if $\nu' \in a$ and otherwise with $a_1 = a_0 \cup \{\nu'\}$,

$$\nu'(B_{a_0}) \le \nu'(B_{a_1}) + \sup_{\mu\in\mathscr{M}} |\mu|(B_{a_0} \backslash B_{a_1}) = 0$$

and the claim would be true with $A_\nu = B_{a_0}$.

Let $\mathfrak{X}_0$ be a maximal linearly ordered subset of $\mathfrak{X}$ and let $\mathbf{A}_0 = \{a \in \mathbf{A} : (a, a') \in \mathfrak{X}_0 \text{ for some } a' \in \mathbf{A}\}$. Given that $\{B_a \backslash B_{a'} : a \in \mathbf{A}_0\}$ is a disjoint collection in $\mathscr{A}$ and that $\sup_{\mu\in\mathscr{M}} |\mu|(B_a \backslash B_{a'}) > 0$ for every $a \in \mathbf{A}_0$, then under the current assumption $\mathbf{A}_0$ may be enumerated as $\{a_n : n \in \mathbb{N}\}$. Set $a_\infty = \bigcup_n a_n$ and

$$A_\nu = \bigcap_n B_{a_n} = B_{a_\infty} \in \mathscr{A}$$

Clearly, $\nu(A_\nu^c) = 0$. If $\nu' \in a_\infty$ then there is $n \in \mathbb{N}$ such that $\nu' \in a_n$ so that $\nu'(A_\nu) \le \nu'(B_{a_n}) = 0$. If $\nu' \notin a_\infty$ then we obtain a pair (a_∞, a_*) with $a_* = a_\infty \cup \{\nu'\}$ which contradicts the maximality of $\mathfrak{X}_0$ unless $\sup_{\mu\in\mathscr{M}} |\mu|(B_{a_\infty} \backslash B_{a_*}) = 0$ so that $0 = \nu'(B_{a_*}) = \nu'(B_{a_\infty}) = \nu'(A_\nu)$. In other words we have found $A_\nu \in \mathscr{A}$ such that

$$\nu(A_\nu^c) = \nu'(A_\nu) = 0 \qquad \nu' \in \mathscr{N}_1 \backslash \{\nu\} \tag{10}$$

[2] I am grateful to Pietro Rigo for suggesting me the claim of Theorem 3 and to an anonymous referee for several corrections to its original proof.

Returning to the family $\mathscr{N}_0 = \{\nu_\alpha : \alpha \in \mathfrak{A}_0\}$ defined in (8), we recall that $\mathfrak{A}_0$ is linearly ordered by the order $<$ and that, as we noticed soon after (8), every $\alpha \in \mathfrak{A}_0$ admits countably many predecessors. Let then

$$H_\alpha = A_{\nu_\alpha} \backslash \bigcup_{\{\alpha' \in \mathfrak{A}_0 : \alpha' < \alpha\}} A_{\nu_{\alpha'}} \in \mathscr{A}$$

Clearly, $\{H_\alpha : \alpha \in \mathfrak{A}_0\}$ is disjoint and $\nu_\alpha(H_\alpha) = \nu_\alpha(A_{\nu_\alpha}) > 0$. By assumption, then, $\mathfrak{A}_0$ must be at most countable. $\square$

In either case, $ba(\mathscr{A})$ or $ca(\mathscr{A})$, the possibility of reducing the corresponding family to a countable subfamily illustrates well the simplification that arises form the existence of a dominating measure.

4. Measure theoretic implications

When $\mathscr{H} \subset \mathscr{A}$, the collection $\{\lambda_H : H \in \mathscr{H}\}$ obtained by restricting λ to H for all $H \in \mathscr{H}$ is a typical example of a dominated family. Theorem 1 may be useful to identify some countable structure in $\mathscr{A}$. A typical application is the following[3]:

Corollary 1. *Let $\mathscr{H}$ be an arbitrary family of subsets of Ω and define the following class:*

$$I(\mathscr{H}) = \left\{ E \subset \Omega : \inf_{\{\alpha \subset \mathscr{H} \text{ finite}\}} \lambda^*\Big(E \backslash \bigcup_\alpha H\Big) = 0 \right\} \tag{11}$$

There exist $H_1, H_2, \ldots \in \mathscr{H}$ such that the limit

$$\lambda^*(E) = \lim_n \inf_{\{A_n \in \mathscr{A}:\ \bigcup_{j \le n} H_j \subset A_n\}} \lambda^*(E \cap A_n) \qquad E \in I(\mathscr{H}) \tag{12}$$

exists uniformly in $E \in I(\mathscr{H})$.

Proof. For each $H \in \mathscr{H}$ define the algebra

$$\mathscr{A}_H = \big\{(H \cap A) \cup (H^c \cap B) : A, B \in \mathscr{A}\big\} \tag{13}$$

and $\bar\lambda_H \in ba(\mathscr{A}_H)$ implicitly (see [2, Theorem 3.3.3]) via

$$\bar\lambda_H(D) = \lambda^*(H \cap D) \qquad D \in \mathscr{A}_H \tag{14}$$

Designate by $\hat\lambda_H$ the restriction of $\bar\lambda_H$ to $\mathscr{A}$. Then, $\hat\lambda_H \ll \lambda$. By Theorem 1 we can extract $H_1, H_2, \ldots$ from $\mathscr{H}$ such that $m = \sum_n \alpha_n \hat\lambda_{H_n} / (\|\hat\lambda_{H_n}\| \vee 1) \gg \hat\lambda_H$ for each $H \in \mathscr{H}$. Write $\bar H_n = \bigcup_{j \le n} H_j$. If $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathscr{A}$ with

$$\bar H_n \subset A_n \quad \text{and} \quad \lambda(A_n) < \lambda^*(\bar H_n) + 2^{-n} \qquad n \in \mathbb{N} \tag{15}$$

then $\lim_n m(A_n^c) = 0$ so that $\lim_n \hat\lambda_H(A_n^c) = 0$ for every $H \in \mathscr{H}$. By subadditivity of λ^* this conclusion extends easily to

$$\lim_n \lambda^*(H_\alpha \cap A_n^c) = 0 \qquad H_\alpha = \bigcup_\alpha H,\ \alpha \subset \mathscr{H} \text{ finite} \tag{16}$$

[3] Remember $\lambda \in ba(\mathscr{A})_+$.

We deduce that choosing $E \in I(\mathscr{H})$ and letting α run over all finite subsets of $\mathscr{H}$,

$$\lambda^*(E) = \sup_\alpha \lambda^*(E \cap H_\alpha) = \sup_\alpha \lim_n \lambda^*(E \cap H_\alpha \cap A_n) \leq \liminf_n \lambda^*(E \cap A_n) \leq \limsup_n \lambda^*(E \cap A_n)$$

while the converse is obvious. Moreover, choose $q > p > n$ and observe that

$$\begin{aligned} \left|\lambda^*(E \cap A_p) - \lambda^*(E \cap A_q)\right| &\leq \lambda(A_p \triangle A_q) \\ &= \lambda(A_p) + \lambda(A_q) - 2\lambda(A_p \cap A_q) \\ &\leq 2^{-p} + \lambda^*(\bar{H}_p) + 2^{-q} + \lambda^*(\bar{H}_q) - 2\lambda^*(\bar{H}_p) \\ &\leq \lim_k \lambda^*(\bar{H}_k) - \lambda^*(\bar{H}_n) + 2^{-(n-1)} \end{aligned} \tag{17}$$

The same uniform bound applies if we replace each A_n with a decreasing net $\langle A_n^d \rangle_{d \in D}$ in $\mathscr{A}$ with each sequence $\langle A_n^d \rangle_{n \in \mathbb{N}}$ satisfying (16). We conclude that

$$\lambda^*(E) = \lim_d \lim_n \lambda^*(E \cap A_n^d) = \lim_n \lim_d \lambda^*(E \cap A_n^d) = \lim_n \inf_{\{A_n \in \mathscr{A}:\ \bigcup_{j \leq n} H_j \subset A_n\}} \lambda^*(E \cap A_n) \tag{18}$$

and

$$\begin{aligned} \lambda^*(E) - \lim_d \lambda^*(E \cap A_n^d) &= \lim_k \lim_d \{\lambda^*(E \cap A_k^d) - \lambda^*(E \cap A_n^d)\} && \text{(by (18))} \\ &\leq \sup_{d \in D,\ p,q \geq n} \left|\lambda^*(E \cap A_p^d) - \lambda^*(E \cap A_q^d)\right| \\ &\leq \lim_k \lambda^*(\bar{H}_k) - \lambda^*(\bar{H}_n) + 2^{-(n-1)} && \text{(by (17))} \end{aligned}$$

which proves uniform convergence. □

For the next result, define the λ-completion of $\mathscr{A}$ as follows

$$\mathscr{A}(\lambda) = \left\{ B \subset \Omega : \inf_{\{A, A' \in \mathscr{A} : A \subset B \subset A'\}} \lambda(A' \backslash A) = 0 \right\} \tag{19}$$

It is clear that λ admits exactly one extension to $\mathscr{A}(\lambda)$ (still denoted by λ), defined by letting

$$\lambda(B) = \sup_{\{A \in \mathscr{A} : A \subset B\}} \lambda(A) = \inf_{\{A' \in \mathscr{A} : B \subset A'\}} \lambda(A') \qquad B \in \mathscr{A}(\lambda) \tag{20}$$

It is also easily seen that $\mathscr{A}(\lambda)$ consists of all λ-measurable sets, i.e. those $B \subset \Omega$ for which the function $\mathbf{1}_B$ is measurable and therefore admits a sequence $\langle h_n \rangle_{n \in \mathbb{N}}$ in $\mathscr{S}(\mathscr{A})$ that λ-converges to $\mathbf{1}_B$. In fact one may then choose $\eta < 1/6$ and $B_n \in \mathscr{A}$ such that $\{|\mathbf{1}_B - h_n| > \eta\} \subset B_n$ and $\lambda(B_n) < 2^{-n}$ and define $A_n = \{h_n > 2\eta\} \cap B_n^c$ and $A'_n = A_n \cup B_n$. Then, $A_n \subset B \subset A'_n$ and $A'_n \backslash A_n \subset B_n$. It is conversely obvious that each $B \in \mathscr{A}(\lambda)$ is λ-measurable. In particular, if $A \in \mathscr{A}$ and $\lambda(A) = 0$ then $B \subset A$ implies $B \in \mathscr{A}(\lambda)$.

The following result establishes that a finitely additive set function is *locally* countably additive.

Theorem 4. *There exists a disjoint sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ in $\mathscr{A}$ such that: (i) $\lambda(B) = \sum_n \lambda(B \cap A_n)$ when $B \in \mathscr{A}(\lambda)$, (ii) let $B \subset \Omega$ be a set with $B \cap A_n \in \mathscr{A}(\lambda)$ for every $n \in \mathbb{N}$, then $B \in \mathscr{A}(\lambda)$, (iii) $\mathscr{A}_1 \equiv \sigma(A_1, A_2, \ldots) \subset \mathscr{A}(\lambda)$, (iv) λ is countably additive in restriction to $\mathscr{A}_1$, (v) if $\mathscr{M} \subset ba(\mathscr{A}(\lambda))$ is weak* compact and $\mathscr{M} \ll \lambda$, then $\mathscr{M}$ is uniformly countably additive on $\mathscr{A}_1$.*

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Proof. Put $\mathscr{H} = \mathscr{A}$ in Corollary 1. We have that $\mathscr{A}(\lambda) \subset I(\mathscr{H})$ and that and there is a sequence $\langle H_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{H}$ satisfying (12). Let $A_n = H_n \backslash \bigcup_{j<n} H_j$. Thus

$$\lambda(B) = \sum_n \lambda(B \cap A_n) \qquad B \in \mathscr{A}(\lambda) \tag{21}$$

Moreover,

$$\inf_{\{A' \in \mathscr{A}:\ \bigcup_n A_n \subset A'\}} \lambda(A') \leq \lambda(\Omega) = \sum_n \lambda(A_n) \leq \sup_{\{A \in \mathscr{A}:\ A \subset \bigcup_n A_n\}} \lambda(A) \tag{22}$$

which proves that $\bigcup_n A_n \in \mathscr{A}(\lambda)$. Let $A_0 = \bigcap_n A_n^c$ and observe that $\lambda(A_0) = 0$. Fix $B \subset \Omega$ and assume that $B \cap A_n \in \mathscr{A}(\lambda)$ for $n = 1, 2, \ldots$ and write $B_k = B \cap \bigcup_{n=0}^k A_n$. Of course $B \cap A_0 \in \mathscr{A}(\lambda)$ so that $B_k \in \mathscr{A}(\lambda)$ for $k = 1, 2, \ldots$. The inclusion $\bigcup_n A_n \in \mathscr{A}(\lambda)$ and (i) imply

$$\lim_N \lambda^* (B \backslash B_N) \leq \lim_N \lambda\Big(\bigcup_{n>N} A_n\Big) = \lim_N \sum_{n>N} \lambda(A_n) = 0$$

Therefore the sequence $\langle B_k \rangle_{k\in\mathbb{N}}$ λ-converges to B, and so $B \in \mathscr{A}(\lambda)$. This proves (ii). Each element in $\mathscr{A}_1$ will be of the form $A_\alpha = \bigcup_{n\in\alpha} A_n$ for some $\alpha \subset \mathbb{N} \cup \{0\}$ so that (iii) follows from (ii). Moreover, from (i),

$$\lambda(A_\alpha) = \sum_n \lambda(A_\alpha \cap A_n) = \sum_{n\in\alpha} \lambda(A_n)$$

from which (iv) follows easily. If $\mathscr{M} \subset ba(\mathscr{A}(\lambda))$ is weak* compact and $\mathscr{M} \ll \lambda$ then $\mathscr{M}_1 = \{\mu|\mathscr{A}_1 : \mu \in \mathscr{M}\}$ is a weak* compact subset of $ba(\mathscr{A}_1)$ and $\mathscr{M}_1 \ll \lambda|\mathscr{A}_1$. (v) then follows from [13, Theorem 1.1]. □

In the last claim, the set obtained by restricting $\mathscr{M}$ to $\mathscr{A}_1$ is weak* closed and uniformly countably additive. It is thus weakly compact as a subset of $ca(\mathscr{A}_1)$ [9, IV.9.1]. The claim requires, however, weak* compactness in the space $ba(\mathscr{A}(\lambda))$, a somewhat unconventional property in applications where it is in fact more common to establish weak* compactness in the space $ba(\mathscr{A})$. The problem is that, even when $\mathscr{M} \ll \lambda$ is weak* compact in $ba(\mathscr{A})$ the family obtained by extending (uniquely) each $m \in \mathscr{M}$ to $\mathscr{A}(\lambda)$, may fail to be weak* compact as a subset of $ba(\mathscr{A}(\lambda))$. In fact we can only establish a partial result, assuming the Seever property. $\mathscr{A}$ is said to possess the Seever property [2, p. 210] whenever for every pair of sequences $\langle C_n \rangle_{n\in\mathbb{N}}$ and $\langle B_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{A}$ with $C_n \subset C_{n+1} \subset B_{n+1} \subset B_n$ there exists $A \in \mathscr{A}$ with $C_n \subset A \subset B_n$. Of course σ-algebras possess the Seever property.

Lemma 2. *Let $\mathscr{M} \subset ba(\mathscr{A})$ be weak* compact and $\mathscr{M} \ll \lambda$. Denote by $\bar{\mu}$ the extension of $\mu \in \mathscr{M}$ from $\mathscr{A}$ to $\mathscr{A}(\lambda)$. If either (i) $\mathscr{A}$ possesses the Seever property or (ii) $\mathscr{M} \subset ba(\mathscr{A})_+$ then $\bar{\mathscr{M}} = \{\bar{\mu} : \mu \in \mathscr{M}\}$ is a weak* compact subset of $ba(\mathscr{A}(\lambda))$ and $\bar{\mathscr{M}} \ll \lambda$.*

Proof. Let $\langle \bar{\mu}_\alpha \rangle_{\alpha\in\mathfrak{A}}$ be a net in $\bar{\mathscr{M}}$. Passing to a subnet if necessary $\langle \mu_\alpha \rangle_{\alpha\in\mathfrak{A}}$ converges weak* to $\mu_0 \in \mathscr{M}$. Let $\bar{\mu}_0$ be the extension of μ_0 to $\mathscr{A}(\lambda)$. Assume that $\mathscr{A}$ has the Seever property. Then it is easily seen that each $B \in \mathscr{A}(\lambda)$ admits $A \in \mathscr{A}$ such that $\lambda(A \triangle B) = 0$. Then,

$$\bar{\mu}_0(B) = \mu_0(A) = \lim_\alpha \mu_\alpha(A) = \lim_\alpha \bar{\mu}_\alpha(B)$$

If, on the other hand, $\mathscr{M} \subset ba(\mathscr{A})_+$, then $A, A' \in \mathscr{A}$, $B \in \mathscr{A}(\lambda)$ and $A \subset B \subset A'$ imply

$$\mu_0(A) \leq \liminf_\alpha \bar{\mu}_\alpha(B) \leq \limsup_\alpha \bar{\mu}_\alpha(B) \leq \mu_0(A')$$

so that $\lim_\alpha \bar{\mu}_\alpha(B)$ exists and coincides with $\bar{\mu}_0(B)$. Setwise convergence of a bounded net is clearly equivalent to weak* convergence. □

5. A theorem of Drewnowski and weak compactness

Theorem 4 delivers an interesting extension of a celebrated result originally due to Drewnowski [8] but popularized by Diestel and Uhl [7, Theorem, p. 38]. The main insight of this theorem is that the sequence mentioned in Theorem 4 may be extracted from a given one.

Theorem 5 *(Drewnowski). A disjoint sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ in $\mathscr{A}$ admits a subsequence $\langle C_n\rangle_{n\in\mathbb{N}}$ such that $\mathscr{A}_1 = \sigma(C_1, C_2, \dots) \subset \mathscr{A}(\lambda)$ and λ is countably additive in restriction to $\mathscr{A}_1$.*

Proof. Denote by $\mathfrak{S}$ the set of rarefying subsequences, i.e. of those functions $\sigma : \mathbb{N} \to \mathbb{N}$ which are strictly increasing and such that $\lim_k \sigma(k+1) - \sigma(k) = \infty$. For each $\sigma \in \mathfrak{S}$ write $B_\sigma = \bigcup_k B_{\sigma(k)}$. In the notation of Corollary 1, let $\mathscr{H} = \{B_\sigma : \sigma \in \mathfrak{S}\}$ and write $\mathscr{A}_{B_\sigma}$ and $\bar{\lambda}_{B_\sigma}$ (as defined in (13) and (14)) more simply as $\mathscr{A}_\sigma$ and $\bar{\lambda}_\sigma$, respectively. Recall that $B_\sigma \in \mathscr{A}_\sigma$ and that $\bar{\lambda}_\sigma(B_\sigma) = \lambda^*(B_\sigma)$. Given that $\mathscr{H} \subset I(\mathscr{H})$, we rewrite (12) as

$$\begin{aligned}
\lambda^*(B_\sigma) &= \lim_n \inf_{\{A_n\in\mathscr{A}:\ \bigcup_{j\le n} B_{\sigma_j}\subset A_n\}} \lambda^*(B_\sigma \cap A_n) \\
&= \lim_n \inf_{\{A_n\in\mathscr{A}:\ \bigcup_{j\le n} B_{\sigma_j}\subset A_n\}} \bar{\lambda}_\sigma(B_\sigma \cap A_n) \\
&= \lim_n \bar{\lambda}^*_\sigma\Big(B_\sigma \cap \bigcup_{j\le n} B_{\sigma_j}\Big)
\end{aligned} \tag{23}$$

where $\bar{\lambda}^*_\sigma$ denotes the outer measure induced by $\bar{\lambda}_\sigma$ as in (3) (and thus relatively to the algebra $\mathscr{A}_\sigma$). Consider first the eventuality that $\bigcup_j B_{\sigma_j} = \bigcup_{j\le N} B_{\sigma_j}$ for some $N \in \mathbb{N}$. Given that the sequences $\sigma_1, \dots, \sigma_N$ are rarefying it would then be easy to construct $\sigma \in \mathfrak{S}$ such that the image $\{\sigma(k) : k \in \mathbb{N}\}$ of σ is disjoint from the union of the images of $\sigma_1, \dots, \sigma_N$ and so, given that the original sequence $\langle B_i\rangle_{i\in\mathbb{N}}$ is disjoint, that $B_\sigma \cap \bigcup_{j\le N} B_{\sigma_j} = \varnothing$. To this end it is enough to observe that if k is sufficiently large so that $\min_{j\le N,\ k'\ge k} \sigma_j(k'+1) - \sigma_j(k') > N$ and if $j_k = \arg\max_{j\le N} \sigma_j(k)$ then there exists at least one integer $\sigma_{j_k}(k) < n \le \sigma_{j_k}(k) + N$ which does not appear in none of the sequences $\sigma_1, \dots, \sigma_N$. For such choice of σ, (23) gives $\lambda^*(B_\sigma) = 0$ and the proof would be complete upon letting $C_n = B_{\sigma(n)}$. Outside of this special case it is possible to form $\sigma \in \mathfrak{S}$ by extracting from each sequence σ_j at most one index. As a consequence, $B_\sigma \cap \bigcup_{j\le n} B_{\sigma_j}$ is the union of at most n sets of the original sequence $\langle B_i\rangle_{i\in\mathbb{N}}$ and is thus an element of $\mathscr{A}$. But then, applying (23) we obtain

$$\lambda^*(B_\sigma) = \lim_n \bar{\lambda}^*_\sigma\Big(B_\sigma \cap \bigcup_{j\le n} B_{\sigma_j}\Big) = \lim_k \lambda\Big(\bigcup_{j\le k} B_{\sigma(j)}\Big) = \sum_j \lambda\big(B_{\sigma(j)}\big) \le \lambda_*(B_\sigma) \tag{24}$$

Let $C_k = B_{\sigma(k)}$. Then, (24) implies that $\bigcup_k C_k \in \mathscr{A}(\lambda)$ and that $\lambda(\bigcup_k C_k) = \sum_k \lambda(C_k)$. If $\alpha \subset \mathbb{N}$ the same conclusion applies to $C_\alpha = \bigcup_{k\in\alpha} C_k$, either trivially (if α is finite) or by representing α in the form of a subsequence of σ, itself an element of $\mathfrak{S}$. □

The original claim of Drewnowski was formulated for the case in which $\mathscr{A}$ is a σ-algebra and, as a consequence, the inclusion $\mathscr{A}_1 \subset \mathscr{A}$ is trivial. The extension obtained in Theorem 5 seems interesting as it essentially allows to assume countable additivity in most proofs involving sequences of sets, in particular those related to weak compactness in the space $ba(\mathscr{A})$. The idea to exploit Drewnowski's result to prove claims on weak compactness is due to Zhang [13]. The following is an extension of his Theorem 1.3.

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Theorem 6 *(Zhang). A set $\mathscr{M} \subset ba(\mathscr{A})$ is weakly compact if and only if the following conditions jointly hold: (i) $\lambda \gg \mathscr{M}$ for some $\lambda \in ba(\mathscr{A})_+$ and (ii) the family $\bar{\mathscr{M}} = \{\bar{\mu} : \mu \in \mathscr{M}\}$, with $\bar{\mu}$ the extension of μ to $\mathscr{A}(\lambda)$, is a weak* compact subset of $ba(\mathscr{A}(\lambda))$.*

Proof. If $\mathscr{M}$ is weakly compact then by [9, IV.9.12] there exists $\lambda \in ba(\mathscr{A})_+$ such that $\mathscr{M}$ is uniformly absolutely continuous with respect to λ. Let $\langle B_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathscr{A}(\lambda)$ and $\lim_n \lambda(B_n) = 0$. Let for each $n = 1, 2, \dots$, $A'_n, A''_n \in \mathscr{A}$ be such that $A'_n \subset B_n \subset A''_n$ and $\sup_{\mu\in\mathscr{M}} |\mu|(A''_n \backslash A'_n) < \varepsilon$. Then, $\lim_n \lambda(A'_n) = 0$ so that $\lim_n \sup_{\mu\in\mathscr{M}} \mu(A'_n) = 0$ and

$$\lim_n \sup_{\mu\in\mathscr{M}} \bar{\mu}(B_n) \le \limsup_n \sup_{\mu\in\mathscr{M}} \mu(A'_n) + \varepsilon = \varepsilon$$

so that $\bar{\mathscr{M}}$ is relatively weakly compact again by [9, IV.9.12]. If $\langle\bar{\mu}_\alpha\rangle_{\alpha\in\mathfrak{A}}$ is a net in $\bar{\mathscr{M}}$ that converges setwise to $\bar{\mu}_0 \in ba(\mathscr{A}(\lambda))$ and $\mu_0 = \bar{\mu}_0|\mathscr{A}$ then necessarily μ_0 is the setwise limit of the net $\langle\mu_\alpha\rangle_{\alpha\in\mathfrak{A}}$, i.e. $\mu_0 \in \mathscr{M}$ and thus $\bar{\mu}_0 \in \bar{\mathscr{M}}$. Thus $\bar{\mathscr{M}}$ is weakly closed and then weakly compact. Weak* compactness follows trivially.

Assume that $\bar{\mathscr{M}}$ is weak* compact. It is then weakly closed and from [3, p. 284] and [7, Proposition 17, p. 8] it is enough to show that for no disjoint sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ in $\mathscr{A}(\lambda)$ one may have

$$\lim_n \sup_{\mu\in\mathscr{M}} |\bar{\mu}(B_n)| > 0 \tag{25}$$

Of course, (25), if true, remains valid upon passing to any subsequence so that, by Theorem 5, we can assume that (λ and therefore) each $\bar{\mu} \in \bar{\mathscr{M}}$ is countably additive in restriction to $\mathscr{A}_1 = \sigma(B_1, B_2, \dots)$. However, the set $\{\bar{\mu}|\mathscr{A}_1 : \mu \in \mathscr{M}\}$ is a weak* compact subset of $ca(\mathscr{A}_1)$ dominated by λ and is thus uniformly countably additive by [13, Theorem 1.1], a fact contradicting (25). □

The extension of $\mathscr{M}$ to $ba(\mathscr{A}(\lambda))$ as a weak* compact set is a delicate property, as we have seen already. The claim of Theorem 6 simplifies considerably in the two special cases considered in Lemma 2 as then condition (*ii*) may be replaced with the assumption that $\mathscr{M}$ is weak* compact. In general, finding conditions on $\mathscr{M}$ which imply weak* compactness of $\bar{\mathscr{M}}$ may be difficult.

The key step in the preceding proof is obtaining a weak* compact set of countably additive measures, such as the set $\{\bar{\mu}|\mathscr{A}_1 : \mu \in \mathscr{M}\}$ above. In general it is easy to construct examples of bounded subsets of $ca(\mathscr{A})$ whose weak* closure contains non-countably additive elements.

Example 3. Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathscr{A}$ with $A_n \ne \varnothing = \bigcap_n A_n$. Define

$$m_n(A) = \frac{\sum_{\{\omega_i\in A\cap A_n\}} 2^{-i}}{\sum_{\{\omega_i\in A_n\}} 2^{-i}} \qquad A \in \mathscr{A}$$

$\langle m_n\rangle_{n\in\mathbb{N}}$ is norm bounded sequence in $ca(\mathscr{A})$ and $m_n(A_j) = 1$ for all $n \ge j$. If m is a weak* cluster point then, necessarily $m(\Omega) = m(A_n) = 1$, for all n and so m is purely finitely additive.

The preceding example may be used to prove the following:

Lemma 3. *The following are equivalent: (i) every bounded sequence in $ca(\mathscr{A})$ has a weak* cluster point in $ca(\mathscr{A})$, (ii) $\mathscr{A}$ is a compact class (see [2, Definition 2.3.3]) and (iii) $ba(\mathscr{A}) = ca(\mathscr{A})$.*

One should note that the implication $(ii) \Rightarrow (iii)$ is just [2, Theorem 2.3.4].

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6. Additional implications

Another possible development of Theorem 1 is the following finitely additive version of a theorem of Yan [12, Theorem 2, p. 220] which is well known in stochastic analysis and mathematical finance:

Corollary 2 *(Yan). Let $\mathcal{K} \subset L^1(\lambda)$ be convex with $0 \in \mathcal{K}$, write $\mathcal{C} = \mathcal{K} - \mathscr{S}(\mathscr{A})_+$ and denote by $\overline{\mathcal{C}}$ the closure of $\mathcal{C}$ in $L^1(\lambda)$. The following are equivalent:*

(i) *for each $f \in L^1(\lambda)_+$ with $\lambda(f) > 0$ there exists $\eta > 0$ such that $\eta f \notin \overline{\mathcal{C}}$;*
(ii) *for each $A \in \mathscr{A}$ with $\lambda(A) > 0$ there exists $d > 0$ such that $d\mathbf{1}_A \notin \overline{\mathcal{C}}$;*
(iii) *there exists a finitely additive probability P on $\mathscr{A}$ such that*
  (a) *$\mathcal{K} \subset L^1(P)$ and $\sup_{k\in\mathcal{K}} P(k) < \infty$,*
  (b) *$\sup_{\{A\in\mathscr{A}:\lambda(A)>0\}} P(A)/\lambda(A) < \infty$ and*
  (c) *$P(A) = 0$ if and only if $\lambda(A) = 0$.*

Proof. The implication $(i) \Rightarrow (ii)$ is obvious. If A and d are as in (ii) there exists a continuous linear functional ϕ^A on $L^1(\lambda)$ and a and b such that

$$\sup_{x\in\overline{\mathcal{C}}} \phi^A(x) < a < b < \phi^A(d1_A)$$

Given that $\mathcal{C}$ contains the convex cone $-\mathscr{S}(\mathscr{A})_+$, that $\mathscr{S}(\mathscr{A})_+$ is dense in $L^1(\lambda)_+$ and that ϕ^A is continuous, we conclude that $\sup_{f\in L^1(\lambda)_+} \phi^A(-f) < \infty$ i.e. that $\phi^A \geq 0$. It follows from [5, Theorem 2] that ϕ^A admits the representation $\phi^A(f) = \mu^A(f)$ for some $\mu^A \in ba(\lambda)_+$. Moreover,

$$\sup_{\{B\in\mathscr{A}:\lambda(B)>0\}} \mu^A(B)/\lambda(B) \leq \sup_{\{f\in L^1(\lambda):\|f\|\leq 1\}} \phi^A(f) = \|\phi^A\| < \infty$$

and $\sup_{f\in\mathcal{C}} \mu^A(f) < a < b < d\mu^A(A)$ so that μ^A meets (a) and (b) above. The inclusion $0 \in \mathcal{C}$ implies $a > 0$ so that $\mu^A(A) > 0$. By normalization we can assume $\|\phi^A\| \vee a \leq 1$. The collection $\mathscr{M} = \{\mu^A : A \in \mathscr{A},\ \lambda(A) > 0\}$ so obtained is dominated by λ and therefore by some $m \in \mathbf{A}(\mathscr{M})$, by Theorem 1. Thus $m \leq \lambda$ and $\sup_{h\in\mathcal{C}} m(h) \leq 1$. If $A \in \mathscr{A}$ and $\lambda(A) > 0$ then $m \gg \mu^A$ implies $m(A) > 0$. The implication $(ii) \Rightarrow (iii)$ follows upon letting P be the finitely additive probability obtained from m by normalization. Let P be as in (iii) so that $L^1(\lambda) \subset L^1(P)$, by (b). If $f \in L^1(\lambda)_+$ and $\lambda(f) > 0$ then $f \wedge n$ converges to f in $L^1(\lambda)$ [9, III.3.6] so that we can assume that f is bounded. Then, by [2, 4.5.7 and 4.5.8], there exists an increasing sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathscr{A})$ with $0 \leq f_n \leq f$ such that f_n converges to f in $L^1(\lambda)$ and therefore in $L^1(P)$ too. For n large enough, then, $\lambda(f_n) > 0$ and, f_n being positive and simple, $P(f_n) > 0$. But then $P(f) = \lim_n P(f_n) > 0$ so that ηf cannot be an element of $\overline{\mathcal{C}}$ for all $\eta > 0$ as $\sup_{h\in\overline{\mathcal{C}}} P(h) < \infty$. □

An application of Corollary 2 is obtained in [4, Lemma 3.1]. Theorem 5 eventually provides a finitely additive version of a useful result of Mukherjee and Summers [11, Lemma 3], illustrating the countable structure of the atoms of an additive set function.[4]

Corollary 3 *(Mukherjee and Summers). Let λ have atoms. There exists a countable, pairwise disjoint collection $G_1, G_2, \ldots$ of λ-atoms of $\mathscr{A}$ such that for any λ-atom $B \in \mathscr{A}$ there exists $n \in \mathbb{N}$ such that $\lambda(B\Delta G_n) = 0$.*

Proof. Apply Theorem 5 with $\mathscr{H}$ the collection of all λ-atoms of $\mathscr{A}$. Let $\langle H_n \rangle_{n\in\mathbb{N}}$ be the corresponding sequence in $\mathscr{H}$ and put $G_n = H_n \backslash \bigcup_{i<n} H_i$. Upon passing to a subsequence if necessary we may assume

[4] I am in debt with an anonymous referee for calling my attention on this paper.

$\lambda(G_n) > 0$ so that $G_n \in \mathscr{H}$ for each $n \in \mathbb{N}$. If $B \in \mathscr{H}$ it follows from (12) that $\lambda(B \cap G_n) > 0$ for some n. Given that B and G_n are atoms then $\lambda(B \backslash G_n) = \lambda(G_n \backslash B) = 0$. □

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