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Global strong solutions for 3D viscous incompressible heat conducting magnetohydrodynamic flows with non-negative density

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ABSTRACT

We study an initial boundary value problem for the nonhomogeneous heat conducting magnetohydrodynamic fluids with non-negative density. Firstly, it is shown that for the initial density allowing vacuum, the strong solution to the problem exists globally if the gradients of velocity and magnetic field satisfy $\|\nabla \mathbf{u}\|_{L^4(0,T;L^2)} + \|\nabla \mathbf{b}\|_{L^4(0,T;L^2)} < \infty$. Then, under some smallness condition, we prove that there is a unique global strong solution to the 3D viscous incompressible heat conducting magnetohydrodynamic flows. Our method relies upon the delicate energy estimates and regularity properties of Stokes system and elliptic equations.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth simply connected domain, the motion of a viscous, incompressible, and heat conducting magnetohydrodynamic fluid in Ω can be described by the following MHD system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div}(2\mu \mathfrak{D}(\mathbf{u})) + \nabla P = \mathbf{b} \cdot \nabla \mathbf{b}, \\ c_v[\partial_t(\rho\theta) + \operatorname{div}(\rho \mathbf{u}\theta)] - \kappa \Delta \theta = 2\mu |\mathfrak{D}(\mathbf{u})|^2 + \nu |\operatorname{curl} \mathbf{b}|^2, \\ \partial_t \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{b} = \nu \Delta \mathbf{b}, \\ \operatorname{div} \mathbf{u} = 0, \ \operatorname{div} \mathbf{b} = 0 \end{cases} \tag{1.1}$$

with the initial condition

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$$(\rho, \mathbf{u}, \theta, \mathbf{b})(0, x) = (\rho_0, \mathbf{u}_0, \theta_0, \mathbf{b}_0)(x), \quad x \in \Omega, \tag{1.2}$$

and the boundary condition

$$\mathbf{u} = \mathbf{0}, \ \frac{\partial \theta}{\partial \mathbf{n}} = 0, \ \mathbf{b} \cdot \mathbf{n} = 0, \ \operatorname{curl} \mathbf{b} \times \mathbf{n} = \mathbf{0}, \ \text{on } \partial\Omega, \tag{1.3}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. Here, $t \geq 0$ is time, $x \in \Omega$ is the spatial coordinate, and ρ, $\mathbf{u}$, θ, P, $\mathbf{b}$ are the fluid density, velocity, absolute temperature, pressure, and the magnetic field, respectively; $\mathfrak{D}(\mathbf{u})$ denotes the deformation tensor given by

$$\mathfrak{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^{tr}).$$

The constant $\mu > 0$ is the viscosity coefficient. Positive constants c_v and κ are respectively the heat capacity, the ratio of the heat conductivity coefficient over the heat capacity, and $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field.

Magnetohydrodynamics studies the dynamics of electrically conducting fluids and the theory of the macroscopic interaction of electrically conducting fluids with a magnetic field. The issues of well-posedness and dynamical behaviors of MHD system are rather complicated to investigate because of the strong coupling and interplay interaction between the fluid motion and the magnetic field. In the absence of magnetic field, that is, $\mathbf{b} = \mathbf{0}$, the MHD system reduces to the Navier–Stokes equations. In general, due to the similarity of the second equation and the third equation in (1.1), the study for MHD system has been along with that for Navier–Stokes one.

When $\mathbf{b} = \mathbf{0}$ and $\theta = 0$, the system (1.1) reduces to the well-known nonhomogeneous incompressible Navier–Stokes equations and there are a lot of results on the existence in the literature. In the case that the viscosity μ is a constant, Kazhikov [16] (see also [2]) proved that when ρ_0 is bounded away from zero, the nonhomogeneous Navier–Stokes equations have at least one global weak solution in the energy space. In addition, he also proved the global existence of strong solutions to this system for small data in three space dimensions and all data in two dimensions. When the initial data may contain vacuum states, Simon [25] obtained the global existence of weak solutions, and Choe–Kim [6] proposed a compatibility condition and investigated the local existence of strong solutions, which was later improved by Craig–Huang–Wang [7] for global strong small solutions. On the other hand, in the case that μ depends on the density ρ, Lions [20, Chapter 2] established the global existence of weak solutions to nonhomogeneous Navier–Stokes equations in any space dimensions for the initial density allowing vacuum. Recently, Cho–Kim [4] proved the local existence of unique strong solutions for initial data satisfying a natural compatibility condition. Very recently, Huang–Wang [15], and independently by Zhang [29], showed the global existence of strong solutions on bounded domains under some smallness assumption. There are also very interesting investigations about the existence of strong solutions to the 2D nonhomogeneous Navier–Stokes equations, refer to [14,19,21,24].

For the study of nonhomogeneous incompressible MHD system (i.e., $\theta = 0$ in (1.1)), Gerbeau and Le Bris [11], Desjardins and Le Bris [8] studied the global existence of weak solutions with finite energy on 3D bounded domains and on the torus, respectively. In the presence of vacuum, under the following compatibility conditions,

$$\operatorname{div} \mathbf{u}_0 = \operatorname{div} \mathbf{b}_0 = 0, \ \ -\mu\Delta\mathbf{u}_0 + \nabla P_0 - (\mathbf{b}_0 \cdot \nabla)\mathbf{b}_0 = \sqrt{\rho_0}\mathbf{g}, \quad \text{in} \ \ \Omega, \tag{1.4}$$

where $(P_0, \mathbf{g}) \in H^1 \times L^2$ and $\Omega = \mathbb{R}^3$, Chen–Tan–Wang [3] obtained the local existence of strong solutions to the 3D Cauchy problem of (1.1). Later, Gong–Li [12] showed that the local strong solution obtained in

[3] is indeed a global one provided that some suitable smallness conditions hold true. Without compatibility conditions as that of (1.4), for small initial velocity and magnetic field in suitable norm, Li–Wang [18] proved the global existence of strong solution for a 3D nonhomogeneous incompressible magnetohydrodynamic flow in a bounded domain $\Omega \subset \mathbb{R}^3$ with $C^{2+\varepsilon}$ boundary for some $\varepsilon > 0$, as well as the weak-strong uniqueness. There are also very interesting investigations about the existence of strong solutions to the 2D nonhomogeneous incompressible MHD system, refer to [9,13,22,23].

Recently, the local and global existence of strong solutions to the 3D viscous incompressible heat conducting Navier–Stokes flows with non-negative density were established, by Choe–Kim [5] and Zhong [30], respectively. At the same time, the local unique strong solution to the 3D viscous incompressible heat conducting magnetohydrodynamic flows with large initial data was obtained by Wu [27]. However, whether the unique local strong solution to the problem (1.1)–(1.3) can exist globally is still unknown. In fact, this is the main aim of this paper.

Before stating our main results, we first explain the notations and conventions used throughout this paper. We denote by

$$\int \cdot \mathrm{d}x = \int_{\Omega} \cdot \mathrm{d}x.$$

For $1 \le p \le \infty$ and integer $k \ge 0$, the standard Sobolev spaces are denoted by:

$$\begin{cases} L^p = L^p(\Omega),\ W^{k,p} = W^{k,p}(\Omega),\ H^k = H^{k,2}(\Omega), \\ H_0^1 = \{u \in H^1 | u = 0 \text{ on } \partial\Omega\},\ H_{\mathbf{n}}^2 = \{u \in H^2 | \nabla u \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}. \end{cases}$$

Now we define precisely what we mean by strong solutions to the problem (1.1)–(1.3).

Definition 1.1. $(\rho, \mathbf{u}, \theta, \mathbf{b})$ is called a strong solution to (1.1)–(1.3) in $\Omega \times (0,T)$, if for some $q_0 > 3$,

$$\begin{cases} \rho \ge 0,\ \rho \in C([0,T]; W^{1,q_0}),\ \rho_t \in C([0,T]; L^{q_0}), \\ \mathbf{u} \in C([0,T]; H_0^1 \cap H^2) \cap L^2(0,T; W^{2,q_0}), \\ \theta \ge 0,\ (\theta, \mathbf{b}) \in C([0,T]; H^2) \cap L^2(0,T; W^{2,q_0}), \\ (\mathbf{b}_t, \mathbf{u}_t, \theta_t) \in L^2(0,T; H^1),\ (\mathbf{b}_t, \sqrt{\rho}\mathbf{u}_t, \sqrt{\rho}\theta_t) \in L^\infty(0,T; L^2). \end{cases}$$

Furthermore, both (1.1) and (1.2) hold almost everywhere in $\Omega \times (0,T)$.

Our main results read as follows:

Theorem 1.1. *For constant $q \in (3,6]$, assume that the initial data $(\rho_0 \ge 0, \mathbf{u}_0, \theta_0 \ge 0, \mathbf{b}_0)$ satisfy*

$$\begin{cases} \rho_0 \in W^{1,q}(\Omega),\ \mathbf{u}_0 \in H_0^1(\Omega) \cap H^2(\Omega),\ \theta_0 \in H_{\mathbf{n}}^2(\Omega),\ \operatorname{div} \mathbf{u}_0 = 0, \\ \mathbf{b}_0 \in H^2(\Omega),\ \mathbf{b}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \operatorname{curl} \mathbf{b}_0 \times \mathbf{n}|_{\partial\Omega} = \mathbf{0},\ \operatorname{div} \mathbf{b}_0 = 0, \end{cases} \tag{1.5}$$

and the compatibility conditions

$$\begin{cases} -\mu\Delta\mathbf{u}_0 + \nabla P_0 - \mathbf{b}_0 \cdot \nabla\mathbf{b}_0 = \sqrt{\rho_0}\mathbf{g}_1, \\ \kappa\Delta\theta_0 + 2\mu|\mathfrak{D}(\mathbf{u}_0)|^2 + \nu|\operatorname{curl}\mathbf{b}_0|^2 = \sqrt{\rho_0}\mathbf{g}_2, \end{cases} \tag{1.6}$$

for some $P_0 \in H^1(\Omega)$ and $\mathbf{g}_1, \mathbf{g}_2 \in L^2(\Omega)$. Let $(\rho, \mathbf{u}, \theta, \mathbf{b})$ be a strong solution to the problem (1.1)–(1.3). If $T^* < \infty$ is the maximal time of existence for that solution, then we have

$$\lim_{T\to T^*} \left(\|\nabla \mathbf{u}\|_{L^4(0,T;L^2)} + \|\nabla \mathbf{b}\|_{L^4(0,T;L^2)}\right) = \infty. \tag{1.7}$$

Remark 1.1. The local existence of a strong solution with initial data as in Theorem 1.1 has been established in [27]. Hence, the maximal time T^* is well-defined. Moreover, the same criterion holds true in the periodic case.

Remark 1.2. When there is no electromagnetic field effect, that is $\mathbf{b} = \mathbf{0}$, (1.1) turns to be the viscous heat conducting Navier–Stokes flows, and Theorem 1.1 is the same as [30, Theorem 1.1]. Roughly speaking, we generalize [30, Theorem 1.1] to the heat conducting MHD flows.

We will prove Theorem 1.1 by contradiction in Section 3. In fact, the proof of the theorem is based on a priori estimates under the assumption that $\|\nabla \mathbf{u}\|_{L^4(0,T;L^2)} + \|\nabla \mathbf{b}\|_{L^4(0,T;L^2)}$ is bounded independent of any $T \in (0, T^*)$. The a priori estimates are then sufficient for us to apply the local existence result repeatedly to extend a local solution beyond the maximal time of existence T^*, consequently, contradicting the maximality of T^*.

Based on Theorem 1.1, we can establish the global existence of strong solutions to (1.1)–(1.3) under some smallness condition.

Theorem 1.2. *Let the conditions in Theorem 1.1 be in force. Then there exists a small positive constant ε_0 depending only on $\|\rho_0\|_{L^\infty}$, μ, ν, and Ω such that if*

$$\left(\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{b}_0\|_{L^2}^2\right)\left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + \|\operatorname{curl} \mathbf{b}_0\|_{L^2}^2\right) \le \varepsilon_0, \tag{1.8}$$

the system (1.1)–(1.3) has a unique global strong solution.

The rest of this paper is organized as follows. In Section 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the proof of Theorem 1.1. Finally, we give the proof of Theorem 1.2 in Section 4.

2. Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used frequently later.

We begin with the following Gronwall's inequality, which plays a central role in proving a priori estimates on strong solutions $(\rho, \mathbf{u}, \theta, \mathbf{b})$.

Lemma 2.1. *Suppose that h and r are integrable on (a, b) and nonnegative a.e. in (a, b). Further assume that $y \in C[a, b]$, $y' \in L^1(a, b)$, and*

$$y'(t) \le h(t) + r(t)y(t) \quad \text{for a.e } t \in (a, b).$$

Then

$$y(t) \le \left[y(a) + \int_a^t h(s)\exp\left(-\int_a^s r(\tau)d\tau\right)ds\right]\exp\left(\int_a^t r(s)ds\right), \quad t \in [a, b].$$

Proof. See [26, pp. 12–13]. □

Next, the following well-known inequalities will be frequently used later.

Lemma 2.2. *For $p \in [2,6]$, $q, m \in [1,\infty)$, $\alpha \in (0,m)$, $\varepsilon > 0$, $a, b \in \mathbb{R}$, and $\theta \in (0,1)$, it holds that*

$$\|f\|_{L^m} \le \|f\|_{L^{\alpha q}}^{\frac{\alpha}{m}} \|f\|_{L^{(m-\alpha)q}}^{1-\frac{\alpha}{m}}, \qquad \text{(Hölder's inequality)}$$

$$|ab| \le \varepsilon |a|^{\frac{1}{\theta}} + \left(\frac{\theta}{\varepsilon}\right)^{\frac{\theta}{1-\theta}} (1-\theta)|b|^{\frac{1}{1-\theta}}, \qquad \text{(Young's inequality)}$$

$$\|g\|_{L^p} \le C(p,\Omega)\|g\|_{H^1} \quad \text{for} \quad g \in H^1(\Omega), \quad \text{(Sobolev's inequality)}$$

and

$$\|h\|_{L^r(\partial\Omega)} \le C(r,\Omega)\|h\|_{L^r(\Omega)}^{1-\frac{1}{r}} \|h\|_{W^{1,r}(\Omega)}^{\frac{1}{r}} \quad \text{for} \quad h \in W^{1,r}(\Omega). \quad \text{(Trace inequality)}$$

Proof. See [17, Chapter 2]. □

The following divergence theorem is useful in the sequel.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain. If $\boldsymbol{\phi} \in H^1(\Omega)$ and $\mathbf{v} \in L^2(\Omega)$ such that $\operatorname{curl} \mathbf{v} \in L^2(\Omega)$, then we have*

$$\int_\Omega \operatorname{curl} \mathbf{v} \cdot \boldsymbol{\phi} dx = \int_\Omega \mathbf{v} \cdot \operatorname{curl} \boldsymbol{\phi} dx + \int_{\partial\Omega} (\mathbf{v} \times \mathbf{n}) \cdot \boldsymbol{\phi} d\sigma.$$

Proof. For any vector field $\mathbf{w} \in H^1(\Omega)$, the well known divergence theorem gives rise to

$$\int_\Omega \operatorname{div} \mathbf{w} dx = \int_{\partial\Omega} \mathbf{w} \cdot \mathbf{n} d\sigma,$$

which combined with the vector identity

$$\operatorname{div}(\mathbf{v} \times \boldsymbol{\phi}) = \boldsymbol{\phi} \cdot \operatorname{curl} \mathbf{v} - \mathbf{v} \cdot \operatorname{curl} \boldsymbol{\phi}$$

yields the desired result. □

Next, the following lemma is useful.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded simply connected domain. There exists a constant C such that for any $\boldsymbol{\psi} \in \{\mathbf{f} \in H^1(\Omega) : \operatorname{div} \mathbf{f} = 0,\ \mathbf{f} \cdot \mathbf{n}|_{\partial\Omega} = 0\}$, it holds*

$$\|\boldsymbol{\psi}\|_{H^1} \le C\|\operatorname{curl} \boldsymbol{\psi}\|_{L^2}.$$

Proof. See [28, p. 1033]. □

Finally, we give some regularity results for the following Stokes system

$$\begin{cases} -\mu \Delta \mathbf{U} + \nabla P = \mathbf{F}, & x \in \Omega, \\ \operatorname{div} \mathbf{U} = 0, & x \in \Omega, \\ \mathbf{U} = \mathbf{0}, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Lemma 2.5. *Let $m \geq 2$ be an integer, r any real number with $1 < r < \infty$ and let Ω be a bounded domain of $\mathbb{R}^3$ of class $C^{m-1,1}$. Let $\mathbf{F} \in W^{m-2,r}(\Omega)$ be given. Then the Stokes system (2.1) has a unique solution $\mathbf{U} \in W^{m,r}(\Omega)$ and $P \in W^{m-1,r}(\Omega)/\mathbb{R}$. In addition, there exists a constant $C > 0$ depending only on m, r, and Ω such that*

$$\|\mathbf{U}\|_{W^{m,r}} + \|P\|_{W^{m-1,r}/\mathbb{R}} \leq C\|\mathbf{F}\|_{W^{m-2,r}}.$$

Proof. See [1, Theorem 4.8]. $\square$

3. Proof of Theorem 1.1

Let $(\rho, \mathbf{u}, \theta, \mathbf{b})$ be a strong solution described in Theorem 1.1. Suppose that (1.7) were false, that is, there exists a constant $M_0 > 0$ such that

$$\lim_{T\to T^*} \left(\|\nabla\mathbf{u}\|_{L^4(0,T;L^2)} + \|\nabla\mathbf{b}\|_{L^4(0,T;L^2)}\right) \leq M_0 < \infty. \tag{3.1}$$

Rewrite the system (1.1) as

$$\begin{cases} \rho_t + \mathbf{u}\cdot\nabla\rho = 0, \\ \rho\mathbf{u}_t + \rho\mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} + \nabla P = \mathbf{b}\cdot\nabla\mathbf{b}, \\ c_v[\rho\theta_t + \rho\mathbf{u}\cdot\nabla\theta] - \kappa\Delta\theta = 2\mu|\mathfrak{D}(\mathbf{u})|^2 + \nu|\operatorname{curl}\mathbf{b}|^2, \\ \mathbf{b}_t - \mathbf{b}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{b} + \nu\operatorname{curl}(\operatorname{curl}\mathbf{b}) = \mathbf{0}, \\ \operatorname{div}\mathbf{u} = 0,\ \operatorname{div}\mathbf{b} = 0. \end{cases} \tag{3.2}$$

In this section, C stands for a generic positive constant which may depend on M_0, μ, c_v, κ, ν, T^*, and the initial data.

First, since $\operatorname{div}\mathbf{u} = 0$, we have the following well-known estimate on the $L^\infty(0,T;L^\infty)$-norm of the density.

Lemma 3.1. *It holds that for any $t \in (0, T^*)$,*

$$\|\rho(t)\|_{L^\infty} = \|\rho_0\|_{L^\infty}. \tag{3.3}$$

Proof. See [20, Theorem 2.1]. $\square$

The following lemma gives the basic energy estimates.

Lemma 3.2. *It holds that for any $T \in (0, T^*)$,*

$$\begin{aligned} &\sup_{0\leq t\leq T}\left(c_v\|\rho\theta\|_{L^1} + \|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2\right) + \int_0^T \left(\mu\|\nabla\mathbf{u}\|_{L^2}^2 + \nu\|\operatorname{curl}\mathbf{b}\|_{L^2}^2\right) dt \\ &\leq c_v\|\rho_0\theta_0\|_{L^1} + \|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{b}_0\|_{L^2}^2. \end{aligned} \tag{3.4}$$

Proof. First, applying standard maximum principle to $(3.2)_3$ along with $\theta_0 \geq 0$ shows (see [10, p. 43])

$$\inf_{\Omega\times[0,T]} \theta \geq 0. \tag{3.5}$$

Multiplying $(3.2)_2$ by $\mathbf{u}$ and integrating (by parts) over Ω, we derive that

$$\frac{1}{2}\frac{d}{dt}\int \rho|\mathbf{u}|^2 dx + \mu \int |\nabla \mathbf{u}|^2 dx = \int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}dx. \tag{3.6}$$

Multiplying $(3.2)_4$ by $\mathbf{b}$ and integrating (by parts) over Ω, we get after using Lemma 2.3 and (1.3) that

$$\frac{1}{2}\frac{d}{dt}\int |\mathbf{b}|^2 dx + \nu \int |\operatorname{curl}\mathbf{b}|^2 dx = \int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx - \int \mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}dx. \tag{3.7}$$

Due to $\operatorname{div}\mathbf{b} = 0$ and $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$, we have

$$\int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}dx = \int b^i\partial_i b^j u^j dx = -\int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx. \tag{3.8}$$

Similarly, one obtains

$$-\int \mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}dx = -\int u^i\partial_i b^j b^j dx = \int \mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}dx,$$

and thus

$$\int \mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}dx = 0. \tag{3.9}$$

Combining (3.6)–(3.9), we deduce that

$$\frac{1}{2}\frac{d}{dt}\int \left(\rho|\mathbf{u}|^2 + |\mathbf{b}|^2\right) dx + \int \left(\mu|\nabla\mathbf{u}|^2 + \nu|\operatorname{curl}\mathbf{b}|^2\right) dx = 0. \tag{3.10}$$

Integrating $(3.2)_3$ with respect to the spatial variable gives rise to

$$c_v\frac{d}{dt}\int \rho\theta dx - 2\mu\int |\mathfrak{D}(\mathbf{u})|^2 dx - \nu\int |\operatorname{curl}\mathbf{b}|^2 dx = 0. \tag{3.11}$$

Substituting (3.11) into (3.10) and noting that

$$\begin{aligned}
-2\mu\int |\mathfrak{D}(\mathbf{u})|^2 dx &= -\frac{\mu}{2}\int (\partial_i u^j + \partial_j u^i)^2 dx \\
&= -\mu\int |\nabla\mathbf{u}|^2 dx - \mu\int \partial_i u^j \partial_j u^i dx \\
&= -\mu\int |\nabla\mathbf{u}|^2 dx,
\end{aligned}$$

we derive

$$\frac{d}{dt}\int \left(c_v\rho\theta + \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{b}|^2\right) dx = 0. \tag{3.12}$$

Integrating (3.10) and (3.12) with respect to time and adding the resulting equations lead to

$$\int \left(c_v\rho\theta + \rho|\mathbf{u}|^2 + |\mathbf{b}|^2\right) dx + \int_0^t\int \left(\mu|\nabla\mathbf{u}|^2 + \nu|\operatorname{curl}\mathbf{b}|^2\right) dxds = \int \left(c_v\rho_0\theta_0 + \rho_0|\mathbf{u}_0|^2 + |\mathbf{b}_0|^2\right) dx.$$

This implies the desired (3.4) and consequently completes the proof. □

Next, the following lemma concerns the key time-independent estimates on the $L^\infty(0,T;L^2)$-norm of the gradients of the velocity and the magnetic field.

Lemma 3.3. *Under the condition (3.1), it holds that for any $T\in(0,T^*)$,*

$$\sup_{0\le t\le T}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\operatorname{curl}\mathbf{b}\|_{L^2}^2+\|\mathbf{b}\|_{L^4}^4\right)+\int_0^T\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2+\|\mathbf{u}\|_{H^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)dt\le C. \tag{3.13}$$

Proof. Multiplying $(3.2)_2$ by $\mathbf{u}_t$ and integrating the resulting equations over Ω, we derive from Cauchy–Schwarz inequality that

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\int|\nabla\mathbf{u}|^2dx+\int\rho|\mathbf{u}_t|^2dx\\
&=\int\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}_tdx-\int\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_tdx\\
&=-\frac{d}{dt}\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\int[\mathbf{b}_t\cdot\nabla\mathbf{u}\cdot\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}_t-\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t]\,dx\\
&\le-\frac{d}{dt}\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\frac12\int\left(\rho|\mathbf{u}_t|^2+|\mathbf{b}_t|^2\right)dx+\int\left(2\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2+8|\mathbf{b}|^2|\nabla\mathbf{u}|^2\right)dx,
\end{aligned}$$

and thus

$$\begin{aligned}
&\frac{d}{dt}\int\left(\mu|\nabla\mathbf{u}|^2+2\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\right)dx+\int\rho|\mathbf{u}_t|^2dx\\
&\le\int|\mathbf{b}_t|^2dx+\int\left(4\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2+16|\mathbf{b}|^2|\nabla\mathbf{u}|^2\right)dx.
\end{aligned} \tag{3.14}$$

Multiplying $(3.2)_4$ by $\mathbf{b}_t$ and integrating by parts yield

$$\begin{aligned}
\nu\frac{d}{dt}\int|\operatorname{curl}\mathbf{b}|^2dx+2\int|\mathbf{b}_t|^2dx&=2\int(\mathbf{b}\cdot\nabla\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{b})\cdot\mathbf{b}_tdx\\
&\le\frac12\int|\mathbf{b}_t|^2dx+8\int\left(|\mathbf{b}|^2|\nabla\mathbf{u}|^2+|\mathbf{u}|^2|\nabla\mathbf{b}|^2\right)dx,
\end{aligned} \tag{3.15}$$

which combined with (3.14) implies

$$\begin{aligned}
&\frac{d}{dt}\int\left(\mu|\nabla\mathbf{u}|^2+\nu|\operatorname{curl}\mathbf{b}|^2+2\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\right)dx+\int\left(\rho|\mathbf{u}_t|^2+|\mathbf{b}_t|^2\right)dx\\
&\le C\int\left(\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2+|\mathbf{b}|^2|\nabla\mathbf{u}|^2+|\mathbf{u}|^2|\nabla\mathbf{b}|^2\right)dx.
\end{aligned} \tag{3.16}$$

Recall that $(\mathbf{u},P)$ satisfies the following Stokes system

$$\begin{cases}
-\mu\Delta\mathbf{u}+\nabla P=-\rho\mathbf{u}_t-\rho\mathbf{u}\cdot\nabla\mathbf{u}+\mathbf{b}\cdot\nabla\mathbf{b}, & x\in\Omega,\\
\operatorname{div}\mathbf{u}=0, & x\in\Omega,\\
\mathbf{u}=\mathbf{0}, & x\in\partial\Omega.
\end{cases}$$

Applying Lemma 2.5 with $\mathbf{F}\triangleq-\rho\mathbf{u}_t-\rho\mathbf{u}\cdot\nabla\mathbf{u}+\mathbf{b}\cdot\nabla\mathbf{b}$, we obtain from (3.3) that

$$\begin{aligned}\|\mathbf{u}\|_{H^2}^2 &\le C\left(\|\rho\mathbf{u}_t\|_{L^2}^2+\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2}^2\right)\\ &\le C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2}^2\right).\end{aligned}\tag{3.17}$$

It follows from the standard L^2-estimates of elliptic system and $(3.2)_4$ that

$$\|\nabla^2\mathbf{b}\|_{L^2}^2\le C\left(\|\mathbf{b}_t\|_{L^2}^2+\|\mathbf{u}\cdot\nabla\mathbf{b}\|_{L^2}^2+\|\mathbf{b}\cdot\nabla\mathbf{u}\|_{L^2}^2\right),\tag{3.18}$$

which together with (3.17) leads to

$$\begin{aligned}\|\mathbf{u}\|_{H^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2 &\le K\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)+C\left(\|\sqrt{\rho}\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2}^2\right)\\ &\quad+C\left(\|\mathbf{u}\cdot\nabla\mathbf{b}\|_{L^2}^2+\|\mathbf{b}\cdot\nabla\mathbf{u}\|_{L^2}^2\right).\end{aligned}\tag{3.19}$$

Adding (3.19) multiplied by $\frac{1}{2K}$ to (3.16), we get from Lemmas 2.2 and 2.4 that

$$\begin{aligned}&A'(t)+\frac{1}{2}\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)+\frac{1}{2K}\left(\|\mathbf{u}\|_{H^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)\\ &\le C\int\left(\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2+|\mathbf{b}|^2|\nabla\mathbf{u}|^2+|\mathbf{u}|^2|\nabla\mathbf{b}|^2+|\mathbf{b}|^2|\nabla\mathbf{b}|^2\right)dx\\ &\le C\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^6}+C\|\mathbf{b}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^6}\\ &\quad+C\|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{b}\|_{L^2}^2+C\|\mathbf{b}\|_{L^\infty}^2\|\nabla\mathbf{b}\|_{L^2}^2\\ &\le C\|\nabla\mathbf{u}\|_{L^2}^3\|\nabla\mathbf{u}\|_{H^1}+C\|\operatorname{curl}\mathbf{b}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\\ &\quad+C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\|\nabla\mathbf{b}\|_{L^2}^2+C\|\nabla\mathbf{b}\|_{L^2}^3\|\nabla\mathbf{b}\|_{H^1}\\ &\le\frac{1}{2}\left(\|\mathbf{u}\|_{H^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)+C+C\left(\|\nabla\mathbf{u}\|_{L^2}^4+\|\nabla\mathbf{b}\|_{L^2}^4\right)\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2\right),\end{aligned}$$

and thus

$$\begin{aligned}&A'(t)+\frac{1}{2}\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)+\frac{1}{2K}\left(\|\mathbf{u}\|_{H^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)\\ &\le C+C\left(\|\nabla\mathbf{u}\|_{L^2}^4+\|\nabla\mathbf{b}\|_{L^2}^4\right)\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2\right),\end{aligned}\tag{3.20}$$

where we have used the following Sobolev's inequality

$$\|\mathbf{v}\|_{L^\infty}\le C\|\nabla\mathbf{v}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{v}\|_{H^1}^{\frac{1}{2}}\quad\text{for}\quad\mathbf{v}\in H_0^1\cap H^2.$$

Here

$$A(t)\triangleq\int\left(\mu|\nabla\mathbf{u}|^2+\nu|\operatorname{curl}\mathbf{b}|^2+2\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\right)dx$$

satisfies

$$\frac{\mu}{2}\|\nabla\mathbf{u}\|_{L^2}^2+\nu\|\operatorname{curl}\mathbf{b}\|_{L^2}^2-\frac{8}{\mu}\|\mathbf{b}\|_{L^4}^4\le A(t)\le\frac{3\mu}{2}\|\nabla\mathbf{u}\|_{L^2}^2+\nu\|\operatorname{curl}\mathbf{b}\|_{L^2}^2+\frac{8}{\mu}\|\mathbf{b}\|_{L^4}^4\tag{3.21}$$

due to

$$\left|\int 2\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx\right|\le\frac{\mu}{2}\|\nabla\mathbf{u}\|_{L^2}^2+\frac{8}{\mu}\|\mathbf{b}\|_{L^4}^4.$$

Multiplying $(3.2)_4$ by $|\mathbf{b}|^2\mathbf{b}$ and integrating (by parts) over Ω, we deduce from Lemma 2.2 that

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\int|\mathbf{b}|^4dx+\frac{1}{2}\int|\nabla|\mathbf{b}|^2|^2dx+\frac{1}{4}\int|\mathbf{b}|^2|\nabla\mathbf{b}|^2dx\\
&=-\int_{\partial\Omega}|\mathbf{b}|^2(\mathbf{b}\cdot\nabla)\mathbf{n}\cdot\mathbf{b}d\sigma+\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot|\mathbf{b}|^2\mathbf{b}dx\\
&=-\int_{\partial\Omega}|\mathbf{b}|^2(\mathbf{b}\cdot\nabla)\mathbf{n}\cdot\mathbf{b}d\sigma-\int b^i\partial_i(|\mathbf{b}|^2b^j)u^jdx\\
&\le C\int_{\partial\Omega}|\mathbf{b}|^4d\sigma+\frac{1}{8}\int|\nabla|\mathbf{b}|^2|^2dx+\frac{1}{4}\int|\mathbf{b}|^2|\nabla\mathbf{b}|^2dx+C\int|\mathbf{b}|^4|\mathbf{u}|^2dx\\
&\le C\int|\mathbf{b}|^4dx+\frac{1}{4}\int|\nabla|\mathbf{b}|^2|^2dx+\frac{1}{4}\int|\mathbf{b}|^2|\nabla\mathbf{b}|^2dx+C\|\mathbf{b}\|_{L^6}^4\|\mathbf{u}\|_{L^6}^2\\
&\le C\int|\mathbf{b}|^4dx+\frac{1}{4}\int|\nabla|\mathbf{b}|^2|^2dx+\frac{1}{4}\int|\mathbf{b}|^2|\nabla\mathbf{b}|^2dx+C\|\nabla\mathbf{b}\|_{L^2}^4\|\nabla\mathbf{u}\|_{L^2}^2,
\end{aligned}
$$

which yields

$$
\frac{d}{dt}\int|\mathbf{b}|^4dx+\int\left(|\nabla|\mathbf{b}|^2|^2+|\mathbf{b}|^2|\nabla\mathbf{b}|^2\right)dx\le C\|\mathbf{b}\|_{L^4}^4+C\|\nabla\mathbf{b}\|_{L^2}^4\|\nabla\mathbf{u}\|_{L^2}^2. \tag{3.22}
$$

Now, adding (3.22) multiplied by $\frac{10}{\mu}$ to (3.20), we obtain that

$$
\begin{aligned}
&\frac{d}{dt}\left(A(t)+\frac{10}{\mu}\|\mathbf{b}\|_{L^4}^4\right)+\frac{1}{2}\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)+\frac{1}{2K}\left(\|\mathbf{u}\|_{H^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)\\
&\le C+C\|\mathbf{b}\|_{L^4}^4+C\left(\|\nabla\mathbf{u}\|_{L^2}^4+\|\nabla\mathbf{b}\|_{L^2}^4\right)\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2\right).
\end{aligned} \tag{3.23}
$$

This combined with (3.21), Gronwall's inequality, and (3.1) implies the desired (3.13). This finishes the proof of Lemma 3.4. □

Finally, the following lemma will deal with the higher order estimates of the solutions which are needed to guarantee the extension of the local strong solution to be a global one.

Lemma 3.4. *Under the condition (3.1), it holds that for any $T\in(0,T^*)$,*

$$
\sup_{0\le t\le T}\left(\|\rho\|_{W^{1,q}}+\|\mathbf{u}\|_{H^2}^2+\|\theta\|_{H^2}^2+\|\mathbf{b}\|_{H^2}^2\right)\le C. \tag{3.24}
$$

Proof. Differentiating $(3.2)_2$ with respect to t and using $(1.1)_1$, we arrive at

$$
\rho\mathbf{u}_{tt}+\rho\mathbf{u}\cdot\nabla\mathbf{u}_t-\mu\Delta\mathbf{u}_t=-\nabla P_t+\operatorname{div}(\rho\mathbf{u})\left(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}\right)-\rho\mathbf{u}_t\cdot\nabla\mathbf{u}+\mathbf{b}_t\cdot\nabla\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{b}_t. \tag{3.25}
$$

Multiplying (3.25) by $\mathbf{u}_t$ and integrating (by parts) over Ω yield

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int\rho|\mathbf{u}_t|^2dx+\mu\int|\nabla\mathbf{u}_t|^2dx&=\int\operatorname{div}(\rho\mathbf{u})|\mathbf{u}_t|^2dx+\int\operatorname{div}(\rho\mathbf{u})\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_tdx\\
&\quad-\int\rho\mathbf{u}_t\cdot\nabla\mathbf{u}\cdot\mathbf{u}_tdx+\int\mathbf{b}_t\cdot\nabla\mathbf{b}\cdot\mathbf{u}_tdx\\
&\quad+\int\mathbf{b}\cdot\nabla\mathbf{b}_t\cdot\mathbf{u}_tdx\triangleq\sum_{k=1}^{5}J_k.
\end{aligned} \tag{3.26}
$$

It should be noted that though the solution $(\rho, \mathbf{u}, P, \theta, \mathbf{b})$ is not regular enough to justify the derivation of (3.26), one can prove it rigorously by an appropriate regularization procedure. By virtue of Hölder's inequality, Sobolev's inequality, (3.3), (3.13), and Lemma 2.4, we find that

$$
\begin{aligned}
|J_1| = & \left| -\int \rho\mathbf{u}\cdot\nabla|\mathbf{u}_t|^2 dx \right| \\
\le & 2\|\rho\|_{L^\infty}^{\frac12}\|\mathbf{u}\|_{L^6}\|\sqrt{\rho}\mathbf{u}_t\|_{L^3}\|\nabla\mathbf{u}_t\|_{L^2} \\
\le & C\|\rho\|_{L^\infty}^{\frac12}\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2} \\
\le & C\|\rho\|_{L^\infty}^{\frac34}\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac32} \\
\le & \frac{\mu}{10}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2; \\
|J_2| = & \left| -\int \rho\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t) dx \right| \\
\le & \int \left(\rho|\mathbf{u}||\nabla\mathbf{u}|^2|\mathbf{u}_t| + \rho|\mathbf{u}|^2|\nabla^2\mathbf{u}||\mathbf{u}_t| + \rho|\mathbf{u}|^2|\nabla\mathbf{u}||\nabla\mathbf{u}_t|\right) dx \\
\le & \|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^6}\|\mathbf{u}_t\|_{L^6} + \|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^6}^2\|\nabla^2\mathbf{u}\|_{L^2}\|\mathbf{u}_t\|_{L^6} \\
& + \|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}_t\|_{L^2} \\
\le & C\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{u}\|_{H^2}\|\nabla\mathbf{u}_t\|_{L^2} \\
\le & \frac{\mu}{10}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2; \\
|J_3| \le & \|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^4}^2 \le \|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac32} \\
\le & C\|\rho\|_{L^\infty}^{\frac34}\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac32} \\
\le & \frac{\mu}{10}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2; \\
|J_4| \le & \|\mathbf{b}_t\|_{L^3}\|\nabla\mathbf{b}\|_{L^2}\|\mathbf{u}_t\|_{L^6} \le C\|\mathbf{b}_t\|_{L^2}^{\frac12}\|\mathbf{b}_t\|_{L^6}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2} \\
\le & \frac{\mu}{10}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C(\delta)\|\mathbf{b}_t\|_{L^2}^2 + \frac{\delta}{2}\|\operatorname{curl}\mathbf{b}_t\|_{L^2}^2; \\
|J_5| = & \left| -\int \mathbf{b}\cdot\nabla\mathbf{u}_t\cdot\mathbf{b}_t dx \right| \\
\le & \|\mathbf{b}\|_{L^6}\|\nabla\mathbf{u}_t\|_{L^2}\|\mathbf{b}_t\|_{L^3} \le C\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}\|\mathbf{b}_t\|_{L^2}^{\frac12}\|\mathbf{b}_t\|_{L^6}^{\frac12} \\
\le & \frac{\mu}{10}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C(\delta)\|\mathbf{b}_t\|_{L^2}^2 + \frac{\delta}{2}\|\operatorname{curl}\mathbf{b}_t\|_{L^2}^2.
\end{aligned}
$$

Substituting the above estimates into (3.26), we derive that

$$
\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \mu\|\nabla\mathbf{u}_t\|_{L^2}^2 \le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}\|_{H^2}^2 + C\|\mathbf{b}_t\|_{L^2}^2 + \delta\|\operatorname{curl}\mathbf{b}_t\|_{L^2}^2. \tag{3.27}
$$

Next, differentiating $(3.2)_4$ with respect to t and multiplying the resulting equations by $\mathbf{b}_t$, we obtain after integration by parts and using (3.13) and Lemma 2.4 that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |\mathbf{b}_t|^2dx+\nu\int|\operatorname{curl}\mathbf{b}_t|^2dx &\le C\left(\||\mathbf{u}_t||\mathbf{b}|\|_{L^2}+\||\mathbf{u}||\mathbf{b}_t|\|_{L^2}\right)\|\nabla\mathbf{b}_t\|_{L^2}\\
&\le C\left(\|\mathbf{u}_t\|_{L^6}\|\mathbf{b}\|_{L^3}+\|\mathbf{u}\|_{L^6}\|\mathbf{b}_t\|_{L^2}^{\frac12}\|\mathbf{b}_t\|_{L^6}^{\frac12}\right)\|\nabla\mathbf{b}_t\|_{L^2}\\
&\le C\left(\|\nabla\mathbf{u}_t\|_{L^2}+\|\mathbf{b}_t\|_{L^2}^{\frac12}\|\operatorname{curl}\mathbf{b}_t\|_{L^2}^{\frac12}\right)\|\operatorname{curl}\mathbf{b}_t\|_{L^2}\\
&\le \frac{\nu}{2}\|\operatorname{curl}\mathbf{b}_t\|_{L^2}^2+C\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{b}_t\|_{L^2}^2,
\end{aligned}
$$

which implies that

$$
\frac{d}{dt}\|\mathbf{b}_t\|_{L^2}^2+\nu\|\operatorname{curl}\mathbf{b}_t\|_{L^2}^2\le C_1\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{b}_t\|_{L^2}^2. \tag{3.28}
$$

Adding (3.27) multiplied by $\frac{2C_1}{\mu}$ to (3.28) and then choosing δ suitably small, we deduce that

$$
\frac{d}{dt}\left(2C_1\mu^{-1}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)+C_1\|\nabla\mathbf{u}_t\|_{L^2}^2+\frac{\nu}{2}\|\operatorname{curl}\mathbf{b}_t\|_{L^2}^2\le C\|\mathbf{u}\|_{H^2}^2+C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right).
$$

Then we obtain from the Gronwall inequality and (3.13) that

$$
\sup_{0\le t\le T}\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)+\int_0^T\left(\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\operatorname{curl}\mathbf{b}_t\|_{L^2}^2\right)dt\le C. \tag{3.29}
$$

Consequently, we derive from the regularity theory of elliptic system, $(3.2)_4$, (3.29), and (3.13) that

$$
\begin{aligned}
\|\mathbf{b}\|_{H^2}^2 &\le C\left(\|\mathbf{b}_t\|_{L^2}^2+\|\mathbf{u}\cdot\nabla\mathbf{b}\|_{L^2}^2+\|\mathbf{b}\cdot\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\|_{H^1}^2\right)\\
&\le C+C\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{b}\|_{L^3}^2+C\|\mathbf{b}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}^2\\
&\le C+C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^6}+C\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{H^1}\|\nabla\mathbf{u}\|_{L^2}^2\\
&\le C+\frac12\|\mathbf{b}\|_{H^2}^2,
\end{aligned}
$$

and thus

$$
\sup_{0\le t\le T}\|\mathbf{b}\|_{H^2}^2\le C. \tag{3.30}
$$

Furthermore, it follows from Lemmas 2.5 and 2.2, (3.3), (3.13), (3.29), and (3.30) that

$$
\begin{aligned}
\|\mathbf{u}\|_{H^2}^2 &\le C\left(\|\rho\mathbf{u}_t\|_{L^2}^2+\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2}^2\right)\\
&\le C\|\rho\|_{L^\infty}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\rho\|_{L^\infty}^2\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^3}^2+C\|\mathbf{b}\|_{L^6}^2\|\nabla\mathbf{b}\|_{L^3}^2\\
&\le C+C\|\nabla\mathbf{u}\|_{L^2}^3\|\nabla\mathbf{u}\|_{L^6}+C\|\nabla\mathbf{b}\|_{L^2}^3\|\nabla\mathbf{b}\|_{L^6}\\
&\le C+C\|\nabla\mathbf{u}\|_{L^2}^3\|\mathbf{u}\|_{H^2}+C\|\nabla\mathbf{b}\|_{L^2}^3\|\mathbf{b}\|_{H^2}\\
&\le C+\frac12\|\mathbf{u}\|_{H^2}^2,
\end{aligned}
$$

which yields

$$
\sup_{0\le t\le T}\|\mathbf{u}\|_{H^2}^2\le C. \tag{3.31}
$$

Now we estimate $\|\nabla\rho\|_{L^q}$. First of all, applying Lemma 2.5 once more, we have

$$\begin{aligned}\|\mathbf{u}\|_{W^{2,6}}^2 &\le C\left(\|\rho\mathbf{u}_t\|_{L^6}^2 + \|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^6}^2 + \|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^6}^2\right)\\ &\le C\|\rho\|_{L^\infty}^2\|\mathbf{u}_t\|_{L^6}^2 + C\|\rho\|_{L^\infty}^2\|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^6}^2 + C\|\mathbf{b}\|_{L^\infty}^2\|\nabla\mathbf{b}\|_{L^6}^2\\ &\le C\|\nabla\mathbf{u}_t\|_{L^2}^2 + C,\end{aligned}$$

which together with (3.29) implies

$$\int_0^T \|\mathbf{u}\|_{W^{2,6}}^2 dt \le C. \tag{3.32}$$

Then taking spatial derivative ∇ on the transport equation $(3.2)_1$ leads to

$$\partial_t\nabla\rho + \mathbf{u}\cdot\nabla^2\rho + \nabla\mathbf{u}\cdot\nabla\rho = \mathbf{0}.$$

Thus standard energy methods yields for any $q\in(3,6]$,

$$\frac{d}{dt}\|\nabla\rho\|_{L^q} \le C(q)\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^q} \le C\|\mathbf{u}\|_{W^{2,6}}\|\nabla\rho\|_{L^q},$$

which combined with Gronwall's inequality and (3.20) gives

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^q} \le C.$$

This along with (3.3) yields

$$\sup_{0\le t\le T}\|\rho\|_{W^{1,q}} \le C. \tag{3.33}$$

Finally, we turn to estimate $\|\theta\|_{H^2}$. To this end, denote by $\bar\theta \triangleq \frac{1}{|\Omega|}\int\theta dx$, the average of θ, then we obtain from (3.3), (3.4), and the Poincaré inequality that

$$|\bar\theta|\int\rho dx \le \left|\int\rho\theta dx\right| + \left|\int\rho(\theta-\bar\theta)dx\right| \le C + C\|\nabla\theta\|_{L^2},$$

which together with the fact that $|\int v dx| + \|\nabla v\|_{L^2}$ is an equivalent norm to the usual one in $H^1(\Omega)$ implies that

$$\|\theta\|_{H^1} \le C + C\|\nabla\theta\|_{L^2}. \tag{3.34}$$

Similarly, one deduces

$$\|\theta_t\|_{H^1} \le C\|\sqrt{\rho}\theta_t\|_{L^2} + C\|\nabla\theta_t\|_{L^2}. \tag{3.35}$$

Multiplying $(3.2)_3$ by θ_t and integrating the resulting equation over Ω yield that

$$\begin{aligned}\frac{\kappa}{2}\frac{d}{dt}\int|\nabla\theta|^2dx + c_v\int\rho|\theta_t|^2dx &= -c_v\int\rho(\mathbf{u}\cdot\nabla\theta)\theta_t dx + 2\mu\int|\mathfrak{D}(u)|^2\theta_t dx\\ &\quad + \nu\int|\operatorname{curl}\mathbf{b}|^2\theta_t dx \triangleq \sum_{i=1}^3 I_i.\end{aligned} \tag{3.36}$$

By Hölder's inequality, (3.3), and (3.31), we get

$$|I_1| \le c_v \|\rho\|_{L^\infty}^{\frac{1}{2}} \|\sqrt{\rho}\theta_t\|_{L^2} \|\mathbf{u}\|_{L^\infty} \|\nabla\theta\|_{L^2} \le \frac{c_v}{2} \|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2. \tag{3.37}$$

From (3.31) and (3.34), one has

$$\begin{aligned} I_2 &= 2\mu \frac{d}{dt} \int |\mathfrak{D}(\mathbf{u})|^2 \theta dx - 2\mu \int (|\mathfrak{D}(\mathbf{u})|^2)_t \theta dx \\ &\le 2\mu \frac{d}{dt} \int |\mathfrak{D}(\mathbf{u})|^2 \theta dx + C \int \theta |\nabla \mathbf{u}| |\nabla \mathbf{u}_t| dx \\ &\le 2\mu \frac{d}{dt} \int |\mathfrak{D}(\mathbf{u})|^2 \theta dx + C\|\theta\|_{L^6} \|\nabla \mathbf{u}\|_{L^3} \|\nabla \mathbf{u}_t\|_{L^2} \\ &\le 2\mu \frac{d}{dt} \int |\mathfrak{D}(\mathbf{u})|^2 \theta dx + C\|\theta\|_{H^1} \|\mathbf{u}\|_{H^2} \|\nabla \mathbf{u}_t\|_{L^2} \\ &\le 2\mu \frac{d}{dt} \int |\mathfrak{D}(\mathbf{u})|^2 \theta dx + C\|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C. \end{aligned} \tag{3.38}$$

Moreover, one infers

$$\begin{aligned} I_3 &= \nu \frac{d}{dt} \int |\operatorname{curl} \mathbf{b}|^2 \theta dx - \nu \int (|\operatorname{curl} \mathbf{b}|^2)_t \theta dx \\ &\le \nu \frac{d}{dt} \int |\operatorname{curl} \mathbf{b}|^2 \theta dx + C \int \theta |\operatorname{curl} \mathbf{b}| |\operatorname{curl} \mathbf{b}_t| dx \\ &\le \nu \frac{d}{dt} \int |\operatorname{curl} \mathbf{b}|^2 \theta dx + C\|\theta\|_{L^6} \|\operatorname{curl} \mathbf{b}\|_{L^3} \|\operatorname{curl} \mathbf{b}_t\|_{L^2} \\ &\le \nu \frac{d}{dt} \int |\operatorname{curl} \mathbf{b}|^2 \theta dx + C\|\theta\|_{H^1} \|\mathbf{b}\|_{H^2} \|\operatorname{curl} \mathbf{b}_t\|_{L^2} \\ &\le \nu \frac{d}{dt} \int |\operatorname{curl} \mathbf{b}|^2 \theta dx + C\|\operatorname{curl} \mathbf{b}_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C. \end{aligned} \tag{3.39}$$

Substituting (3.37)–(3.39) into (3.36), we obtain that

$$\begin{aligned} &\frac{d}{dt} \int \left(\kappa |\nabla\theta|^2 - 4\mu |\mathfrak{D}(\mathbf{u})|^2 \theta - 2\nu |\operatorname{curl} \mathbf{b}|^2 \theta\right) dx + c_v \|\sqrt{\rho}\theta_t\|_{L^2}^2 \\ &\le C\|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\operatorname{curl} \mathbf{b}_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C. \end{aligned} \tag{3.40}$$

Noting that

$$4\mu \int |\mathfrak{D}(\mathbf{u})|^2 \theta dx \le C\|\theta\|_{L^6} \|\nabla \mathbf{u}\|_{L^{\frac{12}{5}}}^2 \le C\|\theta\|_{H^1} \|\mathbf{u}\|_{H^2}^2 \le \frac{\kappa}{4} \|\nabla\theta\|_{L^2}^2 + C,$$

and

$$2\nu \int |\operatorname{curl} \mathbf{b}|^2 \theta dx \le C\|\theta\|_{L^6} \|\operatorname{curl} \mathbf{b}\|_{L^{\frac{12}{5}}}^2 \le C\|\theta\|_{H^1} \|\mathbf{b}\|_{H^2}^2 \le \frac{\kappa}{4} \|\nabla\theta\|_{L^2}^2 + C,$$

which combined with (3.40), Gronwall's inequality, and (3.29) leads to

$$\sup_{0 \le t \le T} \|\nabla\theta\|_{L^2}^2 + \int_0^T \|\sqrt{\rho}\theta_t\|_{L^2}^2 dt \le C.$$

This along with (3.34) gives rise to

$$\sup_{0\le t\le T}\|\theta\|_{H^1}^2+\int_0^T\|\sqrt{\rho}\theta_t\|_{L^2}^2dt\le C. \tag{3.41}$$

Differentiating $(3.2)_3$ with respect to t and using $(1.1)_1$, we arrive at

$$\begin{aligned}&c_v[\rho\theta_{tt}+\rho\mathbf{u}\cdot\nabla\theta_t]-\kappa\Delta\theta_t\\&=c_v\operatorname{div}(\rho\mathbf{u})\,(\theta_t+\mathbf{u}\cdot\nabla\theta)-c_v\rho\mathbf{u}_t\cdot\nabla\theta+2\mu(|\mathfrak{D}(\mathbf{u})|^2)_t+\nu(|\operatorname{curl}\mathbf{b}|^2)_t.\end{aligned} \tag{3.42}$$

Multiplying (3.42) by θ_t and integrating (by parts) over Ω yield

$$\begin{aligned}&\frac{c_v}{2}\frac{d}{dt}\int\rho|\theta_t|^2dx+\kappa\int|\nabla\theta_t|^2dx\\&=c_v\int\operatorname{div}(\rho\mathbf{u})|\theta_t|^2dx+c_v\int\operatorname{div}(\rho\mathbf{u})(\mathbf{u}\cdot\nabla\theta)\theta_tdx\\&\quad-c_v\int\rho(\mathbf{u}_t\cdot\nabla\theta)\theta_tdx+2\mu\int(|\mathfrak{D}(\mathbf{u})|^2)_t\theta_tdx+\nu\int(|\operatorname{curl}\mathbf{b}|^2)_t\theta_tdx\triangleq\sum_{k=1}^5\bar{J}_k.\end{aligned} \tag{3.43}$$

By virtue of Hölder's inequality, Sobolev's inequality, (3.3), (3.30), (3.31), (3.35), and (3.41), we find

$$\begin{aligned}|\bar{J}_1|=&\left|-c_v\int\rho\mathbf{u}\cdot\nabla|\theta_t|^2dx\right|\\\le&2c_v\|\rho\|_{L^\infty}^{\frac12}\|\mathbf{u}\|_{L^\infty}\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\theta_t\|_{L^2}\\\le&\frac{\kappa}{10}\|\nabla\theta_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2;\\|\bar{J}_2|=&\left|-c_v\int\rho\mathbf{u}\cdot\nabla[(\mathbf{u}\cdot\nabla\theta)\theta_t]dx\right|\\\le&c_v\int\left(\rho|\mathbf{u}||\nabla\mathbf{u}||\nabla\theta||\theta_t|+\rho|\mathbf{u}|^2|\nabla^2\theta||\theta_t|+\rho|\mathbf{u}|^2|\nabla\theta||\nabla\theta_t|\right)dx\\\le&c_v\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^3}\|\nabla\theta\|_{L^2}\|\theta_t\|_{L^6}+c_v\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^6}^2\|\nabla^2\theta\|_{L^2}\|\theta_t\|_{L^6}\\&+c_v\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^\infty}^2\|\nabla\theta\|_{L^2}\|\nabla\theta_t\|_{L^2}\\\le&C(1+\|\nabla^2\theta\|_{L^2})(\|\sqrt{\rho}\theta_t\|_{L^2}+\|\nabla\theta_t\|_{L^2})\\\le&\frac{\kappa}{10}\|\nabla\theta_t\|_{L^2}^2+C\|\nabla^2\theta\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C;\\|\bar{J}_3|\le&c_v\|\rho\|_{L^\infty}^{\frac12}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\|\nabla\theta\|_{L^3}\|\theta_t\|_{L^6}\\\le&C(1+\|\nabla^2\theta\|_{L^2})(\|\sqrt{\rho}\theta_t\|_{L^2}+\|\nabla\theta_t\|_{L^2})\\\le&\frac{\kappa}{10}\|\nabla\theta_t\|_{L^2}^2+C\|\nabla^2\theta\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C;\\|\bar{J}_4|\le&C\int|\nabla\mathbf{u}||\nabla\mathbf{u}_t|\theta_tdx\le C\|\nabla\mathbf{u}\|_{L^3}\|\nabla\mathbf{u}_t\|_{L^2}\|\theta_t\|_{L^6}\\\le&C\|\nabla\mathbf{u}_t\|_{L^2}(\|\sqrt{\rho}\theta_t\|_{L^2}+\|\nabla\theta_t\|_{L^2})\\\le&\frac{\kappa}{10}\|\nabla\theta_t\|_{L^2}^2+C\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2;\end{aligned}$$

$$
\begin{aligned}
|\bar{J}_5| \leq & C\int |\operatorname{curl}\mathbf{b}||\operatorname{curl}\mathbf{b}_t|\theta_t dx \leq C\|\operatorname{curl}\mathbf{b}\|_{L^3}\|\operatorname{curl}\mathbf{b}_t\|_{L^2}\|\theta_t\|_{L^6} \\
\leq & C\|\operatorname{curl}\mathbf{b}_t\|_{L^2}(\|\sqrt{\rho}\theta_t\|_{L^2}+\|\nabla\theta_t\|_{L^2}) \\
\leq & \frac{\kappa}{10}\|\nabla\theta_t\|_{L^2}^2 + C\|\operatorname{curl}\mathbf{b}_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2.
\end{aligned}
$$

Substituting the above estimates into (3.43), we derive that

$$
\begin{aligned}
& c_v\frac{d}{dt}\|\sqrt{\rho}\theta_t\|_{L^2}^2 + \kappa\|\nabla\theta_t\|_{L^2}^2 \\
& \leq C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\nabla^2\theta\|_{L^2}^2 + C\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\operatorname{curl}\mathbf{b}_t\|_{L^2}^2 + C.
\end{aligned} \tag{3.44}
$$

The standard H^2-estimate of $(3.2)_3$ gives rise to

$$
\begin{aligned}
\|\theta\|_{H^2}^2 & \leq C\left(\|\rho\theta_t\|_{L^2}^2 + \|\rho\mathbf{u}\cdot\nabla\theta\|_{L^2}^2 + \||\nabla\mathbf{u}|^2\|_{L^2}^2 + \|\theta\|_{L^2}^2\right) \\
& \leq C\|\rho\|_{L^\infty}\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^4}^4 + C\|\theta\|_{L^2}^2 \\
& \leq C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\theta\|_{H^1}^2 + C\|\mathbf{u}\|_{H^2}^4 \\
& \leq C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C
\end{aligned} \tag{3.45}
$$

due to (3.31) and (3.41). Then we obtain from (3.44) and (3.45) that

$$
c_v\frac{d}{dt}\|\sqrt{\rho}\theta_t\|_{L^2}^2 + \kappa\|\nabla\theta_t\|_{L^2}^2 \leq C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\operatorname{curl}\mathbf{b}_t\|_{L^2}^2 + C,
$$

which combined with the Gronwall inequality and (3.29) that

$$
\sup_{0\leq t\leq T}\|\sqrt{\rho}\theta_t\|_{L^2}^2 + \int_0^T \|\nabla\theta_t\|_{L^2}^2 dt \leq C. \tag{3.46}
$$

Consequently, we deduce from (3.45) and (3.46) that

$$
\sup_{0\leq t\leq T}\|\theta\|_{H^2}^2 \leq C\sup_{0\leq t\leq T}\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C \leq C. \tag{3.47}
$$

Hence the desired (3.24) follows from (3.30), (3.31), (3.33), and (3.47). This finishes the proof of Lemma 3.4. □

With Lemmas 3.1–3.4 at hand, we are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We argue by contradiction. Suppose that (1.7) were false, that is, (3.1) holds. Note that the general constant C in Lemmas 3.1–3.4 is independent of $t < T^*$, that is, all the a priori estimates obtained in Lemmas 3.1–3.4 are uniformly bounded for any $t < T^*$. Hence, the function

$$
(\rho,\mathbf{u},\theta,\mathbf{b})(T^*,x) \triangleq \lim_{t\to T^*}(\rho,\mathbf{u},\theta,\mathbf{b})(t,x)
$$

satisfy the initial condition (1.5) at $t = T^*$. Furthermore, standard arguments yield that $\rho\dot{\mathbf{u}}, \rho\dot{\theta} \in C([0,T];L^2)$, here $\dot{f} \triangleq f_t + \mathbf{u}\cdot\nabla f$, which implies

$$
(\rho\dot{\mathbf{u}},\rho\dot{\theta})(T^*,x) \triangleq \lim_{t\to T^*}(\rho\dot{\mathbf{u}},\rho\dot{\theta})(t,x) \in L^2.
$$

Hence,

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla P - \mathbf{b}\cdot\nabla\mathbf{b}|_{t=T^*} = \sqrt{\rho}(T^*,x)\mathbf{g}_1(x), \\ \kappa\Delta\theta + 2\mu|\mathfrak{D}(\mathbf{u})|^2 + \nu|\operatorname{curl}\mathbf{b}|^2|_{t=T^*} = \sqrt{\rho}(T^*,x)\mathbf{g}_2(x), \end{cases}$$

with

$$\mathbf{g}_1(x) \triangleq \begin{cases} \rho^{-\frac{1}{2}}(T^*,x)(\rho\dot{\mathbf{u}})(T^*,x), & \text{for } \ x \in \{x|\rho(T^*,x) > 0\}, \\ \mathbf{0}, & \text{for } \ x \in \{x|\rho(T^*,x) = 0\}, \end{cases}$$

and

$$\mathbf{g}_2(x) \triangleq \begin{cases} c_v\rho^{-\frac{1}{2}}(T^*,x)(\rho\dot{\theta})(T^*,x), & \text{for } \ x \in \{x|\rho(T^*,x) > 0\}, \\ \mathbf{0}, & \text{for } \ x \in \{x|\rho(T^*,x) = 0\}, \end{cases}$$

satisfying $\mathbf{g}_1, \mathbf{g}_2 \in L^2$ due to (3.29), (3.46), and (3.24). Thus, $(\rho, \mathbf{u}, \theta, \mathbf{b})(T^*, x)$ also satisfies (1.6). Therefore, taking $(\rho, \mathbf{u}, \theta, \mathbf{b})(T^*, x)$ as the initial data, one can extend the local strong solution beyond T^*, which contradicts the maximality of T^*. Thus we finish the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

Throughout this section, we denote

$$C_0 \triangleq \|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{b}_0\|_{L^2}^2.$$

To simplify statements, in what follows, we assume $\mu = \nu = 1$.

First, applying [20, Theorem 2.1] and integrating (3.10) with respect to t respectively, one has the following results.

Lemma 4.1. *Let $(\rho, \mathbf{u}, \theta, \mathbf{b})$ be a strong solution to the system (1.1)–(1.3) on $(0, T)$. Then for any $t \in (0, T)$, there holds*

$$\|\rho(t)\|_{L^\infty} = \|\rho_0\|_{L^\infty} \tag{4.1}$$

and

$$\|\sqrt{\rho}\mathbf{u}(t)\|_{L^2}^2 + \|\mathbf{b}(t)\|_{L^2}^2 + \int_0^t \left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}\|_{L^2}^2\right) ds \le C_0. \tag{4.2}$$

Lemma 4.2. *Let $(\rho, \mathbf{u}, \theta, \mathbf{b})$ be a strong solution to the system (1.1)–(1.3) on $(0, T)$. Then there exist positive constants C and $\bar{C}$ depending only on $\|\rho_0\|_{L^\infty}$ and Ω, such that for any $t \in (0, T)$, there holds*

$$\begin{aligned} &\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}\|_{L^2}^2\right) + \bar{C}\int_0^t \left(\|\mathbf{u}\|_{H^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2\right) ds \\ &\le 2\left(\|\nabla\mathbf{u}_0\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}_0\|_{L^2}^2\right) + CC_0\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^4\right) + C\sqrt{C_0}\sup_{0\le s\le t}\|\operatorname{curl}\mathbf{b}\|_{L^2}^3 \\ &\quad + C\sqrt{C_0}\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2} + \|\nabla\mathbf{b}\|_{L^2}\right)\int_0^t \left(\|\mathbf{u}\|_{H^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2\right) ds. \end{aligned} \tag{4.3}$$

Proof. Multiplying $(3.2)_2$ by $\mathbf{u}_t$ and integrating the resulting equations over Ω, we derive from Cauchy–Schwarz inequality that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla\mathbf{u}|^2dx+\int\rho|\mathbf{u}_t|^2dx\\
&=\int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}_tdx-\int\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_tdx\\
&=-\frac{d}{dt}\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\int\left[\mathbf{b}_t\cdot\nabla\mathbf{u}\cdot\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}_t-\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t\right]dx\\
&\le-\frac{d}{dt}\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\frac{1}{2}\int\left(\rho|\mathbf{u}_t|^2+|\mathbf{b}_t|^2\right)dx+\int\left(2\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2+8|\mathbf{b}|^2|\nabla\mathbf{u}|^2\right)dx,
\end{aligned}$$

and thus

$$\begin{aligned}
&\frac{d}{dt}\int\left(|\nabla\mathbf{u}|^2+2\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\right)dx+\int\rho|\mathbf{u}_t|^2dx\\
&\le\int|\mathbf{b}_t|^2dx+\int\left(4\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2+16|\mathbf{b}|^2|\nabla\mathbf{u}|^2\right)dx.
\end{aligned}\tag{4.4}$$

Multiplying $(3.2)_4$ by $\mathbf{b}_t$ and integrating by parts yield

$$\begin{aligned}
\frac{d}{dt}\int|\operatorname{curl}\mathbf{b}|^2dx+2\int|\mathbf{b}_t|^2dx&=2\int(\mathbf{b}\cdot\nabla\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{b})\cdot\mathbf{b}_tdx\\
&\le\frac{1}{2}\int|\mathbf{b}_t|^2dx+8\int\left(|\mathbf{b}|^2|\nabla\mathbf{u}|^2+|\mathbf{u}|^2|\nabla\mathbf{b}|^2\right)dx,
\end{aligned}\tag{4.5}$$

which combined with (4.4) implies

$$\begin{aligned}
&\frac{d}{dt}\int\left(|\nabla\mathbf{u}|^2+|\operatorname{curl}\mathbf{b}|^2+2\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\right)dx+\int\left(\rho|\mathbf{u}_t|^2+|\mathbf{b}_t|^2\right)dx\\
&\le 16\int\left(\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2+|\mathbf{b}|^2|\nabla\mathbf{u}|^2+|\mathbf{u}|^2|\nabla\mathbf{b}|^2\right)dx.
\end{aligned}\tag{4.6}$$

Integrating (4.6) with respect to t gives rise to

$$\begin{aligned}
&\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\operatorname{curl}\mathbf{b}\|_{L^2}^2\right)+\int_0^t\left(\|\sqrt{\rho}\mathbf{u}_s(s)\|_{L^2}^2+\|\mathbf{b}_s(s)\|_{L^2}^2\right)ds\\
&\le\|\nabla\mathbf{u}_0\|_{L^2}^2+\|\operatorname{curl}\mathbf{b}_0\|_{L^2}^2+4\sup_{0\le s\le t}\int|\mathbf{b}|^2|\nabla\mathbf{u}|dx\\
&\quad+16\int_0^t\int\left(\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2+|\mathbf{b}|^2|\nabla\mathbf{u}|^2+|\mathbf{u}|^2|\nabla\mathbf{b}|^2\right)dxds.
\end{aligned}\tag{4.7}$$

Recall that $(\mathbf{u},P)$ satisfies the following Stokes system

$$\begin{cases}
-\Delta\mathbf{u}+\nabla P=-\rho\mathbf{u}_t-\rho\mathbf{u}\cdot\nabla\mathbf{u}+\mathbf{b}\cdot\nabla\mathbf{b}, & x\in\Omega,\\
\operatorname{div}\mathbf{u}=0, & x\in\Omega,\\
\mathbf{u}=\mathbf{0}, & x\in\partial\Omega.
\end{cases}$$

Applying Lemma 2.5 with $\mathbf{F} \triangleq -\rho\mathbf{u}_t - \rho\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b}$, we obtain from Hölder's inequality and (4.1) that

$$\begin{aligned}\|\mathbf{u}\|_{H^2}^2 &\le C\left(\|\rho\mathbf{u}_t\|_{L^2}^2 + \|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2}^2\right)\\ &\le C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\sqrt{\rho}\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2}^2\right).\end{aligned} \tag{4.8}$$

It follows from the standard L^2-estimates of elliptic system and $(3.2)_4$ that

$$\|\nabla^2\mathbf{b}\|_{L^2}^2 \le C\left(\|\mathbf{b}_t\|_{L^2}^2 + \|\mathbf{u}\cdot\nabla\mathbf{b}\|_{L^2}^2 + \|\mathbf{b}\cdot\nabla\mathbf{u}\|_{L^2}^2\right), \tag{4.9}$$

which together with (4.8) leads to

$$\begin{aligned}\|\mathbf{u}\|_{H^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2 &\le L\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2\right) + C\left(\|\sqrt{\rho}\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2}^2\right)\\ &\quad + C\left(\|\mathbf{u}\cdot\nabla\mathbf{b}\|_{L^2}^2 + \|\mathbf{b}\cdot\nabla\mathbf{u}\|_{L^2}^2\right).\end{aligned} \tag{4.10}$$

Integrating (4.10) multiplied by $\frac{1}{2L}$ with respect to t and adding the resulting inequality to (4.7), we derive

$$\begin{aligned}&\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}\|_{L^2}^2\right) + \frac{1}{2L}\int_0^t\left(\|\mathbf{u}\|_{H^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2\right)ds\\ &\le \|\nabla\mathbf{u}_0\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}_0\|_{L^2}^2 + 4\sup_{0\le s\le t}\int|\mathbf{b}|^2|\nabla\mathbf{u}|dx\\ &\quad + \bar{L}\int_0^t\int\left(\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2 + |\mathbf{b}|^2|\nabla\mathbf{u}|^2 + |\mathbf{u}|^2|\nabla\mathbf{b}|^2 + |\mathbf{b}|^2|\nabla\mathbf{b}|^2\right)dxds.\end{aligned} \tag{4.11}$$

By Lemmas 2.2, 2.4, and (4.2), we have

$$\begin{aligned}\int|\mathbf{b}|^2|\nabla\mathbf{u}|dx &\le \|\mathbf{b}\|_{L^4}^4\|\nabla\mathbf{u}\|_{L^2} \le \|\mathbf{b}\|_{L^2}^{\frac12}\|\mathbf{b}\|_{L^6}^{\frac32}\|\nabla\mathbf{u}\|_{L^2}\\ &\le C\|\mathbf{b}\|_{L^2}^{\frac12}\|\operatorname{curl}\mathbf{b}\|_{L^2}^{\frac32}\|\nabla\mathbf{u}\|_{L^2} \le \frac18\|\nabla\mathbf{u}\|_{L^2}^2 + C\|\mathbf{b}\|_{L^2}\|\operatorname{curl}\mathbf{b}\|_{L^2}^3\\ &\le \frac18\|\nabla\mathbf{u}\|_{L^2}^2 + C\sqrt{C_0}\|\operatorname{curl}\mathbf{b}\|_{L^2}^3,\end{aligned}$$

and thus

$$4\sup_{0\le s\le t}\int|\mathbf{b}|^2|\nabla\mathbf{u}|dx \le \frac12\sup_{0\le s\le t}\|\nabla\mathbf{u}\|_{L^2}^2 + C\sqrt{C_0}\sup_{0\le s\le t}\|\operatorname{curl}\mathbf{b}\|_{L^2}^3. \tag{4.12}$$

Similarly, one has

$$\begin{aligned}&\bar{L}\int\left(\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2 + |\mathbf{b}|^2|\nabla\mathbf{u}|^2 + |\mathbf{u}|^2|\nabla\mathbf{b}|^2 + |\mathbf{b}|^2|\nabla\mathbf{b}|^2\right)dx\\ &\le \bar{L}\|\rho\|_{L^\infty}^{\frac12}\|\sqrt{\rho}\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^6}^2 + \bar{L}\|\mathbf{b}\|_{L^2}\|\mathbf{b}\|_{L^6}\|\nabla\mathbf{u}\|_{L^6}^2\\ &\quad + \bar{L}\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^6} + \bar{L}\|\mathbf{b}\|_{L^2}\|\mathbf{b}\|_{L^6}\|\nabla\mathbf{b}\|_{L^6}^2\\ &\le C\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2} + \|\mathbf{b}\|_{L^2}\right)\left(\|\nabla\mathbf{u}\|_{L^2} + \|\nabla\mathbf{b}\|_{L^2}\right)\|\mathbf{u}\|_{H^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{b}\|_{L^2}^2\end{aligned}$$

$$
\begin{aligned}
&+C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{b}\|_{L^2}\|\nabla^2\mathbf{b}\|_{L^2}+C\|\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^2}^3+C\|\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^2}\|\nabla^2\mathbf{b}\|_{L^2}^2\\
\le{}& C\sqrt{C_0}\left(\|\nabla\mathbf{u}\|_{L^2}+\|\nabla\mathbf{b}\|_{L^2}\right)\left(\|\mathbf{u}\|_{H^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)\\
&+C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{b}\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{b}\|_{L^2}\|\nabla^2\mathbf{b}\|_{L^2}+C\sqrt{C_0}\|\nabla\mathbf{b}\|_{L^2}^3,
\end{aligned}
$$

and so

$$
\begin{aligned}
&\bar{L}\int_0^t\int\left(\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2+|\mathbf{b}|^2|\nabla\mathbf{u}|^2+|\mathbf{u}|^2|\nabla\mathbf{b}|^2+|\mathbf{b}|^2|\nabla\mathbf{b}|^2\right)dxds\\
&\le C\sqrt{C_0}\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2}+\|\nabla\mathbf{b}\|_{L^2}\right)\int_0^t\left(\|\mathbf{u}\|_{H^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)ds\\
&\quad+C\sup_{0\le s\le t}\|\nabla\mathbf{u}\|_{L^2}^2\int_0^t\|\nabla\mathbf{b}\|_{L^2}^2ds+C\sup_{0\le s\le t}\|\nabla\mathbf{u}\|_{L^2}^4\int_0^t\|\nabla\mathbf{b}\|_{L^2}^2ds\\
&\quad+\frac{1}{4L}\int_0^t\|\nabla^2\mathbf{b}\|_{L^2}^2ds+C\sqrt{C_0}\sup_{0\le s\le t}\|\nabla\mathbf{b}\|_{L^2}^3\\
&\le C\sqrt{C_0}\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2}+\|\nabla\mathbf{b}\|_{L^2}\right)\int_0^t\left(\|\mathbf{u}\|_{H^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)ds\\
&\quad+CC_0\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^4\right)+\frac{1}{4L}\int_0^t\|\nabla^2\mathbf{b}\|_{L^2}^2ds+C\sqrt{C_0}\sup_{0\le s\le t}\|\operatorname{curl}\mathbf{b}\|_{L^2}^3.
\end{aligned}\tag{4.13}
$$

Substituting (4.12) and (4.13) into (4.11), we deduce that

$$
\begin{aligned}
&\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\operatorname{curl}\mathbf{b}\|_{L^2}^2\right)+\frac{1}{2L}\int_0^t\left(\|\mathbf{u}\|_{H^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)ds\\
&\le 2\left(\|\nabla\mathbf{u}_0\|_{L^2}^2+\|\operatorname{curl}\mathbf{b}_0\|_{L^2}^2\right)+CC_0\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^4\right)+C\sqrt{C_0}\sup_{0\le s\le t}\|\nabla\mathbf{b}\|_{L^2}^3\\
&\quad+C\sqrt{C_0}\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2}+\|\nabla\mathbf{b}\|_{L^2}\right)\int_0^t\left(\|\mathbf{u}\|_{H^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)ds.
\end{aligned}\tag{4.14}
$$

This finishes the proof of Lemma 4.2. □

Lemma 4.3. *Let $(\rho,\mathbf{u},\theta,\mathbf{b})$ be a strong solution to the system (1.1)–(1.3) on $(0,T)$. Then there exists a positive constant ε_0 depending only on $\|\rho_0\|_{L^\infty}$ and Ω such that*

$$
\sup_{0\le t\le T}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\operatorname{curl}\mathbf{b}\|_{L^2}^2\right)\le 4\left(\|\nabla\mathbf{u}_0\|_{L^2}^2+\|\operatorname{curl}\mathbf{b}_0\|_{L^2}^2\right),\tag{4.15}
$$

provided that

$$
\left(\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2+\|\mathbf{b}_0\|_{L^2}^2\right)\left(\|\nabla\mathbf{u}_0\|_{L^2}^2+\|\operatorname{curl}\mathbf{b}_0\|_{L^2}^2\right)\le\varepsilon_0.\tag{4.16}
$$

Proof. Define functions $E(t)$ and $\Phi(t)$ as follows

$$E(t) \triangleq \sup_{0\le s\le t}\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}\|_{L^2}^2\right) + \bar{C}\int_0^t \left(\|\mathbf{u}\|_{H^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2\right) ds,$$

$$\Phi(t) \triangleq C_0 \sup_{0\le s\le t}\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}\|_{L^2}^2\right),$$

where $\bar{C}$ is the same as that in (4.3). In view of the regularities of $\mathbf{u}$ and $\mathbf{b}$, one can easily check that both $E(t)$ and $\Phi(t)$ are continuous functions on $[0,T]$. By (4.3), there is a positive constant M such that

$$E(t) \le 2\left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}_0\|_{L^2}^2\right) + M\left(\sqrt{\Phi(t)} + \Phi(t)\right)E(t). \tag{4.17}$$

We set

$$\varepsilon_0 \triangleq \min\left\{\frac{1}{16M}, \frac{1}{32M^2}\right\},$$

and suppose that

$$\left(\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{b}_0\|_{L^2}^2\right)\left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}_0\|_{L^2}^2\right) \le \varepsilon_0.$$

We claim that

$$\Phi(t) < \min\left\{\frac{1}{2M}, \frac{1}{4M^2}\right\}, \quad 0 \le t \le T.$$

Otherwise, by the continuity and monotonicity of $\Phi(t)$, there is a $T_0 \in (0,T]$ such that

$$\Phi(T_0) = \min\left\{\frac{1}{2M}, \frac{1}{4M^2}\right\}. \tag{4.18}$$

On account of (4.18), it follows from (4.17) that

$$E(T_0) \le 2\left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}_0\|_{L^2}^2\right) + \frac{1}{2}E(T_0),$$

and hence

$$E(T_0) \le 4\left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}_0\|_{L^2}^2\right).$$

Recalling the definition of $E(t)$ and $\Phi(t)$, we deduce from the above inequality that

$$\Phi(T_0) \le C_0 E(T_0) \le 4C_0\left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}_0\|_{L^2}^2\right) \le 4\varepsilon_0 = \min\left\{\frac{1}{4M}, \frac{1}{8M^2}\right\},$$

which contradicts with (4.18).

By virtue of the claim we showed in the above, we derive from (4.17) that

$$E(t) \le 4\left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}_0\|_{L^2}^2\right), \quad 0 < t < T,$$

provided that (4.16) holds true. This implies the desired (4.15) and consequently completes the proof of Lemma 4.3. □

Now, we can give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let ε_0 be the constant stated in Lemma 4.3 and suppose that the initial data $(\rho_0, \mathbf{u}_0, \theta_0, \mathbf{b}_0)$ satisfies (1.5), (1.6), and

$$\left(\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{b}_0\|_{L^2}^2\right)\left(\|\nabla\mathbf{u}_0\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}_0\|_{L^2}^2\right) \le \varepsilon_0.$$

According to [27, Theorem 1], there is a unique local strong solution $(\rho, \mathbf{u}, \theta, \mathbf{b})$ to the system (1.1)–(1.3). Let T^* be the maximal existence time to the solution. We will show that $T^* = \infty$. Suppose, by contradiction, that $T^* < \infty$, then by (1.7), one has

$$\int_0^{T^*} \left(\|\nabla\mathbf{u}\|_{L^2}^4 + \|\nabla\mathbf{b}\|_{L^2}^4\right) dt = \infty. \tag{4.19}$$

By Lemma 4.3, for any $0 < T < T^*$, there holds

$$\sup_{0\le t\le T} \left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}\|_{L^2}^2\right) \le 4\left(\|\nabla\mathbf{u}_0\|_{L^2}^2 + \|\operatorname{curl}\mathbf{b}_0\|_{L^2}^2\right),$$

which together with Lemma 2.4 implies that

$$\int_0^{T^*} \left(\|\nabla\mathbf{u}\|_{L^2}^4 + \|\nabla\mathbf{b}\|_{L^2}^4\right) dt \le 32\left(\|\nabla\mathbf{u}_0\|_{L^2}^4 + \|\operatorname{curl}\mathbf{b}_0\|_{L^2}^4\right) T^* < \infty,$$

contradicting to (4.19). This contradiction provides us that $T^* = \infty$, and thus we obtain the global strong solution. This finishes the proof of Theorem 1.2. □

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References

[1] C. Amrouche, V. Girault, Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension, Czechoslovak Math. J. 44 (1994) 109–140.
[2] S.A. Antontesv, A.V. Kazhikov, V.N. Monakhov, Boundary Value Problems in Mechanics of Nonhomogeneous Fluids, North-Holland, Amsterdam, 1990.
[3] Q. Chen, Z. Tan, Y.J. Wang, Strong solutions to the incompressible magnetohydrodynamic equations, Math. Methods Appl. Sci. 34 (2011) 94–107.
[4] Y. Cho, H. Kim, Unique solvability for the density-dependent Navier–Stokes equations, Nonlinear Anal. 59 (2004) 465–489.
[5] Y. Cho, H. Kim, Existence result for heat-conducting viscous incompressible fluids with vacuum, J. Korean Math. Soc. 45 (2008) 645–681.
[6] H.J. Choe, H. Kim, Strong solutions of the Navier–Stokes equations for nonhomogeneous incompressible fluids, Comm. Partial Differential Equations 28 (2003) 1183–1201.
[7] W. Craig, X.D. Huang, Y. Wang, Global wellposedness for the 3D inhomogeneous incompressible Navier–Stokes equations, J. Math. Fluid Mech. 15 (2013) 747–758.
[8] B. Desjardins, C. Le Bris, Remarks on a nonhomogeneous model of magnetohydrodynamics, Differential Integral Equations 11 (1998) 377–394.
[9] J.S. Fan, F.C. Li, G. Nakamura, Global strong solution to the two-dimensional density-dependent magnetohydrodynamic equations with vaccum, Commun. Pure Appl. Anal. 13 (2014) 1481–1490.
[10] E. Feireisl, Dynamics of Viscous Compressible Fluids, Oxford University Press, Oxford, 2004.
[11] J.F. Gerbeau, C. Le Bris, Existence of solution for a density-dependant magnetohydrodynamic equation, Adv. Differential Equations 2 (1997) 427–452.

[12] H.J. Gong, J.K. Li, Global existence of strong solutions to incompressible MHD, Commun. Pure Appl. Anal. 13 (2014) 1553–1561.

[13] X.D. Huang, Y. Wang, Global strong solution to the 2D nonhomogeneous incompressible MHD system, J. Differential Equations 254 (2013) 511–527.

[14] X.D. Huang, Y. Wang, Global strong solution with vacuum to the two dimensional density-dependent Navier–Stokes system, SIAM J. Math. Anal. 46 (2014) 1771–1788.

[15] X.D. Huang, Y. Wang, Global strong solution of 3D inhomogeneous Navier–Stokes equations with density-dependent viscosity, J. Differential Equations 259 (2015) 1606–1627.

[16] A.V. Kazhikov, Resolution of boundary value problems for nonhomogeneous viscous fluids, Dokl. Akad. Nauk 216 (1974) 1008–1010.

[17] O.A. Ladyženskaja, N.N. Ural'ceva, Linear and Quasilinear Elliptic Equations, Academic Press, New York–London, 1968.

[18] X.L. Li, D.H. Wang, Global strong solution to the three-dimensional density-dependent incompressible magnetohydrodynamic flows, J. Differential Equations 251 (2011) 1580–1615.

[19] Z.L. Liang, Local strong solution and blow-up criterion for the 2D nonhomogeneous incompressible fluids, J. Differential Equations 258 (2015) 2633–2654.

[20] P.L. Lions, Mathematical Topics in Fluid Mechanics, vol. I: Incompressible Models, Oxford University Press, Oxford, 1996.

[21] B.Q. Lü, X.D. Shi, X. Zhong, Global existence and large time asymptotic behavior of strong solutions to the Cauchy problem of 2D density-dependent Navier–Stokes equations with vacuum, http://arxiv.org/abs/1506.03143.

[22] B.Q. Lü, Z.H. Xu, X. Zhong, On local strong solutions to the Cauchy problem of two-dimensional density-dependent magnetohydrodynamic equations with vacuum, http://arxiv.org/abs/1506.02156.

[23] B.Q. Lü, Z.H. Xu, X. Zhong, Global existence and large time asymptotic behavior of strong solutions to the Cauchy problem of 2D density-dependent magnetohydrodynamic equations with vacuum, J. Math. Pures Appl. (2016), in press, http://arxiv.org/abs/1506.03884.

[24] M. Paicu, P. Zhang, Z.F. Zhang, Global unique solvability of inhomogeneous Navier–Stokes equations with bounded density, Comm. Partial Differential Equations 38 (2013) 1208–1234.

[25] J. Simon, Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure, SIAM J. Math. Anal. 21 (1990) 1093–1117.

[26] T. Tao, Nonlinear Dispersive Equations. Local and Global Analysis, American Mathematical Society, Providence, RI, 2006.

[27] H.W. Wu, Strong solutions to the incompressible magnetohydrodynamic equations with vacuum, Comput. Math. Appl. 61 (2011) 2742–2753.

[28] Y.L. Xiao, Z.P. Xin, On the vanishing viscosity limit for the 3D Navier–Stokes equations with a slip boundary condition, Comm. Pure Appl. Math. 60 (2007) 1027–1055.

[29] J.W. Zhang, Global well-posedness for the incompressible Navier–Stokes equations with density-dependent viscosity coefficient, J. Differential Equations 259 (2015) 1722–1742.

[30] X. Zhong, Global strong solutions for 3D viscous incompressible heat conducting Navier–Stokes flows with non-negative density, http://arxiv.org/abs/1605.05007.