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Boundary behavior of solutions of Monge–Ampère equations with singular righthand sides ☆

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ABSTRACT

In this paper, by Karamata regular variation theory and constructing super and subsolutions, we obtain the asymptotic boundary behavior of convex solutions of Monge–Ampère equations with singular righthand sides.

1. Introduction

In this paper we investigate the boundary behavior of strictly convex solutions of the following singular boundary value problem:

$$\begin{cases} \det D^2 u = b(x) f(-u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n$ $(n \geq 2)$ is a smooth bounded convex domain with positive boundary Gauss curvature and f admits a singularity at zero. Precisely we assume that f satisfies:

($\mathbf{f_1}$) $f \in C^1(0,\infty)$, $f(s) > 0$, $f(s) \to \infty$ as $s \to 0$, and is decreasing on $(0,\infty)$;

($\mathbf{f_2}$) There exists $C_f > 0$ such that

$$\lim_{s \to 0^+} H'(s) \int_0^s \frac{d\tau}{H(\tau)} = -C_f, \tag{1.2}$$

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where

$$H(\tau) = ((n+1)F(\tau))^{1/(n+1)}, \ \forall\ 0 < \tau < a$$

and

$$F(\tau) = \int_{\tau}^{a} f(s)ds, \ \forall\ 0 < \tau < a$$

for some constant $a > 0$. Since we will investigate the boundary behavior of u and $u = 0$ on the boundary, we only need concern the behavior of $F(\tau)$ and $f(\tau)$ as $\tau \to 0$. That is, the choice of a is not essential to our results. For convenience, we define φ by

$$\int_{0}^{\varphi(t)} \frac{d\tau}{H(\tau)} = t, \ \forall\ 0 < t < \alpha, \tag{1.3}$$

where $\varphi(\alpha) = a$. Actually, the existence of φ is obvious since $\frac{1}{H}$ is increasing and integrable on $(0, a]$.

We also assume that b satisfies:

($\mathbf{b_1}$) $b \in C^3(\Omega)$ and is positive in Ω;

($\mathbf{b_2}$) There exist $k(t) \in C^1(0, \delta_0)$ (for some $\delta_0 > 0$), which is positive, monotone and integrable, and two positive constants $\overline{b}$ and $\underline{b}$ such that

$$\underline{b} = \liminf_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{b(x)}{k^{n+1}(d(x))} \le \limsup_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{b(x)}{k^{n+1}(d(x))} = \overline{b},$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$, and there exists $C_k \in [0, \infty)$ such that

$$\lim_{t\to 0^+} \left(\frac{K(t)}{k(t)}\right)' = C_k,$$

where $K(t) = \int_0^t k(s)ds$, $0 < t < \delta_0$.

The boundary behavior of solutions of (1.1) may involve the curvatures of $\partial\Omega$. For convenience, we introduce some quantities related to boundary curvatures. For every $\overline{x} \in \partial\Omega$, we denote by $\kappa_1(\overline{x}), \cdots, \kappa_{n-1}(\overline{x})$ the principal curvatures of $\partial\Omega$ at $\overline{x}$, which are positive. We set

$$M_0 = \max_{\overline{x}\in\partial\Omega} \prod_{i=1}^{n-1} \kappa_i(\overline{x}), \quad m_0 = \min_{\overline{x}\in\partial\Omega} \prod_{i=1}^{n-1} \kappa_i(\overline{x}). \tag{1.4}$$

The boundary estimates of solutions of (1.1) are related to M_0 and m_0.

Our main results are summarized as follows.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ ($n \ge 2$) be a smooth bounded convex domain with positive boundary Gauss curvature. Suppose that f satisfies* ($\mathbf{f_1}$) *and* ($\mathbf{f_2}$), *b satisfies* ($\mathbf{b_1}$) *and* ($\mathbf{b_2}$). *If*

$$C_f > 1 - C_k, \tag{1.5}$$

where C_f and C_k are the constants defined in ($\mathbf{f_2}$) *and* ($\mathbf{b_2}$) *respectively, then there exists a unique convex solution u of (1.1) and it holds*

$$1 \leq \liminf_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{-\varphi(\underline{\xi}K(d(x)))}, \quad \limsup_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{-\varphi(\overline{\xi}K(d(x)))} \leq 1, \tag{1.6}$$

where φ is defined by (1.3),

$$\underline{\xi} = \left(\frac{\underline{b}}{M_0(1-C_f^{-1}(1-C_k))}\right)^{1/(n+1)}, \quad \overline{\xi} = \left(\frac{\overline{b}}{m_0(1-C_f^{-1}(1-C_k))}\right)^{1/(n+1)}. \tag{1.7}$$

Remark 1.2. (i_1) The existence of solutions doesn't need the conditions ($\mathbf{f_2}$) and (1.5). We will adopt the following Lemma 2.9, Theorem 2.1 in [16] and Theorem 3.1 in [3] to prove it. For more general existence results, readers can also refer to [17].

(i_2) In ($\mathbf{f_2}$), we assume that $C_f > 0$. If $C_f = 0$, in some cases, our results still hold. Actually, if b is positive and bounded on $\overline{\Omega}$, we can choose $k(t) = 1$ in ($\mathbf{b_2}$). Then, $K(t) = t$, $k'(t) = 0$ and $C_k = 1$. (1.6) is valid for $\underline{\xi} = \left(\frac{\underline{b}}{M_0}\right)^{1/(n+1)}$ and $\overline{\xi} = \left(\frac{\overline{b}}{m_0}\right)^{1/(n+1)}$. If b is unbounded on $\partial\Omega$, there exists a nonincreasing function $k(t)$ such that ($\mathbf{b_2}$) holds. That is, $k'(t) \leq 0$ and $C_k \geq 1$. Then we take $\overline{\xi} = \left(\frac{\overline{b}}{m_0}\right)^{1/(n+1)}$ and the second inequality of (1.6) holds. If $b = 0$ on $\partial\Omega$, there exists a nondecreasing function $k(t)$ such that ($\mathbf{b_2}$) holds. That is, $k'(t) \geq 0$ and $0 \leq C_k \leq 1$. Then we take $\underline{\xi} = \left(\frac{\underline{b}}{M_0}\right)^{1/(n+1)}$ and the first inequality of (1.6) holds. All proofs are similar to those of Theorem 1.1. Note that, in the last case, although the supersolution $-\varphi(\underline{\xi}_\varepsilon K(d(x)))$ may not be convex, the first inequality of (1.6) still holds since the comparison principle (Lemma 3.1) doesn't require the supersolution being convex.

Boundary asymptotic behavior of solutions of singular elliptic boundary value problem has been greatly studied for the classical Laplace operator. We refer the reader to the papers [8,9,12,22,24] and references therein.

Now let us review several important results related to our problem on Monge–Ampère equations.

If $f(s) = s^{-(n+2)}$ and $b(x) = 1$, the existence of solutions of problem (1.1) was obtained for $n = 2$ in [14] and for $n \geq 2$ in [3].

Later in the paper [13], Lazer and McKenna considered the problem (1.1) with $f(s) = s^{-\gamma}$, $\gamma > 1$ and a positive $b \in C^\infty(\overline{\Omega})$, their results are: there exists a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of (1.1). Moreover, there exist two negative constants c_1 and c_2, such that u satisfies

$$c_1 d(x)^\beta \leq u(x) \leq c_2 d(x)^\beta \quad \text{in } \Omega,$$

where $\beta = \frac{n+1}{n+\gamma}$ and $d(x) = \operatorname{dist}(x, \partial\Omega)$.

Next, Mohammed [16] established the existence and estimates of solutions of problem (1.1) with $f \in C^\infty(0,\infty)$ being positive and decreasing, and $b \in C^\infty(\Omega)$ being positive in Ω. The author showed the following results:

(i_1) Problem (1.1) admits a convex solution if and only if the problem

$$\begin{cases} \det D^2 u = b(x), \quad u < 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

admits a convex solution. The existence of convex solutions of problem (1.8) is well known (see [3, Theorem 3]). Based on these results, we will show that there exists a convex solution of (1.1).

(i_2) Let f satisfy ($\mathbf{f_1}$) and $b \in C^\infty(\overline{\Omega})$ be positive. Suppose that u is a convex solution of (1.1). Then there are positive constants C_1 and C_2 such that

$$C_1\varphi(d(x)) \le -u(x) \le C_2\varphi(d(x)) \quad \text{in } \Omega_\alpha,$$
$$\mid Du(x) \mid \le C_2 \frac{\varphi(d(x))}{d(x)} \quad \text{in } \Omega_\alpha,$$

where φ is the solution of (1.3) and $\Omega_\alpha = \{x \in \Omega : \ d(x) < \alpha\}$ for some $\alpha > 0$.

In this paper, we study the asymptotic behavior of solutions of (1.1) for more general b. Here b may be unbounded or may vanish on $\partial\Omega$. The above results in [13,16] are special cases of Theorem 1.1. In particular, if b is bounded on $\overline{\Omega}$, the assumption ($\mathbf{b_2}$) implies that $K(t) = t$. For $f(s) = s^{-\gamma}(\gamma > 1)$, we deduce from (1.3) that $0 < \widetilde{c}_1 t^{\frac{n+1}{n+\gamma}} \le \varphi(t) \le \widetilde{c}_2 t^{\frac{n+1}{n+\gamma}}$ for two positive constants $\widetilde{c}_1$ and $\widetilde{c}_2$. Our results infer that $\widetilde{C}_1 d(x)^{\frac{n+1}{n+\gamma}} \le u(x) \le \widetilde{C}_2 d(x)^{\frac{n+1}{n+\gamma}}$ for two negative constants $\widetilde{C}_1$ and $\widetilde{C}_2$, which is consistent with Theorem 3.1 in [13]. For f only satisfying ($\mathbf{f_1}$), from our results, we can derive that $\overline{C}_1\varphi(d(x)) \le u(x) \le \overline{C}_2\varphi(d(x))$ with $\overline{C}_1$ and $\overline{C}_2$ being negative constants, which is consistent with Theorem 3.2 in [16].

Cîrstea and Rădulescu [4–6] first introduced the Karamata regular variation theory to study the boundary behavior and uniqueness of solutions of boundary blow-up elliptic problems. Since then, this method was developed by several authors to research boundary behavior of solutions of singular elliptic problem, see [1,7,18,19,22–24] and references therein. In this paper, also by Karamata regular variation theory and constructing upper and lower solutions, we investigate the asymptotic behavior of solutions of problem (1.1) near the boundary.

The present paper is organized as follows. In Section 2 we provide some preliminary results that will be needed later. The proof of Theorem 1.1 is provided in Section 3.

2. Preliminaries

Our approach relies on Karamata regular variation theory established by Karamata in 1930 which is a basic tool in stochastic processes (see [2,11,15,20,21] and the references therein). In this section, we first list some basic facts with respect to Karamata regular variation theory. For the proofs, we refer to [2,21].

Definition 2.1. A positive measurable function f defined on $(0, a)$, for some $a > 0$, is called **regularly varying at zero** with index ρ, written $f \in RVZ_\rho$, if for each $\xi > 0$ and some $\rho \in R$,

$$\lim_{s\to 0^+} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{2.1}$$

In particular, when $\rho = 0$, f is called **slowly varying at zero**.

Clearly, if $f \in RVZ_\rho$, then $L(s) = \frac{f(s)}{s^\rho}$ is slowly varying at zero.

Definition 2.2. A positive measurable function f defined on $(0, a)$, for some $a > 0$, is called **rapidly varying at zero**, if $\lim\limits_{s\to 0^+} f(s) = \infty$, and for each $\rho > 1$,

$$\lim_{s\to 0^+} f(s)s^\rho = \infty; \tag{2.2}$$

if $\lim\limits_{s\to 0^+} f(s) = 0$, and for each $\rho > 1$,

$$\lim_{s\to 0^+} \frac{f(s)}{s^\rho} = 0. \tag{2.3}$$

Proposition 2.3. *(Uniform convergence theorem). If $f \in RVZ_\rho$, then (2.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$.*

Proposition 2.4. *(Representation theorem). A function L is slowly varying at zero if and only if it may be written in the form*

$$L(s) = \psi(s) \exp\left(\int_s^{a_1} \frac{y(\tau)}{\tau} d\tau \right), \quad s \in (0, a_1) \tag{2.4}$$

for some $a_1 \in (0, a)$, where the functions ψ and y are measurable and for $s \to 0^+$, $y(s) \to 0$ and $\psi(s) \to c_0$, with $c_0 > 0$.

We say that

$$\hat{L}(s) = c_0 \exp\left(\int_s^{a_1} \frac{y(\tau)}{\tau} d\tau \right), \quad s \in (0, a_1), \tag{2.5}$$

is **normalized** slowly varying at zero and

$$f(s) = s^\rho \hat{L}(s), \quad s \in (0, a_1), \tag{2.6}$$

is **normalized** regularly varying at zero with index ρ (and write $f \in NRVZ_\rho$).

Proposition 2.5. *A function $f \in RVZ_\rho$ belongs to $NRVZ_\rho$ if and only if*

$$f \in C^1(0, a_1), \quad \textit{for some } a_1 > 0 \quad \textit{and} \quad \lim_{s \to 0^+} \frac{sf'(s)}{f(s)} = \rho.$$

Proposition 2.6. *If functions L, L_1 are slowly varying at zero, then*

(i_1) *L^ρ for every $\rho \in R$, $c_1 L + c_2 L_1$ ($c_1 \geq 0$, $c_2 \geq 0$ with $c_1 + c_2 > 0$), $L \circ L_1$ (if $L_1(s) \to 0$ as $s \to 0^+$), are also slowly varying at zero.*
(i_2) *For every $\rho > 0$ and $s \to 0^+$,*

$$s^\rho L(s) \to 0, \quad s^{-\rho} L(s) \to \infty.$$

(i_3) *For $\rho \in R$ and $s \to 0^+$, $\frac{\ln(L(s))}{\ln s} \to 0$ and $\frac{\ln(s^\rho L(s))}{\ln s} \to \rho$.*

Proposition 2.7. *If $f_1 \in RVZ_{\rho_1}$, $f_2 \in RVZ_{\rho_2}$, then $f_1 f_2 \in RVZ_{\rho_1 + \rho_2}$ and $f_1 \circ f_2 \in RVZ_{\rho_1 \rho_2}$.*

Proposition 2.8. *(Asymptotic behavior) If a function L is slowly varying at zero, then for $a > 0$ and $t \to 0^+$,*

(i_1) $\int_0^t s^\rho L(s)\, ds \cong (1+\rho)^{-1} t^{1+\rho} L(t)$, *for $\rho > -1$;*
(i_2) $\int_t^a s^\rho L(s)\, ds \cong (-1-\rho)^{-1} t^{1+\rho} L(t)$, *for $\rho < -1$.*

Based on the above results, we show the following three lemmas that will be used to prove Theorem 1.1.

Lemma 2.9. *Let k and K be the functions given by $(\mathbf{b_2})$. Then*

(i_1) *If k is non-decreasing, $0 \leq C_k \leq 1$; and if k is non-increasing, $C_k \geq 1$;*
(i_2) $\lim_{t \to 0^+} \frac{K(t)}{k(t)} = 0$ *and* $\lim_{t \to 0^+} \frac{K(t)k'(t)}{k^2(t)} = 1 - C_k$*;*

(i_3) *If* $C_k > 0$, $K \in NRVZ_{1/C_k}$ *and* $k \in NRVZ_{(1-C_k)/C_k}$*;*
(i_4) *If* $C_k = 1$, k *is normalized slowly varying at zero;*
(i_5) *If* $C_k = 0$, k *is rapidly varying at zero.*

Proof. (i_1) Since $K'(t) = k(t)$ and then $\left(\frac{K(t)}{k(t)}\right)' = 1 - \frac{K(t)k'(t)}{k^2(t)}$, ($i_1$) holds.

(i_2) If k is non-increasing, it is clear that $\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0$ since $\lim_{t\to 0^+} K(t) = 0$. If k is non-decreasing, by

$$0 \le \frac{K(t)}{k(t)} \le \frac{k(t)t}{k(t)} = t,$$

we deduce

$$\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0.$$

Since

$$\lim_{t\to 0^+} \frac{K(t)k'(t)}{k^2(t)} = 1 - \lim_{t\to 0^+} \left(\frac{K(t)}{k(t)}\right)',$$

the second equality in (i_2) holds.

(i_3) Since

$$\lim_{t\to 0^+} \frac{tk(t)}{K(t)} = \lim_{t\to 0^+} \frac{t}{\frac{K(t)}{k(t)}} = \lim_{t\to 0^+} \frac{1}{\left(\frac{K(t)}{k(t)}\right)'} = \frac{1}{C_k}$$

and

$$\lim_{t\to 0^+} \frac{tk'(t)}{k(t)} = \lim_{t\to 0^+} \frac{tk(t)}{K(t)} \frac{K(t)k'(t)}{k^2(t)} = \lim_{t\to 0^+} \frac{tk(t)}{K(t)} \lim_{t\to 0^+} \frac{K(t)k'(t)}{k^2(t)} = \frac{1-C_k}{C_k}, \tag{2.7}$$

we have, by Proposition 2.5, $K \in NRVZ_{1/C_k}$ and $k \in NRVZ_{(1-C_k)/C_k}$.

(i_4) It can be obtained immediately from (i_3), (2.5) and (2.6).

(i_5) If $C_k = 0$, it follows from (2.7) that $\lim_{t\to 0^+} \frac{tk'(t)}{k(t)} = +\infty$. That is, for any $M > 1$, there exists $t_M > 0$ small enough such that

$$\frac{k'(t)}{k(t)} > \frac{M+1}{t}, \quad \forall\, 0 < t < t_M.$$

Integrating the above inequality with respect to t,

$$\ln(k(t_M)) - \ln(k(t)) > (M+1)(\ln t_M - \ln t), \quad \forall\, 0 < t < t_M.$$

Therefore,

$$0 < \frac{k(t)}{t^M} < \frac{k(t_M)}{(t_M)^{M+1}} t, \quad \forall\, 0 < t < t_M.$$

Let $t \to 0^+$ and then we obtain that k is rapidly varying at zero by Definition 2.2. $\square$

Lemma 2.10. *Let f satisfy* $(\mathbf{f_1})$ *and* $(\mathbf{f_2})$*, and F be defined in* $(\mathbf{f_2})$*. We have*

(i_1) *$C_f \le 1$;*
(i_2) *If $0 < C_f < 1$, f satisfying* $(\mathbf{f_2})$ *is equivalent to $F \in NRVZ_{(n+1)C_f/(C_f-1)}$, and moreover, it implies $f \in RVZ_{(nC_f+1)/(C_f-1)}$;*
(i_3) *If $C_f = 1$, F is rapidly varying to infinity at zero.*

Proof. (i_1) Since

$$H(t) = ((n+1)F(t))^{1/(n+1)} \quad \text{and} \quad F(t) = \int_t^a f(\tau)d\tau, \quad \forall\, 0 < t < a,$$

we have

$$H(t) \ge H(s) \quad \text{and} \quad \int_0^s \frac{dt}{H(t)} \le \frac{s}{H(s)}, \quad \forall\, 0 < t \le s < a.$$

Hence

$$0 \le H(s)\int_0^s \frac{dt}{H(t)} \le H(s)\frac{s}{H(s)} = s.$$

It follows that

$$0 \le \lim_{s\to 0^+} \frac{H(s)\int_0^s \frac{dt}{H(t)}}{s} = \lim_{s\to 0^+} H'(s)\int_0^s \frac{dt}{H(t)} + 1 = 1 - C_f, \tag{2.8}$$

i.e. $C_f \le 1$.

(i_2) If f satisfies $(\mathbf{f_2})$, we see, by (2.8),

$$\lim_{s\to 0^+} \frac{sH'(s)}{H(s)} = \lim_{s\to 0^+} \frac{s}{H(s)\int_0^s \frac{dt}{H(t)}} H'(s)\int_0^s \frac{dt}{H(t)} = \frac{C_f}{C_f - 1}. \tag{2.9}$$

Besides,

$$\frac{C_f}{C_f - 1} = \lim_{s\to 0^+} \frac{sH'(s)}{H(s)} = \lim_{s\to 0^+} \frac{-sf(s)}{(n+1)F(s)} = \lim_{s\to 0^+} \frac{sF'(s)}{(n+1)F(s)}. \tag{2.10}$$

Therefore, $H \in NRVZ_{C_f/(C_f-1)}$ and $F \in NRVZ_{(n+1)C_f/(C_f-1)}$ by Proposition 2.5. That is, f satisfying $(\mathbf{f_2})$ implies that $F \in NRVZ_{(n+1)C_f/(C_f-1)}$.

Now we assume that $F \in NRVZ_{(n+1)C_f/(C_f-1)}$. Then $H \in NRVZ_{C_f/(C_f-1)}$, i.e.,

$$\lim_{s\to 0^+} \frac{sH'(s)}{H(s)} = \frac{C_f}{C_f - 1}$$

by Proposition 2.5. Therefore, from (2.8),

$$\lim_{s\to 0^+} H'(s)\int_0^s \frac{dt}{H(t)} = \lim_{s\to 0^+} \frac{sH'(s)}{H(s)} \lim_{s\to 0^+} \frac{H(s)}{s}\int_0^s \frac{dt}{H(t)}$$
$$= \frac{C_f(1-C_f)}{C_f-1} = -C_f.$$

That is, f satisfies $(\mathbf{f_2})$.

If $F \in NRVZ_{(n+1)C_f/(C_f-1)}$, we see, by (2.5) and (2.6),

$$F(s) = s^{(n+1)C_f/(C_f-1)}\hat{L}(s),\ \forall\ 0<s<a_1$$

for some $a_1 > 0$ and $\hat{L}$ being normalized slowly varying at zero. Taking the derivative with respect to s,

$$f(s) = s^{(nC_f+1)/(C_f-1)}\left(\frac{(n+1)C_f}{1-C_f} + y(s)\right)\hat{L}(s),\ \forall\ 0<s<a_1,$$

where $y(s) \to 0$ as $s \to 0^+$. It follows from Definition 2.1 and Proposition 2.4 that $f \in RVZ_{(nC_f+1)/(C_f-1)}$.

(i_3) If $C_f = 1$, we deduce, from (2.10),

$$\lim_{s\to 0^+} \frac{sF'(s)}{F(s)} = -\infty.$$

That is, for an arbitrary $M > 1$, there exists $l = l(M) > 0$ small enough such that

$$\frac{F'(s)}{F(s)} < -\frac{M+1}{s},\quad \forall\ 0<s<l.$$

Integrating the above inequality with respect to s, we obtain

$$\ln F(l) - \ln F(s) < -(M+1)(\ln l - \ln s),\quad \forall\ 0<s<l.$$

Consequently,

$$\frac{F(l)}{F(s)} < \left(\frac{l}{s}\right)^{-(M+1)},\quad \forall\ 0<s<l,$$

i.e.,

$$F(s)s^M > \frac{F(l)l^{M+1}}{s},\quad \forall\ 0<s<l.$$

Let $s \to 0^+$ and then we see that F is rapidly varying to infinity at zero by Definition 2.2. $\square$

Lemma 2.11. *Let f satisfy $(\mathbf{f_1})$ and $(\mathbf{f_2})$. Recall that φ satisfies*

$$\int_0^{\varphi(t)} ((n+1)F(\tau))^{-1/(n+1)}d\tau = t,\ \forall\ 0<t<\alpha \tag{2.11}$$

with $\varphi(\alpha) = a$. We have

(i_1) $\varphi(0) = 0,\ \varphi(t) > 0,\ \varphi'(t) = ((n+1)F(\varphi(t)))^{1/(n+1)}$,
and $\varphi''(t) = -\left((n+1)F(\varphi(t))\right)^{(1-n)/(n+1)} f(\varphi(t))$;

(i_2) $\lim\limits_{t\to 0^+} \frac{\varphi'(t)}{t\varphi''(t)} = -\frac{1}{C_f}$;
(i_3) $\varphi \in NRVZ_{1-C_f}$;
(i_4) $\varphi' \in NRVZ_{-C_f}$;
(i_5) *If (1.5) holds,* $\lim\limits_{t\to 0^+} \frac{t}{\varphi(\xi K(t))} = 0$ *for* $\xi \in [c_1, c_2]$ *with* $0 < c_1 < c_2$.

Proof. (i_1) By (2.11), it is easy to see that (i_1) holds.

(i_2) Since

$$H(t) = ((n+1)F(t))^{1/(n+1)} \text{ and } H'(t) = -((n+1)F(t))^{-n/(n+1)} f(t),$$

we have, by (2.11),

$$\begin{aligned}\lim_{t\to 0^+} \frac{\varphi'(t)}{t\varphi''(t)} &= \lim_{t\to 0^+} \frac{1}{-t((n+1)F(\varphi(t)))^{-n/(n+1)} f(\varphi(t))} \\ &= \lim_{t\to 0^+} \frac{1}{-\int\limits_0^{\varphi(t)} ((n+1)F(\tau))^{-1/(n+1)} d\tau ((n+1)F(\varphi(t)))^{-n/(n+1)} f(\varphi(t))} \\ &= \lim_{s\to 0^+} \frac{1}{H'(s)\int\limits_0^s (H(\tau))^{-1} d\tau} = -\frac{1}{C_f}.\end{aligned}$$

(i_3) From (i_1), (2.8) and (2.11), we derive

$$\lim_{t\to 0^+} \frac{t\varphi'(t)}{\varphi(t)} = \lim_{t\to 0^+} \frac{tH(\varphi(t))}{\varphi(t)} = \lim_{s\to 0^+} \frac{H(s)}{s} \int\limits_0^s (H(\tau))^{-1} d\tau = 1 - C_f.$$

That is, $\varphi \in NRVZ_{1-C_f}$.

(i_4) It can be directly obtained from (i_2).

(i_5) If (1.5) holds, by Lemma 2.10(i_1), we have $C_k > 0$. Lemma 2.9(i_3) and (i_3) imply that

$$K \in NRVZ_{\frac{1}{C_k}}, \quad \varphi \in NRVZ_{1-C_f}.$$

From Proposition 2.7,

$$\varphi(K(t)) \in NRVZ_\rho,$$

where $\rho = \frac{1-C_f}{C_k} < 1$ and then

$$\frac{t}{\varphi(\xi K(t))} \in NRVZ_{1-\rho}$$

with $1-\rho > 0$. That is, there exists $\hat{L}(t)$ being normalized slowly varying at zero, such that

$$\frac{t}{\varphi(\xi K(t))} = t^{1-\rho}\hat{L}(t), \ 0 < t < a_1$$

for some small $a_1 > 0$. Therefore, it follows from Proposition 2.6(i_2) that

$$\lim_{t\to 0^+} \frac{t}{\varphi(\xi K(t))} = 0. \quad \square$$

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We first present the following Lemma and refer to Lemma 2.1 in [13] for its proof.

Lemma 3.1. *(The comparison principle) Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$, and $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$. Suppose that $g(x, \eta)$ is defined for $x \in \Omega$ and η in some interval containing the ranges of u_1 and u_2. If*

(i$_1$) *$g(x, \eta)$ is increasing in η for all $x \in \Omega$;*
(i$_2$) *the matrix $D^2 u_1$ is positive definite in Ω;*
(i$_3$) *$\det D^2 u_1 \geq g(x, u_1)$ in Ω;*
(i$_4$) *$\det D^2 u_2 \leq g(x, u_2)$ in Ω;*
(i$_5$) *$u_2 \geq u_1$ on $\partial\Omega$,*

then we have

$$u_2 \geq u_1 \quad \textit{in } \Omega.$$

Before giving the proof of Theorem 1.1, we recall some results on the distance function. Let $d(x) = \operatorname{dist}(x, \partial\Omega) = \inf\limits_{y \in \partial\Omega} |x - y|$. For any $\delta > 0$, we define

$$\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}.$$

If Ω is bounded and $\partial\Omega \in C^m$ for $m \geq 2$, by Lemma 14.16 in [10], there exists $\delta_1 > 0$ such that

$$d \in C^m(\Omega_{\delta_1}).$$

Let $\overline{x} \in \partial\Omega$, satisfying $\operatorname{dist}(x, \partial\Omega) = |x - \overline{x}|$, be the projection of the point $x \in \Omega_{\delta_1}$ to $\partial\Omega$, and $\kappa_i(\overline{x})(i = 1, \cdots, n-1)$ be the principal curvatures of $\partial\Omega$ at $\overline{x}$, then, in terms of a principal coordinate system at $\overline{x}$, we have, by Lemma 14.17 in [10],

$$\begin{cases} Dd(x) = (0, 0, \cdots, 1), \\ D^2 d(x) = \operatorname{diag}\left[\frac{-\kappa_1(\bar{x})}{1 - d(x)\kappa_1(\bar{x})}, \cdots, \frac{-\kappa_{n-1}(\bar{x})}{1 - d(x)\kappa_{n-1}(\bar{x})}, 0\right]. \end{cases} \tag{3.1}$$

Proof of Theorem 1.1. The existence of solutions of (1.1) can be obtained by the following way. We consider two cases: $C_k > 0$ and $C_k = 0$, which is defined by $(\mathbf{b_2})$.

If $C_k > 0$, we see from Lemma 2.9(i_3) that

$$k \in NRVZ_{(1-C_k)/C_k}.$$

By (2.5) and (2.6),

$$k(t) = t^{\frac{1}{C_k} - 1} \hat{L}(t), \ \forall \ 0 < t < a_1$$

for some $a_1 > 0$ and $\hat{L}$ being normalized slowly varying at zero. Furthermore, by Proposition 2.6(i_2), we see that for $0 < \widetilde{\alpha} < \frac{1}{C_k}$, $\hat{L}(t) t^{\widetilde{\alpha}} \to 0$ as $t \to 0^+$. Therefore, by $(\mathbf{b_2})$, there exists $\widetilde{\delta} > 0$ sufficiently small such that

$$b(x) \leq 2\bar{b}d(x)^{\left(\frac{1}{C_k}-1\right)(n+1)}\left(\hat{L}(d(x))\right)^{n+1} \leq 2\bar{b}d(x)^{\left(\frac{1}{C_k}-\widetilde{\alpha}-1\right)(n+1)} \text{ in } \Omega_{\widetilde{\delta}}.$$

It follows that

$$b(x) \leq \widetilde{C}d(x)^{(\frac{1}{C_k}-\widetilde{\alpha}-1)(n+1)} \text{ in } \Omega, \tag{3.2}$$

for some positive constant $\widetilde{C}$ since $b(x) \in C^3(\Omega)$. By Theorem 3 in [3], problem (1.8) admits a convex solution. Therefore problem (1.1) admits a convex solution u by Theorem 2.1 in [16].

If $C_k = 0$, by Lemma 2.9(i_5), k is rapidly varying to zero. Definition 2.2 implies that for any $\rho > 1$, $\frac{k(t)}{t^\rho} \to 0$ as $t \to 0^+$. Hence (3.2) holds with $\frac{1}{C_k} - \widetilde{\alpha} - 1$ being replaced by any $\rho > 1$. That is, problem (1.1) also admits a convex solution.

The uniqueness of solutions of (1.1) can be derived immediately by Lemma 3.1.

Now we prove the estimates (1.6).

For any $\varepsilon > 0$, let

$$\underline{\xi}_\varepsilon = \left(\frac{(\underline{b}-\varepsilon)(1-\varepsilon)-\varepsilon}{M_0(1-C_f^{-1}(1-C_k))}\right)^{1/(n+1)} \tag{3.3}$$

and

$$\overline{\xi}_\varepsilon = \left(\frac{(\overline{b}+\varepsilon)(1+\varepsilon)+\varepsilon}{m_0(1-C_f^{-1}(1-C_k))}\right)^{1/(n+1)}, \tag{3.4}$$

where $\overline{b}$, $\underline{b}$ and C_k, M_0 and m_0, and C_f are given by ($\mathbf{b_2}$), (1.4) and ($\mathbf{f_2}$) respectively. Define

$$\overline{u}_\varepsilon(x) = -\varphi(\underline{\xi}_\varepsilon K(d(x))) \text{ and } \underline{u}_\varepsilon(x) = -\varphi(\overline{\xi}_\varepsilon K(d(x))) \text{ in } \Omega. \tag{3.5}$$

Choose $v \in C^2(\overline{\Omega})$ such that

$$D^2 v > 0 \text{ on } \overline{\Omega}, \quad v = 0 \text{ on } \partial\Omega. \tag{3.6}$$

By $\Delta v > 0$ on $\overline{\Omega}$ and the Hopf lemma, there exist negative constants c_1 and c_2 such that

$$c_1 d(x) \leq v(x) \leq c_2 d(x) \text{ on } \overline{\Omega}. \tag{3.7}$$

We claim

$$u + Mv \leq \overline{u}_\varepsilon \text{ in } \Omega_{\delta_\varepsilon} \tag{3.8}$$

and

$$\underline{u}_\varepsilon + Mv \leq u \text{ in } \Omega_{\delta_\varepsilon}, \tag{3.9}$$

where M and δ_ε are positive constants (depending on ε) which will be determined later.

First, we prove that $\overline{u}_\varepsilon$ is a supersolution and $\underline{u}_\varepsilon$ is a subsolution of (1.1) in $\Omega_{\delta_\varepsilon}$. By Lemma 2.9($i_2$),

$$\lim_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{K(d(x))k'(d(x))}{k^2(d(x))} = 1 - C_k. \tag{3.10}$$

Since

$$K \in C^2(0,\delta_0) \cap C[0,\delta_0), \ K(0) = 0$$

and

$$K(d(x)) = \int_0^{\varphi(K(d(x)))} ((n+1)F(\tau))^{-1/(n+1)} d\tau,$$

we see, by Lemma 2.11(i_1) and Lemma 2.11(i_2),

$$\lim_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{((n+1)\,F\,(\varphi\,(K\,(d\,(x)))))^{n/(n+1)}}{K\,(d\,(x))\,f\,(\varphi\,(K\,(d\,(x))))} = \frac{1}{C_f}. \tag{3.11}$$

Since, by (3.3),

$$\underline{\xi}_\varepsilon^{n+1} M_0(1 - C_f^{-1}(1 - C_k)) - (\underline{b} - \varepsilon)(1 - \varepsilon) = -\varepsilon, \tag{3.12}$$

we deduce from (3.10)–(3.12) that there is $\delta_\varepsilon > 0$ sufficiently small such that for $x \in \Omega_{\delta_\varepsilon}$,

$$\begin{aligned}&\underline{\xi}_\varepsilon^{n+1} M_0 \left(1 - \frac{K(d(x))k'(d(x))}{k^2(d(x))} \frac{\left((n+1)F\left(\varphi\left(\underline{\xi}_\varepsilon K(d(x))\right)\right)\right)^{n/(n+1)}}{\underline{\xi}_\varepsilon K(d(x)) f\left(\varphi\left(\underline{\xi}_\varepsilon K(d(x))\right)\right)}\right) \\ &-(\underline{b} - \varepsilon)(1 - \varepsilon) < 0.\end{aligned} \tag{3.13}$$

Similarly, for $x \in \Omega_{\delta_\varepsilon}$,

$$\begin{aligned}&\overline{\xi}_\varepsilon^{n+1} m_0 \left(1 - \frac{K(d(x))k'(d(x))}{k^2(d(x))} \frac{((n+1)F(\varphi(\overline{\xi}_\varepsilon K(d(x)))))^{n/(n+1)}}{\overline{\xi}_\varepsilon K(d(x)) f(\varphi(\overline{\xi}_\varepsilon K(d(x))))}\right) \\ &-(\overline{b} + \varepsilon)(1 + \varepsilon) > 0.\end{aligned} \tag{3.14}$$

Since

$$\lim_{\substack{x\in\Omega \\ d(x)\to 0}} \prod_{i=1}^{n-1} (1 - d\,(x)\,\kappa_i\,(\overline{x})) = 1,$$

where $\overline{x}$ is the point on $\partial\Omega$ such that $d(x) = |x - \overline{x}|$, we also have, for $x \in \Omega_{\delta_\varepsilon}$,

$$1 - \varepsilon < \prod_{i=1}^{n-1} (1 - d\,(x)\,\kappa_i\,(\overline{x})) < 1 + \varepsilon. \tag{3.15}$$

Moreover, it follows from ($\mathbf{b_2}$) that for $x \in \Omega_{\delta_\varepsilon}$,

$$(\underline{b} - \varepsilon)k^{n+1}(d(x)) < b(x) < (\overline{b} + \varepsilon)k^{n+1}(d(x)). \tag{3.16}$$

In view of (3.5), we see, obviously,

$$\overline{u}_\varepsilon(x) < 0 \ \text{ in } \Omega_{\delta_\varepsilon}, \ \ \overline{u}_\varepsilon(x) = 0 \ \text{ on } \partial\Omega.$$

By direct computation,

$$\begin{aligned}
&(\overline{u}_\varepsilon(x))_{ij} = (-\varphi(\underline{\xi}_\varepsilon K(d(x))))_{ij} \\
&= -\underline{\xi}_\varepsilon\left[\underline{\xi}_\varepsilon \varphi''\left(\underline{\xi}_\varepsilon K(d(x))\right)k^2(d(x)) + \varphi'\left(\underline{\xi}_\varepsilon K(d(x))\right)k'(d(x))\right]d_i d_j \\
&\quad - \underline{\xi}_\varepsilon\, \varphi'\left(\underline{\xi}_\varepsilon K(d(x))\right)k(d(x))\, d_{ij} \\
&= \underline{\xi}_\varepsilon^2 k^2(d(x)) f(\varphi(\underline{\xi}_\varepsilon K(d(x))))\Big((n+1)F\left(\varphi\left(\underline{\xi}_\varepsilon K(d(x))\right)\right)\Big)^{(1-n)/(n+1)} \\
&\quad \times\left[1 - \tfrac{K(d(x))k'(d(x))}{k^2(d(x))}\tfrac{\left((n+1)F\left(\varphi\left(\underline{\xi}_\varepsilon K(d(x))\right)\right)\right)^{n/(n+1)}}{\underline{\xi}_\varepsilon K(d(x)) f\left(\varphi\left(\underline{\xi}_\varepsilon K(d(x))\right)\right)}\right]d_i d_j \\
&\quad - \underline{\xi}_\varepsilon k(d(x))\Big((n+1)F\left(\varphi\left(\underline{\xi}_\varepsilon K(d(x))\right)\right)\Big)^{1/(n+1)} d_{ij}.
\end{aligned}$$

The last equality is obtained from Lemma 2.11(i_1). Using (3.10), (3.11) and (1.5),

$$\begin{aligned}
&1 - \lim_{\substack{x\in\Omega \\ d(x)\to 0}} \tfrac{K(d(x))k'(d(x))}{k^2(d(x))}\tfrac{\left((n+1)F\left(\varphi\left(\underline{\xi}_\varepsilon K(d(x))\right)\right)\right)^{n/(n+1)}}{\underline{\xi}_\varepsilon K(d(x)) f\left(\varphi\left(\underline{\xi}_\varepsilon K(d(x))\right)\right)} \\
&= 1 - \tfrac{1-C_k}{C_f} > 0.
\end{aligned}$$

Therefore, for δ_ε sufficiently small,

$$1 - \frac{K(d(x))k'(d(x))}{k^2(d(x))}\frac{\left((n+1)F\left(\varphi\left(\underline{\xi}_\varepsilon K(d(x))\right)\right)\right)^{n/(n+1)}}{\underline{\xi}_\varepsilon K(d(x)) f\left(\varphi\left(\underline{\xi}_\varepsilon K(d(x))\right)\right)} \geq 0 \ \text{ in } \Omega_{\delta_\varepsilon}.$$

Since the matrix $(d_i d_j)$ is nonnegative definite and the matrix (d_{ij}) is nonpositive definite, we have,

$$D^2\overline{u}_\varepsilon \geq 0 \ \text{ in } \Omega_{\delta_\varepsilon}.$$

Therefore, by (3.1), (3.13) and (3.15), we derive that for $x \in \Omega_{\delta_\varepsilon}$,

$$\begin{aligned}
&\det D^2\overline{u}_\varepsilon(x) - (\underline{b}-\varepsilon)k^{n+1}(d(x)) f(-\overline{u}_\varepsilon(x)) \\
&= (-1)^n\left(\underline{\xi}_\varepsilon\varphi'\left(\underline{\xi}_\varepsilon K(d(x))\right)k(d(x))\right)^{n-1}\prod_{i=1}^{n-1}\tfrac{-\kappa_i(\overline{x})}{1-d(x)\kappa_i(\overline{x})} \\
&\quad\times\left[\underline{\xi}_\varepsilon^2\varphi''\left(\underline{\xi}_\varepsilon K(d(x))\right)k^2(d(x)) + \underline{\xi}_\varepsilon\varphi'\left(\underline{\xi}_\varepsilon K(d(x))\right)k'(d(x))\right] \\
&\quad - (\underline{b}-\varepsilon)k^{n+1}(d(x)) f\left(\varphi\left(\underline{\xi}_\varepsilon K(d(x))\right)\right) \\
&\leq (1-\varepsilon)^{-1}k^{n+1}(d(x)) f\left(\varphi\left(\underline{\xi}_\varepsilon K(d(x))\right)\right) \\
&\quad\times\left[\underline{\xi}_\varepsilon^{n+1}M_0\left(1 - \tfrac{K(d(x))k'(d(x))}{k^2(d(x))}\tfrac{\left((n+1)F\left(\varphi\left(\underline{\xi}_\varepsilon K(d(x))\right)\right)\right)^{n/(n+1)}}{\underline{\xi}_\varepsilon K(d(x)) f\left(\varphi\left(\underline{\xi}_\varepsilon K(d(x))\right)\right)}\right) - (\underline{b}-\varepsilon)(1-\varepsilon)\right] \\
&\leq 0,
\end{aligned}$$

i.e.,

$$\det D^2\overline{u}_\varepsilon(x) \leq (\underline{b}-\varepsilon)k^{n+1}(d(x)) f(-\overline{u}_\varepsilon(x)) \leq b(x) f(-\overline{u}_\varepsilon(x)) \quad \text{in } \Omega_{\delta_\varepsilon}. \tag{3.17}$$

Analogously,

$$\det D^2\underline{u}_\varepsilon(x) \geq (\overline{b}+\varepsilon)k^{n+1}(d(x)) f(-\underline{u}_\varepsilon(x)) \geq b(x) f(-\underline{u}_\varepsilon(x)) \quad \text{in } \Omega_{\delta_\varepsilon}. \tag{3.18}$$

Next, we choose a sufficiently large constant $M > 0$, such that

$$u + Mv \le \overline{u}_\varepsilon \quad \text{on } \Gamma = \{x \in \Omega : d(x) = \delta_\varepsilon\}.$$

Since

$$u = v = \overline{u}_\varepsilon = 0 \quad \text{on } \partial\Omega$$

and

$$\det D^2(u + Mv) \ge \det D^2 u = b(x) f(-u) \ge b(x) f(-(u + Mv)) \quad \text{in } \Omega,$$

we deduce, from (3.17) and Lemma 3.1,

$$u + Mv \le \overline{u}_\varepsilon \quad \text{in } \Omega_{\delta_\varepsilon},$$

i.e., (3.8) holds. In the same way, we show (3.9) holds.

Finally, (3.5), (3.7) and (3.8) imply that

$$\frac{u}{-\varphi(\underline{\xi}_\varepsilon K(d(x)))} \ge 1 - \frac{c_1 d(x)}{-\varphi(\underline{\xi}_\varepsilon K(d(x)))} \quad \text{in } \Omega_{\delta_\varepsilon}.$$

Since, by Lemma 2.11 (i_5),

$$\lim_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{d(x)}{\varphi(\underline{\xi}_\varepsilon K(d(x)))} = 0,$$

we have

$$1 \le \liminf_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{-\varphi(\underline{\xi}_\varepsilon K(d(x)))}.$$

Let $\varepsilon \to 0$ and then we conclude

$$1 \le \liminf_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{-\varphi(\underline{\xi} K(d(x)))}.$$

Similarly, we obtain

$$\limsup_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{-\varphi(\overline{\xi} K(d(x)))} \le 1.$$

This completes the proof of (1.6). □

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